REPRESENTATIONS OF INTERNAL CATEGORIES

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Abstract. Adams gave the notion of a Hopf algebroid generalizing the notion of a Hopf algebra and showed that certain generalized homology theories take values in the category of comodules over the Hopf algebroid associated with each homology theory. A Hopf algebra represents an affine group scheme which is a group in the category of a scheme and the notion of comodules over a Hopf algebra is equivalent to the notion of representations of the affine group scheme represented by a Hopf algebra. On the other hand, a Hopf algebroid represents a groupoid in the category of schemes. Therefore, it is natural to consider the notion of comodules over a Hopf algebroid as representations of the groupoid represented by a Hopf algebroid. This motivates the study of representations of groupoids, and more generally categories, for topologists. The aim of this paper is to set a categorical foundation of representations of an internal category which is a category object in a given category, using the notion of a fibered category.

0. Introduction

In [1], Adams generalized the notion of Hopf algebras in the study of generalized homology theories satisfying certain conditions and showed that such a generalized homology theory, say $E_\ast$, takes values in the category of comodules over the ‘generalized Hopf algebra’ associated with $E_\ast$. The notion introduced by Adams is now called a Hopf algebroid which represents a functor taking values in the category of groupoids. Here ‘a groupoid’ means a special category whose morphisms are all isomorphisms. A comodule over a Hopf algebroid $\Gamma$ can be regarded as a representation of the groupoid represented by $\Gamma$. The aim of this paper is to set a categorical foundation of representations of an internal category which is a category object in a given category.

We begin by reviewing the notion of a fibered category following [7] and an internal category in Section 1, we give a detailed description on the relationship between the notions of a fibered category and a 2-category in Section 2, which is originally observed in [7, Section 8]. There, we show that the 2-category of a fibered category over a given category $\mathcal{E}$ is equivalent to the 2-category of ‘lax functors’ from the opposite category of $\mathcal{E}$ to the 2-category of categories. Our construction of fibered categories from lax functors allows us to give the notion of fibered categories represented by internal categories (Example 2.18) and a short definition (Example 2.19) of Grothendieck topoi over a simplicial object in the given site.

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By making use of the notion of fibered category, we give a definition of the representations of internal categories in Section 3 which generalizes the definition given by Deligne in [3]. We give the definition of ‘trivial representation’ and several examples of representations and show that the category of representations of an internal category $G$ on objects of a fibered category represented by an internal category $C$ is isomorphic to the category of internal functors from $G$ to $C$ and internal natural transformations between them (Theorem 3.17). In Section 4, we reformulate the notion of descent theory [6] in terms of representations of special groupoids, namely equivalence relations. We end this note by giving the definition of regular representations and restrictions of representations.

1. Recollections

For a category $C$ and objects $X$, $Y$ of $C$, we denote by $C(X, Y)$ the set of all morphisms from $X$ to $Y$ in $C$.

Let $p : F \to E$ be a functor. For an object $X$ of $E$, we denote by $F_X$ the subcategory of $F$ consisting of objects $M$ of $F$ satisfying $p(M) = X$ and morphisms $\varphi$ satisfying $p(\varphi) = \text{id}_X$. For a morphism $f : X \to Y$ in $E$ and $M \in \text{Ob} F_X$, $N \in \text{Ob} F_Y$, we denote by $F_f(M, N)$ the set of morphisms $\varphi \in F(M, N)$ satisfying $p(\varphi) = f$.

**Definition 1.1.** (Grothendieck [7, Définition 5.1, p. 161]) Let $\alpha : M \to N$ be a morphism in $E$ and set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call $\alpha$ a Cartesian morphism if, for any $M' \in \text{Ob} F_X$, the map $F_X(M', M) \to F_f(M', N)$ defined by $\varphi \mapsto \alpha \varphi$ is bijective.

The following assertion is immediate from the definition.

**Proposition 1.2.** Let $\alpha_i : M_i \to N_i$ $(i = 1, 2)$ be morphisms in $F$ such that $p(M_1) = p(M_2)$, $p(N_1) = p(N_2)$, $p(\alpha_1) = p(\alpha_2)$ and $\lambda : N_1 \to N_2$ a morphism in $F$ such that $p(\lambda) = \text{id}_{p(N_1)}$. If $\alpha_2$ is Cartesian, there is a unique morphism $\mu : M_1 \to M_2$ such that $p(\mu) = \text{id}_{p(M_1)}$ and $\alpha_2 \mu = \lambda \alpha_1$.

**Corollary 1.3.** If $\alpha_i : M_i \to N_i$ $(i = 1, 2)$ are Cartesian morphisms in $F$ such that $p(M_1) = p(M_2)$ and $p(\alpha_1) = p(\alpha_2)$, there is a unique morphism $\mu : M_1 \to M_2$ such that $\alpha_1 = \alpha_2 \mu$ and $p(\mu) = \text{id}_{p(M_1)}$. Moreover, $\mu$ is an isomorphism.

**Definition 1.4.** (Grothendieck [7, Définition 5.1, p. 162]) Let $f : X \to Y$ be a morphism in $E$ and $N \in \text{Ob} F_Y$. If there exists a Cartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, $M$ is called an inverse image of $N$ by $f$. We denote $M$ by $f^*(N)$ and $\alpha$ by $\alpha_f(N) : f^*(N) \to N$.

By (1.3), $f^*(N)$ is unique up to isomorphism.

**Remark 1.5.** For an identity morphism $\text{id}_X$ of $X \in \text{Ob} E$ and $N \in \text{Ob} F_X$, the identity morphism $\text{id}_N$ of $N$ is obviously Cartesian. Hence, the inverse image of $N$ by the identity morphism of $X$ always exists and $\alpha_{\text{id}_X}(N) : \text{id}_X^*(N) \to N$ can be chosen as the identity morphism of $N$. By the uniqueness of $\text{id}_X^*(N)$ up to isomorphism, $\alpha_{\text{id}_X}(N) : \text{id}_X^*(N) \to N$ is an isomorphism for any choice of $\text{id}_X^*(N)$.

The following assertion is also immediate.

**Proposition 1.6.** Let $f : X \to Y$ be a morphism in $E$. If, for any $N \in \text{Ob} F_Y$, there exists a Cartesian morphism $\alpha_f(N) : f^*(N) \to N$, $N \mapsto f^*(N)$ defines a functor $f^* : F_Y \to F_X$.
such that, for any morphism $\varphi : N \to N'$ in $\mathcal{F}_Y$, the following square commutes:

\[
\begin{array}{ccc}
  f^*(N) & \xrightarrow{\alpha_f(N)} & N \\
  \downarrow f^*(\varphi) & & \downarrow \varphi \\
  f^*(N') & \xrightarrow{\alpha_f(N')} & N'
\end{array}
\]

**Definition 1.7.** (Grothendieck [7, Définition 5.1, p. 162]) If the assumption of (1.6) is satisfied, we say that the functor of the inverse image by $f$ exists.

**Definition 1.8.** (Grothendieck [7, Définition 6.1, p. 164]) If a functor $p : \mathcal{F} \to \mathcal{E}$ satisfies the following condition (i), $p$ is called a prefibered category and if $p$ satisfies both (i) and (ii), $p$ is called a fibered category or $p$ is fibrant.

(i) For any morphism $f$ in $\mathcal{E}$, the functor of the inverse image by $f$ exists.

(ii) The composition of Cartesian morphisms is Cartesian.

**Definition 1.9.** (Grothendieck [7, Définition 7.1, p. 170]) Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. A map $\kappa : \text{Mor} \mathcal{E} \to \coprod_{X,Y \in \text{Ob} \mathcal{E}} \text{Funct}(\mathcal{F}_Y, \mathcal{F}_X)$ is called a cleavage if $\kappa(f)$ gives an inverse image functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ for $(f : X \to Y) \in \text{Mor} \mathcal{E}$. A cleavage $\kappa$ is said to be normalized if $\kappa(\text{id}_X) = \text{id}_{\mathcal{F}_X}$ for any $X \in \text{Ob} \mathcal{E}$. A category $\mathcal{F}$ over $\mathcal{E}$ is called a cloven prefibered category (respectively normalized cloven prefibered category) if a cleavage (respectively normalized cleavage) is given.

The functor $p : \mathcal{F} \to \mathcal{E}$ has a cleavage if and only if $p$ is prefibered. If $p$ is prefibered, $p$ has a normalized cleavage by Remark 1.5.

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$ and $N$ an object of $\mathcal{F}_Y$. If $p : \mathcal{F} \to \mathcal{E}$ is a prefibered category, there is a unique morphism $c_{f,g}(N) : g^* f^*(N) \to (fg)^*(N)$ such that the following square commutes and $p(c_{f,g}(N)) = \text{id}_Z$:

\[
\begin{array}{ccc}
g^* f^*(N) & \xrightarrow{\alpha_{f}(f^*(N))} & f^*(N) \\
\downarrow c_{f,g}(N) & & \downarrow \alpha_f(N) \\
(fg)^*(N) & \xrightarrow{\alpha_{fg}(N)} & N
\end{array}
\]

Then, we see the following.
Proposition 1.10. For a morphism $\varphi : M \rightarrow N$ in $\mathcal{F}_Y$, the following square commutes:

$$
g^*f^*(M) \xrightarrow{c_{f,g}(M)} (fg)^*(M)
\quad \xrightarrow{g^*f^*(\varphi)} \quad (fg)^*(\varphi)
\quad \xrightarrow{g^*f^*(N)} \xrightarrow{c_{f,g}(N)} (fg)^*(N)
$$

In other words, $c_{f,g}$ gives a natural transformation $g^*f^* \rightarrow (fg)^*$ of functors from $\mathcal{F}_Y$ to $\mathcal{F}_Z$.

Proposition 1.11. (Grothendieck [7, Proposition 7.2, p. 172]) Let $p : \mathcal{F} \rightarrow \mathcal{E}$ be a cloven prefibered category. Then, $p$ is a fibered category if and only if $c_{f,g}(N)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $N \in \text{Ob} \mathcal{F}_X$.

Proposition 1.12. (Grothendieck [7, Proposition 7.4, p. 172]) Let $p : \mathcal{F} \rightarrow \mathcal{E}$ be a cloven prefibered category. For a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{E}$ and an object $M$ of $\mathcal{F}_W$, we have $c_{h,\text{id}_X}(M) = \alpha_{id_Z}(\text{id}_Z^*f^*h^*(M))$, $c_{\text{id}_W, h}(M) = h^*(\alpha_{id_W}(M))$ and the following diagram commutes:

$$
f^*(g^*h^*)(M) \xrightarrow{c_{f,g}(h^*(M))} (f^*g^*)h^*(M) \xrightarrow{f^*(\alpha_{id_X})} (gf)^*h^*(M)
\quad \xrightarrow{f^*(c_{h,\text{id}_X}(M))} \quad (hg)^*(M) \xrightarrow{c_{h,gf}(M)} (h(gf))^*(M)
$$

We give two examples below which are referred to in the later sections.

Example 1.13. (Grothendieck [7, p. 182, a]) Let $\mathcal{E}$ be a category with pull-backs and $\Delta^1$ a category such that $\text{Ob} \Delta^1 = \{0, 1\}$ and $\text{Mor} \Delta^1 = \{\text{id}_0, \text{id}_1, 0 \rightarrow 1\}$. Set $\mathcal{E}_X(2) = \text{Funct}(\Delta^1, \mathcal{E})$ and let $p : \mathcal{E}_X(2) \rightarrow \mathcal{E}$ be the evaluation functor at 1. Then, for $X \in \text{Ob} \mathcal{E}$, $\mathcal{E}_X(2) = \mathcal{E}/X$ and, for $(f : X \rightarrow Y) \in \text{Mor} \mathcal{E}$, the pull-backs along $f$ define the functor of the inverse image $f^* : \mathcal{E}_Y(2) \rightarrow \mathcal{E}_X(2)$. For morphisms $f : X \rightarrow Y$, $g : Z \rightarrow X$ in $\mathcal{E}$ and an object $\pi : E \rightarrow Y$ of $\mathcal{E}(2)$, $c_{f,g}(\pi : E \rightarrow Y) : (fg)^*(\pi : E \rightarrow Y) \rightarrow g^*f^*(\pi : E \rightarrow Y)$ is the isomorphism induced by $(1 \times g, pr_2) : E \times_Y Z \rightarrow (E \times_Y X) \times_X Z$. Hence, $p : \mathcal{E}_X(2) \rightarrow \mathcal{E}$ is a fibered category.

Example 1.14. Let $\text{Sch}$ be the category of schemes. We define a category $\mathcal{Q}\text{mod}$ as follows. $\text{Ob} \mathcal{Q}\text{mod}$ consists of pairs $(X, \mathcal{M})$ of a scheme $X$ and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$. A morphism $(X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$ in $\mathcal{Q}\text{mod}$ is a pair $(f, \varphi)$ of morphisms $f : X \rightarrow Y$ in $\text{Sch}$ and $\varphi : \mathcal{N} \rightarrow f_*\mathcal{M}$ in the category of $\mathcal{O}_Y$-modules. The composition of morphisms $(f, \varphi) : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$ and $(g, \psi) : (Z, \mathcal{L}) \rightarrow (X, \mathcal{M})$ is defined to be $(fg, \varphi \psi(L))$. Define a functor $p : \mathcal{Q}\text{mod} \rightarrow \text{Sch}$ by $p(X, \mathcal{M}) = X$ and $p(f, \varphi) = f$. For a morphism $f : X \rightarrow Y$ of schemes and an $\mathcal{O}_Y$-module $\mathcal{N}$, we denote by $\eta_f(\mathcal{N}) : \mathcal{N} \rightarrow f_*f^*\mathcal{N}$ the unit of the adjunction of $f^*$ and $f_*$. Then, $(f, \eta_f(\mathcal{N})) : (X, f^*\mathcal{N}) \rightarrow (Y, \mathcal{N})$ is a Cartesian morphism. In fact, for $(f, \varphi) \in \mathcal{Q}\text{mod}_{f}((X, \mathcal{M}), (Y, \mathcal{N}))$, $\varphi : f^*\mathcal{N} 
\rightarrow \mathcal{M}$ denotes the adjoint of $\varphi : \mathcal{N} \rightarrow f_*\mathcal{M}$, then $g((\text{id}_X, \varphi)) : (X, \mathcal{M}) \rightarrow (Y, f^*\mathcal{N})$ is the unique morphism in $\mathcal{Q}\text{mod}_{X}$ such that $(f, \varphi) = (f, (\text{id}_X, \varphi))$. Thus, we define
Let $\mathcal{F}$ be a fibered category. Moreover, $f: X \to Y$, $g: Z \to X$ in $\text{Sch}$ and an object $(Y, N)$ of $\text{Qmod}$, let $\tilde{\mathcal{C}}_{f, g}(N) : (fg)^* N \to g^* f^* N$ be the adjoint of composition

$$\mathcal{N} \xrightarrow{\eta_f(N)} f_* f^* N \xrightarrow{f_*(\eta_g(f^* N)))} f_* g_* g^* f^* N = (fg)^* g^* f^* N.$$ 

Then, $\tilde{\mathcal{C}}_{f, g}(N)$ is an isomorphism of $\mathcal{O}_Z$-modules and

$$c_{f, g}(Y, N) = (\text{id}_Z, \tilde{\mathcal{C}}_{f, g}(N)): (fg)^*(Y, N) \to g^* f^*(Y, N)$$

is an isomorphism in $\text{Qmod}_Z$. Hence, $p: \text{Qmod} \to \text{Sch}$ is a fibered category.

**Definition 1.16.** (Grothendieck [7, Proposition 6.6, p. 167]) A morphism $(f, \alpha): (X, M) \to (Y, N)$ in $\mathcal{D} \times \mathcal{E}$ is Cartesian if and only if $\alpha: M \to N$ is Cartesian.

**Proposition 1.17.** (Grothendieck [7, Corollaire 6.9, p. 168]) If $p: \mathcal{F} \to \mathcal{E}$ is a prefibered (respectively fibered) category, so is the pull-back $p_F: \mathcal{D} \times \mathcal{E} \to \mathcal{F}$ of $p$ along $F: \mathcal{D} \to \mathcal{E}$. Moreover, $\widetilde{F}: \mathcal{D} \times \mathcal{E} \to \mathcal{F}$ maps Cartesian morphisms to Cartesian morphisms.

We need to introduce the notion of a ‘Cartesian section’ in order to define the notion of trivial representation.

**Definition 1.18.** (Grothendieck [7, Définition 5.5, p. 164]) Let $p: \mathcal{F} \to \mathcal{E}$ be a functor. We call a functor $s: \mathcal{E} \to \mathcal{F}$ a Cartesian section if $ps = \text{id}_\mathcal{E}$ and $s(f)$ is Cartesian for any $f \in \text{Mor} \mathcal{E}$. The subcategory of $\text{Funct}(\mathcal{E}, \mathcal{F})$ consisting of Cartesian sections and morphisms $\varphi: s \to s'$ satisfying $p(\varphi_X) = \text{id}_X$ for any $X \in \text{Ob} \mathcal{E}$ is denoted by $\text{Lim}(\mathcal{F}/\mathcal{E})$.

**Proposition 1.19.** (Giraud [5, Lemme 5.7]) If $\mathcal{E}$ has a terminal object $1$, the functor $e: \text{Lim}(\mathcal{F}/\mathcal{E}) \to \mathcal{F}_1$ given by $e(s) = s(1)$ and $e(\varphi) = \varphi_1$ is fully faithful. Moreover, if $p: \mathcal{F} \to \mathcal{E}$ is a fibered category, $e$ is an equivalence of categories.

Next, we briefly review the notion of a bifibered category following [7, Section 10].

**Definition 1.20.** Let $p: \mathcal{F} \to \mathcal{E}$ be a functor and $\alpha: M \to N$ a morphism in $\mathcal{F}$. Set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call $\alpha$ a co-Cartesian morphism if, for any $N' \in \text{Ob} \mathcal{F}_Y$, the map $\mathcal{F}_X(N, N') \to \mathcal{F}_f(M, N')$ defined by $\varphi \mapsto \varphi_\alpha$ is bijective.

The following assertion is the dual of Proposition 1.2.

**Proposition 1.21.** If $\alpha_i : M \to N_i$ ($i = 1, 2$) are co-Cartesian morphisms in $\mathcal{F}$ such that $p(N_1) = p(N_2)$ and $p(\alpha_1) = p(\alpha_2)$, there is a unique morphism $\psi: N_1 \to N_2$ such that $\alpha_1 = \alpha_2 \psi$ and $p(\psi) = \text{id}_{p(N_1)}$. Moreover, $\psi$ is an isomorphism.
**Definition 1.22.** Let $f : X \to Y$ be a morphism in $\mathcal{E}$ and $M \in \text{Ob}\mathcal{F}_X$. If there exists a Cartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, $N$ is called a direct image of $M$ by $f$. We denote $M$ by $f_* (N)$ and $\alpha$ by $\alpha^f (M) : M \to f_* (M)$. By Proposition 1.21, $f_* (N)$ is unique up to isomorphism.

**Proposition 1.23.** Let $\alpha : M \to N$, $\alpha' : M' \to N'$ be morphisms in $\mathcal{F}$ such that $p(M) = p(M')$, $p(N) = p(N')$, $p(\alpha) = p(\alpha')$ ($= f$) and $\lambda : M \to M'$ a morphism in $\mathcal{F}$ such that $p(\lambda) = \text{id}_{p(N)}$. If $\alpha'$ is co-Cartesian, there is a unique morphism $\mu : N \to N'$ such that $p(\mu) = \text{id}_{p(N)}$ and $\alpha' \mu = \lambda \alpha$.

**Corollary 1.24.** Let $f : X \to Y$ be a morphism in $\mathcal{E}$. If, for any $M \in \text{Ob}\mathcal{F}_X$, there exists a co-Cartesian morphism $\alpha^f (M) : M \to f_* (M)$, $M \mapsto f_* (M)$ defines a functor $f_* : \mathcal{F}_X \to \mathcal{F}_Y$.

**Definition 1.25.** If the assumption of Corollary 1.24 is satisfied, we say that the functor of the direct image by $f$ exists.

**Definition 1.26.** If a functor $p : \mathcal{F} \to \mathcal{E}$ satisfies the following condition (i), $p$ is called a precofibered category and if $p$ satisfies both (i) and (ii), $p$ is called a cofibered category or $p$ is cofibrant.

(i) For any morphism $f$ in $\mathcal{E}$, the functor of the direct image by $f$ exists.

(ii) The composition of co-Cartesian morphisms is co-Cartesian.

In other words, $p : \mathcal{F} \to \mathcal{E}$ is a precofibered (respectively cofibered) category if and only if $p : \mathcal{F}^{\text{op}} \to \mathcal{E}^{\text{op}}$ is a prefibered (respectively fibered) category.

Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. A map $\kappa : \text{Mor}\mathcal{E} \to \coprod_{X,Y \in \text{Ob}\mathcal{E}} \text{Funct}(\mathcal{F}_X, \mathcal{F}_Y)$ is called a cocleavage if $\kappa(f)$ is a direct image functor $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ for $(f : X \to Y) \in \text{Mor}\mathcal{E}$. A cocleavage $\kappa$ is said to be normalized if $\kappa(\text{id}_X) = \text{id}_{\mathcal{F}_X}$ for any $X \in \text{Ob}\mathcal{E}$. A category $\mathcal{F}$ over $\mathcal{E}$ is called a cloven precofibered category (respectively normalized cloven precofibered category) if a cocleavage (respectively normalized cocleavage) is given.

The functor $p : \mathcal{F} \to \mathcal{E}$ has a cocleavage if and only if $p$ is precofibered. If $p$ is precofibered, $p$ has a normalized cocleavage.

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$ and $M$ an object of $\mathcal{F}_Z$. If $p : \mathcal{F} \to \mathcal{E}$ is a precofibered category, there is a unique morphism $c^{f,g} (M) : (fg)_* (M) \to f_* g_* (M)$ such that the following square commutes and $p(c^{f,g} (M)) = \text{id}_Z$:

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha^f(M)} & (fg)_* (M) \\
\downarrow{\alpha^g(M)} & & \downarrow{c^{f,g}(M)} \\
g_* (M) & \xrightarrow{c^{f,g}(g_* (M))} & f_* g_* (M)
\end{array}
\]

The following is the dual of Definition 1.9.

**Proposition 1.27.** Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven precofibered category. Then, $p$ is a cofibered category if and only if $c^{f,g} (M)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $M \in \text{Ob}\mathcal{F}_Z$.

**Proposition 1.28.** Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $f : X \to Y$ a morphism in $\mathcal{E}$. 

Suppose that the functor of the direct image by $f$ exists. Then, the inverse image $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ by $f$ has a left adjoint if and only if the functor of the direct image by $f$ exists.

(2) Suppose that the functor of the direct image by $f$ exists. Then, the direct image $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ by $f$ has a right adjoint if and only if the functor of the inverse image by $f$ exists.

\begin{proof}
(1) Suppose that the functor of the inverse image by $f$ exists and that it has a left adjoint $f_* : \mathcal{F}_X \to \mathcal{F}_Y$. For $M \in \text{Ob} \mathcal{F}_X$, we denote by $\eta_M : M \to f^* f_*(M)$ the unit of the adjunction $f_* \dashv f^*$. Set $\alpha^f(M) = \alpha_f(f_*(M)) \eta_M : M \to f_*(M)$. By the assumption, the following composition is bijective for any $M \in \text{Ob} \mathcal{F}_X$, $N \in \text{Ob} \mathcal{F}_Y$:

$$f_*(M), N \xrightarrow{f_*} f^*(f_*(M), f^*(N)) \xrightarrow{\eta^*_M} f^*_X(M, f^*(N)) \xrightarrow{\alpha^f(N)_*} f(M, N).$$

We note that, since $\alpha_f(N) f^*(\varphi) = \varphi \alpha_f(f_*(M))$ for $\varphi \in \mathcal{F}_Y(f_*(M), N)$, the above composition coincides with the map $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ induced by $\alpha^f(M)$. This shows that the functor of the direct image by $f$ exists.

Conversely, assume that the functor of the direct image by $f$ exists. For $M \in \text{Ob} \mathcal{F}_X$, let us denote by $\alpha^f(M) : M \to f_*(M)$ the canonical co-Cartesian morphism. Then, we have bijections $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ and $\alpha_f(M)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ given by $\psi \mapsto \psi \alpha^f(M)$ and $\varphi \mapsto \alpha_f(M) \varphi$, which are natural in $M \in \text{Ob} \mathcal{F}_X$ and $N \in \text{Ob} \mathcal{F}_Y$. Thus, we have a natural bijection $\mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$.

Proof of (2) is the dual of the above.
\end{proof}

We remark that, if $p : \mathcal{E} \to \mathcal{F}$ is a prefibered and precocofibered category, the unit $\eta : \text{id}_{\mathcal{F}_X} \to f^* f_*$ of the adjunction $f_* \dashv f^*$ is the unique natural transformation satisfying $\alpha_f(f_*(M)) \eta_M = \alpha^f(M)$ for any $M \in \text{Ob} \mathcal{F}_X$. Dually, the counit $\varepsilon : f_* f^* \to \text{id}_{\mathcal{F}_Y}$ is the unique natural transformation satisfying $\varepsilon_N \alpha_f^f(f^*(N)) = \alpha_f(N)$ for any $N \in \text{Ob} \mathcal{F}_Y$.

**Proposition 1.29.** (Grothendieck [7, Proposition 10.1, p. 182]) Let $p : \mathcal{E} \to \mathcal{F}$ be a prefibered and precocofibered category. Then, it is a fibered category if and only if it is a cofibered category.

**Definition 1.30.** We call a functor $p : \mathcal{F} \to \mathcal{E}$ a bifibered category if it is a fibered and cofibered category.

**Example 1.31.** Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the fibered category given in Example 1.13. For a morphism $f : X \to Y$ in $\mathcal{E}$, consider the functor $f_* : \mathcal{E}_X^{(2)} \to \mathcal{E}_Y^{(2)}$ given by $f_*(\pi : E \to X) = (f \pi : E \to Y)$. It is easily seen that $f_*$ is a left adjoint of $f^*$.

**Example 1.32.** Let $p : \text{Qmod} \to \text{Sch}$ be the fibered category given in Example 1.14. If $f : X \to Y$ is a quasi-separated and quasi-compact morphism, then for any object $(X, \mathcal{M})$ of $\text{Qmod}$, the direct image $f_* \mathcal{M}$ of $\mathcal{M}$ by $f$ is also quasi-coherent. Hence, the inverse image functor $f^* : \text{Qmod}_Y \to \text{Qmod}_X$ has a left adjoint $f_* : \text{Qmod}_X \to \text{Qmod}_Y$.

We end this section by recalling the definition of internal category and related notions.

**Definition 1.33.** (Johnstone [11]) Let $\mathcal{E}$ be a category with finite limits. An internal category $C$ in $\mathcal{E}$ consists of the following objects and morphisms.
We denote by \((C_0, C_1; \sigma, \tau, \varepsilon, \mu)\) an internal category \(C\) whose object-of-objects and object-of-morphisms are \(C_0\) and \(C_1\), respectively, with structure maps \(\sigma, \tau, \varepsilon, \mu\).

An internal functor \(f : C \to D\) of internal categories consists of two morphisms \(f_0 : C_0 \to D_0\) and \(f_1 : C_1 \to D_1\) in \(\mathcal{E}\) such that the following diagrams commute:

\[
\begin{array}{ccc}
C_0 & \xrightarrow{\sigma} & C_1 \\
\downarrow {f_0} & & \downarrow {f_1} \\
D_0 & \xrightarrow{\sigma} & D_1 \\
\end{array}
\quad \begin{array}{ccc}
C_0 \times_{C_0} C_1 & \xrightarrow{\mu} & C_1 \\
\downarrow {f_1 \times f_1} & & \downarrow {f_1} \\
D_0 \times_{D_0} D_1 & \xrightarrow{\mu} & D_1 \\
\end{array}
\quad \begin{array}{ccc}
C_1 & \xrightarrow{\varepsilon} & C_0 \\
\downarrow {f_0} & & \downarrow {f_0} \\
D_1 & \xrightarrow{\varepsilon} & D_0 \\
\end{array}
\]

The above internal functor \(f\) is denoted by \((f_0, f_1)\). If both \(f_0\) and \(f_1\) are monomorphisms, \(D\) is regarded as an internal subcategory of \(C\).

An internal natural transformation \(\varphi : f \to g\) of internal functors \(f, g : C \to D\) is a morphism \(\varphi : C_0 \to D_1\) in \(\mathcal{E}\) making the following diagrams commute:

\[
\begin{array}{ccc}
D_0 & \xrightarrow{f_0} & C_0 \\
\downarrow {\varphi} & & \downarrow {g_0} \\
D_0 & \xrightarrow{\sigma} & D_1 \\
\end{array}
\quad \begin{array}{ccc}
D_0 \times_{D_0} D_1 & \xrightarrow{\mu} & D_1 \\
\downarrow {(\varphi, \mu)} & & \downarrow {\mu} \\
D_0 \times_{D_0} D_1 & \xrightarrow{\mu} & D_1 \\
\end{array}
\]

We denote by \(\text{cat}(\mathcal{E})\) the category of internal categories in \(\mathcal{E}\).

If \(f = (f_0, f_1) : C \to D\) and \(g = (g_0, g_1) : D \to E\) are internal functors, the composition is defined to be \(gf = (g_0 f_0, g_1 f_1) : C \to E\). Thus, we have the category \(\text{cat}(\mathcal{E})\) of internal categories and internal functors in \(\mathcal{E}\).

**Remark 1.34.** (1) Let \(f, g, h : C \to D\) be internal functors and \(\varphi : f \to g\), \(\psi : g \to h\) internal natural transformations. The composition \(\psi \cdot \varphi : f \to h\) is the morphism \(C_0 \to D_1\)
in \( \mathcal{E} \) given as follows. Since \( \tau \varphi = g_0 = \sigma \psi \), there is a morphism \((\varphi, \psi) : C_0 \to D_1 \times D_0 \) \( D_1 \) satisfying \( \text{pr}_1(\varphi, \psi) = \varphi \) and \( \text{pr}_2(\varphi, \psi) = \psi \). We set \( \psi \cdot \varphi = \mu(\varphi, \psi) : C_0 \to D_1 \). Then, it is easy to verify that \( \psi \cdot \varphi \) is an internal natural transformation from \( f \) to \( h \).

(2) For an internal functor \( f : C \to D \), we set \( \text{id}_f = \varepsilon f_0 : C_0 \to D_1 \). Then, \( \text{id}_f \) is an internal natural transformation from \( f \) to \( f \) and it can be verified that, for internal natural transformations \( \varphi : f \to g \) and \( \psi : g \to f' \), \( \text{id}_f \cdot \psi = \psi \) and \( \varphi \cdot \text{id}_f = \varphi \) hold.

(3) Let \( f, f' : C \to D \), \( g, g' : D \to E \) be internal functors and \( \varphi : f \to f' \), \( \psi : g \to g' \) internal natural transformations. Then, \( \tau g_1 \varphi = \sigma \psi f_0' \) and there exists a morphism \((g_1 \varphi, \psi f_0') : C_0 \to E_1 \times E_0 E_1 \) satisfying \( \text{pr}_1(g_1 \varphi, \psi f_0') = g_1 \varphi \) and \( \text{pr}_2(g_1 \varphi, \psi f_0') = \psi f_0' \). We put \( \varphi \cdot \psi = \mu(g_1 \varphi, \psi f_0') \). It is routine to check that \( \varphi \cdot \psi \) is an internal natural transformation from \( g f \) to \( g' f' \).

**Definition 1.35.** Let \( C = (C_0, C_1; \sigma, \tau, \varepsilon, \mu) \) be an internal category in \( \mathcal{E} \). We call a pair of morphisms \((\pi : X \to C_0, \alpha : X \times C_0 C_1 \to X)\) of \( \mathcal{E} \) an internal diagram on \( C \) if it makes the following diagrams commute, where \( X \times C_0 C_1 \) is the fibered product of \( \pi : X \to C_0 \) and \( \sigma : C_1 \to C_0 \):

\[
\begin{array}{ccccccccc}
X \times C_0 C_1 & \xrightarrow{\alpha} & X & \xrightarrow{\pi} & X \times C_0 C_1 \times C_0 C_1 & \xrightarrow{\alpha \times \text{id}_{C_1}} & X \times C_0 C_1 & \xrightarrow{\alpha} & X \\
\downarrow{\text{pr}_2} & & \downarrow{\pi} & & \downarrow{\text{id}_X \times \mu} & & \downarrow{\alpha} & & \downarrow{\text{id}_X \times \varepsilon} \\
C_1 & \xrightarrow{\tau} & C_0 & \xrightarrow{X \times C_0 C_1} & X & \xrightarrow{X \times C_0 C_1} & X & \xrightarrow{\text{pr}_1} & X
\end{array}
\]

Let \((\pi : X \to C_0, \alpha)\) and \((\rho : Y \to C_0, \beta)\) be internal diagrams on \( C \). A morphism \( f : X \to Y \) in \( \mathcal{E} \) is called a morphism of internal diagrams if it satisfies \( \rho f = \pi \) and \( \beta(f \times \text{id}_{C_0}) = f \alpha \). We denote by \( \mathcal{E}^C \) the category of internal diagrams on \( C \).

### 2. 2-categories and lax functors

First we give the definitions of a 2-category and a lax functor (see [2, 12, 13]).

**Definition 2.1.** A 2-category \( \mathcal{C} \) is determined by the following data.

1. A set \( \text{Ob} \mathcal{C} \) called a set of objects.
2. For each pair of objects \((X, Y)\), a category \( \mathcal{C}(X, Y) \). An object of \( \mathcal{C}(X, Y) \) is called a 1-arrow and a morphism of \( \mathcal{C}(X, Y) \) is called a 2-arrow. The composition of 2-arrows \( S \xrightarrow{f} T, T \xrightarrow{g} U \) in \( \mathcal{C}(X, Y) \) is denoted by \( g f \).
3. For each triple \((X, Y, Z)\) of objects of \( \mathcal{C} \), a functor \( \mu_{X,Y,Z} : \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z) \) called a composition functor (here \( \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \) is the product category of \( \mathcal{C}(X, Y) \) and \( \mathcal{C}(Y, Z) \)) such that the following diagram commutes:

\[
\begin{array}{ccccccccc}
\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \times \mathcal{C}(Z, W) & \xrightarrow{\mu_{X,Y,Z} \times 1} & \mathcal{C}(X, Z) \times \mathcal{C}(Z, W) \\
\downarrow{1 \times \mu_{Y,Z,W}} & & \downarrow{\mu_{X,Z,W}} \\
\mathcal{C}(X, Y) \times \mathcal{C}(Y, W) & \xrightarrow{\mu_{X,Y,W}} & \mathcal{C}(X, W)
\end{array}
\]

We usually denote \( \mu_{X,Y,Z}(S, T) \) by \( T \circ S \) if \((S, T) \in \text{Ob} \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \) and denote \( \mu_{X,Y,Z}(f, g) \) by \( g * f \) if \((f, g) \in \text{Mor} \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \).
(4) For each object $X$ of $\mathcal{C}$, a 1-arrow $1_X$ in $\mathcal{C}(X, X)$ called an identity 1-arrow of $X$ satisfying $S \circ 1_X = S$ and $1_X \circ T = T$ for any $S \in \text{Ob} \mathcal{C}(X, Y)$ and $T \in \text{Ob} \mathcal{C}(Y, X)$.

**Example 2.2.** (1) Let $\mathcal{C}$ be a category. For any pair of objects $(X, Y)$ of $\mathcal{C}$, we regard the set $\mathcal{C}(X, Y)$ of morphisms from $X$ to $Y$ as a discrete category, that is, $\mathcal{C}(X, Y)$ is a category whose set of morphisms consists of only identity morphisms. The composition of morphisms $\mu_{X,Y,Z} : \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ gives a composition functor; that is, $\text{id}_f \ast \text{id}_g = \text{id}_{fg}$. Hence, $\mathcal{C}$ can be regarded as a 2-category.

(2) Let us denote by $\text{cat}$ the category of categories. For a pair $(\mathcal{C}, \mathcal{D})$ of categories, $\text{cat}(\mathcal{C}, \mathcal{D})$ is the functor category $\text{Funct}(\mathcal{C}, \mathcal{D})$. Define the composition functor $\mu_{\mathcal{C}, \mathcal{D}, \mathcal{E}} : \text{cat}(\mathcal{C}, \mathcal{D}) \times \text{cat}(\mathcal{D}, \mathcal{E}) \to \text{cat}(\mathcal{C}, \mathcal{E})$ by $\mu_{\mathcal{C}, \mathcal{D}, \mathcal{E}}(F, G) = G \circ F$ (the composition of functors). $\mu_{\mathcal{C}, \mathcal{D}, \mathcal{E}}(f, g)X = gF(X)G(fX) = G'(fX)gF(X)$ for an object $X \in \text{Ob} \mathcal{C}$ and natural transformations $f : F \to F'$, $g : G \to G'$. For a category $\mathcal{C}$, the identity functor of $\mathcal{C}$ is the identity 1-arrow of $\mathcal{C}$. Hence, $\mathcal{C}$ has a structure of a 2-category.

(3) Let us denote by $\text{pfib}(\mathcal{E})$ the 2-category of cloven prefibered categories over a category $\mathcal{E}$ defined as follows. Objects of $\text{pfib}(\mathcal{E})$ are cloven prefibered categories over $\mathcal{E}$. For cloven prefibered categories $p : \mathcal{F} \to \mathcal{E}$ and $q : \mathcal{D} \to \mathcal{E}$, a 1-arrow of $\text{pfib}(\mathcal{E})(p, q)$ is a functor $F : \mathcal{F} \to \mathcal{D}$ satisfying $qF = p$. A 2-arrow $\varphi : F \to G$ in $\text{pfib}(\mathcal{E})(p, q)$ is a natural transformation of functors such that, for each $M \in \text{Ob} \mathcal{F}$, $q(\varphi_M) = \text{id}_{p(M)}$. Let $r : \mathcal{C} \to \mathcal{E}$ be another cloven prefibered category. The composition functor $\mu_{p,q,r} : \text{pfib}(\mathcal{E})(p, q) \times \text{pfib}(\mathcal{E})(q, r) \to \text{pfib}(\mathcal{E})(p, r)$ is defined in similar way as (2) above. That is, the composition of 1-arrows is just the composition of functors and composition of 2-arrows is given by $\mu_{p,q,r}(f, g)M = g_{F'(M)}G'(fM)M = G'(fM)g_{F(M)}$ for an object $M \in \text{Ob} \mathcal{F}$ and natural transformations $f : F \to F'$, $g : G \to G'$ of functors $F : \mathcal{F} \to \mathcal{D}$, $G, G' : \mathcal{D} \to \mathcal{C}$.

(4) We define 2-categories $\text{fib}(\mathcal{E})$, $\text{pfib}^b(\mathcal{E})$, $\text{fib}^b(\mathcal{E})$ as follows. Here $\text{fib}(\mathcal{E})$ is a full subcategory of $\text{pfib}(\mathcal{E})$ consisting of fibered categories, $\text{pfib}^b(\mathcal{E})$ is a subcategory of $\text{pfib}(\mathcal{E})$ having the same objects as those of $\text{pfib}(\mathcal{E})$ and morphisms which map Cartesian morphisms to Cartesian morphisms and $\text{fib}^b(\mathcal{E})$ is a full subcategory of $\text{pfib}^b(\mathcal{E})$ consisting of fibered categories.

(5) Let $\mathcal{E}$ be a category with finite limits. Using the compositions of internal natural transformations given in Remark 1.34, the category of internal category $\text{cat}(\mathcal{E})$ has a structure of 2-category whose 2-arrows are internal natural transformations.

**Definition 2.3.** Let $\mathcal{D}$ and $\mathcal{C}$ be 2-categories. A lax functor $(\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}$ consists of the following data.

1. A map $\Gamma : \text{Ob} \mathcal{C} \to \text{Ob} \mathcal{D}$.
2. For each pair $(X, Y)$ of objects of $\mathcal{C}$, a functor $\Gamma_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{D}(\Gamma(X), \Gamma(Y))$.
3. For each object $X$ of $\mathcal{C}$, a 2-arrow $\gamma_X : 1_{\Gamma(X)} \to \Gamma_{X,X}(1_X)$.
4. For each triple $(X, Y, Z)$ of objects of $\mathcal{C}$, there is a natural transformation $\gamma_{X,Y,Z} : \mu_{\Gamma(X),\Gamma(Y),\Gamma(Z)}(\Gamma_X \times \Gamma_Y \times \Gamma_Z) \Rightarrow \Gamma_{X,Z} \mu_{X,Y,Z}$

(namely, there is a 2-arrow $\gamma_{X,Y,Z}(S,T) : \Gamma_{Y,Z}(T) \circ \Gamma_{X,Y}(S) = \Gamma_{X,Z}(T \circ S)$ in $\mathcal{D}$ for composable 1-arrows $S : X \to Y$, $T : Y \to Z$) making the following diagrams in the categories $\mathcal{D}(\Gamma(X), \Gamma(Y))$ and $\mathcal{D}(\Gamma(X), \Gamma(W))$, respectively, commute for 1-arrows.
A lax functor \((\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}\) is defined to be the composition in \(\mathcal{D}\) and \(\gamma_x\) is the identity natural transformation.

**Definition 2.4.** Let \((\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}\) and \((\Delta, \delta) : \mathcal{D} \to \mathcal{C}\) be lax functors. We define a lax functor \((\Pi, \pi) : \mathcal{C} \to \mathcal{C}\) as follows. Put \(\Pi(X) = \Delta(\Gamma(X))\) and \(\Pi_{X,Y} = \Delta(\Gamma(X)) \Delta(\Gamma(Y))\) for \(X, Y \in \text{Ob } \mathcal{C}\). \(\pi_X : 1_{\Pi(X)} \to \Pi_{X,X}(1_X)\) is a composition

\[
1_{\Pi(X)} = 1_{\Delta(\Gamma(X))} \xrightarrow{\delta_{\Gamma(X)}} \Delta(\Gamma(X), \Gamma(X))(1_{\Gamma(X)}) \xrightarrow{\Delta(\Gamma(X), \Gamma(X))(\gamma_X)} \Delta(\Gamma(X), \Gamma(X)) \Gamma_X X(1_X) = \Pi_{X,X}(1_X).
\]

For 1-arrows \(S : X \to Y, T : Y \to Z\) in \(\mathcal{C}\),

\[
(\pi_{X,Y,Z})(S,T) = \mu_{\Pi(X), \Pi(Y), \Pi(Z)}(\Pi_{X,Y,Z})(S,T) = \Pi_{X,Z} \mu_{X,Y,Z}(S,T)
\]

is defined to be the composition

\[
\mu_{\Delta(\Gamma(X)), \Delta(\Gamma(Y)), \Delta(\Gamma(Z))}(\Delta(\Gamma(X), \Gamma(Y)) \Gamma_X X \times \Delta(\Gamma(Y), \Gamma(Z)) \Gamma_Y Z)(S, T) \xrightarrow{\Delta(\Gamma(X), \Gamma(Y), \Gamma(Z))(\gamma_X)} \Delta(\Gamma(X), \Gamma(Y), \Gamma(Z))(\Gamma_X X \times \Gamma_Y Z)(S, T) \xrightarrow{\Delta(\Gamma(X), \Gamma(Y), \Gamma(Z))(\gamma_Y)} \Delta(\Gamma(X), \Gamma(Y), \Gamma(Z))(\Gamma_X Z)(S, T).
\]

We call \((\Pi, \pi)\) the composition of \((\Gamma, \gamma)\) and \((\Delta, \delta)\).

We denote by \(I_\mathcal{C} = (I, \iota) : \mathcal{C} \to \mathcal{C}\) the identity lax functor, that is, \(I\) is the identity map of \(\text{Ob } \mathcal{C}\), \(I_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{C}(I(X), I(Y))\) is the identity functor, \(\iota_X : 1_{I(X)} \to I_{X,X}(1_X)\) is the identity 2-arrow and \(\iota_{X,Y,Z} : \mu_{I(X), I(Y), I(Z)}(I_{X,Y} \times I_{Y,Z}) \to I_{X,Z} \mu_{X,Y,Z}\) is the identity natural transformation.

**Definition 2.5.** (1) A lax functor \((\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}\) is called a 2-functor if the 2-arrow \(\gamma_X : 1_{\Gamma(X)} \to \Gamma_{X,X}(1_X)\) is an isomorphism for every \(X \in \text{Ob } \mathcal{C}\) and \(\gamma_{X,Y,Z} : \mu_{\Gamma(X), \Gamma(Y), \Gamma(Z)}(\Gamma_X X \times \Gamma_Y Z) \to \Gamma_{X,Z} \mu_{X,Y,Z}\) is a natural equivalence for every \(X, Y, Z \in \text{Ob } \mathcal{C}\).

(2) If \(\mathcal{C}\) is a category regarded as a 2-category as in (1) of Example 2.2, we call a lax functor \((\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}\) a lax diagram.

(3) A lax diagram which is also a 2-functor is called a pseudo-functor.
Example 2.6. For a functor $F : D \to E$, we define a 2-functor $pFib(F) = (F^*, \gamma_F) : pFib(E) \to pFib(D)$ as follows. For an object $p : F \to E$ of $pFib(E)$, let $F^*(p) = pF : D \times E F \to D$ be the pull-back of $p$ along $F$. If $\kappa$ is a cleavage of $p$, the cleavage $\kappa_F$ of $pF$ is given by $(\kappa_F(f))(Y, N) = (X, \kappa(F(f))(N))$ and $(\kappa_F(f))(\id_Y, \varphi) = (\id_X, \kappa(F(f))(\varphi))$ for a morphism $f : X \to Y$ in $D$ and $N \in \Ob F_{F(Y)}$. For a 1-arrow $\varphi : p \to q$ from an object $p : F \to E$ to an object $q : C \to E$ of $pFib(E)$, let $F^*\varphi : pF \to qF$ be the 1-arrow in $pFib(D)$ induced by $\id_D \times \varphi : D \times F \to D \times C$. It follows from Proposition 1.16 that if $\varphi$ is a 1-arrow in $pFib(E)$, $F^*\varphi$ is a 1-arrow in $pFib(D)$. Let $\psi : p \to q$ be 1-arrows in $pFib(E)$ and $\chi : \varphi \to \psi$ a 2-arrow. Define a 2-arrow $F^*\chi : F^*\varphi \to F^*\psi$ by $F^*\chi(X, M) = (\id_X, \chi_M) : F^*\varphi(X, M) \to F^*\psi(X, M)$ for $(X, M) \in \Ob(D \times E F)$. Thus, we have a functor $F^* : pFib(E)(p, q) \to pFib(D)(F(p), F(q))$. For $p \in \Ob pFib(E)$, since $1_{F^*(p)} = \id_{pF} = F^*1_p : pF \to pF$, let $(\gamma_F)_p : 1_{F^*(p)} \to F^*_p(1_p)$ be the identity 2-arrow. For 1-arrows $\varphi : p \to q$, $\psi : q \to r$ in $pFib(E)$, composition of 1-arrows $F^*_p(\varphi) : pF \to qF$ and $F^*_q(\psi) : qF \to rF$ coincides with $F^*_r(\psi\varphi)$. We define a 2-arrow $((\gamma_F)_p, \gamma_q) : F^*_p(\varphi) F^*_q(\psi) \to F^*_r(\psi\varphi)$ in $pFib(D)$ as the identity 2-arrow. In fact, $pFib(F)$ is a functor.

Definition 2.7. For 2-categories $C$ and $D$, we define the 2-category $\text{Lax}(C, D)$ of lax functors as follows. Objects of $\text{Lax}(C, D)$ are lax functors from $C$ to $D$. Let $(\Gamma, \gamma), (\Delta, \delta) : C \to D$ be lax functors. A 1-arrow $(\Lambda, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)$ consists of the following data.

1. For each $X \in \Ob C$, a 1-arrow $\Lambda_X : \Gamma(X) \to \Delta(X)$ in $D$.
2. For each 1-arrow $S : X \to Y$ in $C$, a 2-arrow $\Lambda_S : \Delta_Y \circ \Gamma_X(Y) \to \Delta_X \circ \Gamma_Y(X)$ in $D$, making the following diagrams commute for every $X \in \Ob C$ and 1-arrows $S : X \to Y$, $T : Y \to Z$ in $C$:

\[
\begin{array}{c}
\Lambda_X \circ 1_{\Gamma(X)} \\
\downarrow \quad \downarrow \lambda_{1_X}^* \\
\Lambda_X \circ \Gamma_X, X(1_X) \quad \Delta_X, X(1_X) \circ \Lambda_X
\end{array}
\]

\[
\begin{array}{c}
\Lambda_X \circ 1_{\Delta(X)} \\
\downarrow \quad \downarrow \delta_X \circ \id_{\Lambda_X} \\
\Lambda_X \circ \Gamma_X, X(1_X) \quad \Delta_X, X(1_X) \circ \Lambda_X
\end{array}
\]

\[
\begin{array}{c}
\Lambda_Z \circ \Gamma_Y, Z(T) \circ \Gamma_X, Y(S) \\
\downarrow \lambda_T \circ \id_{\Gamma_Y}, Y(S) \\
\Delta_Y, Z(T \circ S) \circ \Lambda_X
\end{array}
\]

\[
\begin{array}{c}
\Lambda_Z \circ \Gamma_Y, Z(T) \circ \Gamma_X, Y(S) \\
\downarrow \lambda_T \circ \id_{\Gamma_Y}, Y(S) \\
\Delta_Y, Z(T \circ S) \circ \Lambda_X
\end{array}
\]

\[
\begin{array}{c}
\Delta_Y, Z(T) \circ \Delta_Y \circ \Gamma_X, Y(S) \\
\downarrow \id_{\Gamma_Y}, Z(T) \circ \lambda_S \\
\Delta_Y, Z(T \circ S) \circ \Lambda_X
\end{array}
\]

\[
\begin{array}{c}
\Delta_Y, Z(T) \circ \Delta_Y \circ \Gamma_X, Y(S) \\
\downarrow \id_{\Gamma_Y}, Z(T) \circ \lambda_S \\
\Delta_Y, Z(T \circ S) \circ \Lambda_X
\end{array}
\]

For 1-arrows $(\Lambda, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)$ and $(\Phi, \varphi) : (\Delta, \delta) \to (E, \epsilon)$, we define a 1-arrow $(\Psi, \psi) : (\Gamma, \gamma) \to (E, \epsilon)$ by

\[
\psi_X = \Phi_X \circ \Lambda_X : \Gamma(X) \to E(X),
\]

\[
\psi_S = (\varphi_S \circ \id_{\Lambda_X} \circ \varphi_S \circ \id_{\Lambda_X}) : \Phi_Y \circ \Lambda_Y \circ \Gamma_X, Y(S) \to E_{X, Y}(S) \circ \Phi_X \circ \Lambda_X
\]
for $X \in \text{Ob} \mathcal{C}$ and a 1-arrow $S : X \to Y$ in $\mathcal{C}$. The composition $(\Phi, \varphi) \circ (\Lambda, \lambda)$ is defined to be $(\Psi, \psi)$. The identity 1-arrow $1_{(\Gamma, \gamma)}$ of $(\Gamma, \gamma) \in \text{Ob} \text{Lax}(\mathcal{C}, \mathcal{D})$ is a pair $(I, \iota)$ such that $I_X$ is the identity 1-arrow of $\Gamma(X)$ for $X \in \text{Ob} \mathcal{C}$ and $\iota_S$ is the identity 2-arrow of $\Gamma_{X,Y}(S)$ for a 1-arrow $S : X \to Y$ in $\mathcal{C}$.

Let $(\Lambda, \lambda), (\Phi, \varphi) : (\Gamma, \gamma) \to (\Delta, \delta)$ be 1-arrows in $\text{Lax}(\mathcal{C}, \mathcal{D})$. A 2-arrow $\chi : (\Lambda, \lambda) \to (\Phi, \varphi)$ consists of 2-arrows $\chi_X : \Lambda_X \to \Phi_X$ in $\mathcal{D}$ for $X \in \text{Ob} \mathcal{C}$ such that, for every 1-arrow $S : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$
\begin{array}{ccc}
\Delta_Y \circ \Gamma_{X,Y}(S) & \xrightarrow{\lambda^S} & \Delta_{X,Y}(S) \circ \Lambda_X \\
\chi_Y \circ \text{id}_{\Gamma_{X,Y}(S)} & & \text{id}_{\Delta_{X,Y}(S)} \circ \chi_X \\
\Phi_Y \circ \Gamma_{X,Y}(S) & \xrightarrow{\varphi^S} & \Delta_{X,Y}(S) \circ \Phi_X \\
\end{array}
$$

For 2-arrows $\chi : (\Lambda, \lambda) \to (\Phi, \varphi)$ and $\omega : (\Phi, \varphi) \to (\Psi, \psi)$ $((\Lambda, \lambda), (\Phi, \varphi), (\Psi, \psi) : (\Gamma, \gamma) \to (\Delta, \delta))$, the composition $\omega \chi : (\Lambda, \lambda) \to (\Psi, \psi)$ in $\text{Lax}(\mathcal{C}, \mathcal{D})((\Gamma, \gamma), (\Delta, \delta))$ is defined by $(\omega \chi)_X = \omega_X \chi_X$. If $\chi : (\Lambda, \lambda) \to (\Phi, \varphi)$ and

$$\omega : (\Psi, \psi) \to (\Upsilon, \upsilon)((\Lambda, \lambda), (\Phi, \varphi) : (\Gamma, \gamma) \to (\Delta, \delta), (\Psi, \psi), (\Upsilon, \upsilon) : (\Delta, \delta) \to (E, \epsilon))$$

are 2-arrows, the composition $\omega \ast \chi : (\Psi, \psi) \circ (\Lambda, \lambda) \to (\Upsilon, \upsilon) \circ (\Phi, \varphi)$ is defined by $(\omega \ast \chi)_X = \omega_X \ast \chi_X : \Psi_X \circ \Lambda_X \to \Upsilon_X \circ \Phi_X$. The identity 2-arrow $1_{(\Lambda, \lambda)}$ of a 1-arrow $(\Lambda, \lambda)$ in $\text{Lax}(\mathcal{C}, \mathcal{D})$ is given by $(1_{(\Lambda, \lambda)})_X = (\text{the identity 2-arrow of } \Lambda_X)$ for any $X \in \text{Ob} \mathcal{C}$.

**Definition 2.8.** For later use, we denote by $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ the full subcategory of $\text{Lax}(\mathcal{C}, \mathcal{D})$ consisting of lax functors $(\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}$ such that $\gamma_X : 1_{\Gamma(X)} \to \Gamma_{X,X}(1_X)$ is an isomorphism for each $X \in \text{Ob} \mathcal{C}$. Moreover, $\text{2-Funct}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\text{Lax}(\mathcal{C}, \mathcal{D})$ consisting of 2-functors. We also consider a subcategory $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ of $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ and a subcategory $\text{2-Funct}^2(\mathcal{C}, \mathcal{D})$ of $\text{2-Funct}(\mathcal{C}, \mathcal{D})$ given as follows. Here $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ (respectively $\text{2-Funct}^2(\mathcal{C}, \mathcal{D})$) has the same objects as $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ (respectively $\text{2-Funct}(\mathcal{C}, \mathcal{D})$). A 1-arrow $(\Lambda, \lambda)$ in $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ (respectively $\text{2-Funct}(\mathcal{C}, \mathcal{D})$) belongs to $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ (respectively $\text{2-Funct}^2(\mathcal{C}, \mathcal{D})$) if and only if a 2-arrow $\lambda_S$ in $\mathcal{D}$ is an isomorphism for every 1-arrow $S$ in $\mathcal{C}$. A 2-arrow in $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ (respectively $\text{2-Funct}(\mathcal{C}, \mathcal{D})$) belongs to $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ (respectively $\text{2-Funct}^2(\mathcal{C}, \mathcal{D})$) if and only if its domain and codomain are 1-arrows in $\text{Lax}^2(\mathcal{C}, \mathcal{D})$ (respectively $\text{2-Funct}^2(\mathcal{C}, \mathcal{D})$).

Note that $\text{2-Funct}(\mathcal{C}, \mathcal{D})$ is also a full subcategory of $\text{Lax}^2(\mathcal{C}, \mathcal{D})$.

**Example 2.9.** Let $(A, \alpha) : \mathcal{C}' \to \mathcal{C}$ be a lax functor. Define a lax functor $(A, \alpha)^* : \text{Lax}(\mathcal{C}, \mathcal{D}) \to \text{Lax}(\mathcal{C}', \mathcal{D}')$ as follows. Put $(A, \alpha)^* = (A^*, \alpha_A)$. For a lax functor $(\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}$, we set $A^*((\Gamma, \gamma)) = (\Gamma, \gamma) \circ (A, \alpha)$. For a 1-arrow $(\Lambda, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)$ in $\text{Lax}(\mathcal{C}, \mathcal{D})$, let us define a 1-arrow $(\Lambda_A, \lambda_A) : (\Gamma, \gamma) \circ (A, \alpha) \to (\Delta, \delta) \circ (A, \alpha)$ in $\text{Lax}(\mathcal{C}', \mathcal{D}')$ by $(\Lambda_A)_X = \Lambda_{A(X)}$ for $X \in \text{Ob} \mathcal{C}'$ and $(\lambda_A)_S = (\lambda_A)_S : \Lambda_{A(Y)} \circ \Gamma_{A(X), A(Y)}(A_{X,Y}(S)) = \Delta_{A(X), A(Y)}(A_{X,Y}(S)) \circ \Lambda_{A(X)}$ for a 1-arrow $S : X \to Y$ in $\mathcal{C}'$. We set $A^*_{(\Gamma, \gamma), (\Delta, \delta)}((\Lambda, \lambda)) = (\Lambda_A, \lambda_A)$. For a 2-arrow $\chi : (\Lambda, \lambda) \to (\Phi, \varphi)$ in $\text{Lax}(\mathcal{C}, \mathcal{D})$, let $\chi_A : (\Lambda_A, \lambda_A) \to (\Phi_A, \varphi_A)$ be the 2-arrow given by $(\chi_A)_X = \chi_{A(X)} : (\Lambda_A)_X = \Lambda_{A(X)} \to \Phi_{A(X)} = (\Phi_A)_X$ for $X \in \text{Ob} \mathcal{C}'$. We set $A^*_{(\Gamma, \gamma), (\Delta, \delta)}(\chi) = \chi_A$. 

\image
For a lax functor \((\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}\), since \(A^*_{(\Gamma, \gamma)}(1_{(\Gamma, \gamma)})\) is the identity 1-arrow of \(1^*_{(\Gamma, \gamma)}\), let \(\gamma(\lambda_{(\Gamma, \gamma)}) : 1^*_{(\Gamma, \gamma)} \to A^*_{(\Gamma, \gamma)}(1_{(\Gamma, \gamma)})\) be the identity 2-arrow of \(1^*_{(\Gamma, \gamma)}\). For lax functors \((\Gamma, \gamma), (\Delta, \delta), (E, \epsilon) : \mathcal{C} \to \mathcal{D}\), let \(\gamma(\lambda_{(\Delta, \delta), (E, \epsilon)}(1_{(\Gamma, \gamma)}))\) be the identity natural transformation.

\[
\mu A^*_{(\Gamma, \gamma)}, A^*_{((\Delta, \delta), (E, \epsilon))}(A^*_{(\Gamma, \gamma)}, (\Delta, \delta) \times A^*_{(\Delta, \delta), (E, \epsilon)}) \to A^*_{(\Gamma, \gamma), (\Delta, \delta), (E, \epsilon)}(\mu(\Gamma, \gamma), (\Delta, \delta), (E, \epsilon)).
\]

In fact, for composable 1-arrows \((\Lambda, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)\) and \((\Phi, \varphi) : (\Delta, \delta) \to (E, \epsilon)\) in \(\text{Lax}(\mathcal{C}, \mathcal{D})\), it is easy to verify

\[
A^*_{(\Delta, \delta), (E, \epsilon)}((\Phi, \varphi)) \circ A^*_{(\Gamma, \gamma), (\Delta, \delta)}((\Lambda, \lambda)) = A^*_{(\Gamma, \gamma), (E, \epsilon)}((\Phi, \varphi) \circ (\Lambda, \lambda)).
\]

Let

\[
((\gamma(\lambda_{(\Gamma, \gamma)}), (\Delta, \delta), (E, \epsilon))((\Lambda, \lambda), (\Phi, \varphi)) : A^*_{(\Delta, \delta), (E, \epsilon)}((\Phi, \varphi)) \circ A^*_{(\Gamma, \gamma), (\Delta, \delta)}((\Lambda, \lambda))
\]

be the identity 2-arrow in \(\text{Lax}(\mathcal{C}, \mathcal{D})\). It is routine to verify that \(A, \alpha^* : \text{Lax}(\mathcal{C}, \mathcal{D}) \to \text{Lax}(\mathcal{C}', \mathcal{D})\) is a lax functor.

**Remark 2.10.** Note that the lax functor \((A, \alpha^* : \text{Lax}(\mathcal{C}, \mathcal{D}) \to \text{Lax}(\mathcal{C}', \mathcal{D})\) defined above maps \(\text{Lax}^x(\mathcal{C}, \mathcal{D}) \to \text{Lax}^x(\mathcal{C}', \mathcal{D})\), \(\text{2-Func}(\mathcal{C}, \mathcal{D}) \to \text{2-Func}(\mathcal{C}', \mathcal{D})\) and \(\text{Lax}^c(\mathcal{C}, \mathcal{D}) \to \text{Lax}^c(\mathcal{C}', \mathcal{D})\). Therefore, we have the following lax functors:

\[
(A, \alpha^* : \text{Lax}^x(\mathcal{C}, \mathcal{D}) \to \text{Lax}^x(\mathcal{C}', \mathcal{D}), \quad (A, \alpha^* : \text{Lax}^c(\mathcal{C}, \mathcal{D}) \to \text{Lax}^c(\mathcal{C}', \mathcal{D}),
\]

\[
(A, \alpha^* : \text{2-Func}(\mathcal{C}, \mathcal{D}) \to \text{2-Func}(\mathcal{C}', \mathcal{D}), \quad (A, \alpha^* : \text{2-Func}(\mathcal{C}, \mathcal{D}) \to \text{2-Func}(\mathcal{C}', \mathcal{D}).
\]

**Example 2.11.** Let \((B, \beta) : \mathcal{D} \to \mathcal{D}'\) be a 2-functor. Define a lax functor \((B, \beta)_* : \text{Lax}(\mathcal{C}, \mathcal{D}) \to \text{Lax}(\mathcal{C}', \mathcal{D}')\) as follows. Put \((B, \beta)_* = (B_*, \gamma_B)\). For a lax functor \((\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}\), we set \(B_*(\Gamma, \gamma) = (B, \beta) \circ (\Gamma, \gamma)\). For a 1-arrow \((\Lambda, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)\) in \(\text{Lax}(\mathcal{C}, \mathcal{D})\), let us define a 1-arrow \((\Lambda_B, \lambda_B) : (B, \beta) \circ (\Gamma, \gamma) \to (B, \beta) \circ (\Delta, \delta)\) in \(\text{Lax}(\mathcal{C}', \mathcal{D}')\) by \((\Lambda_B)X = B_{\Gamma(X), \Delta(Y)}(\Lambda X)\) for \(X \in \text{Ob}\mathcal{C}\) and \((\lambda_B)_S\) is given by the following composition for a 1-arrow \(S : X \to Y \in \mathcal{C}:

\[
B_{\Gamma(Y), \Delta(Y)}(\Lambda Y) \circ B_{\Gamma(X), \Gamma(Y)}(\Gamma X, Y(S))
\]

\[
\xrightarrow{(B_{\Gamma(X), \Gamma(Y), \Delta(Y)}(\Lambda Y, \Gamma X, Y(S)))}
B_{\Gamma(X), \Delta(Y)}(\Lambda Y \circ \Gamma X, Y(S))
\]

\[
\xrightarrow{B_{\Gamma(X), \Delta(Y), \Lambda X}(\Lambda S)}
B_{\Gamma(X), \Delta(Y)}(\Delta X, Y(S) \circ \Lambda X)
\]

\[
\xrightarrow{(B_{\Gamma(X), \Delta(Y), \Lambda X})(\Delta X, Y(S), \Lambda X)}
B_{\Delta(X), \Delta(Y)}(\Delta X, Y(S) \circ \Lambda X).
\]

We set \((B_*(\Gamma, \gamma), (\Delta, \delta), (\Lambda, \lambda) = (B \circ \lambda_B, \beta)\). For a 2-arrow \(\chi : (\Lambda, \lambda) \to (\Phi, \varphi)\) in \(\text{Lax}(\mathcal{C}, \mathcal{D})\), let \(\chi_B : (B \circ \lambda_B, \beta) \to (B \circ \varphi(B, \beta))\) be the 2-arrow given by \((\chi_B)_X = B_{\Gamma(X), \Delta(X)}(\chi X) : (B \circ \lambda_B)X = B_{\Gamma(X), \Delta(X)}(\Lambda X) \to B_{\Gamma(X), \Delta(X)}(\Phi X) = (B \circ \varphi(B, \beta))_X\) for \(X \in \text{Ob}\mathcal{C}\). We set \((B_*(\Gamma, \gamma), (\Delta, \delta), (\Lambda, \lambda) = \chi_B\).

For a lax functor \((\Gamma, \gamma) : \mathcal{C} \to \mathcal{D}\), let \((\gamma_B)(\Gamma, \gamma) : 1_{B_*(\Gamma, \gamma)} \to (B_*(\Gamma, \gamma), (\Gamma, \gamma))(1_{(\Gamma, \gamma)})\) be the 2-arrow in \(\text{Lax}(\mathcal{C}, \mathcal{D}')\) given by \(((\gamma_B)(\Gamma, \gamma))X = \beta_{\Gamma(X)}\) for each \(X \in \text{Ob}\mathcal{C}\). For lax functors
each object \(X\) is natural in \((\Lambda, \lambda)\) defined as follows. For composable 1-arrows \((\Lambda, \lambda): (\Gamma, \gamma) \rightarrow (\Delta, \delta)\) and \((\Phi, \phi): (\Delta, \delta) \rightarrow (\epsilon, \epsilon)\) in \(\text{Lax}(\mathcal{C}, \mathcal{D})\), let

\[
\mu_B((\Gamma, \gamma),(\Delta, \delta),(\epsilon, \epsilon))((B_\ast(\Gamma, \gamma),(\Delta, \delta),(\epsilon, \epsilon)) \rightarrow (B_\ast(\Gamma, \gamma),(\Delta, \delta),(\epsilon, \epsilon))
\]
defined as follows. For composable 1-arrows \((\Lambda, \lambda): (\Gamma, \gamma) \rightarrow (\Delta, \delta)\) and \((\Phi, \phi): (\Delta, \delta) \rightarrow (\epsilon, \epsilon)\) in \(\text{Lax}(\mathcal{C}, \mathcal{D})\), let

\[
((\gamma_B(\Gamma, \gamma),(\Delta, \delta),(\epsilon, \epsilon)))((\Lambda, \lambda),(\Phi, \phi)) \rightarrow (B_\ast(\Gamma, \gamma),(\Delta, \delta),(\epsilon, \epsilon))((\Lambda, \lambda),(\Phi, \phi))
\]
be the 2-arrow in \(\text{Lax}(\mathcal{C}, \mathcal{D}')\) given by

\[
(((\gamma_B(\Gamma, \gamma),(\Delta, \delta),(\epsilon, \epsilon)))((\Lambda, \lambda),(\Phi, \phi)))X = (\beta^{\Gamma(X), \Delta(X), E(X)})(\Lambda_X, \Phi_X)
\]
for \(X \in \text{Ob} \mathcal{C}\). Again, it is routine to verify that \((B, \beta)_\ast: \text{Lax}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Lax}(\mathcal{C}, \mathcal{D}')\) is a lax functor.

**Remark 2.12.** The lax functor \((B, \beta)_\ast: \text{Lax}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Lax}(\mathcal{C}, \mathcal{D}')\) defined above maps \(\text{Lax}^s(\mathcal{C}, \mathcal{D})\) to \(\text{Lax}^s(\mathcal{C}, \mathcal{D}')\), \(\text{2-Funct}(\mathcal{C}, \mathcal{D})\) to \(\text{2-Funct}(\mathcal{C}, \mathcal{D}')\) and \(\text{Lax}^c(\mathcal{C}, \mathcal{D})\) to \(\text{Lax}^c(\mathcal{C}, \mathcal{D}')\). Therefore, we have the following lax functors:

\[
(B, \beta)_\ast: \text{Lax}^s(\mathcal{C}, \mathcal{D}) \rightarrow \text{Lax}^s(\mathcal{C}, \mathcal{D}'), \quad (B, \beta)_\ast: \text{Lax}^c(\mathcal{C}, \mathcal{D}) \rightarrow \text{Lax}^c(\mathcal{C}, \mathcal{D}'), \quad (B, \beta)_\ast: \text{2-Funct}(\mathcal{C}, \mathcal{D}) \rightarrow \text{2-Funct}(\mathcal{C}, \mathcal{D}'), \quad (B, \beta)_\ast: \text{2-Funct}'(\mathcal{C}, \mathcal{D}) \rightarrow \text{2-Funct}'(\mathcal{C}, \mathcal{D}').
\]

Let \(p:F \rightarrow E\) be a cloven prefibered category with cleavage \(\kappa\). We associate a lax diagram \((\Gamma(p), \gamma(p)): \mathcal{E}^{op} \rightarrow \text{cat}\) as follows.

**Construction 2.13.** We set \(\Gamma(p)(X) = \mathcal{F}_X\) for \(X \in \text{Ob} \mathcal{E}\). For a morphism \(f:X \rightarrow Y\) in \(\mathcal{E}\), we define \(\Gamma(p)_Y X(f): \mathcal{F}_Y \rightarrow \mathcal{F}_X\) to be the inverse image functor \(f^* = \kappa(f): \mathcal{F}_Y \rightarrow \mathcal{F}_X\). For \(X \in \text{Ob} \mathcal{E}\), since \(\alpha_{id_X}(N) = \id_X^\ast(N) \rightarrow N\) is an isomorphism by Proposition 1.5, a natural transformation \(\gamma(p)_X: 1_{\mathcal{F}_X} \rightarrow \id_X^\ast\) is given by \((\gamma(p)_X)_N = \alpha_{id_X}(N)^{-1} : N \rightarrow \id_X^\ast(N) (N \in \text{Ob} \mathcal{F}_X)\). For each pair \((g, f)\) in \(\mathcal{E}(Z, X) \times \mathcal{E}(X, Y)\), a 2-arrow \((\gamma(p)_Y X Z, (f, g)) = g^* f^* \rightarrow (fg)^*\) is defined to be \(c_{f, g}\). It follows from Proposition 1.12 that \((\Gamma(p), \gamma(p))\) is a lax diagram. We call \((\Gamma(p), \gamma(p))\) the lax diagram associated with \(p:F \rightarrow E\). Moreover, \((\Gamma(p), \gamma(p))\) is a pseudo-functor if and only if \(p:F \rightarrow E\) is a fibered category.

Let \(p: F \rightarrow E, q: D \rightarrow E\) be objects of \(\text{pfi b}(\mathcal{E})\) and \(F: p \rightarrow q\) a morphism in \(\text{pfi b}(\mathcal{E})\). We define a morphism \((\Lambda(F), \lambda(F)): (\Gamma(p), \gamma(p)) \rightarrow (\Gamma(q), \gamma(q))\) as follows.

For \(X \in \text{Ob} \mathcal{E}\), \(\Lambda(F)_X : \mathcal{F}_X \rightarrow \mathcal{D}_X\) is the restriction \(\mathcal{F}_X\) of \(F\). For a morphism \(f:X \rightarrow Y\) in \(\mathcal{E}\) and an object \(N\) of \(\mathcal{F}_Y\), let \((\Lambda(F)_f)_N : \mathcal{F}_X f^*(N) \rightarrow f^*(\mathcal{F}_Y(N))\) be the unique morphism in \(\mathcal{D}_X\) such that \(\alpha_f(\mathcal{F}_Y(N)) (\Lambda(F)_f)_N = F(\alpha_f(N))\). Then, \((\Lambda(F)_f)_N\) is natural in \(N\) and we have a natural transformation \(\lambda(F)_f: \Lambda(F)_X \circ \Gamma(p)_Y X (f) \rightarrow \Gamma(q)_Y X (f) \circ \Lambda(F)_Y\). We note that, if \(F\) is the identity morphism of \(p\), \(\Lambda(F)_X\) is the identity functor of \(\mathcal{F}_X\) for every \(X \in \text{Ob} \mathcal{E}\) and \(\lambda(F)_f = \id_{f^*}\) for every morphism \(f\) in \(\mathcal{E}\). Also note that \(\lambda(F)_f\) is an equivalence for every morphism \(f\) if and only if \(F\) preserves Cartesian morphisms.

Let \(F, G: p \rightarrow q\) be a morphism in \(\text{pfi b}(\mathcal{E})\) and \(\varphi : F \rightarrow G\) a 2-arrow in \(\text{pfi b}(\mathcal{E})\). For each object \(X\) of \(\mathcal{E}\), the natural transformation \(\varphi_X : F_X \rightarrow G_X\) induced by \(\varphi\) defines a 2-arrow \(\chi(\varphi) : (\Lambda(F), \lambda(F)) \rightarrow (\Lambda(G), \lambda(G))\) by \(\chi(\varphi)_X = \varphi_X : F_X \rightarrow G_X\).
For a category $\mathcal{E}$, we define a functor $\Theta = \Theta_\mathcal{E} : \text{pfib}(\mathcal{E}) \to \text{Lax}^2(\mathcal{E}^{op}, \text{cat})$ as follows. Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven prefibered category with cleavage $\kappa$. Put $\Theta(p) = (\Gamma(p), \gamma(p))$. If $F : p \to q : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{D} \to \mathcal{E}$ is a morphism in $\text{pfib}(\mathcal{E})$, put $\Theta(F) = (\Lambda(F), \lambda(F))$. Let $G : q \to r : \mathcal{C} \to \mathcal{E}$ be a morphism in $\text{pfib}(\mathcal{E})$. Then,

$$\Lambda(GF)_X = G_XF_X = \Lambda(G)_X\Lambda(F)_X$$

and

$$\alpha_f(G\gamma F\gamma(N))(\lambda(G)f)_{FY(N)}G_X((\lambda(F)f)N) = ((\lambda(G)f)*\text{id}_{\Lambda(F)Y})\circ(\text{id}_{\Lambda(G)X}*\lambda(F)f))N.$$ 

Hence,

$$\Lambda(GF), \lambda(GF) = (\Lambda(G), \lambda(G)) \circ (\Lambda(F), \lambda(F)).$$

For a 2-arrow $\varphi : F \to G (F, G : p \to q)$ in $\text{pfib}(\mathcal{E})$, we set $\Theta(\varphi) = \chi(\varphi)$, namely $\Theta(\varphi)_X = \varphi_X : \mathcal{F}_X \to \mathcal{D}_X$ for $X \in \text{Ob } \mathcal{E}$. Moreover, if $\xi : G \to H (H : p \to q)$ and $\psi : K \to L (K, L : q \to r)$ are 2-arrows in $\text{pfib}(\mathcal{E})$, then it is straightforward to verify $\Theta(\xi \varphi) = \Theta(\xi) \Theta(\varphi)$ and $\Theta(\psi \varphi) = \Theta(\psi) \circ \Theta(\varphi)$. Thus, $\Theta$ is a functor. In other words, we have a lax functor $(\Theta_\mathcal{E}, \theta_\mathcal{E}) : \text{pfib}(\mathcal{E}) \to \text{Lax}^2(\mathcal{E}^{op}, \text{cat})$, where $(\theta_\mathcal{E})_p$ is the identity 2-arrow of 1$\Theta(p)$ and $(\theta_p)_p,q,r$ is the identity natural transformation for any object $p, q, r$ of $\text{pfib}(\mathcal{E})$.

Let $(\Gamma, \gamma) : \mathcal{E} \to \mathcal{D}$ be an object of $\text{Lax}^2(\mathcal{E}, \mathcal{D})$. For a 1-arrow $S : X \to Y$ in $\mathcal{E}$, we put $R_Y(S) = (\gamma_{X,Y,Y})_{(S,1_X)} : \Gamma_{Y,Y}(1_Y)\circ \Gamma_{X,Y}(S) \to \Gamma_{X,Y}(S), \ L_Y(S) = (\gamma_{X,X,Y})_{(1_X,S)} : \Gamma_{X,Y}(S)\circ \Gamma_{X,X}(1_X) \to \Gamma_{X,Y}(S)$. Since $\gamma_Y : 1_{\Gamma(Y)} \to \Gamma_{Y,Y}(1_Y)$ is an isomorphism, the commutativity of the upper diagram of (4) in Definition 2.3 implies the following assertion.

**Proposition 2.14.** The 2-arrows $R_Y(f)$ and $L_Y(f)$ in $\mathcal{D}$ are isomorphisms.

Let $(\Gamma, \gamma) : \mathcal{E}^{op} \to \text{cat}$ be an object of $\text{Lax}^2(\mathcal{E}^{op}, \text{cat})$. We construct a cloven prefibered category $p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}$ as follows.

**Construction 2.15.** Set $\text{Ob } \mathcal{F}(\Gamma) = \{(X, x) \mid X \in \text{Ob } \mathcal{E}, x \in \text{Ob } \Gamma(X)\}$. For $(X, x), (Y, y) \in \text{Ob } \mathcal{F}(\Gamma)$, we put $\mathcal{F}(\Gamma)(((X, x), (Y, y)) = \{(f, u) \mid f \in \mathcal{E}(X,Y), u \in \Gamma(X)(x, \gamma_{X,Y,X}(f)(y))\}$. Composition of morphisms $(f, u) : (X, x) \to (Y, y)$ and $(g, v) : (Y, y) \to (Z, z)$ is defined to be $(gf, ((\gamma_{Z,Y,Y})(g,f))\gamma_{Y,Y,X}(f)(v)u)$. Note that $(\text{id}_X, (\gamma_{X,Y}))$ is the identity morphism of $(X, x)$. Define a functor $p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}$ by $p(\Gamma)(X, x) = X, \ p(\Gamma)(f, u) = f$. For each $X \in \text{Ob } \mathcal{E}$, there is an isomorphism $f_X : \mathcal{F}(\Gamma)_X \to \Gamma(X)$ of categories given by $f_X(X, x) = x$ and $f_X(\text{id}_X, u) = (\gamma_X)^{-1}_X u ((\text{id}_X, u) : (X, x) \to (X, y))$.

We claim that a morphism $(f, u) : (X, x) \to (Y, y)$ is Cartesian if and only if $u : x \to \Gamma_{Y,X}(f)(y)$ is an isomorphism in $\Gamma(X)$. In fact, since

$$(f, u)(\text{id}_X, v) = (f, R_Y(f)f)(\gamma_{Y,X,X}(u)v) = (f, (R_Y(f)f)(\gamma_X^{-1}X)(u)v) = (f, u(\gamma_X)^{-1}_X v)$$

by the assumption and the commutativity of the upper diagram of (4) of (2.3), the map

$$\mathcal{F}(\Gamma)_X((X, z), (X, x)) \to \mathcal{F}(\Gamma)_Y((X, z), (Y, y))$$

given by $(\text{id}_X, v) \mapsto (f, u)(\text{id}_X, v)$ is bijective for every $z \in \text{Ob } \Gamma(X)$ if and only if $u$ is an isomorphism.
In particular, \((\text{id}_X, u) : (X, x) \to (X, y)\) is an isomorphism if and only if \(u\) is an isomorphism. The inverse of \((\text{id}_X, u)\) is given by \((\text{id}_X, (yX)_x u^{-1}(yX)_y)\).

For a morphism \(f : X \to Y\) in \(\mathcal{E}\), set \(f^*(Y, y) = (X, \Gamma_{Y,X}(f)(y))\) and the canonical morphism \(\alpha_f(Y, y) : f^*(Y, y) \to (Y, y)\) defined to be \((f, \text{id}_{\Gamma_{Y,X}(f)(y)})\).

Then \(\alpha_f(Y, y)\) is Cartesian by the above fact, hence the inverse image functor \(f^* : \mathcal{F}(\Gamma)_Y \to \mathcal{F}(\Gamma)_X\) of \(f\) is given by \(f^*(Y, y) = (X, \Gamma_{Y,X}(f)(y))\) and \(f^*(\text{id}_Y, v) = (\text{id}_X, R_Y(f)_z^{-1}L_Y(f)_z\Gamma_{Y,X}(f)(v))\). Note that, for a morphism \(f : X \to Y\) in \(\mathcal{E}\),

\[
\begin{array}{ccc}
\mathcal{F}(\Gamma)_X & \xrightarrow{f^*} & \mathcal{F}(\Gamma)_Y \\
\xrightarrow{f} & & \xrightarrow{f} \\
\Gamma(X) & \xrightarrow{\Gamma_{Y,X}(f)} & \Gamma(Y)
\end{array}
\]

commutes. For morphisms \(f : X \to Y\) and \(g : Z \to X\) in \(\mathcal{E}\) and \((Y, y) \in \text{Ob} \ \mathcal{F}(\Gamma)\), we define \(c_{f,g}(Y, y) : g^*f^*(Y, y) \to (fg)^*(Y, y)\) by \(c_{f,g}(Y, y) = (\text{id}_Z, (R_Y(fg))_y^{-1}((\gamma_{Y,X,Z}(f,g))_y)\).

Then, the following square commutes:

\[
\begin{array}{ccc}
g^*f^*(Y, y) & \xrightarrow{\alpha_g(f^*(Y, y))} & f^*(Y, y) \\
\xrightarrow{c_{f,g}(Y, y)} & & \xrightarrow{\alpha_f(Y, y)} \\
(fg)^*(Y, y) & \xrightarrow{\alpha_{fg}(Y, y)} & (Y, y)
\end{array}
\]

It follows from (1.11) that \(p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}\) is a fibered category if and only if \((\Gamma, \gamma)\) is a pseudo-functor.

Let \((\Gamma, \gamma), (\Delta, \delta) : \mathcal{E}^{\text{op}} \to \text{cat}\) be objects of \(\text{Lax}^s(\mathcal{E}^{\text{op}}, \text{cat})\). For 1-arrow \((\Lambda, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)\) of lax diagrams, we construct a functor \(F_\Lambda : \mathcal{F}(\Gamma) \to \mathcal{F}(\Delta)\) as follows. For \((X, x) \in \text{Ob} \ \mathcal{F}(\Gamma)\), \(F_\Lambda(X, x) = (X, \Lambda_X(x))\) and, for a morphism \((f, u) : (X, x) \to (Y, y)\) in \(\mathcal{F}(\Gamma)\), \(F_\Lambda(f, u) = (f, (\lambda f)_y \Lambda_x(u))\). It is clear that \(F_\Lambda\) preserves fibers. Since

\[
F_\Lambda(\alpha_f(Y, y)) = F_\Lambda(f, \text{id}_{\Gamma_{Y,X}(f)(y)}) = (f, (\lambda f)_y \Lambda_X(\text{id}_{\Gamma_{Y,X}(f)(y)})) = (f, (\lambda f)_y),
\]

\(F_\Lambda\) preserves Cartesian morphism if and only if \(\lambda f : \Lambda_X \Gamma_{Y,X}(f) \to \Delta_{Y,X}(f)\) is a natural equivalence of functors from \(\Gamma(Y)\) to \(\Delta(X)\) for every morphism \(f : X \to Y\) in \(\mathcal{E}\).

Let \((\Lambda, \lambda), (\Phi, \varphi) : (\Gamma, \gamma) \to (\Delta, \delta)\) be 1-arrows in \(\text{Lax}^s(\mathcal{E}^{\text{op}}, \text{cat})\). For 2-arrow \(\chi : (\Lambda, \lambda) \to (\Phi, \varphi)\), we define a natural transformation \(\tilde{\chi} : F_\Lambda \to F_\Phi\) by \(\tilde{\chi}(X,x) = (\text{id}_X, (\delta_X)_{\chi(x)}(\chi_X)_x)\) for \((X, x) \in \text{Ob} \ \mathcal{F}(\Gamma)\).

Remark 2.16. Let \((\Gamma, \gamma)\) be an object of \(\text{Lax}^s(\mathcal{E}^{\text{op}}, \text{cat})\) and \(f : X \to Y\) a morphism in \(\mathcal{E}\). Then, the pull-back functor \(f^* : \mathcal{F}(\Gamma)_Y \to \mathcal{F}(\Gamma)_X\) has a left adjoint if and only if \(\Gamma_{Y,X}(f) : \Gamma(Y) \to \Gamma(X)\) has a left adjoint. If \((\Gamma, \gamma) : \mathcal{E}^{\text{op}} \to \text{cat}\) is a 2-functor, the fibered category \(p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}\) is a bifibered category if and only if \(\Gamma_{Y,X}(f) : \Gamma(Y) \to \Gamma(X)\) has a left adjoint for every morphism \(f : X \to Y\) in \(\mathcal{E}\).

The following examples are applications of the above construction.
Example 2.17. Let $\mathcal{C}$ be a category and $F$ a presheaf of sets on $\mathcal{C}$, that is a functor from $\mathcal{C}^{\text{op}}$ to the category of sets. Here $\mathcal{C}_F$ denotes a category with objects $(X, x)$ for $X \in \text{Ob} \mathcal{C}$, $x \in F(X)$ and morphisms $\mathcal{C}_F((X, x), (Y, y)) = \{\alpha \in \mathcal{C}(X, Y) \mid F(\alpha)(y) = x\}$. We call $\mathcal{C}_F$ the category of $F$-models. Note that there is a functor $U_F : \mathcal{C}_F \to \mathcal{C}$ given by $U_F(X, x) = X$.

For a morphism $u : F \to G$ of presheaves, we define a functor $u_\sharp : \mathcal{C}_F \to \mathcal{C}_G$ by $u_\sharp((X, x)) = (X, u(x))$ and $u_\sharp(\alpha) = \alpha$. Let us denote by $\hat{\mathcal{C}}$ the category of presheaves of sets on $\mathcal{C}$. Then, $u_\sharp$ induces a functor $u^* : \hat{\mathcal{C}} \to \hat{\mathcal{C}}_F$ by $u^*(S) = S_{u_\sharp}$. We note that, for morphisms $u : F \to G$ and $v : E \to F$ of presheaves, since $(uv)_\sharp = u_{\sharp} v_{\sharp}$, we have $u^*v^* = v^*u^*$. Define a functor $\Gamma : \hat{\mathcal{C}}^{\text{op}} \to \text{cat}$ by $\Gamma(F) = \mathcal{C}_F^{\text{op}}$ and $\Gamma(u) = u^*$. The fibered category $p(\Gamma) : \mathcal{F}(\Gamma) \to \hat{\mathcal{C}}$ associated with $\Gamma$ is called the fibered category of models on $\mathcal{C}$.

Example 2.18. Let $\mathcal{E}$ be a category with finite limits and $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1; \sigma, \tau, \varepsilon, \mu)$ an internal category in $\mathcal{E}$. We denote by $\Gamma_C : \mathcal{E}^{\text{op}} \to \text{cat}$ the functor represented by $\mathcal{C}$. That is, $\Gamma_C$ is described as follows. For $X \in \text{Ob} \mathcal{E}$, put

$$\text{Ob} \Gamma_C(X) = \mathcal{E}(X, C_0) \quad \text{and} \quad \Gamma_C(X)(u, v) = \{\varphi \in \mathcal{E}(X, C_1) \mid \sigma \varphi = u, \tau \varphi = v\}.$$

The composition of morphisms $\varphi : u \to v$ and $\psi : v \to w$ is defined to be a composition $X \xrightarrow{(\varphi, \psi)} C_1 \times_{C_0} C_1 \xrightarrow{\mu} C_1$. For an object $u$ of $\Gamma_C(X)$, $\varepsilon u : X \to C_1$ is the identity morphism $1_u : u \to u$. For a morphism $f : X \to Y$ in $\mathcal{E}$, $\Gamma_C(f) : \Gamma_C(Y) \to \Gamma_C(X)$ is defined by $\Gamma_C(f)(u) = uf$ for an object $u : Y \to C_0$ of $\Gamma_C(Y)$ and $\Gamma_C(f)(\varphi) = \varphi f$ for a morphism $\varphi : Y \to C_1$ in $\Gamma_C(Y)$. We call $p_{\Gamma_C} : \mathcal{F}(\Gamma_C) \to \mathcal{E}$ the fibered category represented by $\mathcal{C}$ and we simply denote this by $p_C : \mathcal{F}(\mathcal{C}) \to \mathcal{E}$.

Example 2.19. Let $(\mathcal{E}, J)$ be a site (see [8, Section 1]). For each object $X$ of $\mathcal{E}$, we give $\mathcal{E}/X$ the topology induced by the functor $\Sigma_X : \mathcal{E}/X \to \mathcal{E}$ given by $\Sigma_X(p : E \to X) = E$ (see [9, Section 3]). If $f : X \to Y$ is a morphism in $\mathcal{E}$, then the functor $\Sigma_f : \mathcal{E}/X \to \mathcal{E}/Y$ defined by $\Sigma_f(p : E \to X) = (fp \circ E \to Y)$ is continuous and cocontinuous by [9, Proposition 5.2]. Define a functor $\Sigma_f^* : \mathcal{E}/Y \to \mathcal{E}/X$ by $\Sigma_f^*(S) = S \Sigma_f$ for a presheaf $S$ on $\mathcal{E}/Y$. Then, $\Sigma_f^* : \mathcal{E}/Y \to \mathcal{E}/X$ induces $\Sigma_f^* : \mathcal{E}/Y \to \mathcal{E}/X$ which is naturally equivalent to the composition $\mathcal{E}/Y \xrightarrow{\Sigma_f} \mathcal{E}/Y \xrightarrow{\Sigma_f^*} \mathcal{E}/X$. Here we denote by $\hat{\mathcal{C}}$ the category of sheaves on $\mathcal{C}$, by $i : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ the inclusion functor and by $a : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ the associated sheaf functor (see [8, Théorème 3.4]). It follows from [9, Proposition 2.3] that $\Sigma_f^*$ is left exact and it has a right adjoint $\tilde{f}_* : \mathcal{E}/X \to \mathcal{E}/Y$. Thus, $(\tilde{f}_*, \Sigma_f^*) : \mathcal{E}/X \to \mathcal{E}/Y$ is a geometric morphism of Grothendieck topoi. Define a functor $\Gamma : \mathcal{E}^{\text{op}} \to \text{cat}$ by $\Gamma(X) = \mathcal{E}/X$ and $\Gamma(f) = \Sigma_f^*$. Applying the construction given in Construction 2.15 to $\Gamma$, we have a bifibered category $p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}$ which is an example of fibered topos [10, Définition 7.1.1].

Let $X$ be a simplicial object in $\mathcal{E}$, namely a functor $\Delta^{op} \to \mathcal{E}$. Consider the pull-back $p_X : \mathcal{F}(X) \to \Delta^{op}$ of $p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}$ along $X$. This fibered category $p_X$, called the Grothendieck topos over $X$. In the case $\mathcal{E} = \text{Sch}$ and $J$ is the étale topology, $p_X$ is the category of sheaves on simplicial scheme $X$ (cf. [4]).

We define a functor $\Xi = \Xi_\mathcal{E} : \text{Lax}^s(\mathcal{E}^{\text{op}}, \text{cat}) \to \text{pfib}(\mathcal{E})$ as follows. For an object $(\Gamma, \gamma) : \mathcal{E}^{\text{op}} \to \text{cat}$ of $\text{Lax}^s(\mathcal{E}^{\text{op}}, \text{cat})$, put $\Xi(\Gamma, \gamma) = (p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E})$ (see Construction 2.15). If $(\Delta, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)$ is a 1-arrow in $\text{Lax}^s(\mathcal{E}^{\text{op}}, \text{cat})$, we put $\Xi(\Delta, \lambda) = (F_\lambda : \mathcal{F}(\Gamma) \to \mathcal{F}(\Delta))$. For 1-arrows $(\Delta, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)$ and $(\Phi, \varphi)$:
\[(\Delta, \delta) \to (\Pi, \pi), \text{ put } (\Psi, \psi) = (\Phi, \varphi) \circ (\Lambda, \lambda) : (\Gamma, \gamma) \to (\Pi, \pi). \] Let \((X, x)\) be an object of \(\mathcal{F}(\Gamma)\) and \((f, u) : (X, x) \to (Y, y)\) a morphism in \(\mathcal{F}(\Gamma)\). Then, it is routine to verify \(F_{\Phi} F_{\Lambda}(X, x) = F_{\Psi}(f, u)\). It follows that \(\Xi((\Phi, \varphi) \circ (\Lambda, \lambda)) = \Xi(\Phi, \varphi) \circ \Xi(\Lambda, \lambda)\). Let \((\Lambda, \lambda), (\Phi, \varphi) : (\Gamma, \gamma) \to (\Delta, \delta)\) be 1-arrows in \(\text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat})\). For a 2-arrow \(\chi : (\Lambda, \lambda) \to (\Phi, \varphi)\), we set \(\Xi(\chi) = \tilde{\chi}\). If \(\omega : (\Phi, \varphi) \to (\Psi, \psi)\) is a 2-arrow in \(\text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat})\), then we have \((\Xi(\omega) \Xi(\chi))(X, x) = \Xi(\omega \chi)(X, x)\), hence \(\Xi(\omega \chi) = \Xi(\omega) \Xi(\chi)\). Let \((\Lambda, \lambda), (\Phi, \varphi) : (\Gamma, \gamma) \to (\Delta, \delta)\) and \((\Psi, \psi), (Y, u) : (\Delta, \delta) \to (\Pi, \pi)\) be 1-arrows in \(\text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat})\).

\[
(\xi, \xi) : \text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat}) \to \text{pfib}(\mathcal{E}), \text{ where } (\xi, \xi)_{(\gamma, \gamma)}, (\delta, \delta), (\Phi, \Phi) \text{ is the identity 2-arrow of } L(\gamma, \gamma). \]

**THEOREM 2.20.** There is an equivalence \(\Theta : \text{pfib}(\mathcal{E}) \to \text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat})\) of 2-categories. This induces the following equivalences:

\[
\text{pfib}^4(\mathcal{E}) \cong \text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat}), \quad \text{fib}(\mathcal{E}) \cong \text{Funct}(\mathcal{E}^{\text{op}}, \text{cat}), \quad \text{fib}'(\mathcal{E}) \cong \text{Funct}'(\mathcal{E}^{\text{op}}, \text{cat}).
\]

**Proof.** Put \((Z, \zeta) = (\xi, \xi)_{(\Theta, \Theta)} : \text{pfib}(\mathcal{E}) \to \text{pfib}(\mathcal{E})\). For an object \(p : \mathcal{F} \to \mathcal{E}\) of \(\text{pfib}(\mathcal{E})\), \(Z(p) = \Xi(\Theta(p)) : \mathcal{F}(\Gamma(p)) \to \mathcal{E}\) is a cloven prefibered category such that

\[
\text{Ob } \mathcal{F}(\Gamma(p)) = \{(X, M) \mid X \in \text{Ob } \mathcal{E}, M \in \text{Ob } \mathcal{F}_X\},
\]

\[
\mathcal{F}(\Gamma(p))(X, M), (Y, N) = \{(f, u) \mid f \in \mathcal{E}(X, Y), u \in \mathcal{F}(M, f^*(N))\}.
\]

Define a 1-arrow \(E_p : Z(p) \to p\) in \(\text{pfib}(\mathcal{E})\) as follows.

If \((X, M), (X, N) \in \text{Ob } \mathcal{F}(\Gamma(p))\) and \((f, u) : (X, M) \to (X, N)\) is a morphism in \(\mathcal{F}(\Gamma(p))\), we put \(E_p(X, M) = M\) and \(E_p(f, u) = \alpha_f(M) u\). Then, \(E_p\) is an isomorphism and its inverse \(E_p^{-1} : p \to Z(\Theta(p))\) is given by \(E_p^{-1}(M) = (p(M), M)\) and \(E_p^{-1}(\rho) = (p(\rho), \rho)\), where \(u : M \to p(\rho)^*(N)\) is the unique morphism satisfying \(\alpha_{p(\rho)}(N) u = \rho\).

If \(F : p \to q\) is a 1-arrow in \(\text{pfib}(\mathcal{E})\), for \((X, M) \in \text{Ob } \mathcal{F}(\Gamma(p))\) and a morphism \((f, u) : (X, M) \to (Y, N)\) in \(\mathcal{F}(\Gamma(p))\), then it is easy to verify \(Z(F)(X, M) = (X, F(M))\). Hence, \(E_q Z(F)(X, M) = F E_p(X, M)\) and \(\epsilon_p : E_q Z(F) \to F E_p\) is defined to be the identity 2-arrow in \(\text{pfib}(\mathcal{E})\). Therefore, we have a 1-arrow \((E, \epsilon) : (Z, \zeta) \to I_{\text{pfib}(\mathcal{E})}\) in \(\text{Lax}(\text{pfib}(\mathcal{E}), \text{pfib}(\mathcal{E}))\) which is an isomorphism.

Put \((\Omega, \omega) = (\Theta, \Theta) : \text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat}) \to \text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat})\) and let \((\Gamma, \gamma)\) be an object of \(\text{Lax}^4(\mathcal{E}^{\text{op}}, \text{cat})\). By the definitions of \(\Theta\) and \(\Xi\), we have \(\Omega(\Gamma, \gamma) = (\Gamma(p(\Gamma)), \gamma(p(\Gamma)))\) and

\[
\text{Ob } \Gamma(p(\Gamma))(X) = \{(X, x) \mid x \in \text{Ob } \Gamma(X)\}, \quad \text{Mor } \Gamma(p(\Gamma))(X) = \{(\text{id}_X, u) \mid u \in \text{Mor } \Gamma(X)\}.
\]

For \(X \in \text{Ob } \mathcal{E}\), let \(H(\Gamma, \gamma)_X : \Gamma(X) \to \Gamma(p(\Gamma))(X)\) be a functor given by

\[
H(\Gamma, \gamma)_X(x) = (X, x) \quad \text{and} \quad H(\Gamma, \gamma)_X(u) = (\text{id}_X, (\gamma_x)_u),
\]

where \(x, y \in \text{Ob } \Gamma(X), (u : x \to y) \in \text{Mor } \Gamma(X)\). Then, \(H(\Gamma, \gamma)_X\) is an isomorphism. In fact, the inverse \(H(\Gamma, \gamma)_X^{-1} : \Gamma(p(\Gamma))(X) \to \Gamma(X)\) is given by \(H(\Gamma, \gamma)_X^{-1}(X, x) = x, \quad (x, u) \in \Gamma(p(\Gamma))(X)\).
For a morphism \( f : X \to Y \) in \( E \) and \( y \in \text{Ob} \Gamma(Y) \), it is routine to verify that
\[
\Gamma(p(\Gamma))(f)_{Y,X} H(\Gamma, \gamma)_Y(y) = H(\Gamma, \gamma)_X \Gamma_Y(f)(y),
\]
that is, \( H(\Gamma, \gamma)_X \) is natural in \( X \). Let \( \eta(\Gamma, \gamma)_f : H(\Gamma, \gamma)_X \Gamma_Y(f) \to \Gamma(p(\Gamma))_Y \). For a 1-arrow \((\Delta, \delta) : (\Gamma, \gamma) \to (\Delta', \delta')\) in \( \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C}) \) by \( H(\Gamma, \gamma) = (H(\Gamma, \gamma), \eta(\Gamma, \gamma)) \). We denote by \( \eta(\Delta, \lambda, \delta)(\Lambda, \lambda) \to \Omega(\Lambda, \lambda) H(\Gamma, \gamma) \) the identity 2-arrow in \( \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C}) \). Now we have a 1-arrow \((H, \eta) : I_{\text{Lax}^s(\mathcal{D}^{op}, \mathcal{C})} \to \Omega \) in \( \text{Lax}(\text{Lax}^s(\mathcal{D}^{op}, \mathcal{C}), \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C})) \) which is an isomorphism. \( \square \)

For a functor \( F : \mathcal{D} \to \mathcal{E} \), we denote by \( F^{op} : \mathcal{D}^{op} \to \mathcal{E}^{op} \) the functor induced by \( F \). Regarding \((F^{op}, \text{id})\) as a lax functor, we have a lax functor \((F^{op}, \text{id})^* : \text{Lax}(\mathcal{E}^{op}, \mathcal{C}) \to \text{Lax}(\mathcal{D}^{op}, \mathcal{C})\).

**Proposition 2.21.** The following diagrams commutes up to natural equivalence:

\[
\begin{array}{ccc}
\text{pfib}(\mathcal{E}) & \xrightarrow{\text{pfib}(F)} & \text{pfib}(\mathcal{D}) \\
\text{Lax}^s(\mathcal{E}^{op}, \mathcal{C}) & \xrightarrow{(F^{op}, \text{id})^*} & \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C}) \\
\xrightarrow{(\Theta_{\mathcal{E}}, \Theta_{\mathcal{E}})} & & \xrightarrow{(\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}})} \\
\text{Lax}^s(\mathcal{E}^{op}, \mathcal{C}) & \xrightarrow{(F^{op}, \text{id})^*} & \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C}) \\
\xrightarrow{(\Xi_{\mathcal{E}}, \Xi_{\mathcal{E}})} & & \xrightarrow{(\Xi_{\mathcal{D}}, \Xi_{\mathcal{D}})} \\
\text{pfib}(\mathcal{E}) & \xrightarrow{\text{pfib}(F)} & \text{pfib}(\mathcal{D})
\end{array}
\]

**Proof.** Let \((\Gamma', \gamma') : \text{pfib}(\mathcal{E}) \to \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C})\) be the composition of \( \text{pfib}(F) : \text{pfib}(\mathcal{E}) \to \text{pfib}(\mathcal{D})\) and \((\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}}) : \text{pfib}(\mathcal{D}) \to \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C})\). For an object \( p : \mathcal{F} \to \mathcal{E}\) of \( \text{pfib}(\mathcal{E}) \), \( \Gamma'(p) : \mathcal{D}^{op} \to \mathcal{C} \) is given by \( \Gamma'(p)(X) = (\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_X \) for \( X \in \text{Ob} \mathcal{D}\) and \( \Gamma'_{X,Y}(f) = \kappa_F(f) \) for \( f : X \to Y \in \text{Mor} \mathcal{D}\). On the other hand, let \((\Delta', \delta') : \text{pfib}(\mathcal{E}) \to \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C})\) be the composition of \((\Theta_{\mathcal{E}}, \Theta_{\mathcal{E}}) : \text{pfib}(\mathcal{E}) \to \text{Lax}^s(\mathcal{E}^{op}, \mathcal{C})\) and \((F^{op}, \text{id})^* : \text{Lax}^s(\mathcal{E}^{op}, \mathcal{C}) \to \text{Lax}^s(\mathcal{D}^{op}, \mathcal{C})\). Then, for an object \( p : \mathcal{F} \to \mathcal{E}\) of \( \text{pfib}(\mathcal{E}) \), \( \Delta'(p) : \mathcal{D}^{op} \to \mathcal{C} \) is given by \( \Delta'(p)(X) = \mathcal{F}_X(p) \) for \( X \in \text{Ob} \mathcal{D}\) and \( \Delta'_{X,Y}(f) = \kappa_F(f) \) for \( f : X \to Y \in \text{Mor} \mathcal{D}\). Since the projection functor \( \widetilde{F} : \mathcal{D} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F}\) induces an isomorphism from \((\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_X \to \mathcal{F}_X(p)\) for each object \( X \) of \( \mathcal{D}\), \( \Gamma' \) and \( \Delta' \) are naturally equivalent. This shows the commutativity of the first diagram. The commutativity of the second diagram can be verified similarly. Details are left to readers. \( \square \)

If we identify \( \text{Lax}^s(\mathcal{E}^{op}, \mathcal{C}) \) with \( \text{Lax}^s(\mathcal{E}, \mathcal{C}^{op})\), then we can say that the functor \( \text{pfib} \) from \( \mathcal{C} \) to the category of prefibered categories is ‘represented’ by \( \mathcal{C}^{op} \) by Theorem 2.20 and the above result.

### 3. Representations of internal categories

Let \( \mathcal{E} \) be a category with finite limits and \( p : \mathcal{F} \to \mathcal{E} \) a cloven fibered category over \( \mathcal{E} \).
Definition 3.1. Let $\mathcal{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$ and $M$ an object of $\mathcal{F}_{C_0}$. A morphism $\xi : \sigma^+(M) \to \tau^+(M)$ in $\mathcal{F}_{C_1}$ is called a representation of $\mathcal{C}$ on $M$ if the following conditions are satisfied:

1. There exists at most one subrepresentation of $\xi$ respectively. A morphism $\phi : \xi \to \tau$ creates colimits.

2. Let $F : \mathcal{C} \to \mathcal{E}$ be a monomorphism, namely, it is easy to verify the following.

Proposition 3.3. Suppose that $\tau^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves monomorphisms. (For example, if $p : \mathcal{F} \to \mathcal{E}$ is a bifibered category, then $\tau^*$ has a left adjoint by (1) of Proposition 1.28, hence $\tau^*$ preserves monomorphisms.) Let $\xi$ and $\xi'$ be representations of $\mathcal{C}$ on $N$ and $\eta : N \to M$ a monomorphism. If $\eta$ is the underlying morphism of monomorphisms $\eta : \xi \to \xi$ and $\eta' : \xi' \to \xi$ of representations, then $\xi = \eta'$.

Proposition 3.4. Let $D : \mathcal{D} \to \mathcal{F}_{C_1}$ be a functor.

1. Let $(\pi_i : M \to UD(i))_{i \in \text{ObD}}$ be a limiting cone of $UD : D \to \mathcal{F}_{C_0}$. Suppose that $(\tau^*(\pi_i) : \tau^*(M)\to\tau^*(UD(i)))_{i \in \text{ObD}}$ is a limiting cone of $\tau^*UD : D \to \mathcal{F}_{C_1}$ and that $(\mu^*\tau^*(\pi_i) : \mu^*\tau^*(M)\to\mu^*\tau^*(UD(i)))_{i \in \text{ObD}}$ is a monomorphic family of $\mathcal{F}_{C_1\times_C_0}C_1$. Then, there exists a unique morphism $\xi : \sigma^+(M) \to \tau^+(M)$ such that $\xi$ is a representation of $\mathcal{C}$ on $M$ and $(\pi_i : (M, \xi) \to D(i))_{i \in \text{ObD}}$ is a limiting cone of $D$. Hence, if $\tau^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves limits and $\mu^* : \mathcal{F}_{C_1} \to \mathcal{F}_{C_1\times_C_0}C_1$ preserves monomorphic families, $U : \mathcal{F}_{C_1} \to \mathcal{F}_{C_0}$ creates limits in the sense of MacLane [14, Ch. V].

2. Let $(\xi_i : (M, \xi) \to D(i))_{i \in \text{ObD}}$ be a colimiting cone of $UD : D \to \mathcal{F}_{C_0}$. Suppose that $(\tau^*(\pi_i) : \sigma^*(M) \to \sigma^*(D(i)))_{i \in \text{ObD}}$ is a colimiting cone of $\tau^*UD : D \to \mathcal{F}_{C_1}$ and that $(\mu^*\sigma^*(\pi_i) : \mu^*\sigma^*(M) \to \mu^*\sigma^*(D(i)))_{i \in \text{ObD}}$ is an epimorphic family. Then, there exists a unique morphism $\xi : \sigma^+(M) \to \tau^+(M)$ such that $\xi$ is a representation of $\mathcal{C}$ on $M$ and $(\xi_i : D(i) \to (M, \xi))_{i \in \text{ObD}}$ is a colimiting cone of $D$. Hence, if $\sigma^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves colimits and $\mu^* : \mathcal{F}_{C_1} \to \mathcal{F}_{C_1\times_C_0}C_1$ preserves epimorphic families, $U : \mathcal{F}_{C_1} \to \mathcal{F}_{C_0}$ creates colimits.

Proof. For $i \in \text{ObD}$, we denote by $\xi_i : \sigma^*UD(i) \to \tau^*UD(i)$ the structure morphism of the representation of $\mathcal{C}$ on $UD(i)$.

1. Since $(\sigma^*(\pi_i) : \sigma^*(M) \to \sigma^*UD(i))_{i \in \text{ObD}}$ is a cone of $\sigma^*UD : D \to \mathcal{F}_{C_1}$, there exists a unique morphism $\xi : \sigma^+(M) \to \tau^+(M)$ in $\mathcal{F}_{C_1}$ satisfying $\tau^*(\pi_i)\xi = \xi_i\sigma^*(\pi_i)$ for every $i \in \text{ObD}$. Thus, we have $f^*\tau^*(\pi_i)\xi f^*(\xi_i) = f^*(\xi_i)\xi f^*(\xi_i)$ for $f = \text{pr}_1, \text{pr}_2, \mu : C_1 \times_C_0 C_1 \to C_1$. Since $\xi_i$ satisfies (A) of Definition 3.1, we can
easily verify that $\mu^*\tau^*(\pi_1(M))\mu^*(\xi) = c_{\alpha, p_1}(M)^{-1}c_{\sigma, \mu}(M)$. Since $(\mu^*\tau^*(\pi_1) : \mu^*\tau^*(M) \rightarrow \mu^*\tau^*(U D(i))_{i \in \text{Ob } \mathcal{D}}$ is a monomorphism, it follows that $\xi$ satisfies (A) of Definition 3.1. Since $\varepsilon^*\tau^*(\pi_1)\varepsilon^*(\xi) = \varepsilon^*(\xi_1)\varepsilon^*\sigma^*(\pi_1)$ and $\xi_1$ satisfies (U) of Definition 3.1, we have $\text{id}_{c_0}(\pi_1)c_{\sigma, \varepsilon}(M) = \text{id}_{c_0}(\pi_1)c_{\sigma, \varepsilon}(M)^*\varepsilon^*(\xi)$. Since $\alpha_{\text{id}_{c_0}(U D(i))\text{id}_{c_0}(\pi_1)} = \pi_i\alpha_{\text{id}_{c_0}(M)}$ by Proposition 1.6, we have $\pi_i\alpha_{\text{id}_{c_0}(M)}c_{\tau, \varepsilon}(M)^*\varepsilon^*(\xi) = \pi_i\alpha_{\text{id}_{c_0}(M)}c_{\sigma, \varepsilon}(M)$ for every $i \in \text{Ob } \mathcal{D}$. Thus, $\alpha_{\text{id}_{c_0}(M)}c_{\tau, \varepsilon}(M)^*\varepsilon^*(\xi) = \alpha_{\text{id}_{c_0}(M)}c_{\sigma, \varepsilon}(M)$. Since $\alpha_{\text{id}_{c_0}(M)}$ is an isomorphism, we see that $\xi$ satisfies (U) of Definition 3.1.

The proof of (2) is similar. □

**Corollary 3.4.** Suppose that $\tau^* : \mathcal{F}_{c_0} \rightarrow \mathcal{F}_{c_1}$ and $\mu^* : \mathcal{F}_{c_1} \rightarrow \mathcal{F}_{c_1 \times c_0 c_1}$ have left adjoints. Then, $U : \text{Rep}(C; \mathcal{F}) \rightarrow \mathcal{F}_{c_0}$ creates limits. In particular, if $p : \mathcal{F} \rightarrow \mathcal{E}$ is a bifibered category, $U : \text{Rep}(C; \mathcal{F}) \rightarrow \mathcal{F}_{c_0}$ creates limits.

**Proposition 3.5.** The forgetful functor $U : \text{Rep}(C; \mathcal{F}) \rightarrow \mathcal{F}_{c_0}$ reflects isomorphisms.

**Proof.** Let $\varphi : \xi \rightarrow \zeta$ be a morphism in $\text{Rep}(C; \mathcal{F})$ such that $U(\varphi)$ is an isomorphism. Since $\tau^*(\varphi^{-1})\xi = \tau^*(\varphi^{-1})\zeta\sigma^*(\varphi)\sigma^*(\varphi^{-1}) = \tau^*(\varphi^{-1})\tau^*(\varphi)\xi\sigma^*(\varphi^{-1}) = \xi\sigma^*(\varphi^{-1})$, $\varphi^{-1}$ is also a morphism in $\text{Rep}(C; \mathcal{F})$. Hence, $\varphi$ is an isomorphism in $\text{Rep}(C; \mathcal{F})$. □

**Proposition 3.6.** If a morphism $\xi : \sigma^*(M) \rightarrow \tau^*(M)$ in $\mathcal{F}_{c_1}$ satisfies (A) and it is a monomorphism or an epimorphism, then $\xi$ is a representation of $(\mathcal{C}, \sigma, \tau, \varepsilon, \mu)$ is an internal groupoid in $\mathcal{E}$ and $\xi$ is a representation of $(\mathcal{C}, \sigma, \tau, \varepsilon, \mu)$ in $\mathcal{E}$.

**Proof.** Assume that $\xi$ satisfies (A). Consider morphisms $\varepsilon_1 = (\text{id}_{c_0}, \varepsilon \tau), \varepsilon_2 = (\varepsilon \sigma, \text{id}_{c_0}) : C_1 \rightarrow C_1 \times c_0 c_1$. Since $\mu_1 = \mu_2 = \text{id}_{c_1}$, pulling back the equality of (A) along $\varepsilon_i$, we have $\varepsilon_i c_{\tau, \sigma}(\varepsilon \sigma)(\xi) c_{\sigma, \varepsilon}(M)^{-1} = c_{\tau, \sigma}(\varepsilon \tau) \xi c_{\sigma, \varepsilon}(M)^{-1} = \xi$. Hence, if $\xi$ is a monomorphism, $c_{\tau, \sigma}(\varepsilon \sigma)(\xi) = c_{\sigma, \varepsilon}(M)$, if $\xi$ is an epimorphism, $c_{\tau, \sigma}(\varepsilon \tau)(\xi) = c_{\sigma, \varepsilon}(M)$. Applying $\tau^*$ to the above equalities, we see that (U) holds in both cases.

Assume that $\mathcal{C}$ is an internal groupoid and that $\xi$ is an representation. We can define the inverse $\xi^{-1} : \tau^*(M) \rightarrow \sigma^*(M)$ by $\xi^{-1} = c_{\tau, \sigma}(\varepsilon \tau) \xi c_{\sigma, \varepsilon}(M)^{-1}$. A morphism $\xi : \sigma^*(M) \rightarrow \tau^*(M)$ in $\mathcal{F}_{c_1}$ is a representation of $\mathcal{C}$ on $\mathcal{M}$ if and only if $\xi$ is an isomorphism and satisfies the condition (A). In fact, consider morphisms $i_1 = (\text{id}_{c_1}, i), i_2 = (i, \text{id}_{c_1}) : C_1 \rightarrow C_1 \times c_0 C_1$. Since $\mu_1 = \varepsilon \sigma$ and $\mu_2 = \varepsilon \tau$, pulling back the equality of (A) along $i_i (i = 1, 2)$, we have $\xi^{-1} = \text{id}_{\sigma(M)}$ and $\xi^{-1} = \text{id}_{\varepsilon(M)}$ by (U). □

The above notion of the representations of internal categories can be generalized as follows.

**Definition 3.7.** Let $\mathcal{C} = (c_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$, $(\pi : X \rightarrow c_0, \alpha)$ an internal diagram on $\mathcal{C}$ and $\mathcal{M}$ an object of $\mathcal{F}_X$. A morphism $\xi : p^*_X(M) \rightarrow \alpha^*(M)$ ($p_X : X \times c_0 C_1 \rightarrow X$ is the projection) in $\mathcal{F}_{X \times c_0 C_1}$ is called a representation of $\mathcal{C}$ on $\mathcal{M}$ over $(X, \alpha)$ if the following conditions are satisfied:

\[ (A) \quad c_{\alpha, \alpha \times \text{id}_{c_1}}(M)(\alpha \times \text{id}_{c_1})^*(\xi)c_{p_X, \alpha \times \text{id}_{c_1}}(M)^{-1}c_{\alpha, p_1}(M)p^*_1(\xi)c_{p_X, p_1}(M)^{-1} = c_{\alpha, \text{id}_X \times \mu}(M)(\text{id}_X \times \mu)^*(\xi)c_{p_X, \text{id}_X \times \mu}(M)^{-1}, \]

where $p_1 : X \times c_0 C_1 \times c_0 C_1 \rightarrow X \times c_0 C_1$ denotes the projection;

\[ (U) \quad c_{p_X, (\text{id}_X, \varepsilon \pi)}(M)(\text{id}_X, \varepsilon \pi)^*(\xi) = c_{\alpha, (\text{id}_X, \varepsilon \pi)}(M). \]
Let $\xi : p_X^*(M) \to \alpha^s(M)$ and $\zeta : p_Y^*(N) \to \alpha^s(N)$ be representations of $C$ on $M$ and $N$ over $(X, \alpha)$, respectively. A morphism $\varphi : M \to N$ in $\mathcal{F}_X$ is called a morphism of representations of $C$ over $(X, \alpha)$ if $\alpha^s(\varphi)^* \xi = \zeta p_Y^*(\varphi)$. We denote by $\text{Rep}(C, X; \mathcal{F})$ the category of the representations of $C$ over $(X, \alpha)$. We denote an object $\xi : p_X^*(M) \to \alpha^s(M)$ of $\text{Rep}(C, X; \mathcal{F})$ by $(M, \xi)$.

For an internal diagram $(\pi : X \to C_0, \alpha)$ on $C$, we define an internal category $C_\alpha = (C_\alpha, X; \sigma_\alpha, \tau_\alpha, \varepsilon_\alpha, \mu_\alpha)$ associated with $(X, \alpha)$ by $C_\alpha = X \times_{C_0} C_1$, $\sigma_\alpha = p_X, \tau_\alpha = \alpha : C_\alpha \to X$, $\varepsilon_\alpha = (\text{id}_X, \varepsilon_\pi) : X \to C_\alpha$ and $\mu_\alpha = (\text{id}_X \times \mu)(\text{id}_{C_\alpha} \times \text{pr}_2) : C_\alpha \times_X C_\alpha \to C_\alpha \times_C 1 \to 1$. Here $\text{pr}_2 : C_\alpha = X \times_{C_0} C_1 \to C_1$ denotes the projection.

Let $M$ be an object of $\mathcal{F}_X$ and $\xi : p_X^*(M) \to \alpha^s(M)$ a morphism in $\mathcal{F}_{C_\alpha}$. Then, $\xi$ is a representation of $C_\alpha$ if and only if it is a representation of $C$ over $(X, \alpha)$. Thus, we see the following result.

**Proposition 3.8.** Let $C$ be an internal category and $(X, \alpha)$ an internal diagram on $C$. Then, the category $\text{Rep}(C, X; \mathcal{F})$ is isomorphic to $\text{Rep}(C_\alpha; \mathcal{F})$.

**Example 3.9.** Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the fibered category given in Example 1.13 and $C$ an internal category in $\mathcal{E}$. For an object $M = (\pi : M \to C_0)$ of $\mathcal{E}^{(2)}_{C_0}$, consider the following pull-back diagrams in $\mathcal{E}$:

\[\begin{array}{ccc}
\sigma^*(M) & \xrightarrow{\pi^\sigma} & C_1 \\
\downarrow \sigma & & \downarrow \tau \\
M & \xrightarrow{\pi} & C_0
\end{array}\]

Here $\bar{\tau}_e : \mathcal{E}(\sigma^*(M), \tau^*(M)) \to \mathcal{E}(\sigma^*(M), M)$ gives a bijection $\bar{\tau}_e : \mathcal{E}^{(2)}_{C_1}(\sigma^*(M), \tau^*(M)) \to \{\alpha \in \mathcal{E}(\sigma^*(M), M) | \pi \cdot \tau = \tau \cdot \pi^\sigma\}$. It is easy to verify that a morphism $\xi : \sigma^*(M) \to \tau^*(M)$ in $\mathcal{E}^{(2)}_{C_1}$ is a representation of $C$ if and only if $\bar{\tau}_e : \sigma^*(M) = M \times_{C_0} C_1 \to M$ is a structure morphism of an internal diagram on $C$. In fact, $\xi$ satisfies the condition (A) (respectively (U)) of Definition 3.1 if and only if the following diagram on the left (respectively right) commutes, where we put $\alpha = \bar{\tau}_e$:

\[\begin{array}{ccc}
M \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\alpha \times \text{id}_{C_1}} & M \times_{C_0} C_1 \\
\downarrow \text{id}_M \times \mu & & \downarrow \alpha \\
M \times_{C_0} C_1 & \xrightarrow{\alpha} & M
\end{array}\]

Hence, $\text{Rep}(C; \mathcal{E}^{(2)})$ is regarded as the category of internal diagrams $\mathcal{E}^C$ on $C$.

**Example 3.10.** Let $p : \text{Qmod} \to \text{Sch}$ be the fibered category given in Example 1.14 and $C$ an internal category in $\text{Sch}$. For an object $(C_0, \mathcal{M})$ of $\text{Qmod}_{C_0}$, a morphism $(\text{id}_{C_1}, \xi) : \sigma^*(C_0, \mathcal{M}) = (C_1, \sigma^*\mathcal{M}) \to (C_1, \tau^*\mathcal{M}) = \tau^*(C_0, \mathcal{M})$ is an representation of $C$ on $(C_0, \mathcal{M})$ if and only if a morphism $\xi : \tau^*\mathcal{M} \to \sigma^*\mathcal{M}$ of $\mathcal{O}_{C_1}$-modules satisfies

\[\tilde{c}_{\sigma, p_1}(\mathcal{M})^{-1} p_{1*}(\xi) \tilde{c}_{\tau, p_1}(\mathcal{M}) \tilde{c}_{\sigma, p_2}(\mathcal{M})^{-1} p_{2*}(\xi) \tilde{c}_{\tau, p_2}(\mathcal{M}) = \tilde{c}_{\sigma, \mu}(\mathcal{M})^{-1} \mu^* (\xi) \tilde{c}_{\tau, \mu}(\mathcal{M})\]
and \( \varepsilon^*(\xi)c_{\sigma,t}(M) = \tilde{c}_{\tau,t}(M) \). Suppose that \( C \) is the internal category in \( \text{Sch} \) associated with a Hopf algebroid \((A, H)\), that is, \( C_0 = \text{Spec} A, C_1 = \text{Spec} H \), and that \( \mathcal{M} \) is the quasi-coherent \( \mathcal{O}_{C_0} \)-module associated with an \( A \)-module \( M \). There is a natural bijection

\[
\Phi : \text{hom}_{\mathcal{O}_{C_1}}(\tau^* \mathcal{M}, \sigma^* \mathcal{M}) \to \text{hom}_A(M, M \otimes_A H),
\]

where \( H \) is regarded as a left \( A \)-module by the left unit \( \eta_L : A \to H \) inducing \( \sigma \). An \( \mathcal{O}_{C_1} \)-module homomorphism \( \xi : \tau^*(M) \to \sigma^*(M) \) defines a representation of \( C \) on \((C_0, \mathcal{M})\) if and only if \( \Phi(\xi) \) is a structure map of a \( H \)-comodule.

For a morphism \( f : X \to Y \) in \( \mathcal{E} \) and a Cartesian section \( s : \mathcal{E} \to \mathcal{F} \) of a fibered category \( p : \mathcal{F} \to \mathcal{E} \), let us denote by \( s_f : s(X) \to f^*(s(Y)) \) the unique morphism in \( \mathcal{F}_X \) satisfying \( \alpha_f(s(Y))sf = s(f) \). Since both \( s(f) \) and \( \alpha_f(s(Y)) \) are Cartesian morphisms, \( s_f \) is necessarily an isomorphism. We put \( s_{f, g} = s_g s_{f}^{-1} : f^*(s(Y)) \to g^*(s(Z)) \) for morphisms \( f : X \to Y \) and \( g : X \to Z \) in \( \mathcal{E} \).

**Proposition 3.11.** Let \( C = (C_0, C_1; \tau, \sigma, \varepsilon, \mu) \) be an internal category in \( \mathcal{E} \) and \( s : \mathcal{E} \to \mathcal{F} \) a Cartesian section. Then, \( s_{\sigma, \tau} : \tau^* s(C_0) \to \tau^* s(C_0) \) is a representation of \( C \) on \( s(C_0) \).

**Proof.** By Proposition 3.6, we only have to verify the condition (A). Since we assumed that \( \mathcal{E} \) has finite limits, in particular, \( \mathcal{E} \) has a terminal object 1, we may assume that \( s = s_T \) for some \( T \in \text{Ob} \mathcal{F}_1 \) by Proposition 1.19. Then, \( s_\sigma = c_{\sigma_0, \sigma}(T)^{-1}, s_\tau = c_{\sigma_0, \tau}(T)^{-1} \) and we have the following equalities by Proposition 1.10 (here \( \sigma_0 \) denotes the unique morphism \( C_0 \to 1 \)):

\[
c_{\tau, p_2}(s(C_0))p_2^*(s_\tau) = c_{\tau, p_2}(\sigma_0^*(T))p_2^*(\sigma_0^*(T))^{-1} = c_{\sigma_0, \tau}^{-1}c_{\sigma_0, \tau}(T),
\]

\[
p_2^*(s_\sigma^{-1})c_{\sigma, p_2}(s(C_0))^{-1} = c_{\sigma, p_2}(\sigma_0^*(T))c_{\sigma, p_2}(\sigma_0^*(T))^{-1} = c_{\sigma_0, \tau}^{-1}c_{\sigma_0, \tau}(T),
\]

\[
c_{\tau, p_1}(s(C_0))p_1^*(s_\tau) = c_{\tau, p_1}(\sigma_0^*(T))p_1^*(\sigma_0^*(T))^{-1} = c_{\sigma_0, \tau}^{-1}c_{\sigma_0, \tau}(T),
\]

\[
c_{\tau, \mu}(s(C_0))\mu^*(s_\tau) = c_{\tau, \mu}(\sigma_0^*(T))\mu^*(\sigma_0^*(T))^{-1} = c_{\sigma_0, \tau}^{-1}c_{\sigma_0, \tau}(T),
\]

\[
\mu^*(s_\sigma^{-1})c_{\sigma, \mu}(s(C_0))^{-1} = \mu^*(\sigma_0^*(T))c_{\sigma, \mu}(\sigma_0^*(T))^{-1} = c_{\sigma_0, \tau}^{-1}c_{\sigma_0, \tau}(T),
\]

\[
c_{\sigma, p_1}(s(C_0))p_1^*(s_\sigma) = c_{\sigma, p_1}(\sigma_0^*(T))p_1^*(\sigma_0^*(T))^{-1} = c_{\sigma_0, \tau}^{-1}c_{\sigma_0, \tau}(T).
\]

Hence, it is straightforward to verify

\[
c_{\tau, p_2}(s(C_0))p_2^*(s_\tau)p_2^*(s_\sigma^{-1})c_{\sigma, p_2}(s(C_0))^{-1}c_{\tau, p_1}(s(C_0))p_1^*(s_\sigma) = c_{\tau, \mu}(s(C_0))\mu^*(s_\tau)\mu^*(s_\sigma^{-1})c_{\sigma, \mu}(s(C_0))^{-1}c_{\sigma, p_1}(s(C_0))p_1^*(s_\sigma).
\]

Therefore, we have

\[
c_{\tau, p_2}(s(C_0))p_2^*(s_\sigma, \tau)c_{\sigma, p_2}(s(C_0))^{-1}c_{\tau, p_1}(s(C_0))p_1^*(s_\sigma, \tau)c_{\sigma, p_1}(s(C_0))^{-1} = c_{\tau, \mu}(s(C_0))\mu^*(s_\sigma, \tau)c_{\sigma, \mu}(s(C_0))^{-1}.
\]

**Definition 3.12.** (1) For a Cartesian section \( s \), we call \( s_{\sigma, \tau} : \sigma^* s(C_0) \to \tau^* s(C_0) \) the trivial representation associated with \( s \). In the case \( s = s_T \) for some \( T \in \text{Ob} \mathcal{F}_1 \), we also call \( s_{\sigma, \tau} \) the trivial representation associated with \( T \).

(2) Let \( \xi : \sigma^* (M) \to \tau^* (M) \) be a representation of \( C \) on \( M \) and \( T \) an object of \( \mathcal{F}_1 \). We call a morphism \( \varphi : (s_T)_{\sigma, \tau} \to \xi \) a primitive element of \( \xi \) with respect to \( T \).
Example 3.13. Let $p : \mathcal{E}^{(2)} \rightarrow \mathcal{E}$ be the fibered category given in Example 1.13 and $\mathcal{C}$ an internal category in $\mathcal{E}$. We note that $\mathcal{E}_1^{(2)}$ is identified with $\mathcal{E}$ by an isomorphism of categories $\mathcal{E}_1^{(2)} \rightarrow \mathcal{E}$ ($X \rightarrow 1 \rightarrow X$). For an object $T$ of $\mathcal{E}_1^{(2)} = \mathcal{E}$, the Cartesian section $s_T : \mathcal{E} \rightarrow \mathcal{E}(2)$ associated with $T$ is given by $s_T(X) = (pr_1 : X \times T \rightarrow X)$. Here $\sigma^*s_T(C_0)$ and $\tau^*s_T(C_0)$ are both identified with $(pr_1 : C_1 \times T \rightarrow C_1)$. Hence, the trivial representation $(s_T)_{\sigma, \tau} : \sigma^*s_T(C_0) \rightarrow \tau^*s_T(C_0)$ associated with $T$ can be regarded as the identity morphism of $C_1 \times T$.

Example 3.14. Let $p : \mathcal{Q} \rightarrow \mathcal{S}$ be the fibered category given in Example 1.14 and $\mathcal{C}$ an internal category in $\mathcal{S}$. In this case, since the terminal object 1 in $\mathcal{S}$ is Spec$\mathbb{Z}$, $\mathcal{Q} \rightarrow \mathcal{S}$ is identified with the category of abelian groups. For an abelian group $G$, the Cartesian section $s_G : \mathcal{S} \rightarrow \mathcal{Q}$ associated with $G$ is given by $s_G(X) = (X, o^*_X \tilde{G})$. The isomorphisms $\tilde{c}_{\sigma, o, c_0} : o^*_C \tilde{G} \rightarrow \sigma^*o^*_C \tilde{G}$ define isomorphisms $c_{\sigma, o, c_0} : \sigma^*o^*_C (\tilde{G}) = \tilde{c}_{\tau, o, c_0} : \tau^*o^*_C (\tilde{G})$ in $\mathcal{Q}$. The trivial representation $(s_G)_{\sigma, \tau} : \sigma^*s_G(C_0) \rightarrow \tau^*s_G(C_0)$ associated with $G$ is $c_{\sigma, o, c_0} c_{\tau, o, c_0}^{-1}$.

We describe the notion of representation of internal categories in terms of 2-categories and lax diagrams.

Let $\mathcal{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$ and $M$ an object of $\Gamma(C_0)$. Recall that $\sigma^*(C_0, M) = (C_1, \Gamma_{C_0, C_1}(\sigma)(M))$, $\tau^*(C_0, M) = (C_1, \Gamma_{C_0, C_1}(\tau)(M))$. For a morphism $\xi : \Gamma_{C_0, C_1}(\sigma)(M) \rightarrow \Gamma_{C_0, C_1}(\tau)(M)$ in $\Gamma(C_1)$, we define a morphism $\xi(\xi) : \sigma^*(C_0, M) \rightarrow \tau^*(C_0, M)$ in $\mathcal{F}(\Gamma)_{C_1}$ by $\xi(\xi) = (id_{C_1}, R_{\tau}^{-1}(\xi))_M$.

**PROPOSITION 3.15.** We have that:

1. $\xi(\xi)$ satisfies (A) of Definition 3.1 if and only if $\xi$ satisfies the equality

$$(((\gamma_{C_0, C_1, C_1 \times C_0, C_1})_{(\tau, p_2)})_M \gamma_{C_1, C_1 \times C_0, C_1} (p_2) \gamma_{C_0, C_1, C_1 \times C_0, C_1} (\sigma, p_2))_M^{-1}$$

$\circ((\gamma_{C_0, C_1, C_1 \times C_0, C_1})_{(\tau, p_1)})_M \gamma_{C_1, C_1 \times C_0, C_1} (p_1) \gamma_{C_0, C_1, C_1 \times C_0, C_1} (\sigma, p_1))_M^{-1}$

$= ((\gamma_{C_0, C_1, C_1 \times C_0, C_1})_{(\tau, \mu)})_M \gamma_{C_1, C_1 \times C_0, C_1} (\mu, \mu)(\gamma_{C_0, C_1, C_1 \times C_0, C_1} (\sigma, \mu))_M^{-1}$;

2. $\xi(\xi)$ satisfies (U) of Definition 3.1 if and only if $\xi$ satisfies

$$((\gamma_{C_0, C_1, C_0})_{(\tau, \epsilon)})_M \gamma_{C_1, C_0} (\epsilon)(\gamma_{C_0, C_1, C_1 \times C_0, C_1})_{(\sigma, \epsilon)}_M.$$

Let $\mathcal{C}$ and $\mathcal{G}$ be internal categories in $\mathcal{E}$. Consider the fibered category $p_\mathcal{C} : \mathcal{F}(\mathcal{C}) \rightarrow \mathcal{E}$ represented by $\mathcal{C}$ given in Example 2.18. The following is immediate from definitions.

**LEMMA 3.16.** We have the following.

1. Let $f_0 : G_0 \rightarrow C_0$ and $f_1 : G_1 \rightarrow C_1$ be morphisms in $\mathcal{E}$, Here $(f_0, f_1) : G \rightarrow C$ is an internal functor if and only if a morphism $(id_{G_1}, f_1) : \sigma^*(G_0, f_0) = (G_1, f_0) \rightarrow (G_1, f_0) = (G_1, f_0) \rightarrow (G_1, f_0)$ is a representation of $G$ on $(G_0, f_0) \in \text{Ob} \mathcal{F}(\mathcal{C})_{G_0}$.

2. Let $f = (f_0, f_1), g = (g_0, g_1) : G \rightarrow C$ be internal functors and $\varphi : G_0 \rightarrow C_1$ a morphism in $\mathcal{E}$. Here $\varphi$ is an internal natural transformation from $f$ to $g$ if and only if $(id_{G_0}, \varphi)$ is a morphism of representations from $(id_{G_1}, f_1) : \sigma^*(G_0, f_0) \rightarrow \tau^*(G_0, f_0)$ to $(id_{G_1}, g_1) : \sigma^*(G_0, g_0) \rightarrow \tau^*(G_0, g_0)$.

Thus, we have the following result.
Theorem 3.17. Define a functor
\[ F : \text{cat}(\mathcal{E})(G, C) \to \text{Rep}(G; \mathcal{F}(C)) \]
by \( F(f) = (\text{id}_{G_{f}}, f) \) for an internal functor \( f = (f_{0}, f_{1}) : G \to C \) and \( F(\varphi) = (\text{id}_{G_{0}}, \varphi) \). Then, \( F \) is an isomorphism of categories.

4. Descent formalism

Definition 4.1. (Grothendieck [6, Définition 1.3]) Let \( p : \mathcal{F} \to \mathcal{E} \) be a cloven fibered category. We say that a diagram \( R \xrightarrow{p_{1}} X \xrightarrow{f} Y \) is \( \mathcal{F} \)-exact if \( fp_{1} = fp_{2} \) and, for any \( M, N \in \text{Ob} \mathcal{F}_{Y} \), the following diagram is an equalizer, where we put \( g = fp_{1} = fp_{2} \) and \( p_{i}^{(i = 1, 2)} \) are maps defined by \( \varphi \mapsto c_{f,p_{i}}(N)p_{i}^{*}(\varphi)c_{f,p_{i}}(M)^{-1} \):

\[ \mathcal{F}_{Y}(M, N) \xrightarrow{f^{*}} \mathcal{F}_{X}(f^{*}(M), f^{*}(N)) \xrightarrow{p_{1}^{*}} \mathcal{F}_{R}(g^{*}(M), g^{*}(N)). \]

Example 4.2. A diagram \( R \xrightarrow{p_{1}} X \xrightarrow{f} Y \) is \( \mathcal{E}^{(2)} \)-exact if and only if, for any \( \pi : M \to Y \), \( \text{id}_{M} \times f : M \times_{Y} X \to M \times_{Y} Y = M \) is a coequalizer of \( \text{id}_{M} \times p_{1} \), \( \text{id}_{M} \times p_{2} : M \times_{Y} R \to M \times_{Y} X \), in other words, \( f \) is a universal strict epimorphism.

Definition 4.3. (Grothendieck [6, Définition 1.4]) Let \( p_{1}, p_{2} : R \to X \) be morphisms in \( \mathcal{E} \) and \( M \) is an object of \( \mathcal{F}_{X} \). An isomorphism \( \xi : p_{1}^{*}(M) \to p_{2}^{*}(M) \) is called a gluing morphism on \( M \) with respect to a pair \( (p_{1}, p_{2}) \). If \( \xi : p_{1}^{*}(M) \to p_{2}^{*}(M) \) and \( \zeta : p_{1}^{*}(N) \to p_{2}^{*}(N) \) are gluing morphisms on \( M, N \in \text{Ob} \mathcal{F}_{X} \), a morphism \( \varphi : M \to N \) in \( \mathcal{F}_{X} \) is said to be compatible with \( \xi \) and \( \zeta \) if the following square commutes:

\[ \begin{array}{ccc}
\quad p_{1}^{*}(M) & \xrightarrow{\xi} & p_{2}^{*}(M) \\
\downarrow & & \downarrow \\
p_{1}^{*}(N) & \xrightarrow{\zeta} & p_{2}^{*}(N)
\end{array} \]

Thus, we can consider the category of gluing morphisms.

Definition 4.4. (Grothendieck [6, Définition 1.5]) Let \( R \xrightarrow{p_{1}} X \xrightarrow{f} Y \) be a diagram in \( \mathcal{E} \) such that \( fp_{1} = fp_{2} \). We say that a gluing morphism \( \xi : p_{1}^{*}(M) \to p_{2}^{*}(M) \) on \( M \in \text{Ob} \mathcal{F}_{X} \) is effective with respect to \( f \) if there exists an isomorphism \( \kappa : M \to f^{*}(N) \) in \( \mathcal{F}_{X} \) for some \( N \in \text{Ob} \mathcal{F}_{Y} \) such that the following diagram commutes:

\[ \begin{array}{ccc}
p_{1}^{*}(M) & \xrightarrow{p_{1}^{*}(\kappa)} & p_{1}^{*}f^{*}(N) \\
\downarrow & & \downarrow \\
p_{2}^{*}(M) & \xrightarrow{p_{2}^{*}(\kappa)} & p_{2}^{*}f^{*}(N)
\end{array} \]

\[ \begin{array}{ccc}
p_{1}^{*}(\kappa)f^{*}(N) & \xrightarrow{c_{f,p_{1}}(N)} & (fp_{1})^{*}(N) \\
\downarrow & & \downarrow \\
p_{2}^{*}(\kappa)f^{*}(N) & \xrightarrow{c_{f,p_{2}}(N)} & (fp_{2})^{*}(N)
\end{array} \]
Assume that \( \mathcal{E} \) is a category with finite limits below.

**Definition 4.5.** Let \( \mathcal{C} = (C_0, C_1; \sigma, \tau, \epsilon, \mu) \) be an internal category in \( \mathcal{E} \).

1. If \( (\sigma, \tau) : C_1 \to C_0 \times C_0 \) is a monomorphism, we call \( \mathcal{C} \) an internal poset.
2. If \( \mathcal{C} \) is an internal groupoid and an internal poset, we call \( \mathcal{C} \) an equivalence relation on \( C_0 \).

For a morphism \( f : X \to Y \) in \( \mathcal{E} \), the kernel pair \( X \times_Y X \rightrightarrows X \) of \( f \) is an equivalence relation on \( X \) with the following structure maps: the domain \( \sigma = p_1 \), the codomain \( \tau = p_2 \), the identity \( \epsilon = \Delta \) (the diagonal morphism), the composition \( \mu = p_1 \times p_2 : (X \times Y) \times_X (X \times Y) \to X \times Y \) and the inverse \( \iota = (p_2, p_1) : X \times_Y X \to X \times_Y X \). We denote this internal groupoid by \( E_f = (X \times_Y X, (p_1, p_2, \Delta, p_1 \times p_2)) \). The notion of descent data is given in terms of representation of groupoids as follows.

**Definition 4.6.** ([Grothendieck, Définition 1.6]) For an object \( M \) of \( \mathcal{F}_X \), a representation \( \xi : p_1^*(M) \to p_2^*(M) \) of \( E_f \) on \( M \) is called a descent data on \( M \) for a morphism \( f : X \to Y \) in \( \mathcal{E} \).

Let \( f : X \to Y \) be a morphism in \( \mathcal{E} \). We denote by \( p_i : X \times_Y X \to X \) \((i = 1, 2)\), \( q_i : X \times_Y Y \times_Y X \to X \) \((i = 1, 2, 3)\) the projections onto the \( i \)-th component. \( \Delta : X \to X \times_X X \) denotes the diagonal morphism. Define \( p_{ij} : X \times_Y X \times_Y X \to X \times_Y X \) \((1 \leq i < j \leq 3)\) by \( p_{ij} = (q_i, q_j) \). We note that \( p_1 p_{12} = p_{13} = q_1 \), \( p_{12} p_2 = q_2 \), \( p_2 p_{13} = q_3 \).

The category of gluing morphisms \( p_1^*(M) \to p_2^*(M) \) is denoted by \( \text{Glue}(\mathcal{F}/\mathcal{E}, f) \). The following assertion is immediate from the definition.

**Proposition 4.7.** Let \( \mathcal{E} \) be a category with finite limits and \( p : \mathcal{F} \to \mathcal{E} \) a cloven fibred category. A gluing morphism \( \xi : p_1^*(M) \to p_2^*(M) \) on \( M \in \text{Ob} \mathcal{F}_X \) with respect to a pair \((p_1, p_2)\) is a descent data on \( M \) for a morphism \( f : X \to Y \) in \( \mathcal{E} \) if and only if \( \xi \) satisfies the following equalities:

\[
c_{p_1, \Delta}(M) \Delta^*(\xi) = c_{p_2, \Delta}(M),
\]

\[
(c_{p_2, p_{13}}(M) p_{13}^*(\xi)) c_{p_1, p_{13}}(M)^{-1} = c_{p_2, p_{23}}(M) p_{23}^*(\xi) c_{p_1, p_{23}}(M)^{-1} c_{p_2, p_{12}}(M) p_{12}^*(\xi) c_{p_1, p_{12}}(M)^{-1}.
\]

**Definition 4.8.** ([Grothendieck, Définition 1.7]) A morphism \( f : X \to Y \) in \( \mathcal{E} \) is called a morphism of \( \mathcal{F} \)-descent if \( X \times_Y X \rightrightarrows X \xrightarrow{f} Y \) is \( \mathcal{F} \)-exact. Moreover, if every descent data on arbitrary object of \( \mathcal{F}_X \) is effective, we say that \( f \) is a morphism of effective \( \mathcal{F} \)-descent.

We set \( \text{Desc}(\mathcal{F}/\mathcal{E}, f) = \text{Rep}(E_f; \mathcal{F}) \) and regard this as a full subcategory of \( \text{Glue}(\mathcal{F}/\mathcal{E}, f) \). We define a functor \( \tilde{D}_f : \mathcal{F}_Y \to \text{Glue}(\mathcal{F}/\mathcal{E}, f) \) as follows. For \( N \in \mathcal{F}_Y \), let \( \tilde{D}_f(N) : p_1^* f^*(N) \to p_2^* f^*(N) \) be the composition

\[
P_1^* f^*(N) \xrightarrow{c_{f,p_1}(N)} (fp_1)^*(N) \xrightarrow{c_{f,p_2}(N)^{-1}} p_2^* f^*(N).
\]

For a morphism \( \phi : N \to N' \), \( \tilde{D}_f(\phi) = f^*(\phi) \). Then, \( \tilde{D}_f \) factors through the inclusion functor \( \text{Desc}(\mathcal{F}/\mathcal{E}, f) \to \text{Glue}(\mathcal{F}/\mathcal{E}, f) \) and we have a functor \( D_f : \mathcal{F}_Y \to \text{Desc}(\mathcal{F}/\mathcal{E}, f) \).
Moreover, a gluing morphism $\xi: p_1^*(M) \to p_2^*(M)$ on $M \in \text{Ob } \mathcal{F}_X$ is effective with respect to $f: X \to Y$ if and only if $\xi$ is isomorphic to an object in the image of $D_f: \mathcal{F}_Y \to \text{Desc}(\mathcal{F}/\mathcal{E}, f)$.

The following fact is also immediate.

**Proposition 4.9.** A morphism $f: X \to Y$ is of $\mathcal{F}$-descent (respectively effective $\mathcal{F}$-descent) if and only if $D_f: \mathcal{F}_Y \to \text{Desc}(\mathcal{F}/\mathcal{E}, f)$ is fully faithful (respectively an equivalence).

**Example 4.10.** Let $\text{Top}$ be the category of topological spaces and continuous maps. Consider the fibered category $p: \text{Top}^{(2)} \to \text{Top}$ (1.13). For a topological space $B$, suppose that an open covering $(U_i)_{i \in I}$ of $B$ is given. Put $X = \bigsqcup_{i \in I} U_i$ and let $f: X \to B$ be the map induced by the inclusion maps $U_i \hookrightarrow B$. Then, $X \times_B X = \bigsqcup_{i,j \in I} U_i \cap U_j$ and the following diagrams commute:

\[
\begin{array}{ccc}
U_i & \xrightarrow{\text{inc}} & U_i \cap U_j & \xrightarrow{\text{inc}} & U_j \\
\downarrow i & & \downarrow i_{ij} & & \downarrow i_j \\
X & \xrightarrow{p_1} & X \times_B X & \xrightarrow{p_2} & X \\
\end{array}
\]

\[
\begin{array}{ccc}
U_i \cap U_j \cap U_k & \xrightarrow{\text{inc}} & U_j \cap U_k & \xrightarrow{\text{inc}} & U_l \cap U_k \\
\downarrow i_{jk} & & \downarrow i_{jk} & & \downarrow i_k \\
X \times_B X \times_B X & \xrightarrow{p_{23}} & X \times_B X & \xrightarrow{p_{13}} & X \times_B X \\
\end{array}
\]

For a topological space $F$, consider an object $\text{pr}_1: X \times F \to X$ of $\text{Top}^{(2)}_X = \text{Top}/X$. Then, the pull-back of $\text{pr}_1: X \times F \to X$ along $p_i: X \times_B X \to X$ ($i = 1, 2$) is the map $q: (X \times_B X) \times F \to X \times_B X$ given by $q(x, y) = x \cdot y \in X \times_B X$, $y \in F$). For a map $\xi: (X \times_B X) \times F = \bigsqcup_{i,j \in I}(U_i \cap U_j) \times F \to \bigsqcup_{i,j \in I}(U_i \cap U_j) \times F = (X \times_B X) \times F$ making the following diagram commute, we denote by $\xi_{ij}: (U_i \cap U_j) \times F \to (U_i \cap U_j \cap U_k) \times F$ the restriction of $\xi$:

\[
\begin{array}{ccc}
\bigsqcup_{i,j \in I}(U_i \cap U_j) \times F & \xrightarrow{\xi} & \bigsqcup_{i,j \in I}(U_i \cap U_j) \times F \\
\downarrow \bigsqcup_{i,j \in I} \text{pr}_1 & & \downarrow \bigsqcup_{i,j \in I} \text{pr}_1 \\
\bigsqcup_{i,j \in I} U_i \cap U_j & \xrightarrow{\xi_{ij}} & \bigsqcup_{i,j \in I} U_i \cap U_j \\
\end{array}
\]

We also denote by $\xi_{ij}^k: (U_i \cap U_j \cap U_k) \times F \to (U_i \cap U_j \cap U_k) \times F$ the restriction of $\xi_{ij}$. Then, a descent data $\xi$ of $(X \times_B X, X; p_1, p_2, \Delta, p_1 \times p_2)$ on $\text{pr}_1: X \times F \to X$ is a homeomorphism $\xi: \bigsqcup_{i,j \in I}(U_i \cap U_j) \times F \to \bigsqcup_{i,j \in I}(U_i \cap U_j) \times F$ which makes the above diagram commute and satisfies $\xi_{jk} \xi_{ij} = \xi_{ik}$.

5. **Regular representations and restrictions**

Let $C$ be an internal category in $\mathcal{C}$ and $p: \mathcal{F} \to \mathcal{C}$ a cloven fibered category.
Definition 5.1. A representation \( \rho : \sigma^*(R) \to \tau^*(R) \) of \( C \) is called a regular representation if there exist an object \( N \) of \( FC_0 \) and, for each representation \( \xi \) of \( C \), a bijection \( \alpha_\xi : \text{Rep}(C;F)(\rho,\xi) \to FC_0(N,U(\xi)) \) which is natural in \( \xi \).

Proposition 5.2. A representation \( \rho : \sigma^*(R) \to \tau^*(R) \) of \( C \) is a regular representation if and only if there exists a morphism \( \eta : N \to R = U(\rho) \) in \( FC_0 \) such that, for any \( \xi \in \text{Ob} FC_0 \), composition \( \text{Rep}(C;F)(\rho,\xi) \to FC_0(U(\rho),U(\xi)) \to FC_0(N,U(\xi)) \) is bijective.

Proof. Suppose that \( \rho : \sigma^*(R) \to \tau^*(R) \) of \( C \) is a regular representation as in Definition 5.1. Put \( \eta = \alpha_\rho(id_\rho) : N \to U(\rho) \). For a morphism \( \theta : \rho \to \xi \), \( U(\theta)\eta = U(\theta)\alpha_\rho(id_\rho) = \alpha_\xi(\theta) \) by the naturality. Hence, the composition \( \eta^*U : \text{Rep}(C;F)(\rho,\xi) \to FC_0(U(\xi),N) \) coincides with \( \alpha_\xi \). The converse is obvious. \( \square \)

By the above result and Theorem 3.17, we have the following.

Corollary 5.3. Let \( C \) and \( G \) be internal categories in \( E \). Consider the fibered category \( pc : FC(C) \to E \) represented by \( C \) given in Example 2.18. A representation \( (id_{G_1},\rho_1) : \sigma^*(G_0,\rho_0) \to \tau^*(G_0,\rho_0) \) of \( G \) is a regular representation if and only if there exists a morphism \( (id_{G_0},\eta) : (G_0,\eta) \to (G_0,\rho_0) \) \( (\eta \in E(G_0,C_1)) \) in \( FC(G_0) \) such that, for any internal functor \( (f_0,f_1) : G \to C \), the map from the set of internal natural transformations \( (\rho_0,\rho_1) \to (f_0,f_1) \) to \( \Gamma_C(G_0)(u,f_0) \) given by \( \varphi \mapsto \mu(\eta,\varphi) \) is bijective.

Proposition 5.4. The forgetful functor \( U : \text{Rep}(C;F) \to FC_0 \) has a left adjoint if and only if, for every \( N \in \text{Ob} FC_0 \), there exist a representation \( \rho_N \) of \( C \) and a morphism \( \eta_N : N \to U(\rho_N) \) in \( FC_0 \) such that, for any \( \xi \in \text{Ob} \text{Rep}(C;F) \), the following composition is bijective:

\[
\text{Rep}(C;F)(\rho_N,\xi) \xrightarrow{U} FC_0(U(\rho_N),U(\xi)) \xrightarrow{\eta_N^*} FC_0(N,U(\xi)).
\]

Proof. Suppose that \( U : \text{Rep}(C;F) \to FC_0 \) has left adjoint \( L : FC_0 \to \text{Rep}(C;F) \). Let \( \eta : id_{FC_0} \to UL \) be the unit of this adjunction. For \( N \in \text{Ob} FC_0 \), a representation \( L(N) \) and a morphism \( \eta_N : N \to UL(N) \) satisfies the condition. In fact, for \( \xi \in \text{Ob} \text{Rep}(C;F) \), the composition \( \text{Rep}(C;F)(L(N),\xi) \xrightarrow{U} FC_0(UL(N),U(\xi)) \xrightarrow{\eta_N^*} FC_0(N,U(\xi)) \) is the adjoint bijection. We show the converse. Define a functor \( L : FC_0 \to \text{Rep}(C;F) \) as follows. For an object \( N \) of \( FC_0 \), put \( L(N) = \rho_N \). For a morphism \( \varphi : N \to M \) in \( FC_0 \), let \( L(\varphi) : \rho_N \to \rho_M \) be the morphism in \( \text{Rep}(C;F) \) which maps to \( \eta_M\varphi \) by the composition \( \text{Rep}(C;F)(\rho_N,\rho_M) \xrightarrow{U} FC_0(U(\rho_N),U(\rho_M)) \xrightarrow{\eta_N^*} FC_0(N,U(\rho_M)) \). It is easy to verify that \( L \) is a functor and that it is a left adjoint of \( U \). \( \square \)

Proposition 5.5. Suppose that \( U : \text{Rep}(C;F) \to FC_0 \) has a left adjoint \( L \). Let us denote by \( \eta \) and \( \epsilon \) the unit and the counit of this adjunction. Put \( T = UL \) and consider the monad \( T = (T,\eta,U(\epsilon_L)) \) associated with this adjunction. Then, the comparison functor \( K : \text{Rep}(C;F) \to FC_0^T \) given by \( K(\xi) = (U(\xi),U(\epsilon_L)) \) is an isomorphism of categories.

Proof. Let \( \xi \to \zeta \) be parallel arrows in \( \text{Rep}(C;F) \) such that \( U(\xi) \to U(\zeta) \) has a split coequalizer in \( FC_0 \). Since \( \sigma^* \) preserves split coequalizers and \( \mu^* \) preserves split epimorphism,
U creates the coequalizer of \( U(\xi) \xrightarrow{U(\psi)} U(\zeta) \) by (2) of Proposition 3.3. Hence, by Beck’s theorem (see [14, p. 151]) the assertion follows.

Let \( C = (C_0, C_1; \sigma, \tau, \varepsilon, \mu) \) and \( D = (D_0, D_1; \sigma, \tau, \varepsilon, \mu) \) be internal categories in \( \mathcal{E} \), \( f = (f_0, f_1) : D \to C \) an internal functor and \( p : \mathcal{F} \to \mathcal{E} \) a fibered category. Suppose that a representation \( \xi : \sigma^* (M) \to \tau^* (M) \) of \( C \) over \( M \in \text{Ob} \mathcal{F}_{C_0} \) is given. We denote by \( \xi_f : \sigma^* f_0^* (M) \to \tau^* f_0^* (M) \) the unique morphism in \( \mathcal{F}_{D_1} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
  f_1^* \sigma^* (M) & \xrightarrow{c_{\sigma, f_1}(M)} & (\sigma f_1)^* (M) = (f_0 \sigma)^* (M) \\
  f_1^* (\xi) & \xrightarrow{c_{\tau, f_1}(M)} & (\tau f_1)^* (M) = (f_0 \tau)^* (M) \\
  f_1^* \tau^* (M) & \xrightarrow{c_{\tau, f_1}(M)} & (\tau f_1)^* (M) = (f_0 \tau)^* (M) \\
\end{array}
\]

Then, it is straightforward to verify that \( \xi_f : \sigma^* f_0^* (M) \to \tau^* f_0^* (M) \) is a representation of \( D \) over \( f_0^* (M) \in \text{Ob} \mathcal{F}_{D_0} \).

**Definition 5.6.** We call \( \xi_f : \sigma^* f_0^* (M) \to \tau^* f_0^* (M) \) the restriction of \( \xi \) along \( f \).

If \( \xi : \sigma^* (N) \to \tau^* (N) \) is a representation of \( C \) over \( N \in \text{Ob} \mathcal{F}_{C_0} \) and \( \psi : \xi \to \zeta \) a morphism of representations, \( f_0^* (U(\psi)) : f_0^* (M) \to f_0^* (N) \) defines a morphism \( f_0^* (\psi) : \xi_f \to \zeta_f \) of representations.

If \( g = (g_0, g_1) : D \to C \) is an internal functor and \( \varphi \) is an internal natural transformation from \( f \) to \( g \), let \( \varphi_\xi : f_0^* (M) \to g_0^* (M) \) be the following composition:

\[
\begin{align*}
  f_0^* (M) & = (\sigma \varphi)^* (M) \xrightarrow{c_{\sigma, \varphi}(M)^{-1}} \varphi^* \sigma^* (M) \xrightarrow{\varphi^* (\xi)} \varphi^* \tau^* (M) \xrightarrow{c_{\tau, \varphi}(M)} (\tau \varphi)^* (M) = f_0^* (M).
\end{align*}
\]

Then, \( \varphi_\xi \) is a morphism of representations \( \xi_f \to \zeta_g \). It can be shown that the following diagram in \( \text{Rep}(D; \mathcal{F}) \) commutes:

\[
\begin{array}{ccc}
  \xi_f & \xrightarrow{f_0^* (\psi)} & \zeta_f \\
  / & \varphi_\xi \downarrow & / \\
  \xi_g & \xrightarrow{g_0^* (\psi)} & \zeta_g
\end{array}
\]

Hence, we have a functor \( \text{Res} : \text{cat}(\mathcal{E})(D, C) \times \text{Rep}(C; \mathcal{F}) \to \text{Rep}(D; \mathcal{F}) \). If \( \mathcal{F} = \mathcal{F}(G) \) for an internal category \( G \), we remark that \( \text{Res} \) is identified with the composition of internal functors by the isomorphism of Theorem 3.17, that is, the following diagram commutes:

\[
\begin{array}{ccc}
  \text{cat}(\mathcal{E})(D, C) \times \text{cat}(\mathcal{E})(C, G) & \xrightarrow{\text{composition}} & \text{cat}(\mathcal{E})(D, G) \\
  \downarrow \text{id} \times F & & \downarrow F \\
  \text{cat}(\mathcal{E})(D, C) \times \text{Rep}(C; \mathcal{F}(G)) & \xrightarrow{\text{Res}} & \text{Rep}(D; \mathcal{F}(G))
\end{array}
\]
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