COMPLETE MINIMAL CYLINDERS PROPERLY IMMERSED IN THE UNIT BALL

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Abstract. The Calabi–Yau conjecture is one of the main problems in the global theory of complete minimal surfaces in $\mathbb{R}^3$. Francisco Martin and Santiago Morales have constructed complete proper minimal surfaces in convex bodies of $\mathbb{R}^3$. In this paper, we modify their technique in the cylindrical case, and construct a complete minimal cylinder properly immersed in the unit ball.

1. Introduction

1.1. Background

In 1965, Calabi posed the following problems on the existence of complete non-flat bounded minimal surfaces in $\mathbb{R}^3$.

- (1) *Are there no complete bounded minimal surfaces in $\mathbb{R}^3$?*
- (2) *Are there no complete non-flat minimal surfaces in a halfspace of $\mathbb{R}^3$?*

As Yau took them up in the millennium lecture in 2000, these are called ‘The Calabi–Yau problem on minimal surfaces’. An affirmative answer was given by Colding and Minicozzi in the embedded case, namely, they showed that embedded complete minimal surfaces must be unbounded [1]. More results in the embedded case were given in [7, 8]. On the other hand, in the immersed case, a negative answer was given by Martin and Morales [5, 6]. In fact, they showed that there exists a complete proper minimal immersion $\varphi : D \rightarrow E$ from an open disk D to a convex domain E in $\mathbb{R}^3$, which is not necessarily bounded or smooth.

In this paper, we apply their technique to the cylindrical case, and prove the following theorem.

MAIN THEOREM (Summary). *There exists a complete minimal cylinder properly immersed in a ball of $\mathbb{R}^3$.*

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Although some examples of complete bounded minimal surfaces with non-trivial topology have already been obtained [4, 9], the properness has not yet been achieved. Note that while the completeness is an intrinsic property decided by the metric, the properness is an extrinsic property.

A construction of minimal surfaces with non-trivial topology is difficult because of the period problem. In order to clear this problem in the cylinder case, we use a special type of the Weierstrass data, the so-called z^2 -type (see Section 1.4).

We now sketch how Martin and Morales constructed such a minimal immersion (disk type). For details, see [5]. The principal tools of their construction were the *Lopez–Ros transformation* of the Weierstrass representation and *Runge’s theorem* (see Section 1.3).

First, for a bounded convex regular[†] domain E , they deformed a given minimal immersion $X : D \rightarrow E$ in such a way that the boundary ∂X changes little by little, but the other part is almost preserved. Thus, they obtained a sequence of minimal immersions $\{X_i\}$:

$$X_1 \rightarrow X_2 \rightarrow \cdots X_n \rightarrow \cdots$$

On the process of this construction, the immersion X_i satisfies the following.

- (1) X_i is close to X over almost all of the domain away from the boundary.
- (2) The more that this step is repeated, the more the intrinsic metric grows near the boundary. On the other hand, the extrinsic distance in $\mathbb{R}^3$ between any two points of X_i is bounded.
- (3) Each X_i is contained in E , and ∂X_i converges to ∂E .

The desired surface is obtained as the limit surface $X_\infty : \Omega \rightarrow \mathbb{R}^3$ ($\Omega \subset D$). Finally, for arbitrary convex domain E , they applied Minkowski’s theorem and obtained the result.

1.2. Weierstrass representation

The theory of complete minimal surfaces is closely related to the complex analysis of one variable. Throughout this section, let M be a Riemann surface. Denote by a triple

$$\phi = (\phi_1, \phi_2, \phi_3),$$

the holomorphic 1-forms defined on M satisfying:

- the conformal condition

$$\sum_{i=1}^3 \phi_i^2 = 0 \quad \text{on } M;$$

[†]A regular domain is a domain whose boundary is a compact analytic surface of $\mathbb{R}^3$.

- the regular condition

$$\sum_{i=1}^3 \|\phi_i\|^2 \neq 0 \quad \text{on } M;$$

- the period condition

$$\operatorname{Re} \int_{\gamma} \phi = 0, \quad \text{for any closed curve } \gamma \subset M.$$

Here we consider ϕ_i to be a holomorphic function times dz in a local parameter z . Then for a given $z_0 \in M$, the map $X : M \rightarrow \mathbb{R}^3$ defined by

$$X(z) = \operatorname{Re} \left(\int_{z_0}^z \phi \right) \quad (1)$$

provides a conformal minimal immersion of M into $\mathbb{R}^3$. Conversely, up to translations, any conformal minimal immersion X of M into $\mathbb{R}^3$ is obtained in this way. Thus, it is easy to define a conformal minimal immersion on a simply connected domain. If we give a holomorphic 1-form $\eta(z) = f(z) dz$ and a meromorphic function $g : M \rightarrow \overline{\mathbb{C}}$, ϕ_k ($k = 1, 2, 3$) which are determined by (g, η) as follows

$$\phi_1 = \frac{1}{2}(1 - g^2)\eta, \quad \phi_2 = \frac{i}{2}(1 + g^2)\eta, \quad \phi_3 = g\eta, \quad (2)$$

will satisfy the above three conditions and the map $X : M \rightarrow \mathbb{R}^3$ given by (1) is a conformal minimal immersion. Indeed, g coincides with the stereographic projection of the Gauss map. By definition, $X(z)$ and (g, η) (or (g, f)) are *the Weierstrass representation* and *the Weierstrass data* of X , respectively. Furthermore,

$$\begin{aligned} G_X^2 &= \frac{1}{2} \|\phi\|^2 \\ &= \left\{ \frac{1}{\sqrt{2}}(1 + |g|^2)|f||dz| \right\}^2 \quad \left(\lambda_X = \frac{1}{\sqrt{2}}(1 + g^2)|f| \right) \\ &= \lambda_X^2 |dz|^2 \end{aligned} \quad (3)$$

is the metric induced by X on M . Throughout this paper, we use several orthonormal bases of $\mathbb{R}^3$. With respect to an orthonormal bases S , we write the Weierstrass data of X as

$$\phi_{(X,S)} = (\phi_{(1,S)}, \phi_{(2,S)}, \phi_{(3,S)}), \quad f_{(X,S)}, \quad g_{(X,S)}, \quad \eta_{(X,S)}.$$

Similarly, for given $v \in \mathbb{R}^3$, we write

$$v = (v_{(1,S)}, v_{(2,S)}, v_{(3,S)}).$$

For a curve α in M , by $\text{len}(\alpha, G_X)$ we mean the length of α with respect to the metric G_X . For $W \subset M$, we define:

- $\text{dist}_{(W, G_X)}(p, q) = \inf\{\text{len}(\alpha, G_X) \mid \alpha : [0, 1] \rightarrow W, \alpha(0) = p, \alpha(1) = q\},$
(for all $p, q \in W$);
- $\text{dist}_{(W, G_X)}(T_1, T_2) = \inf\{\text{dist}_{(W, G_X)}(p, q) \mid p \in T_1, q \in T_2\},$ (for all $T_1, T_2 \subset W$);
- $\text{diam}_{G_X}(W) = \sup\{\text{dist}_{(W, G_X)}(p, q) \mid p, q \in W\}.$

The Euclidean metric on $\mathbb{C}$ will be denoted by $\langle \cdot, \cdot \rangle$. Note that $G_X^2 = \lambda_X^2 \langle \cdot, \cdot \rangle$, where the conformal coefficient λ_X is given by above form.

Given a domain $D \subset \mathbb{C}$, we will say that a function, or a 1-form, is harmonic, holomorphic, meromorphic, on $\overline{D}$, if it is harmonic, holomorphic, meromorphic, on a domain containing $\overline{D}$.

1.3. The Lopez–Ros transformation

The series of transformations in the proof of Lemmas 1 and 2 below is called ‘the Lopez–Ros transformation’. It is very important for a surface to be complete and proper. If (g, f) are the Weierstrass data of a minimal immersion $X : M \rightarrow \mathbb{R}^3$, we can define new data

$$(\tilde{g}, \tilde{f}) = \left(\frac{g}{h}, fh \right) \quad \text{on } M, \quad (4)$$

where $h : M \rightarrow \mathbb{C}$ is holomorphic non-null function. We call a transformation of this type the ‘Lopez–Ros transformation’. Then we can check that $\tilde{\phi}$ defined by $(\tilde{g}, \tilde{f})$ satisfies the above conformal and regular conditions. So, if $\tilde{\phi}$ satisfies the period condition, $\tilde{X} : M \rightarrow \mathbb{R}^3$ given by $\tilde{\phi}$ is well defined. Then $\tilde{X}$ is a new minimal immersion having the same third coordinate as the immersion X . In addition, we can calculate the intrinsic metric of $\tilde{X}$ easily:

$$\begin{aligned} G_{\tilde{X}}^2 &= \left\{ \frac{1}{\sqrt{2}} |fh| \left(1 + \left| \frac{g}{h} \right|^2 \right) |dz| \right\}^2 \\ &= \lambda_{\tilde{X}}^2 |dz|^2. \end{aligned} \quad (5)$$

This means that we can increase the intrinsic distance in a prescribed compact $K \subset M$, by using a suitable function h . This function will be provided by Runge’s theorem.

For instance, let K be a compact set of M and let K' be an open neighborhood of K . From Runge's theorem, there exists a holomorphic function h such that

$$\begin{aligned} |h - \tau| &< \varepsilon, & \text{in } K, \\ |h - 1| &< \varepsilon, & \text{in } M \setminus K', \end{aligned}$$

for a holomorphic function τ on K' , and any $\varepsilon > 0$. If $(\tilde{g}, \tilde{f})$ defines a minimal immersion $\tilde{X}$, then for $\varepsilon > 0$ small enough, $\tilde{X}$ is close to X in $M \setminus K'$, and for $|\tau|$ large enough, the intrinsic metric of $\tilde{X}$ grows with respect to the intrinsic metric of X in K : $\lambda_{\tilde{X}} > \lambda_X$. Moreover, $\tilde{X}$ and X have the same third coordinate. This transformation gives a new surface satisfying the three conditions given in Section 1.1.

1.4. z^2 -type Weierstrass representation

We say that ϕ is a z^2 -type Weierstrass representation when ϕ has the form

$$\phi_j(z) = \hat{\phi}_j(z^2), \quad j = 1, 2, 3, \quad (6)$$

where $\hat{\phi}_j$ are holomorphic functions. Then the residue of ϕ at $z = 0$ disappears, and the immersion is well defined on $D^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

A z^2 -type Weierstrass representation gives rise to certain symmetries on the immersion, i.e.

$$S(X) := X(z) + X(-z) = \text{constant}.$$

In the following, we use certain symmetric domains in $\mathbb{C}$ and the z^2 -type Lopez–Ros function $h(z)$.

1.5. Notation

By D_r , we mean an open disk with radius r and center at the origin. A *polygonal pair* (P, Q) is a pair of closed simple curves in $\mathbb{R}^2$ consisting of a finite number of straight segments satisfying:

- (1) $D_{1/3} \subset \text{Int} Q \subset \overline{\text{Int} Q} \subset D_{2/3} \subset \text{Int} P \subset \overline{\text{Int} P} \subset D_1$;
- (2) $-z \in P$ (for all $z \in P$), $-z \in Q$ (for all $z \in Q$);

where $\text{Int} P$ denotes the bounded connected component of $\mathbb{C} \setminus P$. We denote $T(P, Q) = \text{Int} P \setminus \overline{\text{Int} Q}$ (see Figure 1). Given $\xi > 0$ small enough, we define a new polygonal pair (P^ξ, Q^ξ) , parallel to (P, Q) , such that:

- (i) the Euclidean distance in $\mathbb{R}^2$ from P to P^ξ is ξ ;
- (ii) the Euclidean distance in $\mathbb{R}^2$ from Q to Q^ξ is ξ ;
- (iii) the corresponding set T^ξ associated to (P^ξ, Q^ξ) is contained in T .

Whenever we write T^ξ and (P^ξ, Q^ξ) , we assume that ξ is small enough to define the polygon properly.

FIGURE 1. Polygonal pair (P, Q) .

Given a ball B of $\mathbb{R}^3$ (radius = r), we consider

$$N : \partial B \rightarrow \mathbb{S}^2,$$

the outward pointing unit normal or Gauss map of ∂B . Then we can define $\partial B_t = \{p + t \cdot N(p) : p \in \partial B\}$. We denote by B_t the ball bounded by ∂B_t . The normal projection to B is represented as

$$P_B : \mathbb{R}^3 \setminus B_{-t} \rightarrow \partial B, \quad p + tN(p) \mapsto p \in \partial B.$$

Finally, we define the ‘*extended Gauss map*’

$$N_B : \mathbb{R}^3 \setminus B_{-t} \rightarrow \mathbb{S}^2, \quad x \mapsto N(P_B(x)).$$

2. Proof of Main Theorem

MAIN THEOREM. *Let B and B' be two balls of $\mathbb{R}^3$, whose radii are r and R ($0 < r < R$) with center at the origin, respectively. Let $\varphi : A \rightarrow \mathbb{R}^3$ be a conformal minimal immersion, where $A := \{z \in \mathbb{C} \mid \frac{1}{3} < |z| < 1\}$ is an annulus. Consider the polygonal pair (P, Q) on A such that $\frac{2}{3} < |z| < 1$ for $z \in P$, $\frac{1}{3} < |z| < \frac{2}{3}$ for $z \in Q$, and*

$$\varphi(A \setminus T) \subset B \setminus B_{-d},$$

where $d > 0$, satisfying

$$\sqrt{(d+r)^2 + d^2} - r < \frac{R-r}{2}.$$

Then there exists a domain Ω ($T^\xi \subset \Omega \subset T \subset A$), and a complete properly immersed minimal cylinder $\tilde{\varphi} : \Omega \rightarrow B'$ satisfying, for all $\delta > 0$:

- (1) $\overline{T} \subset \Omega \subset A$;
- (2) $\|\tilde{\varphi} - \varphi\| < \delta$, in T ;
- (3) $\tilde{\varphi}(\Omega \setminus T^\xi) \subset B' \setminus B_{-2d-\delta}$.

First, we prove the following proposition to guarantee the existence of such $h(z) = \hat{h}(z^2)$ on A .

PROPOSITION 1. *Let $\tau > 1$ be a real number and let E_1, E_2 be a finite number of disjoint compact sets of $\mathbb{C}$ such that:*

- (1) $E_i = -E_i$ ($i = 1, 2$), where $-E_i = \{-x \mid x \in E_i\}$;
 - (2) $\mathbb{C} \setminus (E_1 \cup E_2)$ has finite number of arcwise-connected components.
- Then there exists $h : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$, a holomorphic non-null ($h \neq 0$) function, such that:*

- (i) $|h - 1| < 1/\tau$, in E_1 ;
- (ii) $|h - \tau| < 1/\tau$, in E_2 ;
- (iii) $h(z) = \hat{h}(z^2)$ ($\exists \hat{h} : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$: holomorphic function).

This proposition was obtained in [4, Proposition 1] and the proof was led as a consequence of Runge's theorem.

In order to prove the Main Theorem, we need the following two lemmas. We prove Lemma 2 by using Lemma 1 (proved in Section 3), where Lemma 2 will be the key in the proof of the Main Theorem.

LEMMA 1. (Properness) *Let $B \subset B' \subset \mathbb{R}^3$ be two balls, each of which has center 0 and whose radii are r and R , respectively ($0 < r < R$). Let $X : A \rightarrow B'$ be a conformally immersed minimal cylinder defined on an annulus A , satisfying:*

- $X(z)$ has a z^2 -type Weierstrass representation;
- $S(X(z)) = 0$ ($S(X)$ is defined in Section 1.4).

Consider a polygonal pair $(P, Q) \subset A$, such that:

$$X(A \setminus T) \subset B' \setminus \overline{B}.$$

Then, for all $b_1, b_2 > 0$, there exist a polygonal pair $(\tilde{P}, \tilde{Q})$ and a conformal minimal immersion $\tilde{Y} : \tilde{T} \rightarrow \mathbb{R}^3$, such that:

- (1) $T \subset \tilde{T} \subset \overline{\tilde{T}} \subset A$;
- (2) $\|\tilde{Y}(z) - X(z)\| < b_1$, for all $z \in \overline{\tilde{T}}$;
- (3) $\tilde{Y}(\tilde{P}) \subset B' \setminus B'_{-b_2}$, $\tilde{Y}(\tilde{Q}) \subset B' \setminus B'_{-b_2}$;
- (4) $\tilde{Y}(\tilde{T} - T) \subset B' \setminus B_{-b_2}$;
- (5) $\tilde{Y}(z) = \operatorname{Re} \int_{2/3}^z \phi(w) dw + c$, ($c \in \mathbb{R}^3$, $\phi : z^2$ -type);
- (6) $S(\tilde{Y}) = 0$.

LEMMA 2. (Completeness) *Let $B \subset B'$ and (P, Q) be as in Lemma 1. Let $X : \overline{T} \rightarrow \mathbb{R}^3$ be a conformally immersed minimal cylinder, and ε , a and b be positive constants, such that:*

- $X(\overline{T \setminus T^\varepsilon}) \subset B \setminus B_{-a}$;
- $\sqrt{(a+r)^2 + a} - r + \varepsilon < (R-r)/2$.

Then, there exist a polygonal pair $(\tilde{P}, \tilde{Q})$ and a conformal minimal immersion $\tilde{Y} : \overline{\tilde{T}} \rightarrow \mathbb{R}^3$, such that:

- (1) $\overline{T^\varepsilon} \subset \tilde{T} \subset \overline{\tilde{T}} \subset T$;
- (2) $\sigma < \operatorname{dist}_{(\tilde{T}, G_{\tilde{Y}})}(z, P^\varepsilon)$, for all $z \in \tilde{P}$, $\sigma < \operatorname{dist}_{(\tilde{T}, G_{\tilde{Y}})}(z, Q^\varepsilon)$, for all $z \in \tilde{Q}$, where σ is given by the equation

$$\sqrt{(a+r)^2 + (2\sigma+a)^2} - r + \varepsilon = \frac{R-r}{2};$$

- (3) $\tilde{Y}(\tilde{P}) \subset B' \setminus B'_{-b}$, $\tilde{Y}(\tilde{Q}) \subset B' \setminus B'_{-b}$;
- (4) $\tilde{Y}(\tilde{T} \setminus T^\varepsilon) \subset \mathbb{R}^3 \setminus B_{-2(a+b)}$;
- (5) $\|\tilde{Y}(z) - X(z)\| < \varepsilon$, for all $z \in \overline{\tilde{T}^\varepsilon}$;
- (6) $\tilde{Y}(z)$ has a z^2 -type Weierstrass representation;
- (7) $S(\tilde{Y}) = 0$;
- (8) $\tilde{Y}(\tilde{T}) \subset B'$.

Proof. Lemma 1 will be proved in Section 3. The proof of Lemma 2 is essentially the same as in [4] and [5]. We can prove Lemma 2 by combining [4, p. 7 onwards]] with [5]. So we leave these to the readers. First, we admit these Lemmas to construct the sequence of minimal immersion. First of all, we define a sequence of balls $\{B^n = \{x \in \mathbb{R}^3 \mid 0 < |x| < r_n\}\}$ in the following way. Consider $\nu > 0$ small enough, satisfying

$$\sqrt{(d+r)^2 + d^2} - r < \frac{|(R-\tilde{\nu}) - r|}{2},$$

where $0 < \tilde{v} = \sum_{k=2}^{\infty} v/k^2 < r$. Then, define $B^1 \stackrel{\text{def}}{=} B$ and $B^n \stackrel{\text{def}}{=} B'_{-\sum_{k \geq n} v/k^2}$ ($n \geq 2$). Note that $B^n \rightarrow B'$ ($n \rightarrow \infty$).

We also take a decreasing sequence of positive reals $\{b_n\}$ with $b_1 = d$, $b_n < 1/n^3$ ($n \geq 2$), which is related to $\{B^n\}$ in the following way

$$\sqrt{(b_n + r_n)^2 + (b_n)^2} - r_n < \frac{\mu'_n}{2} = \frac{v}{2n^2},$$

where $\mu'_n = \text{dist}(\partial B^n, \partial B^{n+1})$.

Next, construct a sequence

$$\chi_n = (X_n, (P_n, Q_n), \varepsilon_n, \xi_n, \sigma_n)$$

where $X_n : \overline{T_n} \rightarrow \mathbb{R}^3$ are conformal minimal immersions, (P_n, Q_n) is a polygonal pair, $T_n := \text{Int } P_n \setminus \overline{\text{Int } Q_n}$, $\{\varepsilon_n\}$, $\{\xi_n\}$, $\{\sigma_n\}$ are sequences of positive real numbers converging to zero, satisfying $\varepsilon_k < 1/k^3$, $\sum_{k=1}^{\infty} \varepsilon_k < \delta$ and

$$\sqrt{(b_k + r_k)^2 + (b_k + 2\sigma_{k+1})^2} - r_k + \varepsilon_{k+1} < \frac{\mu'_k}{2}. \quad (7)$$

Furthermore, the sequence X_n must satisfy the following properties:

- (1)_n $\sigma_n < \text{dist}_{(T_n^{\xi_n}, G_{X_n})}(P_{n-1}^{\xi_{n-1}}, P_n^{\xi_n})$, $\sigma_n < \text{dist}_{(T_n^{\xi_n}, G_{X_n})}(Q_{n-1}^{\xi_{n-1}}, Q_n^{\xi_n})$;
- (2)_n $\|X_n(z) - X_{n-1}(z)\| < \varepsilon_n$ for all $z \in T_{n-1}^{\varepsilon_n}$;
- (3)_n $\lambda_{X_n}(z) \geq \alpha_n \lambda_{X_{n-1}}(z)$ for all $z \in T_{n-1}^{\xi_{n-1}}$, where $\{\alpha_i\}_{i \in \mathbb{N}}$ is a sequence of real numbers such that $0 < \alpha_i < 1$ and $\prod_{i=1}^n \alpha_i \rightarrow \frac{1}{2}$ ($n \rightarrow \infty$);
- (4)_n $T_{n-1}^{\xi_{n-1}} \subset T_{n-1}^{\varepsilon_n} \subset \overline{T_{n-1}^{\varepsilon_n}} \subset T_n^{\xi_n} \subset \overline{T_n^{\xi_n}} \subset T_n \subset \overline{T_n} \subset T_{n-1}$ ($\because \xi_n < \varepsilon_n < \xi_{n-1}$);
- (5)_n $X_n(z) \in B^n \setminus (B^n)_{-b_n}$ for all $z \in P_n, Q_n$;
- (6)_n $X_n(z) \in \mathbb{R}^3 \setminus (B^{n-1})_{-2(b_{n-1}+b_n)}$ for all $z \in T_n \setminus T_{n-1}^{\varepsilon_n}$;
- (7)_n $X_n(z)$ has a z^2 -type Weierstrass representation ϕ_n ;
- (8)_n $S(X_n) = 0$;
- (9)_n $X_n(T_n) \subset B^n$.

The sequence $\{\chi_n\}$ is constructed in a recursive way. First, we can take χ_1 as follows.

Consider $\varphi : A \rightarrow \mathbb{R}^3$ whose Weierstrass data are $(g, f) = (az^2, 2a/z^2)$ (a is a real constant). Then X is given for $z = u + iv$, (Figure 4) by

$$\varphi(z) = (x_1, x_2, x_3) = \left(-\frac{au}{|z|^2} - \frac{a^3}{3}(u^3 - 3uv^2), -\frac{av}{|z|^2} + \frac{a^3}{3}(v^3 - 3u^2v), 2a^2u \right).$$

Then we can define $b_1 = d$, $X_1 = \varphi|_{\overline{T_1}}$ satisfying the above conditions, where $T_1 \subset A$ consists of (P_1, Q_1) such that $\frac{2}{3} < |z| < 1$ for $z \in P_1$, and $\frac{1}{3} < |z| < \frac{2}{3}$ for $z \in Q_1$.

FIGURE 2. An example of X_1 ($a = 1$, $A = \{z \in \mathbb{C} | \frac{2}{3} \leq |z| \leq 1\}$).

Suppose that we have $\chi_1, \dots, \chi_n$. In order to construct χ_{n+1} , we consider the following data:

$$\begin{aligned} B = B^n \ (r = r_n), \quad B' = B^{n+1} \ (R = r_{n+1}), \quad a = b_n, \quad X = X_n, \\ (P, Q) = (P_n, Q_n) \quad (\text{i.e. } T_n = T), \quad b < b_{n+1}. \end{aligned}$$

Now, properties $(5)_n$ and $(6)_n$ tell us that we can find a small enough positive constant ε , such that Lemma 2 can be applied to the aforementioned data. Here, we write $\varepsilon_0 = \varepsilon$ in order to avoid confusion.

Take a sequence $\{\hat{\varepsilon}_m\} \searrow 0$, with $\hat{\varepsilon}_m < \min\{1/(n+1)^3, \sup \varepsilon_0\}$, for all m . For each m , we consider $(\tilde{P}_m, \tilde{Q}_m)$ and $\tilde{Y}_m$ given in Lemma 2, for the above data and $\varepsilon = b = \hat{\varepsilon}_m$. If m is large enough, the assertions in Lemma 2 tell us that $T_n^{\xi_n} \subset T_n^{\hat{\varepsilon}_m} \subset \tilde{T}_m$ and the sequence $\{\tilde{Y}_m\}$ converges to X_n uniformly in $\overline{T}^{\hat{\varepsilon}_m}$. In particular, $\{\lambda_{Y_m}\}$ converges uniformly to λ_{X_n} in $\overline{T}_n^{\xi_n}$. Therefore, there is $m_0 \in \mathbb{N}$ such that

$$T_n^{\xi_n} \subset T_n^{\hat{\varepsilon}_{m_0}} \subset \tilde{T}_{m_0}, \quad (8)$$

$$\lambda_{\tilde{Y}_{m_0}} \geq \alpha_{n+1} \lambda_{X_n} \text{ in } T_n^{\xi_n}. \quad (9)$$

We define $X_{n+1} = \tilde{Y}_{m_0}$, $(P_{n+1}, Q_{n+1}) = (P_{m_0}, Q_{m_0})$ and $\varepsilon_{n+1} = \hat{\varepsilon}_{m_0}$. The term σ_{n+1} is given by (7) for $k = n+1$. From (2) of Lemma 2 and (8), we infer that

$$\begin{aligned} \sigma_{n+1} &< \text{dist}_{(\overline{T_{n+1}}, G_{X_{n+1}})}(z, P_n^{\varepsilon_{n+1}}), \quad \text{for all } z \in P_{n+1}, \\ \sigma_{n+1} &< \text{dist}_{(\overline{T_{n+1}}, G_{X_{n+1}})}(z, Q_n^{\varepsilon_{n+1}}), \quad \text{for all } z \in Q_{n+1}. \end{aligned}$$

Finally, take ξ_{n+1} small enough such that $(1)_{n+1}$, $(3)_{n+1}$ and $(4)_{n+1}$ hold. The remaining properties directly follow from Lemma 2 and above conditions. This concludes the construction of the sequence $\{\chi_n\}_{n \in \mathbb{N}}$.

Now we define

$$\Omega = \bigcup_n T_n^{\varepsilon_{n+1}} = \bigcup_n T_n^{\xi_n}.$$

Then $\Omega \equiv D_\theta$ for $\theta \in (0, 1)$ (see [2, Theorem IV.69]).

Let $K \subset \Omega$ be a compact set. Then there is n_0 such that $K \subset T_{n-1}^{\xi_n}$ for all $n > n_0$. From $(2)_n$, we have

$$\begin{aligned} \|X_N - X_{n-1}\| &= \|X_N - X_{N-1} + X_{N-1} - \cdots + X_n - X_{n-1}\| \\ &< \sum_{i=n}^N \varepsilon_i \\ &< \sum_{i=n}^{\infty} \frac{1}{i^2} \text{ in } K, \quad N > n > n_0. \end{aligned}$$

Thus, the sequence of minimal immersion $\{X_n\}$ is a Cauchy sequence in the space $\text{Har}(\Omega)$ of harmonic maps from Ω in $\mathbb{R}^3$. Consequently, Harnack's theorem implies that $\{X_n\}$ converges in $\text{Har}(\Omega)$.

Let $\tilde{\varphi} : \Omega \rightarrow \mathbb{R}^3$ be the limit of $\{X_n\}$. Then $\tilde{\varphi}$ has the following properties. (The proof is almost the same as that in [4].)

- (1) $\tilde{\varphi}$ satisfies the conclusion of the theorem. ($\cdot$), $(2)_n$, $(5)_n$, $(6)_n$.
- (2) $\tilde{\varphi}$ is a conformal minimal immersion.
- (3) $\tilde{\varphi}$ is proper. Consider a compact subset $K \subset B'$. Let $n_0 \in \mathbb{N}$ be such that

$$K \subset (B^{n-1})_{-2(b_{n-1}+b_n)-\sum_{k \geq n} \varepsilon_k} \quad \text{for all } n \geq n_0.$$

From $(6)_n$, we have $X_n(z) \in \mathbb{R}^3 \setminus (B^{n-1})_{-2(b_{n-1}+b_n)}$ (for all $z \in T_n \setminus T_{n-1}^{\varepsilon_n}$). So we obtain

$$X_n^{-1}(K) \not\subset T_n \setminus T_{n-1}^{\varepsilon_n}.$$

Moreover, taking $(2)_k$ into account for $k > n$, we get

$$\begin{aligned} \|\varphi(z) - X_n(z)\| &= \left\| \lim_{k \rightarrow \infty} X_k(z) - X_n(z) \right\| \\ &= \lim_{k \rightarrow \infty} \sum_{i=n+1}^k \|X_i - X_{i-1}\| \\ &< \sum_{k=n+1}^{\infty} \varepsilon_k. \end{aligned}$$

So we obtain $\varphi(z) \in \mathbb{R}^3 \setminus (B^{n-1})_{-2(b_{n-1}+b_n)-\sum_{k \geq n} \varepsilon_k}$ (for all $z \in T_n \setminus T_{n-1}^{\varepsilon_n}$). Then we have $\varphi^{-1}(K) \cap (T_n \setminus T_{n-1}^{\varepsilon_n}) = \emptyset$ for $n \geq n_0$. This implies that $\varphi^{-1}(K) \subset T_{n_0-1}^{\varepsilon_{n_0}}$ is compact.

- (4) $\tilde{\varphi}$ is complete with respect to the metric G_φ . From (7), σ_n can be expressed as

$$\sigma_n = \frac{1}{2} \left\{ \sqrt{\left(\frac{\mu'_{n-1}}{2} \right)^2 + \varepsilon_n^2 + r_{n-1} \mu'_{n-1} - 2r_{n-1} \varepsilon_n - \mu'_{n-1} \varepsilon_n - b_{n-1}^2 - 2r_{n-1} b_{n-1} - b_{n-1}} \right\}. \quad (10)$$

Note that $\mu'_{n-1} = v/n^2 \Rightarrow \mu_{n-1}^{2'} = O(n^{-4})$, $\varepsilon_n < 1/n^3$, and $b_{n-1} < 1/(n-1)^3$. Then (10) leads us to $\sigma_n \geq O(n^{-1})$. The completeness of φ follows from the fact that $\sum_{n \geq 2} \sigma_n \rightarrow \infty$. In short, for α which is a connected curve with $\alpha(0) \in S_{2/3}$ and $\alpha(1) \in \partial T$, we have

$$\begin{aligned} \text{len}(\alpha) &= \int \lambda |dz| \\ &\geq \frac{1}{2} M \int_{2/3}^z |dz| \quad (M > 0 \text{ and constant}) \\ &> \frac{1}{2} M (\text{dist}(P_1^{\xi_1}, P_2^{\xi_2}) + \text{dist}(P_2^{\xi_2}, P_3^{\xi_3}) + \cdots) \\ &> \frac{1}{2} M \sum_{n \geq 2} \sigma_n \rightarrow \infty \quad (n \rightarrow \infty). \end{aligned} \quad \square$$

3. Proof of Lemma 1

3.1. The first deformation stage

The proof of Lemma 1 is almost the same as that in [5]. We start by choosing the aforementioned points $\{p_1^1, \dots, q_n^2\}$.

CLAIM 1. There exists an annulus W , with $\overline{T} \subset W \subset \overline{W} \subset A$, and a set of points $\{p_1^i, \dots, p_n^i, q_{n+1}^i, \dots, q_{2n}^i\}$ ($p_j^1 = -p_j^2$, $q_j^1 = -q_j^2$, $i = 1, 2$) included in $W \setminus \overline{T}$, satisfying the following properties.

- (1) The segments $\overline{p_1^1 p_2^1}, \overline{p_2^1 p_3^1}, \dots, \overline{p_n^1 p_1^2}, \overline{p_1^2 p_2^2}, \dots, \overline{p_n^2 p_1^1}$ form a polygon symmetric with respect to the origin $\hat{P}$ in $W \setminus \text{Int } P$. Similarly, the segments $\overline{q_{n+1}^1 q_{n+2}^1}, \overline{q_{n+2}^1 q_{n+3}^1}, \dots, \overline{q_{2n}^1 q_{n+1}^2}, \overline{q_{n+1}^2 q_{n+2}^2}, \dots, \overline{q_{2n}^2 q_{n+1}^1}$ form a polygon symmetric with respect to the origin $\hat{Q}$ in $W \cap \text{Int } Q$.
- (2) There exist open disks B_j^i satisfying

$$\|X(z) - X(w)\| < \varepsilon_0 \quad \text{for all } z, w \in B_j^i;$$

FIGURE 3. The points $p_1^1, \dots, q_n^2$.

for $j = 1, \dots, n$, $p_j^i, p_{j+1}^i \in B_j^i$ ($p_{n+1}^1 = p_1^2, p_{n+1}^2 = p_1^1$). Here, we adopt $B_j^1 = -B_j^2$ (i.e. for all $z \in B_j^1 \Rightarrow -z \in B_j^2$).

- (3) For each p_j^i or q_j^i , ($i = 1, 2, j = 1, \dots, 2n$), there exists an orthonormal basis of $\mathbb{R}^3$, $S_j^i = \{(e_j^i)_1, (e_j^i)_2, (e_j^i)_3\}$, satisfying:

- $(e_j^i)_1 = N(X(p_j^i))$ ($i = 1, 2, j = 1, \dots, n$);
- $\|(e_j^i)_k - (e_{j+1}^i)_k\| < \varepsilon_0/3\mu$ ($i = 1, 2, j = 1, \dots, 2n, k = 1, 2, 3$), for $j = 1, \dots, n$, $(e_{n+1}^1)_k = (e_1^2)_k, (e_{n+1}^2)_k = (e_1^1)_k$;
- $f_{(X, S_j^i)}(p_j^i) \neq 0$ ($i = 1, 2, j = 1, \dots, n$).

- (4) For each pair of symmetric points, there exists a complex number $e^{i\theta_k}$ ($k = 1, \dots, 2n$) satisfying:

- $|e^{i\theta_k}| = 1$ ($i^2 = -1$);
- $\text{Im}\{e^{i\theta_k}\} \neq 0$;

- $|e^{i\theta_k} f_{(X, S_k^1)}(p_k^1)/|f_{(X, S_k^1)}(p_k^1)| - 1| < \varepsilon_0/3\mu$ ($k = 1, \dots, n$), $\mu = \text{dist}(\partial B, \partial B') = r' - r$: constant.

Here, for j or $k = n+1, \dots, 2n$, we adopt the conditions at q_{j-n}^i or q_{k-n}^i . The arguments below is the same as in [5, (p 6)] and we omit it[†].

Next, a small enough real number $\delta \in (0, 1)$ exists and satisfies the following list of properties:

- (A1) $\hat{T} \setminus \bigcup_{k=1, i=1, 2}^{2n} (D(p_k^i, \delta) \cup D(q_k^i, \delta))$ is simply connected, where $D(p_k^i, \delta)$ means the disk centered at p_k^i with radius δ ;
- (A2) $D(p_j^i, \delta) \cup D(p_{j+1}^i, \delta) \subset B_j^i$ (for all $i = 1, 2$, $j = 1, \dots, n$, $p_{n+1}^1 = p_1^2$, $p_{n+1}^2 = p_1^1$);
- (A3) $D(p_j^i, \delta) \cap D(p_l^k, \delta) = \emptyset$ (for all $i, k = 1, 2$, $j, l = 1, \dots, n$, $(i, j) \neq (k, l)$);
- (A4) for all $i = 1, 2$ and $j = 1, \dots, n$

$$\delta \max_{\overline{D(p_j^i, \delta)}} \{|f_{(X, S_j^i)}|\} < 2\varepsilon_0;$$

- (A5) For all $i = 1, 2$ and $j = 1, \dots, n$

$$\delta \frac{\max_{\overline{D(p_j^i, \delta)}} \{|f_{(X, S_j^i)} g_{(X, S_j^i)}^2|\}}{|\text{Im}\{e^{i\theta_j}\}|} < 2\varepsilon_0;$$

- (A6) for all $i = 1, 2$, and $j = 1, \dots, n$ $\delta \max_{\overline{D(p_j^i, \delta)}} \{\|\phi^0\|\} < \varepsilon_0$;

- (A7) for all $i = 1, 2$ and $j = 1, \dots, n$

$$3\mu \frac{\max_{w \in \overline{D(p_j^i, \delta)}} \{|f_{(\phi^0, S_j^i)}(w) - f_{(\phi^0, S_j^i)}(p_j^i)|\}}{|f_{(\phi^0, S_j^i)}(p_j^i)|} < \varepsilon_0.$$

Owing to the symmetry, it is enough to consider the conditions for only $i = 1$. If we put

$$\Xi = \hat{T} \setminus \bigcup_{k=1, i=1, 2}^{2n} (D(p_k^i, \delta) \cup D(q_k^i, \delta)),$$

then we define the constant l as

$$l = \sup_{z \in \Xi} \{\text{dist}_{(\Xi, \langle \cdot, \cdot \rangle)}(z', z) | z' \in S_{2/3}\} + 2\pi\delta + \delta + 1. \quad (11)$$

[†]We had better change [5, equation (11)] into $\|\hat{e}_k(p_j^i) - \hat{e}_k(p_{j+1}^i)\| < \varepsilon_0/6\mu$ ($i = 1, 2$, $j = 1, \dots, n$).

We are now ready to construct a sequence $X_1, \dots, X_n$ in a recursive process. The Weierstrass data ϕ^i , in the orthogonal frame S_k^i , are determined by the Lopez–Ros transformation:

$$(g_{(\phi_k^1, S_k^1)}, f_{(\phi_k^1, S_k^1)}) : z^2\text{-type} \rightarrow (g_{(\phi_{k+1}^1, S_k^1)}, f_{(\phi_{k+1}^1, S_k^1)}) = \left(\frac{g_{(\phi_k^1, S_k^1)}}{h_{k+1}}, f_{(\phi_k^1, S_k^1)} h_{k+1} \right) \quad (12)$$

where

$$h_{k+1}(z) = \begin{cases} \frac{k_{k+1} e^{i\theta_{k+1}}}{(z - p_{k+1}^1)(z - p_{k+1}^2)} + 1 & (k = 0, \dots, n-1) \\ \frac{k_{k+1} e^{i\theta_{k+1}}}{(z - q_{k+1}^1)(z - q_{k+1}^2)} + 1 & (k = n, \dots, 2n-1). \end{cases}$$

Then h_{k+1} is of z^2 -type. Indeed, the poles satisfy $p_{k+1}^1 = -p_{k+1}^2$, so

$$\begin{aligned} h_{k+1}(z) &= \frac{k_{k+1} e^{i\theta_{k+1}}}{(z - p_{k+1}^1)(z - p_{k+1}^2)} + 1 \\ &= \frac{k_{k+1} e^{i\theta_{k+1}}}{z^2 - (p_{k+1}^1)^2} + 1, \quad \text{for } k = 0, \dots, n-1. \end{aligned}$$

Note that the new Weierstrass representation ϕ_{k+1}^1 has poles $p_1^i, \dots, p_{k+1}^i$, and the points $w_1^i, \dots, w_{k+1}^i$, $i = 1, 2$, which satisfy $h_{k+1}(w_{k+1}^i) = 0$ ($i = 1, 2$) are either zeros or poles of ϕ_{k+1}^1 . We define a_j^i , C_j^i , G_j^i , D_j^i by:

- a_j^i is a point in either the segment $\overline{p_j^i r_j^i}$ or $\overline{q_j^i r_j^i}$, where $a_j^1 = -a_j^2$, $r_j^1 = p_j^1 + \delta \in \partial D(p_j^1, \delta)$ ($j = 1, \dots, n$), or $r_j^1 = q_j^1 + \delta \in \partial D(q_j^1, \delta)$ ($j = n+1, \dots, 2n$), and $r_j^1 = -r_j^2$;
- C_j^i is an arc of a circumference centered at either p_j^i ($j = 1, \dots, n$) or q_j^i ($j = n+1, \dots, 2n$) and containing a_j^i , where $C_j^1 = -C_j^2$ means that $-z \in C_j^2$ for all $z \in C_j^1$;
- G_j^i is a closed annular sector bounded by C_j^i , a piece of either $\partial D(p_j^i, \delta)$ ($j = 1, \dots, n$) or $\partial D(q_j^i, \delta)$ ($j = n+1, \dots, 2n$) and two radii of these circumferences, as Figure 4 indicates, where $G_j^1 = -G_j^2$ holds;

FIGURE 4. The disk $D(p_j^i, \delta)$.

- D_j^i is a simply connected domain of $\mathbb{C}$ verifying $\overline{D_j^i} \cap G_j^i = \emptyset$ and $p_j^i, w_j^i \in D_j^i \subset \overline{D_j^i} \subset D(p_j^i, \delta)$ ($j = 1, \dots, n$) or $q_j^i, w_j^i \in D_j^i \subset \overline{D_j^i} \subset D(q_j^i, \delta)$ ($j = n+1, \dots, 2n$).

Moreover $X_1, \dots, X_{2n}$ satisfy the following.

(B1)_j For all $j < n$, $k = j+1, \dots, n$

$$\delta \max_{\overline{D(p_k^1, \delta)}} \{|f_{(\phi^j, S_k^1)}|\} < 2\varepsilon_0.$$

(B2)_j For all $j < n$, $k = j+1, \dots, n$

$$\delta \frac{\max_{\overline{D(p_k^1, \delta)}} \{|f_{(\phi^j, S_k^1)}| g_{(\phi^j, S_k^1)}^2\}}{|\operatorname{Im}\{e^{i\theta_k}\}|} < 2\varepsilon_0.$$

(B3)_j For all $j < n$, $k = j+1, \dots, n$

$$3\mu \frac{\max_{w \in \overline{D(p_k^1, \delta)}} \{|f_{(\phi^j, S_k^1)}(w) - f_{(\phi^0, S_k^1)}(p_k^1)|\}}{|f_{(\phi^0, S_k^1)}(p_k^1)|} < \varepsilon_0.$$

- (B4)_j For all $z \in C_j^1$; $|\operatorname{Re} \int_{\alpha_z^1} \phi^j| < \varepsilon_0$ (α_z^1 is a piece of C_j^1 connecting a_j^1 with z). Here, we can easily check

$$\left| \operatorname{Re} \int_{\alpha_z^1} \phi^j \right| = \left| \operatorname{Re} \int_{\alpha_z^2} \phi^j \right| < \varepsilon_0.$$

- (B5)_j One has $\phi_{(3, S_j^1)}^j = \phi_{(3, S_j^1)}^{j-1}$, where $\phi_{(k, S_j^1)}^j$ represents the k th coordinate of the triple ϕ^j in the frame S_j^1 .

- (B6)_j For all $z \in \overline{W} \setminus (D(p_j^i, \delta) \cup \bigcup_{k=1}^{j-1} D_k^i)$, $i = 1, 2$, $j \leq n$

$$\|\phi^j(z) - \phi^{j-1}(z)\| < \frac{\varepsilon_0}{4nl}.$$

- (B7)_j For $j = 2, \dots, 2n+1$

$$\left| \operatorname{Re} \int_{r_j^1 a_j^1} \phi^j - \operatorname{Re} \int_{r_{j-1}^1 a_{j-1}^1} \phi^{j-1} \right| < 12\varepsilon_0$$

where $\overline{r_{n+1}^1 a_{n+1}^1} = \overline{r_1^2 a_1^2}$, ($1 < j \leq n+1$), $\overline{r_{2n+1}^1 a_{2n+1}^1} = \overline{r_{n+1}^2 a_{n+1}^2}$ ($j \geq n+2$).

- (B8)_j For all $z \in G_j^1$, one has

$$\left\| \left(\operatorname{Re} \int_{r_j^1 z} \phi_1^j \right) (e_j^1)_1 + \left(\operatorname{Re} \int_{r_j^1 z} \phi_2^j \right) (e_j^1)_2 - \frac{1}{2} \left(\int_{r_j^1 z} \frac{k_j dw}{(w - p_j^1)(w - p_j^2)} \right) |f_{(\phi^0, S_j^1)}(p_j^1)| (e_j^1)_1 \right\| < 4\varepsilon_0.$$

- (B9)_j One has $S(X_j) = 0$.

Let ϕ_0 be the Weierstrass representation of the immersion $X = X_0$. In this case, these properties are a consequence of the conditions of δ . Assume that we have defined X_{j-1} satisfying the above properties. To construct X_j starting from X_{j-1} , we will use properties (B1)_{j-1}–(B9)_{j-1}.

Here, the constant $k_j > 0$ is taken small enough to satisfy (B1)_j, (B2)_j, (B3)_j and (B6)_j. Note that this is possible thanks to the fact that $\phi^j \rightarrow \phi^{j-1}$ ($k_j \rightarrow 0$), uniformly on $\overline{W} \setminus (D_{i=1,2}(p_j^i, \delta) \cup \bigcup_{k=1, i=1,2}^{j-1} D_k^i)$. Furthermore, (B5)_j trivially follows from the definition of ϕ_j^1 .

The point a_j^1 is the first point in the (oriented) segment $\overline{r_j^1 p_j^1}$, such that

$$\frac{1}{2} \left(\int_{r_j^1 a_j^1} \frac{k_j dz}{(z - p_j^1)(z - p_j^2)} \right) |f_{(X, S_j^1)}(p_j^1)| = 3\mu (= 3(r' - r)).$$

FIGURE 5. The domain Ω .

Then we can check $(B8)_j$, $(B4)_j$ and $(B7)_j$ in the same way as in [5, pp. 8–10].

Note that the Weierstrass representations ϕ^j have simple poles and zeros in W . Our next job is to describe a domain $\Omega \subset W$ where the above Weierstrass representations determine minimal immersions.

For each $j = 1, \dots, 2n$, let O_j^i be an open disk centered at p_j^i (q_j^i) containing $D(p_j^i(q_j^i), \delta)$. We assume that $O_1^i, \dots, O_{2n}^i$ ($i = 1, 2$) are pairwise disjoint. Let $\alpha_j^i \subset O_j^i \setminus \overline{D(p_j^i, \delta)}$ be a simple curve connecting $\partial O_j^i \cap \hat{T}$ with r_j^i , and finally let N_j^i be a small open neighborhood of $\alpha_j^i \cup \overline{r_j^i a_j^i}$ in $\overline{G_j^i} \cup (O_j^i \setminus D(p_j^i, \delta))$. The domain Ω is defined as

$$\Omega = \left(\hat{T} \setminus \bigcup_{k=1, i=1, 2}^{2n} O_k^i \right) \cup \left(\bigcup_{k=1, i=1, 2}^{2n} N_k^i \right).$$

(See Figure 5.)

If O_j^i , α_j^i and N_j^i are suitably chosen, then the following properties of Ω can be guaranteed:

- $\overline{\Omega}$ is annulus;
- $\overline{r_j^i a_j^i} \subset \overline{\Omega}$, $\overline{T} \subset \Omega$;
- $\overline{\Omega}$ contains neither the poles, p_j^i, q_j^i , nor the zeros, w_j^i , of the functions h_j ($j = 1, \dots, 2n$);

- $\sup_{z \in \overline{\Omega}} \{\text{dist}_{(\Omega, \langle \cdot, \cdot \rangle)}(z', z) \mid z' \in S_{2/3}\} < l$;
- $\overline{\Omega} \cap \overline{D(p_j^i, \delta)} \subset G_j^i$ (it is same as for $D(q_j^i, \delta)$).

Owing to the z^2 -type transformation, the 1-forms ϕ_j^1 do not have real periods in Ω . So we can find a domain $\overline{\Omega} \subset \Omega'$ where the immersions $\psi_j : \Omega' \rightarrow \mathbb{R}^3$, $\psi_j(z) = \text{Re} \int_{2/3}^z \phi_j^1 + X(2/3)$ ($j = 1, \dots, 2n$) are well defined. We define

$$X_j(z) = \psi_j(z) - \frac{S(\psi_j)}{2}.$$

Then we can easily check (B9)_j, that is $S(X_j) = 0$.

We leave the proof of the following conditions for the immersions X_j to [5, p. 11].

(C1)_j For all $z \in \Omega' \setminus (D(p_j^1, \delta) \cup D(p_j^2, \delta))$

$$\|X_j(z) - X_{j-1}(z)\| < \frac{\varepsilon_0}{4n}$$

(same as for $j > n$).

(C2)_j $(X_j)_{(3, S_j^1)} = (X_{j-1})_{(3, S_j^1)}$.

(C3)_j $\|X_{2n}(a_j^i) - X_{2n}(a_{j+1}^i)\| < 16\varepsilon_0$.

(C4)_j $X_{2n}(a_j^i) \in \mathbb{R}^3 \setminus B_{3\mu}$.

3.2. The second deformation stage

In this part of the proof, we employ new orthogonal frames. In this case, we take orthonormal basis $T_j^i = ((w_j^i)_1, (w_j^i)_2, (w_j^i)_3 = N(X_j(a_j^i)))$ ($i = 1, 2$, $j = 1, \dots, 2n$).

First we define Q_j^i , $\hat{C}_j^i$, $(Q_j^i)^\xi$ for $j \in \{1, \dots, 2n\}$, where Q_j^i is a curve which is the connected component of $\partial\Omega \setminus (C_j^i \cup C_{j+1}^i)$ satisfying (see Figures 6 and 7):

- Q_j^i does not cut C_k^i ($i = 1, 2$, $k \notin \{j, j+1\}$);
- Q_j^i ($i = 1, 2$, $j = 1, \dots, 2n$) are pairwise disjoint;
- $Q_j^i \subset B_j^i$;
- $Q_j^i \cap \overline{D(p_l^k, \delta)} = \emptyset$, for $i, k = 1, 2$, $l \notin \{j, j+1\}$.

Up to a small perturbation, we can assume that the curves Q_j^i satisfy $f_{(X_{2n}, T_j^i)}(z) \neq 0$ for all $z \in Q_j^i$. Let $\hat{C}_j^i$ be an open set containing C_j^i and sufficiently small to fulfill

$$\|X_{2n}(z) - X_{2n}(a_j^i)\| < 2\varepsilon_0 \quad \text{for all } z \in \hat{C}_j^i \cap \overline{\Omega}. \quad (13)$$

FIGURE 6. $\hat{C}_j^i$. FIGURE 7. $(Q_j^i)^\xi$.

By the same reason as in [5, p. 11], it is possible to choose $\hat{C}_j^i$ as above. From now, we write $\Omega' \setminus (D(p_j^1, \delta) \cup D(p_j^2, \delta)) = \Omega' \setminus D(p_j^i, \delta)_{i=1,2}$.

We define $(Q_j^i)^\xi = \{z \in \mathbb{C} \mid \text{dist}_{(\mathbb{R}^2, \langle \cdot, \cdot \rangle)}(z, Q_j^i) \leq \xi\}$, where we are assuming that $0 < \xi$ is small enough so that:

- (D1) $(Q_j^i)^\xi \subset \Omega'$;
- (D2) $(Q_j^i)^\xi \cap (Q_l^k)^\xi = \emptyset$, for $(i, j) \neq (k, l)$;
- (D3) $(Q_j^i)^\xi \cap \overline{D(p_k^l, \delta)} = \emptyset$, for $i, k = 1, 2$, $k \notin \{j, j+1\}$;

- (D4) $(Q_j^i)^\xi \subset B_j^i$;
 (D5) $(Q_j^i)^{\xi/2}$ and $\overline{\Omega \setminus (Q_j^i)_{i=1,2}^\xi}$ are simply connected;
 (D6) given $z \in Q_j^i$, one has $|f_{(X_{2n}, T_j^i)}(z) - f_{(X_{2n}, T_j^i)}(x)| < \varepsilon_1$, for all $x \in B(z, \xi/2)$,
 where $\varepsilon_1 = \frac{1}{4} \min_{Q_j^i} \{|f_{(X_{2n}, T_j^i)}|\}$;
 (D7) $\sup\{\text{dist}_{(\Omega \setminus (Q_j^i)_{i=1,2}^\xi, \langle \cdot, \cdot \rangle)}(z', z); z \in \overline{\Omega \setminus (Q_j^i)_{i=1,2}^\xi}, z' \in S_{2/3}\} < l$.

Put $Y_0 \stackrel{\text{def}}{=} X_n$. We construct a new family of minimal immersions

$$\varphi_j : \Omega' \rightarrow \mathbb{R}^3, z \mapsto \text{Re} \int_{2/3}^z \phi^j + X\left(\frac{2}{3}\right), \quad Y_j(z) = \varphi_j(z) - \frac{S(\varphi(z))}{2}.$$

Associated to these, we will have two families of real parameters $\tau_1, \dots, \tau_{2n-1}, \nu_1, \dots, \nu_{2n-1}$. These families will satisfy the following properties:

$$(F1)_j \quad (\varphi_j)_{(3, T_j^1)} = (Y_{j-1})_{(3, T_j^1)};$$

$$(F2)_j \quad \text{for all } z \in \overline{\Omega \setminus (Q_j^i)_{i=1,2}^\xi}$$

$$\|\varphi_j(z) - Y_{j-1}(z)\| < \frac{\varepsilon_0}{4n};$$

$$(F3)_j \quad \text{for all } z \in \overline{\Omega \setminus (Q_j^i)_{i=1,2}^\xi}, k = j+1, \dots, 2n$$

$$|f_{(\varphi_j, T_k^1)}(z) - f_{(Y_{j-1}, T_k^1)}(z)| < \frac{\varepsilon_1}{2n};$$

$$(F4)_j \quad \text{one has}$$

$$\left(\frac{1}{\tau_j - \nu_j}\right) \max_{(Q_j^1)^\xi} \{|f_{(Y_{j-1}, T_j^1)} g_{(Y_{j-1}, T_j^1)}^2|\} + \nu_j \max_{(Q_j^1)^\xi} \{|f_{(Y_{j-1}, T_j^1)}|\} < \frac{2}{\xi};$$

$$(F5)_j \quad \text{one has}$$

$$\frac{1}{2} \left(\frac{\tau_j \xi}{4} \min_{Q_j^1} \{|f_{(Y_0, T_j^1)}|\} - 1 \right) > \text{diam}(B') + 1 = r' + 1.$$

Once again, the sequence $Y_1, \dots, Y_n$ is defined by recursion. Suppose that we have constructed $Y_0, Y_1, \dots, Y_{i-1}$. Then, we define φ_j whose Weierstrass data are given by

$$(g_{(\varphi_j, T_j^1)}, f_{(\varphi_j, T_j^1)}) = \left(\frac{g_{(Y_{j-1}, T_j^1)}}{l_j}, f_{(Y_{j-1}, T_j^1)} l_j \right),$$

where $l_j : \mathbb{C} \rightarrow \mathbb{C}$ are holomorphic functions satisfying:

$$(E1)_j \quad l_j(z) \neq 0 \text{ (for all } z \in \mathbb{C});$$

$$(E2)_j \quad |l_j(z) - \tau_j| < \nu_j \text{ (for all } z \in (Q_j^1)^{\xi/2} \cup (Q_j^2)^{\xi/2});$$

$$(E3)_j \quad |l_j(z) - 1| < \nu_j \text{ (for all } z \in \overline{\Omega \setminus (Q_j^i)^{\xi}}_{i=1,2}).$$

Runge's theorem gives us such functions. Note that Y_j has been obtained from Y_{j-1} by a Lopez–Ros transformation. Properties (F1)_j–(F5)_j hold if the constant ν_j is sufficiently small and τ_j is large enough.

Define the minimal immersion $Y_{2n} = \tilde{Y} : \Omega' \rightarrow \mathbb{R}^3$. Then we can check that $\tilde{Y}$ satisfies the statements of Lemma 1. We leave the details to the reader, see [5, pp. 13–16].

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