ENDOMORPHISMS OF THE SPACE OF HIGHER-ORDER ENTIRE FUNCTIONS AND INFINITE-ORDER DIFFERENTIAL OPERATORS

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Abstract. In this paper, we consider the space of entire functions for a given proximate order and a semi-norm, and show that a continuous endomorphism of this space is represented naturally as a partial differential operator of infinite order with entire function coefficients. Then, in the case of higher order, we study the relationship between the continuous endomorphisms and the linear partial differential operators of infinite order by means of the growth of the symbol of the operators.

1. Introduction

In [7], Martineau studied systematically the convolution operators with analytic functional kernel operating on the space of entire functions of type ρ . In addition, more generally, Lelong and Gruman [5] defined the spaces of entire functions with proximate order $\rho(r)$ and considered the operation of some convolution operators on such spaces. These convolution operators may, of course, be considered as linear partial differential operators of infinite order with constant coefficients, in some sense. In our previous papers [1–4], we characterized the continuous endomorphisms of the sheaves or the stalks of holomorphic functions as the partial differential operators of infinite order with variable coefficients in each sense.

In the present paper, we consider the spaces of entire functions defined in [5] and study the continuous endomorphisms of such spaces and their connection with

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linear partial differential operators of infinite order with entire function coefficients (Theorem 4.1)[†].

2. Notation and recall

In this paper, for multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}^n$ with $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ and a point $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, we will employ the usual notation:

$$|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n, \quad \alpha! := \alpha_1! \alpha_2! \dots \alpha_n!, \quad \binom{\alpha}{\beta} := \frac{\alpha!}{(\alpha - \beta)! \beta!},$$

$$|z| := \sqrt{|z_1|^2 + |z_2|^2 + \dots + |z_n|^2}, \quad D_z^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \dots \partial z_n^{\alpha_n}}.$$

Furthermore, we will also put $|z| := (|z_1|, |z_2|, \dots, |z_n|)$.

In this section, we recall several notions and properties concerning the spaces of entire functions with growth order. Principally, we will follow [5].

For a real-valued continuous function $w(z)$ defined on $\mathbb{C}^n$, we define the space of entire functions with growth with respect to w :

$$B_w := \{f \in \mathcal{O}(\mathbb{C}^n) \mid \|f\|_w := \sup |f(z)e^{-w(z)}| < +\infty\}.$$

This is a Banach space with the norm $\|\cdot\|_w$. In this paper, we will make use of the notion of proximate order introduced by Valiron [8]: a differentiable function $\rho(r) \geq 0$ defined for $r \geq 0$ is said to be a *proximate order* if it satisfies:

- (i) $\lim_{r \rightarrow +\infty} \rho(r) = \rho$;
- (ii) $\lim_{r \rightarrow +\infty} \rho'(r)r \ln r = 0$.

It is well known that in the case where $\rho > 0$, there exists $r_0 \gg 1$ such that for $r > r_0$, the function $r^{\rho(r)}$ is strictly increasing and if $\rho > 1$, so is $r^{\rho(r)-1}$ (see, e.g., [6, Section 12]).

In this article, we will impose the following additional condition on $\rho(r)$:

- (iii) there exist $C, r_1 > 1$ such that we have $\rho \leq \rho(r) \leq \rho(\ln(Cr)/\ln r)$ for any $r > r_1$.

[†]In the original version, as the referee pointed out, Theorem 4.1 was not correct at all and we revised it as the present form (Condition 4.2). He also showed an illustrative example. We would like to express our sincere gratitude to the referee who read the original version very attentively and also indicated several inaccurate grammatical usages.

For a semi-norm $p(z)$ on $\mathbb{C}^n$, we put $p_m(z) := p(z) + |z|/m$ ($m = 1, 2, 3, \dots$). Then $p_m(z)$ is a norm on $\mathbb{C}^n$. So, there exists $C_m > 0$ such that we have for any z ,

$$C_m^{-1} p_m(z) \leq |z| \leq C_m p_m(z). \quad (2.1)$$

We can take $C_m = m$ for large m . We remark that for any m , there exists $\delta_m > 0$ such that

$$p_m(z) \geq (1 + \delta_m) p_{m+1}(z) \quad (2.2)$$

for any z . (Take, e.g., $\delta_m := C_{m+1}^{-1}/m(m+1)$.)

Now as defined in [5], setting $w_m(z) := p_m(z)^{\rho(p_m(z))}$, we define the space of entire functions with respect to the proximate order $\rho(r)$

$$E_p^{\rho(r)} := \varprojlim_m B_{w_m}$$

which is an (FS)-space if $\rho > 1$.

Remark. The above definition of the space $E_p^{\rho(r)}$ seems slightly different from the definition in [5], but they coincide in the case where $\rho > 1$ (see, for example, Lemma 4.4 below).

3. Endomorphisms of the space $E_p^{\rho(r)}$

LEMMA 3.1. *For any $c \in \mathbb{C}^n$, the translation $\tau_c : z \mapsto z + c$ gives an automorphism of the (FS)-space $E_p^{\rho(r)}$.*

Proof. We assume that z satisfies $p_{m+1}(z) > r_0$. We recall that $r^{\rho(r)}$ is strictly increasing for $r > r_0$. Set $v_m(z) := w_m(z + c)$. We know, $p_{m+1}(z)$ being a norm, that

$$p_{m+1}(z + c) \leq p_{m+1}(z) + p_{m+1}(c).$$

Then we have

$$v_{m+1}(z) = w_{m+1}(z + c) \leq [p_{m+1}(z) + p_{m+1}(c)]^{\rho(p_{m+1}(z) + p_{m+1}(c))}.$$

Taking δ_m as in (2.2), for z with $p_{m+1}(c) < \delta_m p_{m+1}(z)$, we have

$$\begin{aligned} v_{m+1}(z) &\leq [p_{m+1}(z) + p_{m+1}(c)]^{\rho(p_{m+1}(z) + p_{m+1}(c))} \\ &< [(1 + \delta_m) p_{m+1}(z)]^{\rho((1 + \delta_m) p_{m+1}(z))} \leq p_m(z)^{\rho(p_m(z))} = w_m(z). \end{aligned}$$

Therefore, there exists $B = B_m > 0$ such that we have

$$\|f\|_{w_m} < B \|f\|_{v_{m+1}}$$

for any $f \in E_p^{\rho(r)}$, which proves the lemma. $\square$

By this lemma and [5, Lemma 9.2], we have the following.

LEMMA 3.2. *Assume that $\rho > 0$. For any function $f(z) \in E_p^{\rho(r)}$, any point $z' \in \mathbb{C}^n$ and any $v \in \mathbb{N}$, we set*

$$F_v(z, z') := \sum_{|\alpha|=v} \frac{(z - z')^\alpha}{\alpha!} D_z^\alpha f(z').$$

Then the expansion

$$f(z) = \sum_{v=0}^{\infty} F_v(z, z')$$

converges in the topology of $E_p^{\rho(r)}$.

Next, we will consider the endomorphisms of the space $E_p^{\rho(r)}$. Let $P : E_p^{\rho(r)} \rightarrow E_p^{\rho(r)}$ be a continuous linear operator. For any α , we set

$$b_\alpha(z, z') := P_z \left(\frac{(z - z')^\alpha}{\alpha!} \right) \quad \text{and} \quad a_\alpha(z) := b_\alpha(z, z).$$

Then for any $f(z) \in E_p^{\rho(r)}$, taking account of the Taylor expansion, we have

$$Pf(z) = \sum_{v=0}^{\infty} \left[\sum_{|\alpha|=v} a_\alpha(z) D_z^\alpha f(z) \right],$$

where the sum is convergent in the topology of $E_p^{\rho(r)}$ and so the endomorphism P is of the form of differential operator of infinite order:

$$P = P(z, D_z) = \sum_{v=0}^{\infty} \sum_{|\alpha|=v} a_\alpha(z) D_z^\alpha. \quad (3.1)$$

PROPOSITION 3.3. *In the above expression (3.1), every $a_\alpha(z)$ is uniquely determined by P and belongs to $E_p^{\rho(r)}$.*

Proof. For any α , we have

$$Pz^\alpha = \sum_{\beta \leq \alpha} a_\beta(z) \frac{\alpha!}{(\alpha - \beta)!} z^{\alpha - \beta} = a_\alpha(z) + \sum_{\beta < \alpha} a_\beta(z) \frac{\alpha!}{(\alpha - \beta)!} z^{\alpha - \beta}.$$

Thus, remarking $a_0(z) = P1 \in E_p^{\rho(r)}$, we know, by induction and by using Lemma 4.4 below, that every $a_\alpha(z) \in E_p^{\rho(r)}$ is uniquely determined by P . $\square$

To study the differential operator in detail, we need several lemmas.

LEMMA 3.4. Let $f(z) \in E_p^{\rho(r)}$. Then for any α , any m and any $R = (R_1, R_2, \dots, R_n)$ with $R_1, R_2, \dots, R_n > 0$, we have the estimate

$$|D_z^\alpha f(z)| \leq \|f\|_{w_m} \frac{\alpha!}{R^\alpha} \max_{|y|=R} e^{w_m(z+y)}. \quad (3.2)$$

Proof. This is obvious from Cauchy's integral formula. $\square$

LEMMA 3.5. Assume that $\rho > 0$. Then there exists $A = A_m > 0$ such that for any α and σ with $0 < \sigma \leq 1$, we have

$$\|z^\alpha\|_{\sigma w_m} \leq \sigma^{-|\alpha|/\rho} A^{|\alpha|} (\alpha!)^{1/\rho}.$$

Proof. Setting $r = p_m(z)$ and $v = |\alpha|$, we have

$$\begin{aligned} \|z^\alpha\|_{\sigma w_m} &= \sup_z |z^\alpha e^{-\sigma w_m(z)}| \leq \sup_r (C_m r)^v \exp(-\sigma r^{\rho(r)}) \\ &= \sup_{r \geq 0} \exp(v(\ln C_m + \ln r) - \sigma r^{\rho(r)}). \end{aligned}$$

By condition (iii), for $r > r_1$, we have

$$y(r) := v(\ln C_m + \ln r) - \sigma r^{\rho} \geq v(\ln C_m + \ln r) - \sigma r^{\rho(r)}$$

and $y(r)$ attains its maximum at r with

$$y'(r) = v \frac{1}{r} - \sigma \rho r^{\rho-1} = \frac{\sigma \rho}{r} \left[\frac{v}{\sigma \rho} - r^{\rho} \right] = 0, \quad \text{that is } r = \left(\frac{v}{\sigma \rho} \right)^{1/\rho}.$$

Thus, we have

$$\begin{aligned} y\left(\left(\frac{v}{\sigma \rho}\right)^{1/\rho}\right) &= v\left(\ln C_m + \frac{1}{\rho} \ln \frac{v}{\sigma \rho}\right) - \frac{v}{\rho} \\ &= v\left(\ln C_m + \frac{1}{\rho} \ln \frac{v}{\sigma \rho} - \frac{1}{\rho}\right) \\ &\geq v(\ln C_m + \ln r) - \sigma r^{\rho(r)} \quad (\forall r > r_1) \end{aligned}$$

and so

$$\begin{aligned} \|z^\alpha\|_{\sigma w_m} &= \max \left[\sup_{r \leq r_1} (C_m r)^{|\alpha|} \exp(-\sigma r^{\rho(r)}), \sup_{r > r_1} \exp(|\alpha|(\ln C_m + \ln r) - \sigma r^{\rho(r)}) \right] \\ &\leq \max[(C_m r_1)^{|\alpha|}, e^{|\alpha|(\ln C_m + \rho^{-1} \ln(|\alpha|/\sigma \rho) - \rho^{-1})}] \\ &= \max \left[(C_m r_1)^{|\alpha|}, \left(\frac{C_m}{(\sigma \rho e)^{1/\rho}} \right)^{|\alpha|} (|\alpha|^{|\alpha|})^{1/\rho} \right]. \end{aligned}$$

Then by using Stirling's formula and the estimation $|\alpha|! \leq n^{|\alpha|} \alpha!$, we obtain the lemma. $\square$

LEMMA 3.6. *Assume that $\rho > 1$. Then, there exists $r_2 > 0$ such that for $a, b, c > 0$ with $b \geq ac$ and $a > r_2$, we have*

$$(a + b)^{\rho(a+b)} > (1 + c)a^{\rho(a)}.$$

Proof. There exists $t \in]a, a + b[$ such that $(a + b)^{\rho(a+b)} - a^{\rho(a)} = b\{t\rho'(t) \ln t + \rho(t)\}t^{\rho(t)-1}$. By (i), (ii) and the fact that $r^{\rho(r)-1}$ is strictly increasing for $r > r_0$, there exists $r_2 > r_0$ so that, for $a > r_2$, we have

$$b\{t\rho'(t) \ln t + \rho(t)\}t^{\rho(t)-1} > ca^{\rho(a)}. \quad \square$$

4. The characterization of endomorphisms

Now we will prove the main theorem of this article:

THEOREM 4.1. *Assume that $\rho > 1$.*

(I) *Let a family of functions $(a_\alpha(z))_\alpha$ satisfy the following condition.*

CONDITION 4.2. *$(a_\alpha(z)) \subset E_p^{\rho(r)}$ and for any m and any $\varepsilon > 0$, the function*

$$S(\zeta) := \sum_{\alpha} \|a_\alpha\|_{\varepsilon w_m} (\alpha!)^{1-1/\rho} \zeta^\alpha$$

is holomorphic for any $\zeta \in \mathbb{C}^n$.

Then

$$P := P(z, D_z) = \sum_{|\alpha|=0}^{\infty} a_\alpha(z) D_z^\alpha$$

gives a continuous endomorphism of the space $E_p^{\rho(r)}$.

(II) *Conversely, let $P : E_p^{\rho(r)} \rightarrow E_p^{\rho(r)}$ be an endomorphism. Then P is represented uniquely as the linear partial differential operator of infinite order*

$$P = P(z, D_z) = \sum_{|\alpha|=0}^{\infty} a_\alpha(z) D_z^\alpha$$

satisfying the following condition.

CONDITION 4.3. $(a_\alpha(z)) \subset E_p^{\rho(r)}$ and for any m , the function

$$S(\zeta) := \sum_{\alpha} \|a_\alpha\|_{w_m} (\alpha!)^{1-1/\rho} \zeta^\alpha$$

is holomorphic in a neighbourhood of $\zeta = 0$.

Proof. (I). Setting $R = R(r) := (r, r, \dots, r)$ with $r > 0$ in (3.2), we have

$$|D_z^\alpha f(z)| \leq \|f\|_{w_{m+2}} \frac{\alpha!}{r^{|\alpha|}} \max_{|y|=R(r)} e^{w_{m+2}(z+y)}. \quad (4.1)$$

Then by the fact that there exists $r_0 > 0$ so that $r^{\rho(r)}$ is strictly increasing for $r > r_0$, we have the following.

LEMMA 4.4. For $|z| > C_{m+1}r_2 =: r_3$, we have $w_m(z) \geq (1 + \delta_m)w_{m+1}(z)$, where C_{m+1} has been defined in (2.1).

Proof. Let $a = p_{m+1}(z)$, $b = |z|/m(m+1)$. Then $p_m(z) = a + b$ and $b \geq \delta_m a$. Thus, for $a > r_2$,

$$w_m(z) = (a + b)^{\rho(a+b)} > (1 + \delta_m)a^{\rho(a)} = (1 + \delta_m)w_{m+1}(z)$$

by Lemma 3.6. □

Using this lemma and (4.1), we have for $|z| > r_3$,

$$\begin{aligned} |a_\alpha(z) D_z^\alpha f(z) e^{-w_m(z)}| &\leq |a_\alpha(z) D_z^\alpha f(z) e^{-(1+\delta_m)w_{m+1}(z)}| \\ &\leq \|f\|_{w_{m+2}} \|a_\alpha\|_{\delta_m w_{m+1}} \frac{\alpha!}{r^{|\alpha|}} \max_{|y|=R(r)} e^{w_{m+2}(z+y) - w_{m+1}(z)}. \end{aligned} \quad (4.2)$$

Because $C_{m+2}^{-1}|z+y| \leq p_{m+2}(z+y) \leq p_{m+2}(z) + p_{m+2}(y)$, for $|z+y| > C_{m+2}r_0 =: r_4$, we also have

$$w_{m+2}(z+y) = p_{m+2}(z+y)^{\rho(p_{m+2}(z+y))} \leq [p_{m+2}(z) + p_{m+2}(y)]^{\rho(p_{m+2}(z) + p_{m+2}(y))}.$$

(II) First we assume that $p_{m+2}(y) < \delta_{m+1}p_{m+2}(z)$ and so $|y| < \delta_{m+1}C_{m+2}^2|z|$. Then we have $p_{m+2}(z) + p_{m+2}(y) < (1 + \delta_{m+1})p_{m+2}(z) \leq p_{m+1}(z)$. Taking m large enough, we may assume that $\delta_{m+1}C_{m+2}^2 \ll 1$. Then, if

$$|z| > \frac{C_{m+2}}{1 - \delta_{m+1}C_{m+2}^2} r_0 := r_5,$$

we have $|z + y| \geq |z| - |y| > (1 - \delta_{m+1} C_{m+2}^2)|z| > C_{m+2} r_0 = r_4$ and

$$\begin{aligned} w_{m+2}(z + y) &\leq [p_{m+2}(z) + p_{m+2}(y)]^{\rho(p_{m+2}(z) + p_{m+2}(y))} \\ &\leq p_{m+1}(z)^{\rho(p_{m+1}(z))} = w_{m+1}(z). \end{aligned}$$

Therefore, for $|z| > r_5$, we have (with $|y| = R(r)$)

$$e^{w_{m+2}(z+y) - w_{m+1}(z)} \leq 1. \quad (4.3)$$

(I2) In the case where $p_{m+2}(y) \geq \delta_{m+1} p_{m+2}(z)$, setting $a := p_{m+2}(z)$, $b := p_{m+2}(y)$, we have, for some $t \in]a, a + b[$,

$$\begin{aligned} w_{m+2}(z + y) - w_{m+1}(z) &< w_{m+2}(z + y) - w_{m+2}(z) \\ &\leq [a + b]^{\rho(a+b)} - a^{\rho(a)} \\ &= b\{t\rho'(t) \ln t + \rho(t)\}t^{\rho(t)-1}. \end{aligned}$$

By (i), (ii) and the fact that in the case $\rho > 1$, the function $r^{\rho(r)-1}$ is strictly increasing for large $r > 0$, there exists $r_6 > 0$ such that, for $|z| > r_6$, we have

$$\begin{aligned} w_{m+2}(z + y) - w_{m+1}(z) &< 2b\rho \cdot (a + b)^{\rho(a+b)-1} \\ &\leq 2b\rho \cdot ((1 + \delta_{m+1}^{-1})b)^{\rho((1 + \delta_{m+1}^{-1})b)-1} \\ &\leq \frac{2\rho}{(1 + \delta_{m+1}^{-1})} ((1 + \delta_{m+1}^{-1})b)^{\rho((1 + \delta_{m+1}^{-1})b)} \\ &\leq B_0 w_{m+2}(y) \leq B_0 (C_{m+2} \sqrt{n}r)^{\rho(C_{m+2} \sqrt{n}r)} \\ &\leq Br^{\rho(r)} \end{aligned}$$

with some $B_0, B > 0$. The last two inequalities follow from the fact that $r^{\rho(r)-\rho}$ is slowly increasing (see, for example, [6]). Therefore, in this case, we have, for $|z| > r_6$,

$$e^{w_{m+2}(z+y) - w_{m+1}(z)} \leq e^{Br^{\rho(r)}}. \quad (4.4)$$

(I3) By (4.2), (4.3) and (4.4), in any case, we have, for $|z| > \max(r_5, r_6) =: r_7$, any $r > 0$ and large m , the estimate

$$\begin{aligned} |a_\alpha(z) D_z^\alpha f(z) e^{-w_m(z)}| \\ \leq \|f\|_{w_{m+2}} \|a_\alpha\|_{\delta_m w_{m+1}} \frac{\alpha!}{r^{|\alpha|}} \max_{|y|=R(r)} e^{w_{m+2}(z+y)} \cdot \sup_{|z| \leq r_7} e^{\delta_m w_{m+1}(z) - w_m(z)} \end{aligned} \quad (4.5)$$

and, for $r > r_7$, we have

$$w_{m+2}(z + y) \leq (p_{m+2}(z) + p_{m+2}(y))^{\rho(p_{m+2}(z) + p_{m+2}(y))} \leq (2C_{m+2}r)^{\rho(2C_{m+2}r)}.$$

This is also true for $|z| \leq r_7$ when $r > 0$ is large enough: in fact, by (4.1), we have

$$|a_\alpha(z) D_z^\alpha f(z) e^{-w_m(z)}| \leq \|f\|_{w_{m+2}} \|a_\alpha\|_{w_m} \frac{\alpha!}{r^{|\alpha|}} \max_{|y|=R(r)} e^{w_{m+2}(z+y)}$$

and, for $r > r_0$, we have

$$w_{m+2}(z+y) \leq (p_{m+2}(z) + p_{m+2}(y))^{\rho(p_{m+2}(z) + p_{m+2}(y))} \leq (2r)^{\rho(2r)}.$$

By using the fact that $r^{\rho(r)-\rho}$ is slowly increasing, for large $r > 0$, taking $B > 0$ larger, if necessary, we have thus for large $r > 0$,

$$\sup_z |a_\alpha(z) D_z^\alpha f(z) e^{-w_m(z)}| \leq \|f\|_{w_{m+2}} \|a_\alpha\|_{\delta_m w_{m+1}} \frac{\alpha!}{r^{|\alpha|}} e^{Br^{\rho(r)}}. \quad (4.6)$$

(I4) Now in $r^{-|\alpha|} e^{Br^{\rho(r)}} = e^{Br^{\rho(r)} - |\alpha| \ln r}$, we set $v := |\alpha|$ and $z(r) := Br^{\rho(r)} - v \ln r$ for the sake of simplicity. In (4.6), if we take $r = r_v := (v/B\rho)^{1/\rho}$ for $v \gg 1$, then we have $z(r_v) = Br_v^{\rho(r_v)} - v \ln r_v$. By condition (iii), we verify that for $v \gg 1$,

$$\ln r_v^{\rho(r_v)} = \rho \frac{\rho(r_v)}{\rho} \ln \left(\left(\frac{v}{B\rho} \right)^{1/\rho} \right) \leq \rho \ln \left(C \left(\frac{v}{B\rho} \right)^{1/\rho} \right) = \ln \left(C^\rho \frac{v}{B\rho} \right)$$

and so $z(r_v) \leq (C^\rho/\rho)v + v(1/\rho)(\ln(B\rho) - \ln v)$. Thus, if m is large enough, we have, for α large enough,

$$\begin{aligned} \sup_z |a_\alpha(z) D_z^\alpha f(z) e^{-w_m(z)}| &\leq \|f\|_{w_{m+2}} \|a_\alpha\|_{\delta_m w_{m+1}} \alpha! e^{(C^\rho/\rho)|\alpha| + (|\alpha|/\rho)(\ln B\rho - \ln |\alpha|)} \\ &= \|f\|_{w_{m+2}} \|a_\alpha\|_{\delta_m w_{m+1}} \frac{\alpha!}{|\alpha|^{|\alpha|/\rho}} e^{(C^\rho/\rho)|\alpha|} (B\rho)^{|\alpha|/\rho}. \end{aligned}$$

Then, by Stirling's formula, there exists $B' > 0$ and $v_0 > 0$ such that we have, for α with $|\alpha| > v_0$,

$$\sup_z |a_\alpha(z) D_z^\alpha f(z) e^{-w_m(z)}| \leq \|f\|_{w_{m+2}} \|a_\alpha\|_{\delta_m w_{m+1}} (\alpha!)^{1-1/\rho} B'^{|\alpha|}. \quad (4.7)$$

When $|\alpha| \leq v_0$, by (4.6), there exists $C > 1$ such that we have

$$\sup_z |a_\alpha(z) D_z^\alpha f(z) e^{-w_m(z)}| \leq C \|f\|_{w_{m+2}} \|a_\alpha\|_{\delta_m w_{m+1}} (\alpha!)^{1-1/\rho} B'^{|\alpha|}. \quad (4.8)$$

Thus, by (4.7), (4.8) and Condition 4.2, we have, for large m ,

$$\begin{aligned} \|Pf\|_{w_m} &\leq \sum_\alpha \sup_z |a_\alpha(z) D_z^\alpha f(z) e^{-w_m(z)}| \\ &\leq C \|f\|_{w_{m+2}} \sum_\alpha \|a_\alpha\|_{\delta_m w_{m+1}} (\alpha!)^{1-1/\rho} B'^{|\alpha|} < \infty \end{aligned}$$

for all $f \in E_p^{\rho(r)}$.

(II) $P : E_p^{\rho(r)} \rightarrow E_p^{\rho(r)}$ being continuous, for any m , there exist $C > 0$ and $M = M_m \geq m$ such that

$$\|Pf\|_{w_m} \leq C\|f\|_{w_M} \quad (4.9)$$

for any $f \in E_p^{\rho(r)}$. For any α , we have

$$b_\alpha(z, z') = P_z \left(\frac{(z - z')^\alpha}{\alpha!} \right) = P_z \sum_{\beta \leq \alpha} \frac{z^\beta}{\beta!} \frac{(-z')^{\alpha-\beta}}{(\alpha - \beta)!} = \sum_{\beta \leq \alpha} b_\beta(z, 0) \frac{(-z')^{\alpha-\beta}}{(\alpha - \beta)!}$$

and so

$$a_\alpha(z) = b_\alpha(z, z) = \sum_{\beta \leq \alpha} b_\beta(z, 0) \frac{(-z)^{\alpha-\beta}}{(\alpha - \beta)!}.$$

Then we have, by (4.9), Lemmas 3.5 and 4.4, that there exists $C > 0$ such that, for $|z| > r_3$,

$$\begin{aligned} |a_\alpha(z)e^{-w_m(z)}| &\leq \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \left| \left((-z)^{\alpha-\beta} P \frac{z^\beta}{\beta!} \right) e^{-w_m(z)} \right| \\ &\leq \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} |z^{\alpha-\beta} e^{-(w_m(z) - w_{m+1}(z))}| \left| \frac{1}{\beta!} (Pz^\beta) e^{-w_{m+1}(z)} \right| \\ &\leq \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} |z^{\alpha-\beta} e^{-\delta_m w_{m+1}(z)}| \sup_z \left| \frac{1}{\beta!} (Pz^\beta) e^{-w_{m+1}(z)} \right| \\ &\leq C \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)! \beta!} \|z^{\alpha-\beta}\|_{\delta_m w_{m+1}} \|z^\beta\|_{w_M} \\ &\leq C \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)! \beta!} \delta_m^{-|\alpha-\beta|/\rho} A_{m+1}^{|\alpha-\beta|} ((\alpha - \beta)!)^{1/\rho} A^{|\beta|} (\beta!)^{1/\rho} \\ &\leq \frac{C}{\alpha!} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(\frac{A_{m+1}}{\delta_m^{1/\rho}} \right)^{|\alpha-\beta|} A^{|\beta|} (\alpha!)^{1/\rho} \\ &= \frac{C}{\alpha!} \left(\frac{A_{m+1}}{\delta_m^{1/\rho}} + A \right)^{|\alpha|} (\alpha!)^{1/\rho} \end{aligned}$$

and, for $|z| \leq r_3$,

$$\begin{aligned} |a_\alpha(z)e^{-w_m(z)}| &\leq \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \left| \left((-z)^{\alpha-\beta} P \frac{z^\beta}{\beta!} \right) e^{-w_m(z)} \right| \\ &\leq \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \sup_{|z| \leq r_3} |z^{\alpha-\beta}| \sup_z \left| \frac{1}{\beta!} (Pz^\beta) e^{-w_m(z)} \right| \end{aligned}$$

$$\begin{aligned}
&\leq Cr_3^{|\alpha|} \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)! \beta!} \|z^\beta\|_{w_M} \\
&\leq Cr_3^{|\alpha|} \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)! \beta!} A^{|\beta|} (\beta!)^{1/\rho} \\
&\leq \frac{Cr_3^{|\alpha|}}{\alpha!} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} A^{|\beta|} (\alpha!)^{1/\rho} = \frac{Cr_3^{|\alpha|}}{\alpha!} (1 + A)^{|\alpha|} (\alpha!)^{1/\rho}
\end{aligned}$$

with $A = \max(A_{M_m}, A_{M_{m+1}})$. Therefore, we have Condition 4.3. $\square$

Example. Let $n = 1$ and let the proximate order to be constant and equal to two: $\rho(r) \equiv 2$. Assume $p(z) := |z|$. Then we have $w_m(z) = (1 + 1/m)^2 |z|^2$. Consider a constant coefficient differential operator of infinite order $P := \sum_{|\alpha|=0}^{\infty} a_\alpha D_z^\alpha$ with $a_\alpha := 1/(\alpha!)^{3/2}$. Then (a_α) satisfies Condition 4.2 and so P gives a continuous endomorphism of the space E_p^2 .

Remark. In the above theorem, Condition 4.2 cannot be replaced by the following: $(a_\alpha(z)) \subset E_p^{\rho(r)}$ and there is $\varepsilon_0 > 0$ such that, for any m , the function

$$S(\zeta) := \sum_{\alpha} \|a_\alpha\|_{\varepsilon_0 w_m} (\alpha!)^{1-1/\rho} \zeta^\alpha$$

is holomorphic for any $\zeta \in \mathbb{C}^n$.

In fact, in the assumption of the preceding example, we define $(a_\alpha(z))$ as follows: $a_0(z) := e^{\varepsilon_0 z^2}$, $a_\alpha(z) := 0$ ($\alpha \neq 0$). We can easily verify that $(a_\alpha(z))$ satisfies the above condition, but not Condition 4.2. Then $P := a_0(z)$ cannot give an endomorphism of E_p^2 . (To see this, it is enough to take $f(z) = e^{z^2} \in E_p^2$.) We remark that this example was pointed out by the referee.

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