REPRESENTATIONS OF CUNTZ ALGEBRAS ON FRACTAL SETS

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Abstract. We consider representations of Cuntz algebras on self-similar fractal sets for proper/improper systems of contractions. Natural representations, called Hausdorff representations, are associated with self-similar sets and Hausdorff measures in the case of similitudes in $\mathbb{R}^n$. We completely classify the Hausdorff representations up to unitary equivalence. The complete invariant is the list $(\lambda D_1, \ldots, \lambda D_N)$, where $\lambda_j$ is the Lipschitz constant of the $j$th contraction and $D$ is the Hausdorff dimension of the fractal set. Any non-trivial list can be realized by similitudes on the unit interval. There exists an improper system of contractions such that its representation of a Cuntz algebra on the self-similar fractal set is not unitarily equivalent to any Hausdorff representation for a proper system of similitudes in $\mathbb{R}^n$.

1. Introduction

There is a close relation between the ergodic theory of measure-preserving transformations and the spectral theory of the associated unitary operators. For example, let $T : X \to X$ be a measure-preserving transformation on a probability measure space $(X, \mathcal{B}, \mu)$. The Hilbert space $H = L^2(X, \mathcal{B}, \mu)$ has the one-dimensional subspaces $CI$ of constant functions. We can associate a unitary operator $U : H \to H$ by $(Uf)(x) = f(T^{-1}x)$ for $f \in H$ and $x \in X$. Then $T$ is ergodic if and only if 1 is a simple eigenvalue of $U$. The associated unitary $U$ preserves the orthogonal complement $CI^\perp$ of $CI$. Then $T$ is weak-mixing if and only if the restricted unitary operator $U|_{CI^\perp} : CI^\perp \to CI^\perp$ has no eigenvalues.

In this paper we consider systems of $N$ contractions instead of measure-preserving transformations and consider associated $N$ isometries instead of

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associated unitaries. We show that there is a close relation between the fractal theory of $N$ contractions and theory of operator algebras of the associated $N$ isometries. See [F] for the fractal theory. In many cases the ranges of these $N$ isometries are orthogonal to each other and they span the whole Hilbert space. Then the $C^*$-algebra generated by these $N$ isometries turns to be isomorphic to the Cuntz algebra $O_N$ studied in [C]. In particular, we consider the case of proper contractions of similitudes in $\mathbb{R}^n$. Then the natural representations, called Hausdorff representations, are associated on the self-similar fractal sets using Hausdorff $D$-dimensional measures, where $D$ is the Hausdorff dimension of the fractal set.

If two measure-preserving transformations are conjugate, then the associated unitary operators are unitarily equivalent. Therefore, invariants of operators up to unitary equivalence are important. One of the most typical is the spectrum of the operator. Similarly, we study the representations of Cuntz algebras up to unitary equivalence. In the case of the Hausdorff representations as above, we completely classify the representations of Cuntz algebras up to unitary equivalence. The complete invariant is the list $(\lambda_1^D, \ldots, \lambda_N^D)$, where $\lambda_j$ is the Lipschitz constant of the $j$th contraction. It is clear that the invariant is computable and simple.

There exists an improper system of contractions such that its representation of a Cuntz algebra on the self-similar fractal set is not unitarily equivalent to any Hausdorff representation for a proper system of similitudes in $\mathbb{R}^n$. The fact is shown by examining the spectral structure of the associated $N$ isometries.

Several interesting works have been already done on relations among Cuntz algebras, endomorphisms of $B(H)$, fractal geometry and wavelet theory by Bratteli and Jorgnesen in a series of papers [BJ1, BJ2, BJ3], and by Jorgensen and Pedersen in [JP]. In [Kw], Kawamura studied Cuntz algebras to implement $^*$-endomorphisms associated with chaotic maps.

2. The representations of Cuntz algebras on fractal sets

Recall that a Cuntz algebra $O_N$ is the $C^*$-algebra generated by $N$ operators $\{s_j; j = 1, 2, \ldots, N\}$ satisfying the following commutation relations:

$$s_j^* s_j = 1 \quad (j = 1, 2, \ldots, N) \quad \text{and} \quad \sum_j s_j s_j^* = 1.$$ 

These commutation relations give an algebraic description of the division of the total space into self-similar $N$ parts in the Hilbert space level. Taking this fact into account, we construct representations of the Cuntz algebras on the fractal sets.
The representation depends on the choice of invariant measure on the fractal set in general. However, in the case of proper contractions of similitudes in $\mathbb{R}^n$, we can choose the natural representations, called Hausdorff representations, on the self-similar fractal sets using Hausdorff $D$-dimensional measures, where $D$ is the Hausdorff dimension of the fractal set.

Let $X$ be a (separable) complete metric space with a metric $d$. A map $f$ on $X$ is called a contraction if its Lipschitz constant $\text{Lip}(f) \leq 1$. We call a system of contractions $\{\sigma_j : j = 1, 2, \ldots, N\}$ on $X$ proper if there exist positive numbers $\{\lambda_i\}$ and $\{\Lambda_i\}$ with $(0 < \lambda_i \leq \Lambda_i < 1)$ satisfying the condition:

$$\lambda_i d(x, y) \leq d(\sigma_i(x), \sigma_i(y)) \leq \Lambda_i d(x, y) \quad \text{for any } x, y \in X, \ i = 1, 2, \ldots, N.$$ 

We call the system improper if it is not proper.

We say that a non-empty compact set $K \subset X$ is self-similar (in a weak sense) with respect to $\{\sigma_j : j = 1, 2, \ldots, N\}$ if $K$ is a finite union of its small copies $\sigma_j(K)$, that is,

$$K = \bigcup_{i=1}^{N} \sigma_i(K).$$

If a system is proper, then such a self-similar set $K \subset X$ is uniquely determined. However, it is not unique for an improper system. So we consider a typical construction of a self-similar set. We need to assume that $X$ is compact or choose a compact subset $Y \subset X$ with $\sigma_j(Y) \subset Y$ for $j = 1, \ldots, N$ and let $Y$ be a new $X$. Then a self-similar fractal set $K$ is constructed in the following manner. Let $K_0 = X$. Define a decreasing sequence $(K_n)_n$ of compact subsets of $X$ inductively by

$$K_n = \bigcup_{j=1}^{N} \sigma_j(K_{n-1}) \quad (n = 1, 2, \ldots).$$

Then a fractal set $K$ is defined by

$$K = \bigcap_n K_n$$

It is easy to check that $K$ is self-similar in the above weak sense. In the following we denote by $K(\sigma_1, \sigma_1, \ldots, \sigma_N)$ the self-similar fractal set $K$ constructed in the above.

Let $\mu$ be a regular Borel measure on $K$. We always assume that a regular Borel measure $\mu$ on $K$ satisfies that

$$\mu(\sigma_i(K)) \neq 0 \quad \text{for any } i = 1, 2, \ldots, N.$$ 

to avoid triviality.
As a representation space, we consider the Hilbert space \( H = L^2(K, \mu) \) of square integrable functions with the inner product \( (f | g) = \int_K f \overline{g} d\mu \) on the fractal set \( K \) and the Borel \( \sigma \)-field \( B_K \).

We construct a system of \( N \) operators \( \{S_1, \ldots, S_N\} \) on the Hilbert space \( H = L^2(K, \mu) \) for a system of contractions \( \{\sigma_j : j = 1, 2, \ldots, N\} \) on the self-similar fractal set \( K = K(\sigma_1, \sigma_1, \ldots, \sigma_N) \).

Suppose that each \( \sigma_j : K \to \sigma_j(K) \) is one to one (after removing null sets) and is a Borel isomorphism. Then the induced measure of \( \mu \) by \( \sigma_j^{-1} \) is just given by \( (\mu \circ \sigma_j^{-1})(A) = \mu(\sigma_j(A)) \) for any Borel set \( A \) of \( K \). Assume that the measures \( \mu \circ \sigma_j \) and \( \mu \) are mutually absolutely continuous. So we have the almost everywhere non-vanishing Radon–Nikodym derivative \( J_j = d(\mu \circ \sigma_j)/d\mu \) on \( K \) satisfying
\[
\int_{\sigma_j(K)} g(y) d\mu(y) = \int_K g(\sigma_j(x)) J_j(x) d\mu(x)
\]
for any \( g \in L^1(\sigma_j(K), \mu) \). Thus, \( J_j(x) \) is the ‘Jacobian’ of \( \sigma_j \) at \( x \in K \) under the change of variables \( y = \sigma_j(x) \) and \( J_j(\sigma_j^{-1}(y))^{-1} \) is the ‘Jacobian’ of the \( \sigma_j^{-1} \) at \( y \in \sigma_j(K) \) satisfying
\[
\int_{\sigma_j(K)} f(\sigma_j^{-1}(y)) J_j(\sigma_j^{-1}(y))^{-1} d\mu(y) = \int_K f(x) d\mu(x)
\]
for any \( f \in L^1(K, \mu) \). In this situation, bounded operators \( S_j \) on a Hilbert space \( H = L^2(K, \mu) \) are given by
\[
(S_j f)(y) = \begin{cases} 
(J_j(\sigma_j^{-1}(y))^{-1/2} f(\sigma_j^{-1}(y)) & y \in \sigma_j(K), \\
0 & y \notin \sigma_j(K),
\end{cases}
\]
for \( f \in H = L^2(K, \mu) \) and \( y \in K \). It is immediate that
\[
\|S_j f\|^2 = \int_{\sigma_j(K)} (J_j(\sigma_j^{-1}(y))^{-1} f(\sigma_j^{-1}(y)))^2 d\mu(y) = \int_K |f(x)|^2 d\mu(x) = \|f\|^2.
\]
Thus, \( S_j^* S_j = I \). We say that \( S_j \) is the associated isometry for \( \sigma_j \). It is clear that \( S_j^* \) is given by
\[
(S_j^* f)(x) = (J_j(x))^{1/2} f(\sigma_j(x))
\]
It is easy to see that
\[
S_j S_j^* f = \chi_{\sigma_j(K)} f.
\]

We say that a regular Borel measure \( \mu \) on \( K \) satisfies the separation condition if
\[
\mu(\sigma_i(K) \cap \sigma_j(K)) = 0 \quad \text{for any } i \neq j, \ i, j = 1, 2, \ldots, N.
\]
We say that a regular Borel measure \( \mu \) on \( K \) is an invariant measure with respect to \( \{ \sigma_j : j = 1, 2, \ldots, N \} \) if for any Borel set \( A \subset K \), \( \sigma_j(A) \) is Borel for \( j = 1, 2, \ldots, N \) and
\[
\mu(A) = \sum_{j=1}^{N} \mu(\sigma_j(A)).
\]
It is clear that if
\[
\mu(A) = \mu\left( \bigcup_{j=1}^{N} \sigma_j(A) \right)
\]
for any Borel set \( A \subset K \) and \( \mu \) satisfies the separation condition, then \( \mu \) is an invariant measure.

Suppose that there exists an expanding continuous map \( f : K \to K \) with
\[
f^{-1}(x) = \bigcup_{j=1}^{N} \sigma_j(x) \quad \text{for any } x \in K.
\]
If \( \mu \) satisfies the separation condition, then \( \mu \) is an invariant measure with respect to \( \{ \sigma_j : j = 1, 2, \ldots, N \} \) if and only if \( \mu \) is an invariant measure with respect to \( f \) in the usual sense, that is, for any Borel set \( A \subset K \),
\[
\mu(A) = \mu(f^{-1}(A)).
\]

Now it is straightforward to get a representation of Cuntz algebra, which we summarize as a proposition.

**Proposition 2.1.** Let \( \{ \sigma_j : j = 1, 2, \ldots, N \} \) be a system of contractions on a compact metric space \( X \). Let \( K = K(\sigma_1, \sigma_2, \ldots, \sigma_N) \) be a self-similar set. Let \( \mu \) be a regular Borel measure on \( K \). Suppose that each \( \sigma_j : K \to \sigma_j(K) \) is a Borel isomorphism (after removing null sets) and the measures \( \mu \circ \sigma_j \) and \( \mu \) are mutually absolutely continuous. Assume that \( \mu \) satisfies the separation condition. Then the \( C^* \)-algebra \( C^*(S_1, \ldots, S_N) \) generated by the associated \( N \) isometries \( S_1, \ldots, S_N \) is isomorphic to a Cuntz algebra \( \mathcal{O}_N = C^*(s_1, \ldots, s_N) \) and there exists a representation \( \pi_\mu : \mathcal{O}_N \to B(L^2(K, \mu)) \) such that \( \pi_\mu(s_i) = S_i \).

**Proof.** Self-similarity and the separation condition imply that
\[
\sum_j S_j S_j^* f = \sum_j \chi_{\sigma_j(K)} f = f.
\]
Thus, \( \{ S_1, \ldots, S_N \} \) generates a Cuntz algebra \( \mathcal{O}_N \). \( \square \)
We say that $\pi_\mu$ is the associated representation of Cuntz algebra $\mathcal{O}_N$ for a measure $\mu$. In this manner we may treat fractal sets as the representation spaces of the Cuntz algebras.

**Example 1. (Cantor set)** Let $X = [0, 1]$ and $\sigma_1$ and $\sigma_2$ be two contractions defined by

$$\sigma_1(x) = \frac{1}{3}x \quad \text{and} \quad \sigma_2(x) = \frac{1}{3}x + \frac{2}{3}.$$ 

Then a self-similar set $K = K(\sigma_1, \sigma_2)$ is the Cantor set. The Hausdorff dimension is given by $D = \dim_H K = \log 2 / \log 3$. Let $\mu = \mathcal{H}^D$ be the Hausdorff $D$-dimensional measure on $K$ restricted to the Borel $\sigma$-field. Then the associated $\pi_\mu$ is a representation of the Cuntz algebra $\mathcal{O}_2$.

**Example 2. (Bernoulli representation)** Let $X = \prod_{n \in \mathbb{N}} \{1, 2, \ldots, N\}$ be the space of all sequences $x = (x_n)_{n \in \mathbb{N}}$ taking values $\{1, 2, \ldots, N\}$. For $x \neq y \in X$, we put an integer

$$n(x, y) = \min\{n \geq 1; x_n \neq y_n\}.$$ 

We define a metric $d$ on $X$ by

$$d(x, y) = \begin{cases} \frac{1}{2^n(x, y)} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then $X$ is a compact metric space. Define a system $\{\alpha_j : j = 1, 2, \ldots, N\}$ of $N$ contractions on $X$ by

$$\alpha_j(x_1, x_2, \ldots) = (j, x_1, x_2, \ldots).$$ 

Then each $\alpha_j$ is a proper contraction with the Lipschitz constant $\text{Lip}(\alpha_j) = \frac{1}{2}$. The self-similar set $K(\alpha_1, \alpha_2, \ldots, \alpha_N) = X$.

Let $p = (p_1, \ldots, p_N)$ be a probability vector, i.e. $p_1, \ldots, p_N > 0$ and $p_1 + \cdots + p_N = 1$. A Bernoulli measure $\nu$ is given on cylinder sets

$$[u_1, \ldots, u_m] = \{x \in X; x_i = u_i \text{ for } i = 1, \ldots, m\}$$

by

$$\nu([u_1, \ldots, u_m]) = p_{u_1} \cdots p_{u_m}.$$ 

Each $\alpha_j : X \to \alpha_j(X)$ is a Borel isomorphism and the measures $\nu \circ \alpha_j$ and $\nu$ are mutually absolutely continuous. The Radon–Nikodym derivative $J_j = d(\nu \circ \alpha_j)/d\nu = p_j$ is a constant. We see that $\nu$ is an invariant measure with respect
to \( \{\alpha_j : j = 1, 2, \ldots, N\} \) and \( v \) satisfies the separation condition. Therefore, we have a representation \( \pi_v : \mathcal{O}_N \to B(L^2(X, v)) \) of the Cuntz algebra \( \mathcal{O}_N \) with \( \pi_v(s_j) = S_j \) such that

\[
(S_j f)(y_1, y_2, \ldots) = \begin{cases} 
(p_j)^{-1/2} f((y_2, y_3, \ldots)) & \text{if } y_1 = j, \\
0 & \text{if } y_1 \neq j,
\end{cases}
\]

for \( f \in H = L^2(X, v) \) and \( y \in X \).

In addition, \( S_j^* \) is given by

\[
(S_j^* f)(x_1, x_2, \ldots) = (p_j)^{1/2} f((j, x_1, x_2, \ldots))
\]

for \( x \in X \). We call the representation \( \pi_v \) the Bernoulli representation for a probability vector \( p = (p_1, \ldots, p_N) \).

We note that the \( C^* \)-subalgebra of \( \mathcal{O}_N \) generated by \( \{S_{u_1} \ldots S_{u_m} S_{v_1}^* \ldots S_{v_1}^* ; u_1, \ldots, u_m, v_1, \ldots, v_m \in \{1, \ldots, N\}, m \in \mathbb{N}\} \)

is isomorphic to the uniformly hyperfinite (or UHF) algebra \( M_{N^\infty} \) of type \( N^\infty \). Hence, we have inclusions

\[
C(X) \subset M_{N^\infty} \subset \mathcal{O}_N.
\]

by identifying the characteristic function \( \chi[u_1, \ldots, u_m] \) on the cylinder set \([u_1, \ldots, u_m]\) with \( S_{u_1} \ldots S_{u_m} S_{v_1}^* \ldots S_{v_1}^* \). Then it is important to see that we can identify \( \pi_v(f) \) with exactly the multiplication operator by \( f \) for any \( f \in C(X) \), i.e., \( \pi_v(f)(g) = fg \) for \( g \in L^2(X) \).

Next we recall that two representations \( \pi_i : \mathcal{O}_N \to B(L^2(K_i)) \) \( (i = 1, 2) \) of the Cuntz algebra \( \mathcal{O}_N \) are unitarily equivalent if we can find a unitary operator \( U : L^2(K_1) \to L^2(K_2) \) such that \( \pi_1(T)U = U\pi_2(T) \) holds for any \( T \in \mathcal{O}_N \). It is enough to check the formula for the generators \( T = s_1, \ldots, s_N \).

We see that the representation of the Cuntz algebra \( \mathcal{O}_2 \) on a Cantor set in Example 1 and the Bernoulli representation of \( \mathcal{O}_2 \) for a probability vector \( p = (\frac{1}{2}, \frac{1}{2}) \) in Example 2 are unitarily equivalent. However, two Bernoulli representations of \( \mathcal{O}_2 \) for a probability vector \( p = (\frac{1}{2}, \frac{1}{2}) \) and \( q = (\frac{1}{4}, \frac{3}{4}) \) are not unitarily equivalent.

### 3. Hausdorff representations

In this section we discuss the case of similitudes in \( \mathbb{R}^n \) with the usual Euclidian metric \( d \). Recall that \( \sigma : \mathbb{R}^n \to \mathbb{R}^n \) is a similitude if there exists a constant \( \lambda \) such that

\[
d(\sigma(x), \sigma(y)) = \lambda d(x, y) \quad \text{for any } x, y \in \mathbb{R}^n.
\]
It is known that \( \sigma : \mathbb{R}^n \to \mathbb{R}^n \) is a similitude if and only if \( \sigma = \mu_r \circ \tau_b \circ A \) for some homothety \( \mu_r(x) = rx \) (\( r \geq 0 \)), translation \( \tau_b(x) = x + b \) and orthonormal transformation \( A \).

In the rest of the section we assume that \( \{\sigma_j : j = 1, 2, \ldots, N\} \) is a system of \( N \) proper contractions consisting of similitudes in \( \mathbb{R}^n \). Let \( \lambda_i \) be the Lipschitz constant of \( \sigma_j \). Then we have that \( 0 < \lambda_i < 1 \) and 

\[
d(\sigma_j(x), \sigma_j(y)) = \lambda_j d(x, y) \quad \text{for any } x, y \in \mathbb{R}^n.
\]

Then there exists a unique compact set \( K \subset \mathbb{R}^n \) (called a self-similar fractal set) such that 

\[
K = \bigcup_{i=1}^{N} \sigma_i(K).
\]

Suppose that \( K_0 \supset \mathbb{R}^n \) is a compact set such that \( K_0 \supset \bigcup_{i=1}^{N} \sigma_i(K_0) \). Then a decreasing sequence \( (K_n)_n \) of compact subsets of \( K_0 \) is defined inductively by 

\[
K_n = \bigcup_{j=1}^{N} \sigma_j(K_{n-1}) \quad (n = 1, 2, \ldots). \tag{1}
\]

The self-similar fractal set \( K = K(\sigma_1, \sigma_2, \ldots, \sigma_N) \) is constructed by \( K = \bigcap_n K_n \).

We also assume that a system \( \{\sigma_j : j = 1, 2, \ldots, N\} \) satisfies the open set condition, i.e. there exists a non-empty bounded open set \( V \) such that 

\[
\bigcup_{i=1}^{N} \sigma_i(V) \subset V
\]

with the union disjoint. Then the Hausdorff dimension \( D = \dim_H K \) of the fractal set \( K \) is equal to the similarity dimension \( s \) given by a unique \( s \) such that 

\[
\sum_{j=1}^{N} \lambda_j^s = 1.
\]

The standard examples include the middle third Cantor set, the Sierpinski gasket and the von Koch curve.

Let \( \mathcal{H}^D \) be the \( D \)-dimensional Hausdorff measure. Then 

\[
0 < \mathcal{H}^D(K) < \infty.
\]

Consider a regular Borel probability measure \( \mu \) on the self-similar fractal set \( K \) given by the normalized \( D \)-dimensional Hausdorff measure restricted on the Borel \( \sigma \)-field \( \mathcal{B}_K \) of \( K \), that is, \( \mu = (\mathcal{H}^D(K))^{-1} \mathcal{H}^D|\mathcal{B}_K \).

Each \( \sigma_j : K \to \sigma_j(K) \) is a Borel isomorphism and the measures \( \mu \circ \sigma_j \) and \( \mu \) are mutually absolutely continuous. The Radon–Nikodym derivative \( J_j = d(\mu \circ \sigma_j)/d\mu = \lambda_j^D \) is a constant, i.e. 

\[
\mu(\sigma_j(A)) = \lambda_j^D \mu(A)
\]
for any Borel set \( A \). We see that \( \mu \) is an invariant measure with respect to \( \{ \sigma_j : j = 1, 2, \ldots, N \} \) and \( \mu \) satisfies the separation condition. Therefore, we have a representation \( \pi_\mu : \mathcal{O}_N \to B(L^2(K, \mu)) \) of the Cuntz algebra \( \mathcal{O}_N \) with \( \pi_\mu(s_j) = S_j \) such that

\[
(S_j f)(y) = \begin{cases} (\lambda^{D_j})^{-1/2} f(\sigma_j^{-1}(y)) & y \in \sigma_j(K), \\ 0 & y \notin \sigma_j(K), \end{cases}
\]

for \( f \in H = L^2(K, \mu) \) and \( y \in K \).

In addition, \( S_j^* \) is given by

\[
(S_j^* f)(x) = (\lambda^{D_j})^{1/2} f(\sigma_j(x))
\]

for \( x \in K \). We call the representation \( \pi_\mu \) of the Cuntz algebra \( \mathcal{O}_N \) the Hausdorff representation for a system \( \{ \sigma_j : j = 1, 2, \ldots, N \} \) of \( N \) proper contractions consisting of similitudes in \( \mathbb{R}^n \).

We now classify the Hausdorff representations up to the unitary equivalence.

**Theorem 3.1.** Let \( \{ \sigma_j : j = 1, 2, \ldots, N \} \) and \( \{ \sigma'_j : j = 1, 2, \ldots, N \} \) be two systems of \( N \) proper contractions consisting of similitudes in \( \mathbb{R}^n \). Let \( \lambda_j \) and \( \lambda'_j \) be the Lipschitz constants of \( \sigma_j \) and \( \sigma'_j \), respectively. We assume that they satisfy the open set condition. Let \( K \) and \( K' \) be their self-similar fractal sets and \( D \) and \( D' \) be the Hausdorff dimensions of \( K \) and \( K' \). Consider regular Borel probability measures \( \mu \) and \( \mu' \) on \( K \) and \( K' \) given by the normalized \( D \)-dimensional and \( D' \)-dimensional Hausdorff measures, respectively. Let \( \pi_\mu \) and \( \pi_{\mu'} \) be their Hausdorff representations of the Cuntz algebra \( \mathcal{O}_N \). Then the following are equivalent:

1. the Hausdorff representations \( \pi_\mu \) and \( \pi_{\mu'} \) are unitarily equivalent;
2. \( (\lambda_1^D, \ldots, \lambda_N^D) = ((\lambda'_1)^D, \ldots, (\lambda'_N)^D) \).

**Proof.** Let \( \nu \) be a Bernoulli measure on \( X = \prod_{n \in \mathbb{N}} \{1, 2, \ldots, N\} \) with a probability vector

\[
p = (p_1, \ldots, p_N) = (\lambda_1^D, \ldots, \lambda_N^D).
\]

Consider the Bernoulli representation \( \pi_\nu : \mathcal{O}_N \to B(L^2(X, \nu)) \) for the probability vector \( p = (p_1, \ldots, p_N) \) such that \( \pi_\nu(s_j) = S_j \). In the same way, we consider the Bernoulli representation \( \pi_{\nu'} : \mathcal{O}_N \to B(L^2(X, \nu')) \) for the probability vector \( p' = (p'_1, \ldots, p'_N) = ((\lambda'_1)^D, \ldots, (\lambda'_N)^D) \).

We first show that the Hausdorff representation \( \pi_\mu \) and the Bernoulli representation \( \pi_{\nu} \) are unitarily equivalent. By [H, Section 3.1], there is a continuous
onto map $\psi : X \to K$ such that
\[
\bigcap_{n=1}^{\infty} (\sigma x_1 \circ \sigma x_2 \circ \cdots \circ \sigma x_n)(K) = \{\psi(x)\}
\]
is a singleton for any $x = (x_1, x_2, \ldots) \in X$ and the following diagram is commutative:
\[
\begin{array}{ccc}
X & \xrightarrow{\alpha_j} & X \\
\downarrow \psi & & \downarrow \psi \\
K & \xrightarrow{\sigma_j} & K
\end{array}
\]
In particular, the image of the cylinder set $\{u_1, \ldots, u_m\} = (\alpha_{u_1} \circ \cdots \circ \alpha_{u_m})(X)$ by $\psi$ is exactly $(\sigma_{u_1} \circ \cdots \circ \sigma_{u_m})(K)$.

By [H, Section 4.4, Theorem (4) and Section 5.3, Theorem (1)], we have $\mu = \psi^* \nu$, that is, for any Borel set $A \subset K$, we have $\mu(A) = \nu(\psi^{-1}(A))$.

Let $i = (i_1, i_2, \ldots, i_n) \in \{1, 2, \ldots, N\}^n$ be a finite word of length $n$. We denote by $\sigma_i$ the finite composition $\sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_n}$. Then we have
\[
K = \bigcup_i \sigma_i(K)
\]
where $i$ runs all finite words of length $n$. Since
\[
\mu(\sigma_i(K)) = \lambda^D_{i_1} \cdots \lambda^D_{i_n} \mu(K),
\]
and $\sum_i \lambda^D_{i_1} \cdots \lambda^D_{i_N} = 1$, where $i$ runs all finite words of length $n$, we have that
\[
\mu(\sigma_i(K) \cap \sigma_j(K)) = 0
\]
for any finite word $i$ and $j$ of length $n$ with $i \neq j$. Let
\[
M := \bigcup_{n=1}^{\infty} \bigcup_{i \neq j: \text{finite words of length } n} (\sigma_i(K) \cap \sigma_j(K))
\]
and $Y = \psi^{-1}(M)$. Then $\mu(M) = 0$ and also we have $\nu(Y) = \nu(\psi^{-1}(M)) = \mu(M) = 0$.

Consider the restriction
\[
\psi|_{X \setminus Y} : X \setminus Y \to K \setminus M.
\]
Since $\psi$ is onto, so is $\psi|_{X \setminus Y}$. Suppose that $x \neq y \in X$ and $\psi(x) = \psi(y)$. Then there exists a natural number $n$ such that $x_n \neq y_n$. Let $i = (x_1, \ldots, x_n)$ and $j = (y_1, \ldots, y_n)$. Then $i \neq j$. Since $\psi(x) \in \sigma_i(K)$ and $\psi(y) \in \sigma_j(K)$, we have
\[
\psi(x) = \psi(y) \in \sigma_i(K) \cap \sigma_j(K) \subset M.
\]
Therefore, $x, y \in Y$. This means that the restriction $\psi|_{X \setminus Y}$ is one to one.
For any cylinder set \([u_1, \ldots, u_m] \subset X\), we have that 
\[ \psi([u_1, \ldots, u_m] \cap Y^c) = \sigma_{(u_1, \ldots, u_m)}(K) \cap M^c \]
is measurable and
\[ \nu([u_1, \ldots, u_m] \cap Y^c) = \mu(\psi([u_1, \ldots, u_m] \cap Y^c)) = p_{u_1} \cdots p_{u_m}. \]

Hence, the restriction \(\psi|_{X \setminus Y}\) is bimeasurable and measure preserving. Therefore, there exists a unitary operator \(U : L^2(K, \mu) \to L^2(X, \nu)\) such that
\[ (Uf)(x) = f(\psi(x)) \]
for any \(f \in L^2(K, \mu)\) and \(x \in X\). Then on the generators \(s_j \in \mathcal{O}_N\), we have
\[ (\pi_{\nu}(s_j)^*Uf)(x) = p_j^{1/2}(Uf)(\alpha_j(x)) = p_j^{1/2}f(\psi(\alpha_j(x))) \]
\[ = p_j^{1/2}f(\sigma_j(\psi(x))) = (\pi_{\mu}(s_j)^*f)(\psi(x)) = (U\pi_{\mu}(s_j)^*f)(x). \]

We conclude that the Hausdorff representation \(\pi_{\mu}\) and the Bernoulli representation \(\pi_{\nu}\) are unitarily equivalent. Similarly \(\pi_{\mu}'\) and \(\pi_{\nu}'\) are also unitarily equivalent.

Therefore, the proof is completed by showing that the following are equivalent:
1. the Bernoulli representations \(\pi_{\nu}\) and \(\pi_{\nu}'\) of \(\mathcal{O}_N\) are unitarily equivalent;
2. \((p_1, \ldots, p_N) = (p'_1, \ldots, p'_N)\).

If \((p_1, \ldots, p_N) = (p'_1, \ldots, p'_N)\), it is clear that \(\pi_{\nu} = \pi_{\nu}'\). Conversely, suppose that the Bernoulli representations \(\pi_{\nu}\) and \(\pi_{\nu}'\) are unitarily equivalent. Recall the following inclusions:
\[ C(X) \subset M_{\infty} \subset \mathcal{O}_N. \]
Then the restrictions \(\pi_{\nu}|_{C(X)}\) and \(\pi_{\nu}'|_{C(X)}\) are also unitarily equivalent as representations of \(C(X)\). Since we can identify \(\pi_{\nu}(f)\) and \(\pi_{\nu}'(f)\) with exactly the multiplication operators by \(f\) on \(L^2(X, \nu)\) and \(L^2(X, \nu')\) for any \(f \in C(X)\), two Bernoulli measures \(\nu\) and \(\nu'\) are absolutely continuous with each other. By Kakutani’s dichotomy theorem [Kk], we have that \((p_1, \ldots, p_N) = (p'_1, \ldots, p'_N)\).

This completes the proof.

Next we give an example of representations which help us to understand the above main theorem.

**Example 3.** Let \(X = [0, 1]\) and \(\sigma_1\) and \(\sigma_2\) be two contractions defined by \(\sigma_1(x) = \frac{1}{3}x\) and \(\sigma_2(x) = \frac{1}{3}x + \frac{2}{3}\). Then the self-similar set \(K = K(\sigma_1, \sigma_2)\) is the Cantor set. The Hausdorff dimension is given by \(D = \dim_H K = \log 2/\log 3\). Then \((\lambda_1^D, \lambda_2^D) = (\frac{1}{2}, \frac{1}{2})\) and the Hausdorff representation \(\pi_{\mu}\) is a representation of the Cuntz algebra \(\mathcal{O}_2\).
Similarly let $X' = [0, 1]$ and $\sigma_1'$ and $\sigma_2'$ be two contractions defined by $\sigma_1'(x) = \frac{1}{2}x$ and $\sigma_2'(x) = \frac{1}{2}x + \frac{1}{2}$. Then the self-similar set $K = K(\sigma_1', \sigma_2') = [0, 1]$. The Hausdorff dimension $D' = 1$ and $(\lambda_1'^D, \lambda_2'^D) = (\frac{1}{2}, \frac{1}{2})$ and the Hausdorff representation $\pi_{\mu'}$ is a representation of the Cuntz algebra $O_2$. By the above theorem the two representations $\pi_{\mu}$ and $\pi_{\mu'}$ are unitarily equivalent.

**Remark.** The unitary equivalence does not preserve the Hausdorff dimensions themselves of the fractal sets.

Any non-trivial list can be realized by $N$-similitudes on the unit interval.

**Proposition 3.2.** For any probability vector $p = (p_1, \ldots, p_N)$ with $\sum_{i=1}^{N} p_i = 1$ and $p_i > 0$, and $0 < D \leq 1$, there exist a systems of $N$ proper contractions $\{\sigma_j : j = 1, 2, \ldots, N\}$ consisting of similitudes on the unit interval $[0, 1]$ such that $p = (p_1, \ldots, p_N) = (\lambda_1^D, \ldots, \lambda_N^D)$.

where $\lambda_j$ are the Lipschitz constants of $\sigma_j$ and $D$ is the Hausdorff dimension $\dim_H K$ of the fractal set $K = K(\sigma_1, \sigma_2, \ldots, \sigma_N)$.

**Proof.** Define $\lambda_i = p_i^{1/D}$. Then $0 < \lambda_i < 1$ and $\sum_{i=1}^{N} \lambda_i \leq \sum_{i=1}^{N} p_i = 1$. Let $\sigma_i(x) = \lambda_i x + (\lambda_1 + \cdots + \lambda_{i-1})$ for $x \in [0, 1]$. Then the system of $N$ proper contractions $\{\sigma_j : j = 1, 2, \ldots, N\}$ is the desired one. $\square$

We consider the spectral structure of the Hausdorff representations.

**Proposition 3.3.** Let $\{\sigma_j : j = 1, 2, \ldots, N\}$ be a systems of $N$ proper contractions consisting of similitudes in $\mathbb{R}^n$. We assume that they satisfy the open set condition. Let $\pi_{\mu}$ be the Hausdorff representation of the Cuntz algebra $O_N$. Then for any finite word $i = (i_1, i_2, \ldots, i_n) \in \{1, 2, \ldots, N\}^n$ of length $n$, the operator $\pi_{\mu}(s_{i_1} \cdots s_{i_n})$ has no eigenvalues.

**Proof.** Let $\lambda_i$ be the Lipschitz constants of $\sigma_j$. Let $K$ be the self-similar fractal sets and $D$ be the Hausdorff dimension of $K$. Recall that $\mu$ is a regular Borel probability measure on $K$ given by the normalized $D$-dimensional Hausdorff measure. Suppose that the operator $\pi_{\mu}(s_i)$ has an eigenvalue $c$ with an eigenfunction $f \in L^2(K, \mu)$. Since $\pi_{\mu}(s_i)$ is an isometry, $|c| = 1$. For $x \notin \sigma_i(K)$, we have $cf(x) = \pi_{\mu}(s_i)f(x) = 0$ by definition. Similarly for any natural number $k$ we have that $c^k f(x) = \pi_{\mu}(s_i^k)f(x) = 0$ if $x \notin \sigma_i^k(K)$. Since $\bigcap_{k=1}^{\infty} \sigma_i^k(K) = \{a\}$ is
a singleton, the support of \( f \) is exactly \([a]\). Therefore, \( f = 0 \) almost everywhere with respect to the (normalized) Hausdorff measure \( \mu \). This is a contradiction. \( \square \)

**Proposition 3.4.** Let \( \{ \sigma_j : j = 1, 2, \ldots, N \} \) be a system of \( N \) proper contractions consisting of similitudes in \( \mathbb{R}^n \) with open set condition. Let \( \pi_\mu \) be its Hausdorff representation of the Cuntz algebra \( \mathcal{O}_N \). For any \( \varepsilon > 0 \) there exists a system \( \{ \sigma'_j : j = 1, 2, \ldots, N \} \) of \( N \) proper contractions consisting of similitudes in \( \mathbb{R}^n \) such that \( 0 < D' < \varepsilon \) and \( \pi_\mu \) and \( \pi_{\mu'} \) are unitarily equivalent, where \( D' \) is the Hausdorff dimension of the corresponding self-similar fractal set \( K' \) and \( \pi_{\mu'} \) is its Hausdorff representation.

**Proof.** Let \( K \) and \( K' \) be their self-similar fractal sets and \( D \) and \( D' \) be the Hausdorff dimension of \( K \) and \( K' \). Choose a positive integer \( p \) with \( 0 < D/p < \varepsilon \). Define \( \{ \sigma'_j : j = 1, 2, \ldots, N \} \) by the \( p \)-times compositions \( (\sigma_1^p, \sigma_2^p, \ldots, \sigma_N^p) \). Let \( \lambda_i \) and \( \lambda'_i \) be the Lipschitz constants of \( \sigma_j \) and \( \sigma'_j \). Then

\[
\lambda'_i = \lambda_i^p \quad \text{and} \quad D' = D/p < \varepsilon.
\]

Since

\[
((\lambda'_1)^{D'}, \ldots, (\lambda'_N)^{D'}) = (\lambda_1^{D}, \ldots, \lambda_N^{D}),
\]

\( \pi_\mu \) and \( \pi_{\mu'} \) are unitarily equivalent by the above theorem. \( \square \)

### 4. Representations for improper system of contractions

In this section we show that there exists an improper system of contractions such that its representation of the Cuntz algebra on the self-similar fractal set is not unitarily equivalent to any Hausdorff representation for a proper system of similitudes in \( \mathbb{R}^n \). This fact is shown by examining the spectral structure of the associated \( N \) isometries.

**Example 4.** Let \( X = [0, 1] \). Define two contractions \( \sigma_1 \) and \( \sigma_2 \) by

\[
\sigma_1(x) = \begin{cases} 
\frac{1}{2}x & \text{if } x \in [0, 1], \\
\frac{1}{2}x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}], \\
\frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1].
\end{cases}
\]

The self-similar fractal set \( K = K(\sigma_1, \sigma_2) = [0, 1] \). Let \( \mu \) be the restriction of Lebesgue measure on \( K \) to the Borel \( \sigma \)-field. Then each \( \sigma_j : K \to \sigma_j(K) \) is a Borel isomorphism and the measures \( \mu \circ \sigma_j \) and \( \mu \) are mutually absolutely continuous. It is clear that \( \mu \) satisfies the separation condition. Hence, there exists
a representation $\pi_\mu : O_2 \to B(L^2(K, \mu))$. Let $\chi_E$ be the characteristic function on $E = [\frac{3}{4}, 1]$. Since $\pi_\mu(S_2)\chi_E = \chi_E$, $1$ is an eigenvalue of the operator $\pi_\mu(S_2)$. By Proposition 3.3, $\pi_\mu$ is not unitarily equivalent to any Hausdorff representation for a proper system of similitudes in $\mathbb{R}^n$.

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