ON THE ZEROS OF THE EISENSTEIN SERIES FOR $\Gamma_0^*(5)$ AND $\Gamma_0^*(7)$

Junichi SHIGEZUMI

(Received 19 July 2006 and revised 17 January 2007)

Abstract. We locate almost all of the zeros of the Eisenstein series associated with the Fricke groups of level 5 and 7 in their fundamental domains by applying and extending the method of Rankin and Swinnerton-Dyer (On the zeros of Eisenstein series. Bull. London Math. Soc. 2 (1970), 169–170). We also use the arguments of some terms of the Eisenstein series in order to improve existing error bounds.

1. Introduction

Rankin and Swinnerton-Dyer considered the problem of locating the zeros of the Eisenstein series $E_k(z)$ in the standard fundamental domain $\mathbb{F}$ (see [RSD]). They proved that all of the zeros of $E_k(z)$ in $\mathbb{F}$ lie on the unit circle. They also stated towards the end of their study that ‘This method can equally well be applied to Eisenstein series associated with subgroups of the modular group’. However, it seems unclear how widely this claim holds.

Subsequently, Miezaki, Nozaki, and the present author considered the same problem for the Fricke group $\Gamma_0^*(p)$ (see [K, Q]), and proved that all of the zeros of the Eisenstein series $E_{k,p}^*(z)$ in a certain fundamental domain lie on a circle whose radius is equal to $1/\sqrt{p}$, $p = 2, 3$ (see [MNS]).

The Fricke group $\Gamma_0^*(p)$ is not a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, but it is commensurable with $\mathrm{SL}_2(\mathbb{Z})$. For a fixed prime p , we define $\Gamma_0^*(p) := \Gamma_0(p) \cup \Gamma_0(p)W_p$, where $\Gamma_0(p)$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Let $k \geq 4$ be an even integer. For $z \in \mathbb{H} := \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$, let

$$E_{k,p}^*(z) := \frac{1}{p^{k/2} + 1}(p^{k/2}E_k(pz) + E_k(z)) \quad (1)$$

be the Eisenstein series associated with $\Gamma_0^*(p)$ (cf. [SG]).

2000 Mathematics Subject Classification: Primary 11F11.

Keywords and Phrases: Eisenstein series; Fricke group; locating zeros; modular forms.

Henceforth, we assume that $p = 5$ or 7 . The region

$$\begin{aligned} \mathbb{F}^*(p) := & \{|z| \geq 1/\sqrt{p}, |z| \geq 1/(2\sqrt{p}), -1/2 \leq \operatorname{Re}(z) \leq 0\} \\ & \cup \{|z| > 1/\sqrt{p}, |z| > 1/(2\sqrt{p}), 0 \leq \operatorname{Re}(z) < 1/2\} \end{aligned} \quad (2)$$

is a fundamental domain for $\Gamma_0^*(p)$. (cf. [SH, SE]). Define $A_p^* := \mathbb{F}^*(p) \cap \{z \in \mathbb{C}; |z| = 1/\sqrt{p} \text{ or } |z| = 1/(2\sqrt{p})\}$.

In the present paper, we will apply the method of Rankin and Swinnerton-Dyer (RSD method) to the Eisenstein series associated with $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$. We have the following conjectures.

CONJECTURE 1.1. *Let $k \geq 4$ be an even integer. Then all of the zeros of $E_{k,5}^*(z)$ in $\mathbb{F}^*(5)$ lie on the arc A_5^* .*

CONJECTURE 1.2. *Let $k \geq 4$ be an even integer. Then all of the zeros of $E_{k,7}^*(z)$ in $\mathbb{F}^*(7)$ lie on the arc A_7^* .*

First, we prove that all but at most two zeros of $E_{k,p}^*(z)$ in $\mathbb{F}^*(p)$ lie on the arc A_p^* (see Sections 4.1 and 5.1). Second, if $(24/(p+1)) \mid k$, we prove that all of the zeros of $E_{k,p}^*(z)$ in $\mathbb{F}^*(p)$ lie on A_p^* (see Sections 4.2 and 5.2).

We can then prove that if $(24/(p+1)) \nmid k$, all but one of the zeros of $E_{k,p}^*(z)$ in $\mathbb{F}^*(p)$ lie on A_p^* . Furthermore, let $\alpha_5 \in [0, \pi]$ (respectively $\alpha_7 \in [0, \pi]$) be the angle which satisfies $\tan \alpha_5 = 2$ (respectively $\tan \alpha_7 = 5/\sqrt{3}$), and let $\alpha_{p,k} \in [0, \pi]$ be the angle which satisfies $\alpha_{p,k} \equiv k(\pi/2 + \alpha_p)/2 \pmod{\pi}$. Then, since α_p is an irrational multiple of π , $\alpha_{p,k}$ appear in the interval $[0, \pi]$ uniformly for all even integers $k \geq 4$. In Section 4.3, we prove that all of the zeros of $E_{k,5}^*(z)$ in $\mathbb{F}^*(5)$ are on A_5^* if $\alpha_{5,k} < (116/180)\pi$ or $(117/180)\pi < \alpha_{5,k}$. That is, we prove about 179/180 of Conjecture 1.1. Similarly, in Sections 5.3 and 5.4, we prove that all of the zeros of $E_{k,7}^*(z)$ in $\mathbb{F}^*(7)$ are on A_7^* if ' $\alpha_{7,k} < (127.68/180)\pi$ or $(128.68/180)\pi < \alpha_{7,k}$ for $k \equiv 2 \pmod{6}$ ' or ' $\alpha_{7,k} < (108.5/180)\pi$ or $(109.5/180)\pi < \alpha_{7,k}$ for $k \equiv 4 \pmod{6}$ '. Thus, we can also prove about 179/180 of Conjecture 1.2.

In [RSD], the authors considered a bound for the error terms R_1 (see (9)) in terms only of their absolute values. However, in the present paper, we also use the arguments of some terms in the series. We can then approach the exact value of the Eisenstein series.

A more detailed account of the material in the present study may be found in [SJ].

2. General theory

2.1. Preliminaries

Let $v_p(f)$ be the order of a modular function f at a point p .

2.1.1. $\Gamma_0^*(5)$. We define

$$A_{5,1}^* := \{z; |z| = 1/\sqrt{5}, \pi/2 < \text{Arg}(z) < \pi/2 + \alpha_5\},$$

$$A_{5,2}^* := \{z; |z + 1/2| = 1/(2\sqrt{5}), \alpha_5 < \text{Arg}(z) < \pi/2\}.$$

Then, $A_5^* = A_{5,1}^* \cup A_{5,2}^* \cup \{i/\sqrt{5}, \rho_{5,1}, \rho_{5,2}\}$, where $\rho_{5,1} := -1/2 + i/(2\sqrt{5})$ and $\rho_{5,2} := -2/5 + i/5$.

Let f be a modular form for $\Gamma_0^*(5)$ of weight k , and let k be an even integer such that $k \equiv 2 \pmod{4}$, then

$$f(i/\sqrt{5}) = f(W_5 i/\sqrt{5}) = i^k f(i/\sqrt{5}) = -f(i/\sqrt{5}).$$

Thus, we have $f(i/\sqrt{5}) = 0$. Similarly, we have $f(\rho_{5,1}) = f(\rho_{5,2}) = 0$. Thus, we have $v_{i/\sqrt{5}}(f) \geq 1$, $v_{\rho_{5,1}}(f) \geq 1$, and $v_{\rho_{5,2}}(f) \geq 1$.

On the other hand, if $k \equiv 0 \pmod{4}$, then we have $v_{i/\sqrt{5}}(E_{k,5}^*) = v_{\rho_{5,1}}(E_{k,5}^*) = v_{\rho_{5,2}}(E_{k,5}^*) = 0$.

2.1.2. $\Gamma_0^*(7)$. We define

$$A_{7,1}^* := \{z; |z| = 1/\sqrt{7}, \pi/2 < \text{Arg}(z) < \pi/2 + \alpha_7\},$$

$$A_{7,2}^* := \{z; |z + 1/2| = 1/(2\sqrt{7}), \alpha_7 - \pi/6 < \text{Arg}(z) < \pi/2\}.$$

Then, $A_7^* = A_{7,1}^* \cup A_{7,2}^* \cup \{i/\sqrt{7}, \rho_{7,1}, \rho_{7,2}\}$, where $\rho_{7,1} := -1/2 + i/(2\sqrt{7})$ and $\rho_{7,2} := -5/14 + \sqrt{3}i/14$.

Let f be a modular form for $\Gamma_0^*(7)$ of weight k . If $k \equiv 2 \pmod{4}$, then we have $v_{i/\sqrt{7}}(f) \geq 1$ and $v_{\rho_{7,1}}(f) \geq 1$. On the other hand, if $k \equiv 0 \pmod{4}$, then we have $v_{i/\sqrt{7}}(E_{k,7}^*) = v_{\rho_{7,1}}(E_{k,7}^*) = 0$.

Similarly, if $k \not\equiv 0 \pmod{6}$, then we have $v_{\rho_{7,2}}(f) \geq 1$, while if $k \equiv 0 \pmod{6}$, then we have $v_{\rho_{7,2}}(E_{k,7}^*) = 0$.

2.2. Valence formula

In order to determine the location of zeros of $E_{k,p}^*(z)$ in $\mathbb{F}^*(p)$, we require the valence formula for $\Gamma_0^*(p)$.

PROPOSITION 2.1. *Let f be a modular function of weight k for $\Gamma_0^*(5)$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_{i/\sqrt{5}}(f) + \frac{1}{2}v_{\rho_{5,1}}(f) + \frac{1}{2}v_{\rho_{5,2}}(f) + \sum_{\substack{p \in \Gamma_0^*(5) \setminus \mathbb{H} \\ p \neq i/\sqrt{5}, \rho_{5,1}, \rho_{5,2}}} v_p(f) = \frac{k}{4}. \quad (3)$$

PROPOSITION 2.2. *Let f be a modular function of weight k for $\Gamma_0^*(7)$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_{i/\sqrt{7}}(f) + \frac{1}{2}v_{\rho_{7,1}}(f) + \frac{1}{3}v_{\rho_{7,2}}(f) + \sum_{\substack{p \in \Gamma_0^*(7) \setminus \mathbb{H} \\ p \neq i/\sqrt{7}, \rho_{7,1}, \rho_{7,2}}} v_p(f) = \frac{k}{3}. \quad (4)$$

The proofs of the above propositions are very similar to that for the valence formula for $\mathrm{SL}_2(\mathbb{Z})$ (cf. [SE]).

2.3. Some Eisenstein series of low weights

By means of a straightforward calculation, we have the following propositions.

PROPOSITION 2.3. *The location of the zeros of the Eisenstein series $E_{k,5}^*$ in $\mathbb{F}^*(5)$ for $4 \geq k \geq 10$ are given by the following table:*

k	v_∞	$v_{i/\sqrt{5}}$	$v_{\rho_{5,1}}$	$v_{\rho_{5,2}}$	$V_{5,1}^*$	$V_{5,2}^*$
4	0	0	0	0	1	0
6	0	1	1	1	0	0
8	0	0	0	0	1	1
10	0	1	1	1	1	0

where $V_{5,n}^*$ denotes the number of simple zeros of the Eisenstein series $E_{k,5}^*$ on the arc $A_{5,n}^*$ for $n = 1, 2$.

PROPOSITION 2.4. *The location of the zeros of the Eisenstein series $E_{k,7}^*$ in $\mathbb{F}^*(7)$ for $k = 4, 6$, and 12 are given by the following table:*

k	v_∞	$v_{i/\sqrt{7}}$	$v_{\rho_{7,1}}$	$v_{\rho_{7,2}}$	V_7^*
4	0	0	0	1	1
6	0	1	1	0	1
12	0	0	0	0	4

where V_7^* denotes the number of simple zeros of the Eisenstein series $E_{k,7}^*$ on $A_{7,1}^* \cup A_{7,2}^*$.

2.4. The space of modular forms

Let $M_{k,p}^*$ be the space of modular forms for $\Gamma_0^*(p)$ of weight k , and let $M_{k,p}^{*0}$ be the space of cusp forms for $\Gamma_0^*(p)$ of weight k . Upon considering the map $M_{k,p}^* \ni f \mapsto f(\infty) \in \mathbb{C}$, it is clear that the kernel of this map is given by $M_{k,p}^{*0}$. So $\dim(M_{k,p}^*/M_{k,p}^{*0}) \leq 1$, and $M_{k,p}^* = \mathbb{C}E_{k,p}^* \oplus M_{k,p}^{*0}$. Let $\eta(z)$ be the Dedekind's η -function.

2.4.1. $\Gamma_0^*(5)$. Note that $\Delta_5 = \eta^4(z)\eta^4(5z)$ is a cusp form for $\Gamma_0^*(5)$ of weight 4. We have the following theorem.

THEOREM 2.1. *Let k be an even integer.*

- (1) *For $k < 0$ and $k = 2$, $M_{k,5}^* = 0$.*
- (2) *For $k = 0$ and 6, we have $M_{k,5}^{*0} = 0$, and $\dim(M_{k,5}^*) = 1$ with a base $E_{k,5}^*$.*
- (3) *$M_{k,5}^{*0} = \Delta_5 M_{k-4,5}^*$.*

The proof of this theorem is very similar to that for $\mathrm{SL}_2(\mathbb{Z})$. Furthermore, we have

$$\begin{aligned} M_{4n,5}^* &= \mathbb{C}(E_{4,5}^*)^n \oplus \mathbb{C}(E_{4,5}^*)^{n-1} \Delta_5 \oplus \cdots \oplus \mathbb{C}\Delta_5^n, \\ M_{4n+6,5}^* &= E_{6,5}^* ((E_{4,5}^*)^n \oplus \mathbb{C}(E_{4,5}^*)^{n-1} \Delta_5 \oplus \cdots \oplus \mathbb{C}\Delta_5^n). \end{aligned}$$

Thus, we have the following proposition.

PROPOSITION 2.5. *Let $k \geq 4$ be an even integer. For every $f \in M_{k,5}^*$, we have*

$$\begin{aligned} v_{i/\sqrt{5}}(f) &\geq s_k, \quad v_{\rho_{5,1}}(f) \geq s_k, \quad v_{\rho_{5,2}}(f) \geq s_k \\ (s_k = 0, 1 \text{ such that } 2s_k &\equiv k \pmod{4}). \end{aligned} \tag{5}$$

2.4.2. $\Gamma_0^*(7)$. We define $\Delta_7 := \eta^6(z)\eta^6(7z)$ and $E_{2,7}'(z) := (7E_2(7z) - E_2(z))/6$. We then have the following theorem.

THEOREM 2.2. *Let k be an even integer, and we define $\Delta_{7,4} := (5/16)((E_{2,7}')^2 - E_{4,7}^*)$ and $\Delta_{7,10}^0 := (559/690)((41\,065/137\,592)(E_{4,7}^*E_{6,7}^* - E_{10,7}^*) - E_{6,7}^*\Delta_{7,4})$.*

- (1) *$M_{k,7}^{*0} = M_{12,7}^{*0} M_{k-12,7}^*$.*
- (2) *For $k < 0$ and $k = 2$, $M_{k,7}^* = 0$. We have $M_{0,7}^* = \mathbb{C}$.*

(3) We have

$$\begin{aligned} M_{4,7}^{*0} &= \mathbb{C}\Delta_{7,4}, \quad M_{6,7}^{*0} = \mathbb{C}\Delta_{7,10}^0/\Delta_{7,4}, \\ M_{8,7}^{*0} &= \mathbb{C}(\Delta_{7,4})^2 \oplus \mathbb{C}E_{4,7}^*\Delta_{7,4}, \quad M_{10,7}^{*0} = \mathbb{C}\Delta_{7,10}^0 \oplus \mathbb{C}E_{6,7}^*\Delta_{7,4}, \\ M_{12,7}^{*0} &= \mathbb{C}(\Delta_7)^2 \oplus \mathbb{C}(\Delta_{7,4})^3 \oplus \mathbb{C}E_{4,7}^*(\Delta_{7,4})^2 \oplus \mathbb{C}(E_{4,7}^*)^2\Delta_{7,4}, \\ M_{14,7}^{*0} &= \mathbb{C}\Delta_{7,4}\Delta_{7,10}^0 \oplus \mathbb{C}E_{6,7}^*(\Delta_{7,4})^2 \oplus \mathbb{C}E_{4,7}^*E_{6,7}^*\Delta_{7,4}. \end{aligned}$$

We thus have the following table.

k	f	v_∞	$v_{i/\sqrt{7}}$	$v_{\rho_{7,1}}$	$v_{\rho_{7,2}}$	V_7^*
4	$(E_{2,7}')^2$	0	0	0	4	0
	$\Delta_{7,4}$	1	0	0	1	0
10	$\Delta_{7,10}^0$	2	1	1	1	0
12	$(\Delta_7)^2$	4	0	0	0	0

Furthermore, we have

$$\begin{aligned} M_{k,7}^* &= E_{k-12n,7}^* \{ \mathbb{C}(E_{4,7}^*)^{3n} \oplus (E_{4,7}^*)^{3(n-1)} M_{12,7}^{*0} \oplus (E_{4,7}^*)^{3(n-2)} (M_{12,7}^{*0})^2 \oplus \dots \\ &\quad \oplus (M_{12,7}^{*0})^n \} \oplus M_{k-12n,7}^{*0} (M_{12,7}^{*0})^n. \end{aligned}$$

Thus, we have the following proposition.

PROPOSITION 2.6. *Let $k \geq 4$ be an even integer. For every $f \in M_{k,7}^*$, we have*

$$\begin{aligned} v_{i/\sqrt{7}}(f) &\geq s_k, \quad v_{\rho_{7,1}}(f) \geq s_k \quad (s_k = 0, 1 \text{ such that } 2s_k \equiv k \pmod{4}), \\ v_{\rho_{7,2}}(f) &\geq t_k \quad (s_k = 0, 1, 2 \text{ such that } -2t_k \equiv k \pmod{6}). \end{aligned} \quad (6)$$

3. The RSD method

3.1. The method

Let $k \geq 4$ be an even integer. For $z \in \mathbb{H}$, we have

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k}. \quad (7)$$

Moreover, we have $\mathbb{F} = \{|z| \geq 1, -1/2 \leq \operatorname{Re}(z) \leq 0\} \cup \{|z| > 1, 0 \leq \operatorname{Re}(z) < 1/2\}$.

At the beginning of their proof in [RSD], Rankin and Swinnerton-Dyer considered the following function:

$$F_k(\theta) := e^{ik\theta/2} E_k(e^{i\theta}), \quad (8)$$

which is real for all $\theta \in [0, \pi]$. Considering the four terms with $c^2 + d^2 = 1$, they proved that

$$F_k(\theta) = 2 \cos(k\theta/2) + R_1, \quad (9)$$

where R_1 denotes the remaining terms of the series. Moreover, they showed that $|R_1| < 2$ for all $k \geq 12$. If $\cos(k\theta/2)$ is $+1$ or -1 , then $F_k(2m\pi/k)$ is positive or negative, respectively, and we can show the existence of the zeros. In addition, we can prove that all of the zeros of $E_k(z)$ in $\mathbb{F}$ lie on the unit circle using the *valence formula* and the theory on the space of modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

3.2. The function $F_{k,p,n}^*$

We expect all of the zeros of the Eisenstein series $E_{k,p}^*(z)$ in $\mathbb{F}^*(p)$ to lie on the arcs $e^{i\theta}/\sqrt{p}$ and $e^{i\theta}/(2\sqrt{p}) - 1/2$, which form the boundary of the fundamental domain defined by (2).

We define

$$F_{k,p,1}^*(\theta) := e^{ik\theta/2} E_{k,p}^*(e^{i\theta}/\sqrt{p}), \quad (10)$$

$$F_{k,p,2}^*(\theta) := e^{ik\theta/2} E_{k,p}^*(e^{i\theta}/2\sqrt{p} - 1/2). \quad (11)$$

We can write

$$F_{k,p,1}^*(\theta) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c}} (ce^{i\theta/2} + \sqrt{p}de^{-i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c}} (ce^{-i\theta/2} + \sqrt{p}de^{i\theta/2})^{-k}, \quad (12)$$

$$\begin{aligned} F_{k,p,2}^*(\theta) &= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \ 2 \nmid cd}} (ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \ 2 \nmid cd}} (ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2})^{-k} \\ &\quad + \frac{2^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \ 2 \nmid cd}} (ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2})^{-k} \\ &\quad + \frac{2^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \ 2 \nmid cd}} (ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2})^{-k}. \end{aligned} \quad (13)$$

Hence, we can use these expressions as definitions. Note that $(ce^{i\theta/2} + \sqrt{p}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2} + \sqrt{p}de^{i\theta/2})^{-k}$ are conjugates of each other for any pair (c, d) . Thus, we have the following proposition.

PROPOSITION 3.1. $F_{k,p,1}^*(\theta)$ is real, for all $\theta \in [0, \pi]$.

PROPOSITION 3.2. $F_{k,p,2}^*(\theta)$ is real, for all $\theta \in [0, \pi]$.

Now, we define

$$F_{k,5}^*(\theta) = \begin{cases} F_{k,5,1}^*(\theta) & \pi/2 \leq \theta \leq \pi/2 + \alpha_5, \\ F_{k,5,2}^*(\theta - \pi/2) & \pi/2 + \alpha_5 \leq \theta \leq \pi. \end{cases}$$

Then, $F_{k,5}^*$ is continuous in the interval $[\pi/2, \pi]$. Note that $F_{k,5,1}^*(\pi/2 + \alpha_5) = e^{i(\pi/2)k/2} F_{k,5,2}^*(\alpha_5)$.

Similarly,

$$F_{k,7}^*(\theta) = \begin{cases} F_{k,7,1}^*(\theta) & \pi/2 \leq \theta \leq \pi/2 + \alpha_7, \\ F_{k,7,2}^*(\theta - 2\pi/3) & \pi/2 + \alpha_7 \leq \theta \leq 7\pi/6. \end{cases}$$

Whereupon $F_{k,7}^*$ is continuous in the interval $[\pi/2, 7\pi/6]$. Note also that $F_{k,7,1}^*(\pi/2 + \alpha_7) = e^{i(2\pi/3)k/2} F_{k,7,2}^*(\alpha_7 - \pi/6)$.

3.3. Application of the RSD method

We introduce $N := c^2 + d^2$. First, we consider the case $N = 1$. For this case, we can write

$$F_{k,p,1}^*(\theta) = 2 \cos(k\theta/2) + R_{p,1}^*, \quad (14)$$

$$F_{k,p,2}^*(\theta) = 2 \cos(k\theta/2) + R_{p,2}^*, \quad (15)$$

where $R_{p,1}^*$ and $R_{p,2}^*$ denote the terms satisfying $N > 1$ of $F_{k,p,1}^*$ and $F_{k,p,2}^*$, respectively.

3.3.1. For $\Gamma_0^*(5)$. For $R_{5,1}^*$, we consider the cases $N = 2, 5, 10, 13, 17$, and $N \geq 25$. Considering $-2/\sqrt{5} \leq \cos \theta \leq 0$, we have

$$|R_{5,1}^*| \leq 2 + 4\left(\frac{1}{2}\right)^{k/2} + 2\left(\frac{1}{3}\right)^{k/2} + \cdots + 2\left(\frac{1}{9}\right)^k + \frac{384\sqrt{6}}{k-3}\left(\frac{1}{2}\right)^k. \quad (16)$$

Similarly, for $R_{5,2}^*$, we consider the cases $N = 2, 5, 10, 13, 17, 25, 26, 29$, and $N \geq 34$. For these cases we have

$$|R_{5,2}^*| \leq 2 + 2\left(\frac{2}{3}\right)^{k/2} + 2\left(\frac{1}{2}\right)^{k/2} + \cdots + 2\left(\frac{1}{129}\right)^{k/2} + \frac{2112\sqrt{33}}{k-3}\left(\frac{8}{33}\right)^{k/2}. \quad (17)$$

We want to show that $|R_{5,1}^*| < 2$ and $|R_{5,2}^*| < 2$. Note that the case for which $(c, d) = \pm(2, 1)$ (respectively $(c, d) = \pm(1, -1)$) yields a bound equal to 2 for $|R_{5,1}^*|$ (respectively $|R_{5,2}^*|$).

3.3.2. $\Gamma_0^*(7)$. For $R_{7,1}^*$, we consider the cases $N = 2, 5, 10, \dots, 61$, and $N \geq 65$. Then, we have

$$|R_{7,1}^*| \leq 4 + 6\left(\frac{1}{3}\right)^{k/2} + 4\left(\frac{1}{7}\right)^{k/2} + \dots + 2\left(\frac{1}{352}\right)^{k/2} + \frac{28\,160}{7(k-3)}\left(\frac{11}{64}\right)^{k/2}. \quad (18)$$

Similarly, for $R_{7,2}^*$, we consider the cases $N = 2, 5, 10, \dots, 89$, and $N \geq 97$. For these cases we have

$$|R_{7,2}^*| \leq 4 + 2\left(\frac{1}{2}\right)^{k/2} + 2\left(\frac{1}{3}\right)^{k/2} + \dots + 2\left(\frac{1}{571}\right)^{k/2} + \frac{62\,464\sqrt{6}}{21(k-3)}\left(\frac{1}{8}\right)^{k/2}. \quad (19)$$

Note that the cases $(c, d) = \pm(2, 1)$ and $\pm(3, 1)$ (respectively $(c, d) = \pm(1, -1)$ and $\pm(3, -1)$) yield a bound equal to 4 for $|R_{7,1}^*|$ (respectively $|R_{7,2}^*|$).

3.4. Arguments of some terms

3.4.1. $\Gamma_0^*(5)$. In the previous section, the important point was the fact that the cases $(c, d) = \pm(2, 1)$ and $(c, d) = \pm(1, -1)$ do not yield good bounds for $|R_{5,1}^*|$ and $|R_{5,2}^*|$, respectively.

Let $\theta_1' := 2 \operatorname{Arg}\{2e^{i\theta_1/2} + \sqrt{5}e^{-i\theta_1/2}\}$ and $\theta_2' := 2 \operatorname{Arg}\{-e^{i\theta_2/2} + \sqrt{5}e^{-i\theta_2/2}\}$, then we have

$$\tan \theta_1'/2 = -\frac{\sqrt{5}-2}{\sqrt{5}+2} \tan \theta_1/2, \quad \tan \theta_2'/2 = -\frac{\sqrt{5}+1}{\sqrt{5}-1} \tan \theta_2/2.$$

Furthermore, it is easy to show that

$$\begin{aligned} \tan(-\pi + \pi/2 + \alpha_5 + d_1(t\pi/k))/2 &< -\frac{\sqrt{5}-2}{\sqrt{5}+2} \tan(\pi/2 + \alpha_5 - (t\pi/k))/2, \\ &< \tan(-\pi + \pi/2 + \alpha_5 + (t\pi/k))/2, \end{aligned}$$

where $d_1 < 1/(1 + 4 \tan(t/2)(\pi/k))$. Thus,

$$\begin{aligned}\theta_1 &= \pi/2 + \alpha_5 - (t\pi/k) \\ \Rightarrow -\pi + \pi/2 + \alpha_5 + d_1(t\pi/k) &< \theta_1' < -\pi + \pi/2 + \alpha_5 + (t\pi/k), \\ k\theta_1/2 &= k(\pi/2 + \alpha_5)/2 - (t/2)\pi \\ \Rightarrow -(k/2)\pi + k(\pi/2 + \alpha_5)/2 + d_1(t/2)\pi &< k\theta_1'/2 \\ &< -(k/2)\pi + k(\pi/2 + \alpha_5)/2 + (t/2)\pi.\end{aligned}$$

Similarly,

$$\begin{aligned}k\theta_2/2 &= k\alpha_5/2 + (t/2)\pi \\ \Rightarrow -(k/2)\pi + k\alpha_5/2 - (t/2)\pi &< k\theta_2'/2 < -(k/2)\pi + k\alpha_5/2 - d_2(t/2)\pi,\end{aligned}$$

where $d_2 < 1/(1 + \tan(t/2)(\pi/k))$.

Note that

$$-(k/2)\pi \equiv \begin{cases} 0 & (k \equiv 0 \pmod{4}) \\ \pi & (k \equiv 2 \pmod{4}) \end{cases} \pmod{2\pi},$$

and both d_1 and d_2 tend to 1 in the limit as k tends to ∞ , or in the limit as t tends to 0.

Recall that $\alpha_{5,k} \equiv k(\pi/2 + \alpha_5)/2 \pmod{\pi}$, then we can write $k(\pi/2 + \alpha_5)/2 = \alpha_{5,k} + m\pi$ for some integer m . We define $\alpha_{5,k}' \equiv k\theta_1'/2 - m\pi \pmod{2\pi}$ for $\theta_1 = \pi/2 + \alpha_5 - (t\pi/k)$.

Similarly, we define $\beta_{5,k} \equiv k\alpha_5/2 \pmod{\pi}$ and $\beta_{5,k}' \equiv k\theta_2'/2 - (k\alpha_5/2 - \beta_{5,k}) \pmod{2\pi}$ for $\theta_2 = \alpha_5 + (t\pi/k)$.

3.4.2. $\Gamma_0^*(7)$. Similarly to the previous section, we consider the arguments of some terms such that $(c, d) = \pm(2, 1)$ and $\pm(3, 1)$ for $|R_{7,1}^*|$, and $(c, d) = \pm(1, -1)$ and $\pm(3, -1)$ for $|R_{7,2}^*|$.

Let $\theta_{1,1}' := 2 \operatorname{Arg}\{2e^{i\theta_1/2} + \sqrt{7}e^{-i\theta_1/2}\}$, $\theta_{1,2}' := 2 \operatorname{Arg}\{3e^{i\theta_1/2} + \sqrt{7}e^{-i\theta_1/2}\}$, $\theta_{2,1}' := 2 \operatorname{Arg}\{-e^{i\theta_2/2} + \sqrt{7}e^{-i\theta_2/2}\}$, and $\theta_{2,2}' := 2 \operatorname{Arg}\{-3e^{i\theta_2/2} + \sqrt{7}e^{-i\theta_2/2}\}$.

We have

$$\begin{aligned}k\theta_1/2 &= k(\pi/2 + \alpha_7)/2 - (t/2)\pi \\ \Rightarrow \begin{cases} (k/3)\pi + k(\pi/2 + \alpha_7)/2 + d_{1,1}(t/2)\pi \\ < k\theta_{1,1}'/2 < (k/3)\pi + k(\pi/2 + \alpha_7)/2 + (3t/2)\pi, \\ -(k/3)\pi + k(\pi/2 + \alpha_7)/2 - t\pi \\ < k\theta_{1,2}'/2 < -(k/3)\pi + k(\pi/2 + \alpha_7)/2 - d_{1,2}(t/2)\pi, \end{cases}\end{aligned}$$

$$\begin{aligned}
k\theta_2/2 &= k(\alpha_7 - \pi/6)/2 + (t/2)\pi \\
&\Rightarrow \begin{cases} -(k/3)\pi + k(\alpha_7 - \pi/6)/2 - (3t/4)\pi \\ < k\theta_{2,1}'/2 < -(k/3)\pi + k(\alpha_7 - \pi/6)/2 - d_{2,1}(t/2)\pi, \\ (k/3)\pi + k(\alpha_7 - \pi/6)/2 + d_{2,2}(t/2)\pi \\ < k\theta_{2,2}'/2 < (k/3)\pi + k(\alpha_7 - \pi/6)/2 + (t/4)\pi, \end{cases}
\end{aligned}$$

where $d_{1,1} < 3/(1 + 2\sqrt{3}\tan(t/2)(\pi/k))$, $d_{1,2} < 2/(1 + \sqrt{3}\tan(t/2)(\pi/k))$, $d_{2,1} < 3/(2 + \sqrt{3}\tan(t/2)(\pi/k))$, and $d_{2,2} < 1/(2 + 3\sqrt{3}\tan(t/2)(\pi/k))$.

Note that

$$(k/3)\pi \equiv \begin{cases} 0 & (k \equiv 0 \pmod{6}) \\ 2\pi/3 & (k \equiv 2 \pmod{6}) \\ 4\pi/3 & (k \equiv 4 \pmod{6}) \end{cases}, \quad -(k/3)\pi \equiv \begin{cases} 0 & (k \equiv 0 \pmod{6}) \\ 4\pi/3 & (k \equiv 2 \pmod{6}) \\ 2\pi/3 & (k \equiv 4 \pmod{6}) \end{cases},$$

modulo 2π . Furthermore, in the limit as k tends to ∞ or in the limit as t tends to 0, $d_{1,1}$ (respectively $d_{1,2}$, $d_{2,1}$, and $d_{2,2}$) tends to 3 (respectively 2, $3/2$, and $1/2$).

Recall that $\alpha_{7,k} \equiv k(\pi/2 + \alpha_7)/2 \pmod{\pi}$. Then, we define $\alpha_{7,k,n}' \equiv k\theta_{1,n}'/2 - (k(\pi/2 + \alpha_7)/2 - \alpha_{7,k}) \pmod{2\pi}$ for $n = 1, 2$ and for $\theta_1 = \pi/2 + \alpha_7 - (t\pi/k)$.

Similarly, we define $\beta_{7,k} \equiv k(\alpha_7 - \pi/6)/2 \pmod{\pi}$ and $\beta_{7,k,n}' \equiv k\theta_{2,n}'/2 - (k(\alpha_7 - \pi/6)/2 - \beta_{7,k}) \pmod{2\pi}$ for $n = 1, 2$ and for $\theta_2 = \alpha_7 - \pi/6 + (t\pi/k)$.

3.5. Algorithm

In this section, we consider the bound

$$|R_{p,n}^*| < 2c_0 \quad \text{for every } k \geq k_0, \quad (20)$$

for some $c_0 > 0$ and an even integer k_0 . Furthermore, we detail an algorithm that can be used to derive the above bound.

Let Λ be an index set, and, applying the RSD method, let us write

$$|R_{p,n}^*| \leq 2 \sum_{\lambda \in \Lambda} e_{\lambda}^k v_k(c_{\lambda}, d_{\lambda}, \theta),$$

where the factor ‘2’ comes from the relation $v_k(c, d, \theta) = v_k(-c, -d, \theta)$. Furthermore, let I be a finite subset of Λ such that $e_i^k v_k(c_i, d_i, \theta)$ does not tend to 0 in the limit as k tends to ∞ for all $i \in I$, and assume $I \subset \mathbb{N}$. Then, we define $X_i := e_i^{-2} v_k(c_i, d_i, \theta)^{-2/k}$ for every $i \in I$.

Assume that for every $i \in I$ and $k \geq k_0$ and for some c_i' and u_i ,

$$\begin{aligned} |\operatorname{Re}\{e_i^k(c_i e^{i\theta/2} + \sqrt{p}d_i e^{-i\theta/2})^{-k}\}| &\leq c_i' X_i^{-k/2}, \\ X_i^{-k/2} &\geq 1 + u_i(\pi/k), \quad 2 \sum_{\lambda \in \Lambda \setminus I} e_\lambda^k v_k(c_\lambda, d_\lambda, \theta) \leq b(1/s)^{k/2}, \end{aligned}$$

and let the number $t > 0$ be given.

Step 1. ‘Determine the number a_1 ’.

First, in order to show the bound (20), we wish to use the bound

$$\sum_{i \in I} c_i' X_i^{-k/2} < c_0 - a_1(t\pi/k)^2 \quad (21)$$

for every $i \in I$ and $k \geq k_0$ and for some $a_1 > 0$.

To show the bound (20) by the above bound (21), we need $b(1/s)^{k/2} < a_1(t\pi/k)^2$ for every $k \geq k_0$. Define $f(k) := s^{k/2}/b - k^2/(2a_1 t^2 \pi^2)$. If we have $k_0 \log s > 4$ and

$$a_1 > (bk_0^2)/(2s^{k_0/2} t^2 \pi^2), \quad (22)$$

then we have $f(k_0) > 0$, $f'(k_0) > 0$, and $f''(k_0) > 0$. In the present paper, we always have $k_0 \log s > 4$. Thus, it is enough to consider the bound (22).

Step 2. ‘Determine the number $c_{0,i}$ and $a_{1,i}$ ’.

Second, to show bound (21), we wish to use the bounds

$$c_i' X_i^{-k/2} < c_{0,i} - a_{1,i}(t\pi/k)^2 \quad (23)$$

for every $i \in I$ and $k \geq k_0$ and for some $c_{0,i} > 0$, $a_{1,i} > 0$.

We determine $c_{0,i}$ and $a_{1,i}$ such that

$$c_{0,i} > 0, \quad a_{1,i} > 0, \quad c_0 = \sum_{i \in I} c_{0,i}, \quad \text{and} \quad a_1 = \sum_{i \in I} a_{1,i}. \quad (24)$$

Step 3. ‘Determine a discriminant Y_i for every $i \in I$ ’.

Finally, for the bound (23), we consider following sufficient conditions:

$$X_i^{k/2} > c_i + a_{2,i}(t\pi/k)^2, \quad X_i > a_{3,i} + a_{4,i}(t\pi/k)^2.$$

For the former bound, it is enough to show that

$$c_i = c_i'/c_{0,i}, \quad a_{2,i} > c_i^2(a_{1,i}/c_i')/\{1 - c_i(a_{1,i}/c_i')(t\pi/k_0)^2\}, \quad (25)$$

while for the latter bound, it is enough to show that

$$a_{3,i} = c_i^{2/k_0}, \quad a_{4,i} = ((2a_{2,i})/(c_i k_0))c_i^{2/k_0}.$$

Because we have

$$\begin{aligned}
 c_i^{2/k} &\leq 1 + 2(\log c_i)/k + 2(\log c_i)^2 c_i^{2/k}/k^2, \\
 X_i - \left(a_{3,i} + a_{4,i} \left(t \frac{\pi}{k} \right)^2 \right) \\
 &\geq \frac{1}{k} \left\{ u_i \pi - 2 \log c_i - 2(\log c_i)^2 c_i^{2/k} \frac{1}{k_0} - \frac{2a_{2,i} t^2 \pi^2}{c_i} c_i^{2/k_0} \frac{1}{k_0^2} \right\} \\
 &=: \frac{1}{k} \times Y_i.
 \end{aligned} \tag{26}$$

In conclusion, if we have $Y_i > 0$, then the bounds (23), (21), and (20) hold.

Note that the above bounds are sufficient conditions; they are not always necessary.

4. $\Gamma_0^*(5)$ (for Conjecture 1.1)

The proof of Conjecture 1.1 is significantly more difficult than the proof of the theorems for $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$. The most difficult point concerns the argument $\text{Arg}(\rho_{5,2})$, which is not a rational multiple of π .

4.1. All but at most two zeros

We have the following lemma.

LEMMA 4.1. *We have the following bounds.*

‘We have $|R_{5,1}^*| < 2 \cos(c_0' \pi)$ for $\theta_1 \in [\pi/2, \pi/2 + \alpha_5 - t\pi/k]$ ’.

(1) For $k \geq 12$, $(c_0', t) = (1/3, 1/6)$.

(2) For $k \geq 58$, $(c_0', t) = (33/80, 9/40)$.

‘We have $|R_{5,2}^*| < 2 \cos(c_0' \pi)$ for $\theta_2 \in [\alpha_5 + t\pi/k, \pi/2]$ ’.

(3) For $k \geq 12$, $(c_0', t) = (0, 1/2)$.

(4) For $k \geq 22$, $(c_0', t) = (1/3, 1/2)$.

(5) For $k \geq 46$, $(c_0', t) = (7/30, 1/5)$.

Proof. (3) Let $k \geq 12$ and $x = \pi/(2k)$, then $0 \leq x \leq \pi/24$, and so $1 - \cos x \geq (32/33)x^2$. Thus, we have

$$\begin{aligned}
 \frac{1}{4} |e^{i\theta/2} - \sqrt{5} e^{-i\theta/2}|^2 &\geq \frac{1}{4} (6 - 2\sqrt{5} \cos(\alpha_5 + x)) \geq 1 + \frac{16}{11} x^2, \\
 \frac{1}{2^k} |e^{i\theta/2} + \sqrt{5} e^{-i\theta/2}|^k &\geq 1 + \frac{96}{11} x^2 (k \geq 12), \\
 2^k \cdot 2v_k(1, 1, \theta) &\leq 2 - \frac{288\pi^2}{\pi^2 + 66} \frac{1}{k^2}.
 \end{aligned}$$

In inequality(17), replace 2 with the bound

$$2 - \frac{288\pi^2}{\pi^2 + 66} \frac{1}{k^2}.$$

Then

$$|R_{5,2}^*| \leq 2 - \frac{288\pi^2}{\pi^2 + 66} \frac{1}{k^2} + 2\left(\frac{2}{3}\right)^{k/2} + \cdots + 2\left(\frac{1}{129}\right)^{k/2} + \frac{2112\sqrt{33}}{k-3} \left(\frac{8}{33}\right)^{k/2}.$$

Furthermore, $(2/3)^{k/2}$ decreases more rapidly in k than $1/k^2$, and for $k \geq 12$, we have

$$|R_{5,2}^*| \leq 1.9821 \dots$$

(1), (2), (4), (5) We use the algorithm in the Section 3.5. Furthermore, we have $X_1 = v_k(2, 1, \theta_1)^{-2/k} \geq 1 + 4t(\pi/k)$ in the proof of (1) and (2), and we have $X_1 = (1/4) v_k(1, -1, \theta_2)^{-2/k} \geq 1 + t(\pi/k)$ in the proof of (4) and (5). We have $c_0 = c_{0,1} \leq \cos(c_0'\pi)$. We can show $Y_1 > 0$ for every item. $\square$

When $4 \mid k$, by the valence formula for $\Gamma_0^*(5)$ and Proposition 2.5, we have at most $k/4$ zeros on the arc A_5^* . We have $k/4 + 1$ integer points (i.e. $\cos(k\theta/2) = \pm 1$) in the interval $[\pi/2, \pi]$. By the above lemma's conditions (1) and (3), we can prove $|R_{5,1}^*| < 2$ or $|R_{5,2}^*| < 2$ at all but at most one integer point. Then, we have all but at most two zeros on A_5^* .

On the other hand, when $4 \nmid k$, we have at most $(k-6)/4$ zeros on the arc A_5^* . Similarly to the previous case, we have all but at most two zeros on A_5^* .

Thus, we have the following proposition.

PROPOSITION 4.1. *Let $k \geq 4$ be an even integer. Then all but at most two of the zeros of $E_{k,5}^*(z)$ in $\mathbb{F}^*(5)$ lie on the arc A_5^* .*

4.2. The case $4 \mid k$

For $\pi/12 < \alpha_{5,k} < 3\pi/4$, by Lemma 4.1(1) and (3), we can prove $|R_{5,1}^*| < 2$ or $|R_{5,2}^*| < 2$ at all of the integer points.

Now, we can write

$$\begin{aligned} F_{k,5,1}^*(\theta_1) &= 2 \cos(k\theta_1/2) + 2 \operatorname{Re}(2e^{-i\theta_1/2} + \sqrt{5}e^{i\theta_1/2})^{-k} + R_{5,1}^{*'} \\ F_{k,5,2}^*(\theta_2) &= 2 \cos(k\theta_2/2) + 2^k \cdot 2 \operatorname{Re}(e^{-i\theta_2/2} - \sqrt{5}e^{i\theta_2/2})^{-k} + R_{5,2}^{*'} \end{aligned}$$

For $0 < \alpha_{5,k} < \pi/12$, the last integer point of $F_{k,5,1}^*(\theta_1)$ is in the interval $[\pi/2 + \alpha_5 - \pi/(6k), \pi/2 + \alpha_5]$. We have $|R_{5,1}^{*'}| < 2$ for $\theta_1 \in [\pi/2, \pi/2 + \alpha_5]$.

Furthermore, because $0 < \alpha_{5,k}' < \pi/6$ for $0 < t < 1/6$, we have $\text{Sign}\{\cos(k\theta_1/2)\} = \text{Sign}\{\text{Re}(2e^{-i\theta_1/2} + \sqrt{5}e^{i\theta_1/2})^{-k}\}$ for $\theta_1 \in [\pi/2 + \alpha_5 - \pi/(6k), \pi/2 + \alpha_5]$.

For $3\pi/4 < \alpha_{5,k} < \pi$, the first integer point of $F_{k,5,2}^*(\theta_2)$ is in the interval $[\alpha_5, \alpha_5 + \pi/(2k)]$. We have $|R_{5,2}^*| < 2$ and $\text{Sign}\{\cos(k\theta_2/2)\} = \text{Sign}\{\text{Re}(e^{-i\theta_2/2} - \sqrt{5}e^{i\theta_2/2})^{-k}\}$ for $\theta_2 \in [\alpha_5, \alpha_5 + \pi/(2k)]$.

Thus, we have the following proposition.

PROPOSITION 4.2. *Let $k \geq 4$ be an integer which satisfies $4 \mid k$. Then all of the zeros of $E_{k,5}^*(z)$ in $\mathbb{F}^*(5)$ lie on the arc A_5^* .*

4.3. The case $4 \nmid k$

4.3.1. *The case $0 < \alpha_{5,k} < \pi/2$.* Now, at most two zeros remain. At the point such that $k\theta_1/2 = k(\pi/2 + \alpha_5)/2 - \alpha_{5,k} - \pi/3$, we have $|R_{5,1}^*| < 1$ by Lemma 4.1(1), and we have $2\cos(k\theta_1/2) = \pm 1$. Then, we have at least one zero between the second to last integer point for $A_{5,1}^*$ and the point $k\theta_1/2$. Similarly, by Lemma 4.1(4), we have at least one zero between the second integer point and the point $k\theta_2/2 = k\alpha_5/2 + \beta_{5,k} + \pi/3$.

4.3.2. *The case $\pi/2 < \alpha_{5,k} < \pi$.* For this case, we expect one more zero between the last integer point for $A_{5,1}^*$ and the first integer point for $A_{5,2}^*$. We consider the following cases.

(i) ‘The case $7\pi/10 < \alpha_{5,k} < \pi$ ’.

- For $3\pi/4 < \alpha_{5,k} < \pi$, we can use Lemma 4.1(1).
- For $7\pi/10 < \alpha_{5,k} < 3\pi/4$, we can use Lemma 4.1(2).

For each case, we consider the point such that $k\theta_1/2 = k(\pi/2 + \alpha_5)/2 - \alpha_{5,k} + \pi - c_0'\pi$. We have $\alpha_{5,k} - \pi + c_0'\pi > (t/2)\pi$ and $|R_{5,1}^*| < 2\cos(c_0'\pi)$, and we have $2\cos(k\theta_1/2) = \pm 2\cos(c_0'\pi)$. Then, we have at least one zero between the second to last integer point for $A_{5,1}^*$ and the point $k\theta_1/2$.

(ii) ‘The case $\pi/2 < \alpha_{5,k} < 19\pi/30$ ’.

- For $\pi/2 < \alpha_{5,k} < 7\pi/12$, we can use Lemma 4.1(4).
- For $7\pi/12 < \alpha_{5,k} < 19\pi/30$, we can use Lemma 4.1(5).

Similar to the case (i) above, we consider the point such that $k\theta_2/2 = k\alpha_5/2 - \beta_{5,k} + c_0'\pi$ for each case.

(iii) ‘The case $13\pi/20 < \alpha_{5,k} < 7\pi/10$ ’.

We have $X_1 = v_k(2, 1, \theta_1)^{-2/k} \geq 1 + 4t(\pi/k)$, and let $\cos(c_0'\pi) = -\cos((x/180)\pi - (t/2)\pi)$. Then, using the algorithm of Section 3.5, we prove ‘For $(x/180)\pi < \alpha_{5,k} < (y/180)\pi$, we have $|R_{5,1}^*| < 2\cos(c_0'\pi)$ for $\theta_1 = \pi/2 + \alpha_5 - t\pi/k$ ’ for ten cases, namely, $(x, y, t) = (121, 126, 3/20)$, $(120, 121, 1/10)$, $(118.8, 120, 1/10)$, $(118.1, 118.8, 2/25)$, $(117.7, 118.1, 1/15)$, $(117.45, 117.7, 3/50)$, $(117.27, 117.45, 1/20)$, $(117.15, 117.27, 9/200)$, $(117.06, 117.15, 1/25)$, $(117, 117.06, 1/25)$.

For each case, we consider the point such that $k\theta_1/2 = k(\pi/2 + \alpha_5)/2 - (t/2)\pi$. We have $\alpha_{5,k} - \pi + c_0'\pi > (t/2)\pi$ and $|R_{5,1}^*| < 2\cos(c_0'\pi)$, and we have $|2\cos(k\theta_1/2)| > 2\cos(c_0'\pi)$. Then, we have at least one zero between the second to last integer point for $A_{5,1}^*$ and the point $k\theta_1/2$.

(iv) ‘The case $19\pi/30 < \alpha_{5,k} < 29\pi/45$ ’.

We have $X_1 = (1/4) v_k(1, -1, \theta_2)^{-2/k} \geq 1 + t(\pi/k)$ and $\cos(c_0'\pi) = \cos((y/180)\pi - \pi/2 + (t/2)\pi)$. Then, we prove ‘For $(x/180)\pi < \alpha_{5,k} < (y/180)\pi$, we have $|R_{5,2}^*| < 2\cos(c_0'\pi)$ for $\theta_2 = \alpha_5 + t\pi/k$ ’ for three cases, namely, $(x, y, t) = (114, 115.4, 4/25)$, $(115.4, 115.8, 3/25)$, $(115.8, 116, 1/10)$.

Similar to the case (iii) above, we consider the point such that $k\theta_2/2 = k\alpha_5/2 + (t/2)\pi$ for each case.

In conclusion, we have the following proposition.

PROPOSITION 4.3. *Let $k \geq 4$ be an integer which satisfies $4 \nmid k$, and let $\alpha_{5,k} \in [0, \pi]$ be the angle which satisfies $\alpha_{5,k} \equiv k(\pi/2 + \alpha_5)/2 \pmod{\pi}$. If we have $\alpha_{5,k} < 29\pi/45$ or $13\pi/20 < \alpha_{5,k}$, then all of the zeros of $E_{k,5}^*(z)$ in $\mathbb{F}^*(5)$ lie on the arc A_5^* . Otherwise, all but at most one zero of $E_{k,5}^*(z)$ in $\mathbb{F}^*(5)$ lie on A_5^* .*

4.4. The remaining case ‘ $4 \nmid k$ and $29\pi/45 < \alpha_{5,k} < 13\pi/20$ ’

In the previous section, we left one zero between the last integer point for $A_{5,1}^*$ and the first one for $A_{5,2}^*$ for the case of ‘ $4 \nmid k$ and $29\pi/45 < \alpha_{5,k} < 13\pi/20$ ’. For the cases of $13\pi/20 < \alpha_{5,k} < 7\pi/10$ and $19\pi/30 < \alpha_{5,k} < 29\pi/45$, the width $|x - y|$ becomes smaller as the intervals of the bounds approach the interval $[29\pi/45, 13\pi/20]$. It seems that the width $|x - y|$ needs to be smaller still if we are to prove Conjecture 1.1 for the remaining interval $[29\pi/45, 13\pi/20]$. Furthermore, we may need to split infinite cases (x, y) such as we saw in the previous section. Thus, we cannot prove the conjecture for this remaining case in a similar manner. However, when k is large enough, there is a possibility that we can prove the conjecture for this remaining case.

Let $29\pi/45 < \alpha_{5,k} < 13\pi/20$, and let $t > 0$ be small enough. Then, we have $\pi/2 < \alpha_{5,k} - (t/2)\pi < \pi$ and $3\pi/2 < \pi + \alpha_{5,k} + d_1(t/2)\pi < \alpha_{5,k}' < \pi + \alpha_{5,k} + (t/2)\pi < 2\pi$. Moreover, we can easily show that $1 + 4t(\pi/k) \leq v_k(2, 1, \theta_1)^{-2/k} \leq e^{4t(\pi/k)}$. Thus, we have

$$\begin{aligned} & -\cos(\alpha_{5,k} - (t/2)\pi) - \cos(\pi + \alpha_{5,k} + d_1(t/2)\pi) \cdot e^{-2\pi t} \\ & > |\cos(k\theta_1/2)| - |\operatorname{Re}\{(2e^{i\theta_1/2} + \sqrt{5}e^{-i\theta_1/2})^{-k}\}| \\ & > -\cos(\alpha_{5,k} - (t/2)\pi) - \cos(\pi + \alpha_{5,k} + (t/2)\pi) \cdot (1 + 4t(\pi/k))^{-k/2}. \end{aligned}$$

We denote the upper bound by A and the lower bound by B . Furthermore, we define $A' := A/\cos(\pi + \alpha_{5,k} + d_1(t/2)\pi)$ and $B' := B/\cos(\pi + \alpha_{5,k} + (t/2)\pi)$. First, we have $A|_{t=0} = B|_{t=0} = 0$. Second, we have

$$\frac{\partial}{\partial t} A'|_{t=0} = \frac{\partial}{\partial t} B'|_{t=0} = \pi(\tan \alpha_{k,5} + 2).$$

Finally, we have $B > 0$ if $\alpha_{5,k} > \pi - \alpha_5$, and we have $A < 0$ if $\alpha_{5,k} < \pi - \alpha_5$ for small enough t .

Similarly, we consider the lower and the upper bounds of $|\cos(k\theta_2/2)| - |\operatorname{Re}\{2^k(e^{i\theta_2/2} - \sqrt{5}e^{-i\theta_2/2})^{-k}\}|$. The lower bound is positive if $\alpha_{5,k} < \pi - \alpha_5$, while the upper bound is negative if $\alpha_{5,k} > \pi - \alpha_5$ for small enough t .

In conclusion, if $4 \nmid k$ is large enough, then $|R_{5,1}^{*'}|$ and $|R_{5,2}^{*'}|$ are small enough, and then we have one more zero on the arc $A_{5,1}^*$ when $\alpha_{5,k} > \pi - \alpha_5$, and one more zero on the arc $A_{5,2}^*$ when $\alpha_{5,k} < \pi - \alpha_5$. However, if k is small, a method of proving the conjecture for this case is not clear.

5. $\Gamma_0^*(7)$ (for Conjecture 1.2)

Similar to the case of $\Gamma_0^*(5)$, proving Conjecture 1.2 is also difficult. The most difficult point is again the argument $\operatorname{Arg}(\rho_{7,2})$.

5.1. All but at most two zeros

We have the following lemma.

LEMMA 5.1. *We have the following bounds.*

'We have $|R_{7,1}^| < 2 \cos(c_0'\pi)$ for $\theta_1 \in [\pi/2, \pi/2 + \alpha_7 - t\pi/k]$ '.*

- (1) For $k \geq 10$, $(c_0', t) = (1/3, 1/3)$.
- (2) For $k \geq 80$, $(c_0', t) = (41/100, 8/25)$.

- (3) For $k \geq 22$, $(c_0', t) = (13/36, 1/3)$.
 ‘We have $|R_{7,2}^*| < 2 \cos(c_0' \pi)$ for $\theta_2 \in [\alpha_7 - \pi/6 + t\pi/k, \pi/2]$ ’.
- (4) For $k \geq 8$, $(c_0', t) = (1/6, 1/2)$.
- (5) For $k = 26, k \geq 44$, $(c_0', t) = (1/3, 2/3)$.
- (6) For $k \geq 70$, $(c_0', t) = (1/4, 1/2)$.
- (7) For $k \geq 200$, $(c_0', t) = (5/18, 1/2)$.

Proof. We use the algorithm given in Section 3.5. Furthermore, we have $X_1 = v_k(2, 1, \theta_1)^{-2/k} \geq 1 + 2\sqrt{3}t(\pi/k)$ and $X_2 = v_k(3, 1, \theta_1)^{-2/k} \geq 1 + 3\sqrt{3}t(\pi/k)$ in the proofs of (1), (2), and (3), and we have $X_1 = (1/4) v_k(1, -1, \theta_2)^{-2/k} \geq 1 + (\sqrt{3}/2)t(\pi/k)$ and $X_2 = (1/4) v_k(3, -1, \theta_2)^{-2/k} \geq 1 + (3\sqrt{3}/2)t(\pi/k)$ in the proofs of (4), (5), (6), and (7). We also have $c_0 \leq \cos(c_0' \pi)$. We can show $X_i - (a_{3,i} + a_{4,i}(t\pi/k)^2) > 0$ in the algorithm given in Section 3.5 for the cases of ‘(4), $k = 8$ ’ and ‘(5), $k = 26$ ’. For the other cases, we can show $Y_1 > 0$ and $Y_2 > 0$. $\square$

Similarly to Proposition 4.1, by the above lemma’s conditions (1), (4), and (5), we have the following proposition.

PROPOSITION 5.1. *Let $k \geq 4$ be an even integer. Then all but at most two of the zeros of $E_{k,7}^*(z)$ in $\mathbb{F}^*(7)$ lie on the arc A_7^* .*

5.2. The case $6 \mid k$

We can write

$$\begin{aligned} F_{k,7,1}^*(\theta_1) &= 2 \cos(k\theta_1/2) \\ &\quad + 2 \operatorname{Re}(2e^{-i\theta_1/2} + \sqrt{7}e^{i\theta_1/2})^{-k} + 2 \operatorname{Re}(3e^{-i\theta_1/2} + \sqrt{7}e^{i\theta_1/2})^{-k} + R_{7,1}^{*'} , \\ F_{k,7,2}^*(\theta_2) &= 2 \cos(k\theta_2/2) + 2^{k+1} \operatorname{Re}(e^{-i\theta_2/2} - \sqrt{7}e^{i\theta_2/2})^{-k} \\ &\quad + 2^{k+1} \operatorname{Re}(3e^{-i\theta_2/2} - \sqrt{7}e^{i\theta_2/2})^{-k} + R_{7,2}^{*'} . \end{aligned}$$

Similarly to Section 4.2, we consider the signs of some of these terms.

For $0 < \alpha_{7,k} < \pi/8$, we have $|R_{7,1}^{*'}| < 2$ and

$$\begin{aligned} \operatorname{Sign}\{\cos(k\theta_1/2)\} &= \operatorname{Sign}\{\operatorname{Re}(2e^{-i\theta_1/2} + \sqrt{7}e^{i\theta_1/2})^{-k}\} \\ &= \operatorname{Sign}\{\operatorname{Re}(3e^{-i\theta_1/2} + \sqrt{7}e^{i\theta_1/2})^{-k}\} \\ \theta_1 &\in [\pi/2 + \alpha_7 - \pi/(8k), \pi/2 + \alpha_7] . \end{aligned}$$

For $\pi/8 < \alpha_{7,k} < \pi/6$ or $\pi/4 < \alpha_{7,k} < 5\pi/6$, we can use Lemma 5.1(1) and (4).

For $5\pi/6 < \alpha_{7,k} < \pi$, we have $|R_{7,2}^*| < 2$ and

$$\begin{aligned}\text{Sign}\{\cos(k\theta_2/2)\} &= \text{Sign}\{\text{Re}(e^{-i\theta_2/2} - \sqrt{7}e^{i\theta_2/2})^{-k}\} \\ &= \text{Sign}\{\text{Re}(3e^{-i\theta_2/2} - \sqrt{7}e^{i\theta_2/2})^{-k}\} \\ \theta_2 &\in [\alpha_7 - \pi/6 + \pi/(6k), \pi/2].\end{aligned}$$

Thus, we have the following proposition.

PROPOSITION 5.2. *Let $k \geq 4$ be an integer which satisfies $6 \mid k$. Then all of the zeros of $E_{k,7}^*(z)$ in $\mathbb{F}^*(7)$ lie on the arc A_7^* .*

5.3. The case $k \equiv 2 \pmod{6}$

We can prove this case in a similar way to that of Section 4.3.

5.3.1. The case $0 < \alpha_{7,k} < 2\pi/3$. We can use Lemma 5.1(1), (4), and (5). When $\alpha_{7,k} < \pi/6$, we consider the point $k\theta_1/2 = k(\pi/2 + \alpha_7)/2 - \alpha_{7,k} - \pi/3$ instead of the last integer point for $A_{7,1}^*$. Similarly, instead of the first integer point for $A_{7,2}^*$, we consider the points $k\theta_2/2 = k(\alpha_7 - \pi/6)/2 + (\pi - \beta_{7,k}) + \pi/6$ and $k(\alpha_7 - \pi/6)/2 + (\pi - \beta_{7,k}) + \pi/3$ for $5\pi/12 < \alpha_{7,k} < 7\pi/12$ and $7\pi/12 < \alpha_{7,k} < 2\pi/3$, respectively.

5.3.2. The case $2\pi/3 < \alpha_{7,k} < \pi$. For this case, we expect one more zero between the last integer point for $A_{7,1}^*$ and the first one for $A_{7,2}^*$. Then, we consider the following cases.

(i) ‘The case $3\pi/4 < \alpha_{7,k} < \pi$ ’.

For $3\pi/4 < \alpha_{7,k} < 5\pi/6$ and $5\pi/6 < \alpha_{7,k} < \pi$, we can use Lemma 5.1(2) and (1), respectively.

(ii) ‘The case $3217\pi/4500 < \alpha_{7,k} < 3\pi/4$ ’.

We have $X_1 = v_k(2, 1, \theta_1)^{-2/k} \geq 1 + 2\sqrt{3}t(\pi/k)$ and $X_2 = v_k(3, 1, \theta_1)^{-2/k} \geq 1 + 3\sqrt{3}t(\pi/k)$, and let $\cos(c_0'\pi) = -\cos((x/180)\pi - (t/2)\pi)$. Then, by the algorithm given in Section 3.5, we prove ‘For $(x/180)\pi < \alpha_{7,k} < (y/180)\pi$, we have $|R_{7,1}^*| < 2\cos(c_0'\pi)$ for $\theta_1 = \pi/2 + \alpha_7 - t\pi/k$ ’ for nine cases, namely,

$$\begin{aligned}(x, y, t) &= (131.5, 135, 1/4), (130.1, 131.5, 83/400), \\ &\quad (129.5, 130.1, 7/40), (129.18, 129.5, 47/300),\end{aligned}$$

(129, 129.18, 71/500), (128.86, 129, 263/2000),
 (128.77, 128.86, 61/500), (128.71, 128.77, 143/1250),
 (128.68, 128.71, 109/1000).

(iii) ‘The case $2\pi/3 < \alpha_{7,k} < 266\pi/375$ ’.

We have $X_1 = (1/4) v_k(1, -1, \theta_2)^{-2/k} \geq 1 + (\sqrt{3}/2)t(\pi/k)$ and $X_2 = (1/4) v_k(3, -1, \theta_2)^{-2/k} \geq 1 + (3\sqrt{3}/2)t(\pi/k)$, and let $\cos(c_0'\pi) = \cos((y/180)\pi - 2\pi/3 - (t/2)\pi)$. Then, we prove ‘For $(x/180)\pi < \alpha_{7,k} < (y/180)\pi$, we have $|R_{7,2}^*| < 2\cos(c_0'\pi)$ for $\theta_2 = \alpha_7 - \pi/6 + t\pi/k$ ’ for four cases, namely,

$(x, y, t) = (120, 126.7, 93/200), (126.7, 127.3, 17/50), (127.3, 127.63, 22/75),$
 $(127.63, 127.68, 13/50).$

In conclusion, we have the following proposition.

PROPOSITION 5.3. *Let $k \geq 4$ be an integer which satisfies $k \equiv 2 \pmod{6}$, and let $\alpha_{7,k} \in [0, \pi]$ be the angle which satisfies $\alpha_{7,k} \equiv k(\pi/2 + \alpha_7)/2 \pmod{\pi}$. If we have $\alpha_{7,k} < 266\pi/375$ or $3217\pi/4500 < \alpha_{7,k}$, then all of the zeros of $E_{k,7}^*(z)$ in $\mathbb{F}^*(7)$ lie on the arc A_7^* . Otherwise, all but at most one zero of $E_{k,7}^*(z)$ in $\mathbb{F}^*(7)$ lie on A_7^* .*

5.4. The case $k \equiv 4 \pmod{6}$

With the exception of some specific cases, we can prove this case in a similar way to the proof of Section 4.3 and the previous section.

5.4.1. *The case $0 < \alpha_{7,k} < \pi/3$.* We can use Lemma 5.1(1) for the case $\alpha_{7,k} < \pi/6$, and we can use Lemma 5.1(4) and (6) for the cases $0 < \alpha_{7,k} < \pi/4$ and $\pi/4 < \alpha_{7,k} < \pi/3$, respectively.

5.4.2. *The case $\pi/3 < \alpha_{7,k} < \pi$.* (i) ‘The case $3\pi/4 < \alpha_{7,k} < \pi$ ’.

For $5\pi/6 < \alpha_{7,k} < \pi$ (respectively $29\pi/36 < \alpha_{7,k} < 5\pi/6, 3\pi/4 < \alpha_{7,k} < 5\pi/6$), we can use Lemma 5.1(1) (respectively (3), (2)).

(ii) ‘The case $\pi/3 < \alpha_{7,k} < 13\pi/36$ ’.

We can use Lemma 5.1(7).

(iii) ‘The case $2\pi/3 < \alpha_{7,k} < 3\pi/4$ ’.

We define $\cos(c_0'\pi) = -\cos((x/180)\pi - (t/2)\pi)$. Then, we prove ‘For $(x/180)\pi < \alpha_{7,k} < (y/180)\pi$, we have $|R_{7,1}^*| < 2\cos(c_0'\pi)$ for $\theta_1 = \pi/2 + \alpha_7 - t\pi/k$ ’ for two cases, namely, $(x, y, t) = (127.6, 135, 59/250), (120, 127.6, 1/4).$

(iv) ‘The case $13\pi/36 < \alpha_{7,k} < 5\pi/9$ ’.

We define $\cos(c_0'\pi) = \cos((y/180)\pi - \pi/3 + (t/2)\pi)$. Then, we prove ‘For $(x/180)\pi < \alpha_{7,k} < (y/180)\pi$, we have $|R_{7,2}^*| < 2\cos(c_0'\pi)$ for $\theta_2 = \alpha_7 - \pi/6 + t\pi/k$ ’ for two cases, namely, $(x, y, t) = (65, 90, 2/5), (90, 100, 2/5)$.

Now, we can write

$$\begin{aligned} F_{k,7,1}^*(\theta_1) &= 2\cos(k\theta_1/2) + 2\operatorname{Re}(3e^{-i\theta_1/2} + \sqrt{7}e^{i\theta_1/2})^{-k} + R_{7,1}^{*''}, \\ F_{k,7,2}^*(\theta_2) &= 2\cos(k\theta_2/2) + 2^k \cdot 2\operatorname{Re}(3e^{-i\theta_2/2} - \sqrt{7}e^{i\theta_2/2})^{-k} + R_{7,2}^{*''}. \end{aligned}$$

(v) ‘The case $73\pi/120 < \alpha_{7,k} < 2\pi/3$ ’.

We have $X_1 = v_k(2, 1, \theta_1)^{-2/k} \geq 1 + 2\sqrt{3}t(\pi/k)$ and $X_2 = v_k(3, 1, \theta_1)^{-2/k} \leq e^{-3\sqrt{3}(t/2)\pi}$. Then, $\operatorname{Sign}\{\cos(k\theta_1/2)\} = \operatorname{Sign}\{\operatorname{Re}(3e^{-i\theta_1/2} + \sqrt{7}e^{i\theta_1/2})^{-k}\}$, and we prove ‘For $(x/180)\pi < \alpha_{7,k} < (y/180)\pi$, we have $|R_{7,1}^{*''}| < |2\cos(k\theta_1/2) + 2\operatorname{Re}(3e^{-i\theta_1/2} + \sqrt{7}e^{i\theta_1/2})^{-k}|$ for $\theta_1 = \pi/2 + \alpha_7 - t\pi/k$ ’ for four cases, namely,

$$\begin{aligned} (x, y) &= (111.6, 120, 23/150), (110.1, 111.6, 1/10), \\ &= (109.65, 110.1, 43/625), (109.5, 109.65, 21/400). \end{aligned}$$

For each case, we consider the point such that $k\theta_1/2 = k(\pi/2 + \alpha_7)/2 - (t/2)\pi$. We can show $\operatorname{Sign}\{\cos(k\theta_1/2)\} = \operatorname{Sign}\{F_{k,7,1}^*(\theta_1)\}$, and then we have at least one zero between the second to last integer point for $A_{7,1}^*$ and the point $k\theta_1/2$.

(vi) ‘The case $5\pi/9 < \alpha_{7,k} < 217\pi/360$ ’.

We have $X_1 = (1/4) v_k(1, -1, \theta_2)^{-2/k} \geq 1 + t(\pi/k)$ and $X_2 = (1/4) v_k(3, -1, \theta_2)^{-2/k} \leq e^{-(3\sqrt{3}/2)(t/2)\pi}$. Then, we have $\operatorname{Sign}\{\cos(k\theta_2/2)\} = \operatorname{Sign}\{\operatorname{Re}(3e^{-i\theta_2/2} - \sqrt{7}e^{i\theta_2/2})^{-k}\}$, and we prove ‘For $(x/180)\pi < \alpha_{7,k} < (y/180)\pi$, we have $|R_{7,2}^{*''}| < |2\cos(k\theta_2/2) + 2^k \cdot 2\operatorname{Re}(3e^{-i\theta_2/2} - \sqrt{7}e^{i\theta_2/2})^{-k}|$ for $\theta_2 = \alpha_7 - \pi/6 + t\pi/k$ ’ for five cases, namely,

$$\begin{aligned} (x, y) &= (100, 106, 3/10), (106, 107.7, 11/50), (107.7, 108.21, 33/200), \\ &= (108.21, 108.42, 2/15), (108.42, 108.5, 113/1000). \end{aligned}$$

In conclusion, we have the following proposition.

PROPOSITION 5.4. *Let $k \geq 4$ be an integer which satisfies $k \equiv 4 \pmod{6}$, and let $\alpha_{7,k} \in [0, \pi]$ be the angle which satisfies $\alpha_{7,k} \equiv k(\pi/2 + \alpha_7)/2 \pmod{\pi}$. If we have $\alpha_{7,k} < 217\pi/360$ or $73\pi/120 < \alpha_{7,k}$, then all of the zeros of $E_{k,7}^*(z)$ in $\mathbb{F}^*(7)$ lie on the arc A_7^* . Otherwise, all but at most one zero of $E_{k,7}^*(z)$ in $\mathbb{F}^*(7)$ are on A_7^* .*

5.5. The remaining cases ' $k \equiv 2 \pmod{6}$, $266\pi/375 < \alpha_{7,k} < 3217\pi/4500$ ' and ' $k \equiv 4 \pmod{6}$, $217\pi/360 < \alpha_{7,k} < 73\pi/120$ '

Similar to the problem described in Section 4.4, it is difficult to prove Conjecture 1.2 for the remaining cases. However, when k is large enough, we have the following observation.

5.5.1. The case ' $k \equiv 2 \pmod{6}$ and $266\pi/375 < \alpha_{7,k} < 3217\pi/4500$ '. Let $t > 0$ be small enough, then we have $\pi/2 < \alpha_{7,k} - (t/2)\pi < \pi$, $\pi < 2\pi/3 + \alpha_{7,k} + d_{1,1}(t/2)\pi < \alpha_{7,k,1}' < 2\pi/3 + \alpha_{7,k} + (t/2)\pi < 3\pi/2$, and $2\pi < 4\pi/3 + \alpha_{7,k} - t\pi < \alpha_{7,k,2}' < 4\pi/3 + \alpha_{7,k} - d_{1,2}(t/2)\pi < 5\pi/2$. Thus, we have

$$\begin{aligned} & -\cos(\alpha_{7,k} - (t/2)\pi) - \cos(2\pi/3 + \alpha_{7,k} + d_{1,1}(t/2)\pi) \cdot (1 + 2\sqrt{3}t(\pi/k))^{-k/2} \\ & \quad - \cos(4\pi/3 + \alpha_{7,k} - d_{1,2}(t/2)\pi) \cdot e^{-(3\sqrt{3}/2)\pi t} \\ & > |\cos(k\theta_1/2)| + |\operatorname{Re}\{(2e^{i\theta_1/2} + \sqrt{7}e^{-i\theta_1/2})^{-k}\}| \\ & \quad - |\operatorname{Re}\{(3e^{i\theta_1/2} + \sqrt{7}e^{-i\theta_1/2})^{-k}\}| \\ & > -\cos(\alpha_{7,k} - (t/2)\pi) - \cos(2\pi/3 + \alpha_{7,k} + (3t/2)\pi) \cdot e^{-\sqrt{3}\pi t} \\ & \quad - \cos(4\pi/3 + \alpha_{7,k} - t\pi) \cdot (1 + 3\sqrt{3}t(\pi/k))^{-k/2}. \end{aligned}$$

We denote the upper bound by A and the lower bound by B . First, we have $A|_{t=0} = B|_{t=0} = 0$ and

$$\frac{\partial}{\partial t}A|_{t=0} = \frac{\partial}{\partial t}B|_{t=0} = 0.$$

Second, let $C = (5\sqrt{3}/2)\pi^2(-\cos \alpha_{7,k})(\tan \alpha_{7,k} + 11/(5\sqrt{3}))$, then we have

$$\frac{\partial^2}{\partial t^2}A|_{t=0} = C + 6\pi^2(-\cos(2\pi/3 + \alpha_{7,k}))/k$$

and

$$\frac{\partial^2}{\partial t^2}B|_{t=0} = C - (27/2)\pi^2 \cos(4\pi/3 + \alpha_{7,k})/k.$$

Finally, we have $B > 0$ if $\alpha_{7,k} > 3\pi/2 - 2\alpha_7$, and we have $A < 0$ if $\alpha_{7,k} < 3\pi/2 - 2\alpha_7$ for large enough k and small enough t .

Similarly, we consider the lower and the upper bounds of $|\cos(k\theta_2/2)| - |\operatorname{Re}\{2^k \cdot (e^{i\theta_2/2} - \sqrt{7}e^{-i\theta_2/2})^{-k}\}| - |\operatorname{Re}\{2^k \cdot (3e^{i\theta_2/2} - \sqrt{7}e^{-i\theta_2/2})^{-k}\}|$. The lower bound is positive if $\alpha_{7,k} < 3\pi/2 - 2\alpha_7$, while the upper bound is negative if $\alpha_{7,k} > 3\pi/2 - 2\alpha_7$ for large enough k and for small enough t .

In conclusion, if k is large enough, then $|R_{7,1}^{*'}|$ and $|R_{7,2}^{*'}|$ is small enough, and then we have one more zero on the arc $A_{7,1}^*$ when $\alpha_{7,k} > 3\pi/2 - 2\alpha_7$, and we have one more zero on the arc $A_{7,2}^*$ when $\alpha_{7,k} < 3\pi/2 - 2\alpha_7$. However, if k is small, a method of proving the conjecture for this case is not clear.

5.5.2. *The case ‘ $k \equiv 4 \pmod{6}$ and $217\pi/360 < \alpha_{7,k} < 73\pi/120$.’* Let $t > 0$ be small enough. Similar to Section 4.4, if k is large enough, then we have one more zero on the arc $A_{7,1}^*$ when $\alpha_{7,k} > \pi - \alpha_7$, and we have one more zero on the arc $A_{7,2}^*$ when $\alpha_{7,k} < \pi - \alpha_7$.

Remark 1. Getz considered a similar problem for the zeros for extremal modular forms of $\mathrm{SL}_2\mathbb{Z}$ (see [G]). It seems that similar results do not hold for extremal modular forms of $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$.

Acknowledgement. I would like to thank Professor Eiichi Bannai for suggesting these problems.

REFERENCES

- [G] J. Getz. A generalization of a theorem of Rankin and Swinnerton-Dyer on zeros of modular forms. *Proc. Amer. Math. Soc.* **132**(8) (2004), 2221–2231.
- [K] A. Krieg. Modular forms on the Fricke Group. *Abh. Math. Sem. Univ. Hamburg* **65** (1995), 293–299.
- [MNS] T. Mieziaki, H. Nozaki and J. Shigezumi. On the zeros of Eisenstein series for $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$. *J. Math. Soc. Japan* **59** (2007), 693–706.
- [Q] H.-G. Quebbemann. Atkin–Lehner eigenforms and strongly modular lattices. *Enseign. Math.* (2) **43**(1–2) (1997), 55–65.
- [RSD] F. K. C. Rankin and H. P. F. Swinnerton-Dyer. On the zeros of Eisenstein series. *Bull. London Math. Soc.* **2** (1970), 169–170.
- [SE] J.-P. Serre. *A Course in Arithmetic* (Graduate Texts in Mathematics, 7). Springer, Heidelberg, 1973.
- [SG] G. Shimura. *Introduction to the arithmetic theory of automorphic functions*. Kanô Memorial Lectures, No. 1 (Publications of the Mathematical Society of Japan, 11). Iwanami Shoten Publishers, Tokyo, Princeton University Press, Princeton, NJ, 1971.
- [SH] H. Shimizu. *Hokei Kansu. I–III* (Iwanami Shoten Kiso Sugaku, 8) (Automorphic Functions. I–III (Iwanami Lectures on Fundamental Mathematics, 8)). Iwanami Shoten Publishers, Tokyo, 1977–1978 (in Japanese).
- [SJ] J. Shigezumi. A detailed note on the zeros of Eisenstein series for $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$. Preprint, 2006, math.NT/0607247.

Junichi Shigezumi
 Graduate School of Mathematics
 Kyushu University
 Hakozaki 6-10-1 Higashi-ku
 Fukuoka 812-8581
 Japan

(E-mail: j.shigezumi@math.kyushu-u.ac.jp)