PERIOD DEFORMATIONS AND RAABE’S FORMULAS FOR GENERALIZED GAMMA AND SINE FUNCTIONS

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(Received 2 January 2007)

Abstract. Deformations of the multiple gamma and sine functions with respect to their periods are studied. To describe such deformations explicitly, a new class of generalized gamma and sine functions are introduced. In particular, we study the deformations from the viewpoint of multiplication formulas and Raabe’s integral formulas for these gamma and sine functions. This new class of gamma functions contains Milnor’s type multiple gamma functions as a special case.

1. Introduction

Almost 100 years ago, Barnes [B] introduced the multiple gamma function $\Gamma_r(x, (\omega_1, \ldots, \omega_r))$ for a period $\underline{\omega} = (\omega_1, \ldots, \omega_r)$ as

$$\Gamma_r(x, (\omega_1, \ldots, \omega_r)) = \exp(\zeta_r'(0, x, (\omega_1, \ldots, \omega_r))).$$

where

$$\zeta_r(s, x, (\omega_1, \ldots, \omega_r)) = \sum_{n_1, \ldots, n_r \geq 0} (x + n_1 \omega_1 + \cdots + n_r \omega_r)^{-s}$$

is the multiple Hurwitz zeta function. Similarly to the case of the usual sine function, this multiple gamma function produces the multiple sine function $S_r(x, (\omega_1, \ldots, \omega_r))$ via the formula (a ‘reflection equation’ for the multiple gamma function)

$$S_r(x, (\omega_1, \ldots, \omega_r)) = \Gamma_r(x, (\omega_1, \ldots, \omega_r))^{-1} \Gamma_r(\omega_1 + \cdots + \omega_r - x, (\omega_1, \ldots, \omega_r))^{(-1)^r}$$

$$= \exp(-\zeta'_r(0, x, \omega) + (-1)^r \zeta'_r(0, |\omega| - x, \omega)),$$

where $|\omega| = \omega_1 + \cdots + \omega_r$ (see [Sh, Ku]). We refer to [KK] for a general theory of multiple sine functions. Multiple sine functions have applications in physics, e.g. as shown in [JM] for describing the solutions of $q$-KZ equations, as well as a number of arithmetic applications such as Kronecker’s Jugendtraum constructing class fields [Sh], expressions of special values of the Riemann zeta and Dirichlet $L$-functions (see [KOW]), etc. See also [SC] for the relevant formulas for the particular period $(1, \ldots, 1)$. The original purpose of this paper was to study deformations of $S_r(x, (\omega_1, \ldots, \omega_r))$ when some period $\omega_j$ goes

2000 Mathematics Subject Classification: Primary 11M36.

Keywords and Phrases: multiple Hurwitz’s zeta function; multiple gamma function; multiple sine function; multiplication formulas; Raabe’s formula.
to zero. For instance, if we let the parameter $\tau$ tend to infinity for the elliptic function $\theta(z, \tau)$ appropriately, $\theta(z, \tau)$ becomes essentially the trigonometric function as well as the Jackson $q$-gamma function tends to the classical gamma function when $q \to 1$. It turns out, however, that the category of the multiple sine functions is not sufficient for describing these deformations. Thus, in order to understand the situation explicitly, we employ a new class of generalizations of the multiple gamma and sine functions defined as

$$
\Gamma_{r,k}(x, (\omega_1, \ldots, \omega_r)) = \exp(\zeta'_{r}(-k, x, (\omega_1, \ldots, \omega_r)))
$$

and

$$
S_{r,k}(x, (\omega_1, \ldots, \omega_r)) = \exp(-\zeta'_{r}(-k, x, \omega) + (-1)^{r+k} \zeta'_{r}(-k, |\omega| - x, \omega))
$$

$$
= \frac{1}{\Gamma_{r,k}(x, \omega)^{(-1)^{r+k}}}(\frac{1}{\Gamma_{r}(x, (\omega_1, \ldots, \omega_r))})
$$

for integers $r > 0$ and $k \geq 0$, which have been introduced in [KOW] when $r = 1$. We call them the generalized gamma function and generalized sine function, respectively. Obviously, the multiple sine function is given as $S_r(x, (\omega_1, \ldots, \omega_r)) = S_{r,0}(x, (\omega_1, \ldots, \omega_r))$. Moreover, we have studied $\Gamma_r(x) = \Gamma_{1,r-1}(x, 1)$ and $S_r(x) = S_{1,r-1}(x, 1)$ as ‘Milnor’s type multiple gamma and sine functions’ in [KOW] (see [M]). In this paper, we exclusively consider $\Gamma_{r,k}(x, (\omega_1, \ldots, \omega_r))$ and $S_{r,k}(x, (\omega_1, \ldots, \omega_r))$ for $\omega_1, \ldots, \omega_r > 0$ without further mention. Moreover, in the case of the generalized sine functions, we restrict the region to $0 < x < \omega_1 + \cdots + \omega_r$ throughout the paper.

Our main result below shows that period deformations of generalized gamma and sine functions are essentially captured within their framework. Specifically, such a period deformation of a generalized sine function is again given by a generalized sine function.

**Theorem 1.1.** We have

$$
\lim_{N_1, \ldots, N_{\ell} \to \infty} \Gamma_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_{\ell}} \alpha_{\ell}\right))^{1/N_1 \cdots N_{\ell}}
$$

$$
= \text{GMF} \exp\left(\int_0^1 \cdots \int_0^1 \log \Gamma_{r+\ell,k} \times (x + t_1 \alpha_1 + \cdots + t_{\ell} \alpha_{\ell}, (\omega_1, \ldots, \omega_r, \alpha_1, \ldots, \alpha_{\ell})) dt_1 \cdots dt_{\ell}\right)
$$

$$
= \text{GRF} \left\{ \Gamma_{r,\ell+k}(x, (\omega_1, \ldots, \omega_r)) \times \exp\left(\left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{k+\ell}\right) \zeta_r(-k - \ell, x, \omega)\right)\right\}^{(-1)^{\ell+1}(\ell+k)!\alpha_1 \cdots \alpha_{\ell}},
$$

(1.1a)
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\[ \lim_{N_1, \ldots, N_\ell \to \infty} S_{r+\ell,k}(x, (\omega_1, \ldots, \omega_r, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell))^{1/N_1 \cdots N_\ell} \]

\[ = \exp \left( \int_0^1 \cdots \int_0^1 \log S_{r+\ell,k}(x) \times \left( x + t_1 \alpha_1 + \cdots + t_\ell \alpha_\ell, (\omega_1, \ldots, \omega_r, \alpha_1, \ldots, \alpha_\ell) \right) dt_1 \cdots dt_\ell \right) \]

\[ = S_{r,\ell+k}(x, (\omega_1, \ldots, \omega_r))^{(-1)^{\ell+k} \alpha_1 \cdots \alpha_\ell}. \quad (1.1b) \]

We note that each of former equalities is derived from a generalized multiplication formula (GMF) in Theorem 3.1 and the corresponding latter equality is a generalized Raabe’s formula (GRF) established in Theorem 5.1.

It would be valuable to recall the simplest case of the multiplication formula and Raabe’s formula for the sine function. It deals with \( S_1(x) = S_1(x, 1) = 2 \sin(\pi x) \); the multiplication formula means

\[ S_1(Nx) = \prod_{h=0}^{N-1} S_1 \left( x + \frac{h}{N} \right) \]

for any positive integer \( N \), and Raabe’s formula \([R]\) is

\[ \int_0^1 \log S_1(t) \, dt = 0, \]

which is equivalent to the famous Euler’s tricky integral \([E]\) (cf. \([FR, KW]\)).

We remark that a deformation of a multiple sine function is not necessarily a multiple sine function as the following \( k = 0 \) case indicates:

\[ \lim_{N_1, \ldots, N_\ell \to \infty} S_{r+\ell}(x, (\omega_1, \ldots, \omega_r, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell))^{1/N_1 \cdots N_\ell} \]

\[ = S_{r,\ell}(x, (\omega_1, \ldots, \omega_r))^{(-1)^{\ell} \alpha_1 \cdots \alpha_\ell}. \]

Moreover, this shows that every generalized sine function \( S_{r,\ell}(x, (\omega_1, \ldots, \omega_r)) \) can be obtained as a deformation of a multiple sine function. We refer to the above text concerning such examples. The present result could be generalized in a future study from the viewpoint of gamma and sine functions in the scheme of (representations of) groups treated in \([W]\) and \([KW]\).

This paper is organized as follows. In Section 2 we give a new explanation of the classical work of Raabe together with that of Kinkelin from our viewpoint: the period deformation of the multiple gamma and sine functions. In Section 3 we derive the multiplication formulas of the generalized gamma and sine functions (Theorem 3.1). Using these results, in Section 4, we show the first part of the respective claims in Theorem 1.1. In Section 5, we devote ourselves to the Raabe-type theorems for multiple Hurwitz zeta, generalized gamma and sine functions (Theorem 5.1). Then the latter part of the respective claims in Theorem 1.1 follows immediately from Theorem 5.1. In the finial section, we give some examples when the period is \((1, \ldots, 1)\).
2. A visit to Raabe’s work from Theorem 1.1

Before starting the actual proof of the main theorem, we visit the work of Raabe [R] from our point of view. This is because the heart of the argument in Raabe [R] is quite similar to a special case of (1.1a) of Theorem 1.1 and this presentation allows us to provide a new understanding of several works for multiple gamma and sine functions that have subsequently arisen after Raabe, e.g. Kinkelin’s work on double gamma functions [Ki], the determination of the gamma factor of the Selberg zeta function [Sa, V], etc.

Let \( r = 0, \ell = 1, k = 1 \) and \( \alpha_1 = 1 \) in (1.1a) in Theorem 1.1. Then it states that

\[
\lim_{N \to \infty} \left( \frac{1}{N} \log \Gamma_1 \left( x, \frac{1}{N} \right) \right) = \int_0^1 \log \Gamma_1 (x + t) \, dt
\]

\[
= - \log \Gamma_{0,1}(x) - \zeta_0(-1, x),
\]

where \( \zeta_0(s, x) = x - s \). Now recall that

\[
\Gamma_1 \left( x, \frac{1}{N} \right) = \frac{\Gamma(Nx)}{\sqrt{2\pi}} N^{-(N-1/2)},
\]

\[
\Gamma_{0,1}(x) = \exp(\zeta_0'(-1, x)) = x^x \quad \text{and} \quad \zeta_0(-1, x) = x.
\]

Thus, our result implies that

\[
\int_0^1 \log \Gamma(x + t) \, dt = \lim_{N \to \infty} \left( \frac{1}{N} \log \Gamma(Nx) - x \log N \right) + \frac{1}{2} \log(2\pi)
\]

\[
= x \log x - x + \frac{1}{2} \log(2\pi).
\]

This is exactly the description Raabe made in [R, p. 11]. More precisely, what he wrote was

\[
\int_0^1 \log \Gamma(x + t) \, dt = \omega \log \Gamma \left( \frac{x}{\omega} \right) + x \log \omega + \frac{1}{2} \log(2\pi)
\]

\[
= x \log x - x + \frac{1}{2} \log(2\pi),
\]

where by ‘\( \omega \)’ he meant the ‘infinitesimal’ (‘unendlich-klein werdenden Grösse’). He starts from the Gauss–Legendre multiplication formula

\[
\Gamma(Nx) = \Gamma(x) \Gamma \left( x + \frac{1}{N} \right) \cdots \Gamma \left( x + \frac{N-1}{N} \right) N^{N-\frac{1}{2}} (2\pi)^{1-N/2},
\]

which is equivalent to our formulation expressed as

\[
\Gamma_1 \left( x, \frac{1}{N} \right) = \Gamma_1(x) \Gamma_1 \left( x + \frac{1}{N} \right) \cdots \Gamma_1 \left( x + \frac{N-1}{N} \right).
\]

By this multiplication formula, Raabe obtains

\[
\int_0^1 \log \Gamma(x + t) \, dt = \lim_{N \to \infty} \frac{1}{N} \sum_{h=0}^{N-1} \log \Gamma \left( x + \frac{h}{N} \right)
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \left\{ \log \Gamma(Nx) - \left( Nx - \frac{1}{2} \right) \log N - \frac{1-N}{2} \log(2\pi) \right\}
\]

\[
= \lim_{N \to \infty} \left( \frac{1}{N} \log \Gamma(Nx) - x \log N \right) + \frac{1}{2} \log(2\pi).
\]
Consequently, Raabe obtains his celebrated formula

\[ \int_0^1 \log \Gamma(x + t) \, dt = x \log x - x + \frac{1}{2} \log(2\pi) \]

by showing

\[ \lim_{N \to \infty} \left( \frac{1}{N} \log \Gamma(Nx) - x \log N \right) = x \log x - x. \]

We remark that Raabe’s formula suggested that Kinkelin [Ki] obtained the ‘double gamma function’

\[ G(x) = \exp \left( \int_0^x \log \Gamma(t) \, dt + \frac{x^2 - x}{2} - \frac{x}{2} \log(2\pi) \right). \]

See formula (7) of [Ki, p. 124]; on that page, Kinkelin refers to Raabe’s work explicitly. Kinkelin proved

\[ G(0) = G(1) = 1, \]
\[ G(x + 1) = G(x)x^x \quad \text{for } x > 0, \]
\[ G(n + 1) = 1^1 2^2 \cdots n^n \quad \text{for each integer } n \geq 1, \]

and obtained the famous formula (when \( x = 1 \) it is the aforementioned Euler integral)

\[ \int_0^x \log(2 \sin \pi t) \, dt = \log \left( \frac{G(1 - x)}{G(x)} \right). \]

See formulas (10) and (11) of [Ki, pp. 125–126] and formula (24) of [Ki, p.135]. Kinkelin’s formula \( G(x + 1) = G(x)x^x \) is equivalent to Raabe’s formula, because the definition of the double gamma function \( G(x) \) above gives

\[ \log G(x + 1) - \log G(x) = \int_x^{x+1} \log \Gamma(t) \, dt + x - \frac{1}{2} \log(2\pi). \]

In other words, Kinkelin ‘integrated’ the difference equation given by Raabe.

Later, the formula of Kinkelin above for the integral of the logarithmic sine function motivated mathematicians to study generalized gamma and sine functions. Specifically, Sarnak [Sa] and Voros [V] in 1987 used that formula to calculate the gamma factor of the Selberg zeta function of a Riemann surface of genus \( g > 1 \). See [Ku, KK] for the cases of higher-dimensional locally symmetric Riemannian spaces.

We note that, as seen below, Theorem 1.1 for \( r = 1, \ell = 1, \omega_1 = 1, \alpha_1 = 1, \) and \( k = 0 \) provides

\[ G(x) = \Gamma_{1,1}(x)e^{-\zeta'(1)} \quad \text{(A)} \]

and

\[ \int_0^x \log S_1(t) \, dt = \log S_{1,1}(x). \quad \text{(B)} \]

Here we write \( \Gamma_{1,1}(x) = \Gamma_{1,1}(x, 1) \) and \( S_{1,1}(x, 1) = S_{1,1}(x) \). From \( S_1(x) = 2 \sin(\pi x) \) and

\[ S_{1,1}(x) = \frac{\Gamma_{1,1}(1 - x)}{\Gamma_{1,1}(x)} = \frac{G(1 - x)}{G(x)}. \]
Thus, we see (B). The present explanation sheds new light on old results of Raabe [R] and Kinkelin [Ki] from the viewpoint of our generalized Raabe formulas. One finds that formula (B) is exactly the Kinkelin formula. Let us show (A) first: Theorem 1.1 claims that

\[ \int_0^1 \log \Gamma_2(x + t) \, dt = - \log \Gamma_{1,1}(x) + \frac{x^2 - x}{2} + \frac{1}{12}, \]

because

\[ \zeta(-1, x) = - \frac{x^2 - x}{2} - \frac{1}{12}. \]

Here we omit the period (1, 1) as \( \Gamma_2(x) = \Gamma_2(x, (1, 1)) \). Hence, the differentiation yields

\[ \frac{\Gamma_1'(x)}{\Gamma_1(x)} + x - \frac{1}{2} = \int_0^1 \frac{\Gamma_2'}{\Gamma_2(x + t)} \, dt = \left[ \log \frac{\Gamma_2(x + t)}{\Gamma_2(x)} \right]_{t=0}^{t=1} = \log \frac{\Gamma_2(x + 1)}{\Gamma_2(x)} = - \log \Gamma_1(x), \]

where we used the periodicity \( \Gamma_2(x + 1) = \Gamma_2(x) \Gamma(x)^{-1} \). Thus, we obtain

\[ \int_0^x \left( \log \Gamma_1(t) + \left( t - \frac{1}{2} \right) \right) \, dt = \int_0^x \frac{\Gamma_1'(t)}{\Gamma_1(t)} \, dt = \log \frac{\Gamma_{1,1}(x)}{\Gamma_{1,1}(0)}. \]

Here, the left-hand side is identical with

\[ \log G(x) = \int_0^x \log \Gamma(t) \, dt + \frac{x^2 - x}{2} - \frac{x}{2} \log(2\pi). \]

Hence, it follows that

\[ G(x) = \Gamma_{1,1}(x) \Gamma_{1,1}(0)^{-1}. \]

Thus, it remains to see that

\[ \Gamma_{1,1}(0) = \Gamma_{1,1}(1) = e^{\zeta'(1)}. \]

This follows from the relation

\[ \zeta(s, x + 1) = \zeta(s, x) - x^{-s}. \]

In fact, from this we have \( \Gamma_{1,1}(x + 1) = \Gamma_{1,1}(x)x^x \) and, hence, \( \Gamma_{1,1}(0) = \Gamma_{1,1}(1) = \exp(\zeta'(1, 1)) = e^{\zeta'(1)}. \) Thus, we have (A). We can show (B) in the same way as follows. Equation (1.1b) in Theorem 1.1 states that

\[ \int_0^1 \log S_2(x + t) \, dt = - \log S_{1,1}(x), \]

and the differentiation leads to

\[ \frac{S_1'(x)}{S_{1,1}} = \frac{S_2'(x + t)}{S_2(x + t)} \, dt = \left[ \log \frac{S_2(x + t)}{S_2(x)} \right]_{t=0}^{t=1} = \log \frac{S_2(x + 1)}{S_2(x)} = - \log S_1(x) \]

by the periodicity \( S_2(x + 1) = S_2(x) S_1(x)^{-1} \). Hence, we have

\[ \int_0^x \log S_1(t) \, dt = \int_0^x \frac{S_1'(t)}{S_{1,1}} \, dt = \left[ \log \frac{S_{1,1}(t)}{S_{1,1}(0)} \right]_{t=0}^{t=x} = \log \frac{S_{1,1}(x)}{S_{1,1}(0)} = \log S_{1,1}(x). \]

Here we used the fact

\[ S_{1,1}(0) = \frac{\Gamma_{1,1}(1)}{\Gamma_{1,1}(0)} = \frac{G(1)}{G(0)} = 1. \]

Thus, we see (B). The present explanation sheds new light on old results of Raabe [R] and Kinkelin [Ki] from the viewpoint of our generalized Raabe formulas.
3. Generalized multiplication formulas

In this section, we prove the following multiplication formulas of $\Gamma_{r,k}(x, \omega)$ and $S_{r,k}(x, \omega)$.

**Theorem 3.1.** We have

\[
\Gamma_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell\right)) = \prod_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} \Gamma_{r+\ell,k}(x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell, (\omega_1, \ldots, \omega_r, \alpha_1, \ldots, \alpha_\ell)).
\]

(3.1a)

\[
S_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell\right)) = \prod_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} S_{r+\ell,k}(x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell, (\omega_1, \ldots, \omega_r, \alpha_1, \ldots, \alpha_\ell)).
\]

(3.1b)

In a simple situation where $r = 0$, $k = 0$ and $N_1 = \cdots = N_\ell = N$, formula (3.1b) gives

\[
S_{\ell}(x, \frac{1}{N} \alpha) = \prod_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} S_{\ell}(x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell, \alpha).
\]

This is equivalent to the multiplication formula of $S_{\ell}(x, \alpha)$ (for a fixed weight $\alpha$) given by

\[
S_{\ell}(Nx, \alpha) = \prod_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} S_{\ell}(x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell, \alpha)
\]

proved in [KK] because $S_{\ell}(x, (1/N)\alpha) = S_{\ell}(Nx, \alpha)$ holds by the homogeneity.

**Proof of Theorem 3.1.** In the expression

\[
\zeta_{r+\ell}(s, x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell\right)) = \sum_{n_1, \ldots, n_r \geq 0, m_1, \ldots, m_\ell \geq 0} (x + n_1 \omega_1 + \cdots + n_r \omega_r + \frac{m_1}{N_1} \alpha_1 + \cdots + \frac{m_\ell}{N_\ell} \alpha_\ell)^{-s},
\]

setting $m_j = N_j m'_j + h_j$ with $m'_j \geq 0$ and $h_j = 0, \ldots, N_j - 1$, we obtain

\[
\zeta_{r+\ell}(s, x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell\right)) = \sum_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} \zeta_{r+\ell}(s, x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell, (\omega_1, \ldots, \omega_r, \alpha_1, \ldots, \alpha_\ell)).
\]
Hence, the differentiation at $s = -k$ gives

$$
\log \Gamma_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1}\alpha_1, \ldots, \frac{1}{N_\ell}\alpha_\ell\right))
= \sum_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} \log \Gamma_{r+\ell,k}(x + \frac{h_1}{N_1}\alpha_1 + \cdots + \frac{h_\ell}{N_\ell}\alpha_\ell, (\omega_1, \ldots, \omega_r, \alpha_1, \ldots, \alpha_\ell)).
$$

This proves (3.1a). Now we show claim (3.1b). From (3.1a) we have

$$
\log S_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1}\alpha_1, \ldots, \frac{1}{N_\ell}\alpha_\ell\right))
= -\log \Gamma_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1}\alpha_1, \ldots, \frac{1}{N_\ell}\alpha_\ell\right))
+ (-1)^{r+\ell+k} \log \Gamma_{r+\ell,k}\left(\omega_1 + \cdots + \omega_r + \frac{1}{N_1}\alpha_1 + \cdots + \frac{1}{N_\ell}\alpha_\ell - x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1}\alpha_1, \ldots, \frac{1}{N_\ell}\alpha_\ell\right)\right)
= -\sum_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} \log \Gamma_{r+\ell,k}(x + \frac{h_1}{N_1}\alpha_1 + \cdots + \frac{h_\ell}{N_\ell}\alpha_\ell, (\omega, \alpha))
+ (-1)^{r+\ell+k} \sum_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} \log \Gamma_{r+\ell,k}\left(|\omega| - x + \frac{h_1+1}{N_1}\alpha_1 + \cdots + \frac{h_\ell+1}{N_\ell}\alpha_\ell, (\omega, \alpha)\right),
$$

and replacing $h_j$ in the latter sum by $N_j - 1 - h_j$, we obtain

$$
\log S_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_r, \frac{1}{N_1}\alpha_1, \ldots, \frac{1}{N_\ell}\alpha_\ell\right))
= -\sum_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} \log \Gamma_{r+\ell,k}(x + \frac{h_1}{N_1}\alpha_1 + \cdots + \frac{h_\ell}{N_\ell}\alpha_\ell, (\omega, \alpha))
+ (-1)^{r+\ell+k} \sum_{h_j=0, 1 \leq j \leq \ell}^{N_j-1} \log \Gamma_{r+\ell,k}\left(|\omega| - x + \frac{N_1-h_1}{N_1}\alpha_1 + \cdots + \frac{N_\ell-h_\ell}{N_\ell}\alpha_\ell, (\omega, \alpha)\right).
$$
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\[ - \sum_{h_j=0, j=1}^{N_j-1} \log \Gamma_{r+\ell,k}(x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell, (\omega, \alpha)) \]

\[ + (-1)^{r+\ell+k} \sum_{h_j=0, j=1}^{N_j-1} \log \Gamma_{r+\ell,k}(|\omega| + |\alpha| - \left(x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell\right), (\omega, \alpha)) \]

\[ = \sum_{h_j=0, j=1}^{N_j-1} \log S_{r+\ell,k}(x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell, (\omega_1, \ldots, \omega_\ell, \alpha_1, \ldots, \alpha_\ell)) \]

This proves the theorem.

\[ \square \]

4. Period deformations

From Theorem 3.1 we easily obtain the former equalities of respective assertions in Theorem 1.1. In fact, we show the following.

**THEOREM 4.1.** We have

\[ \lim_{N_1, \ldots, N_\ell \to \infty} \frac{1}{N_1 \cdots N_\ell} \log \Gamma_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_\ell, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell\right))^{1/N_1 \cdots N_\ell} \]

\[ = \exp\left(\int_0^1 \cdots \int_0^1 \log \Gamma_{r+\ell,k}(x + t \cdot \alpha, (\omega, \alpha)) \, dt\right). \tag{4.1a} \]

\[ \lim_{N_1, \ldots, N_\ell \to \infty} \frac{1}{N_1 \cdots N_\ell} \log S_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_\ell, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell\right))^{1/N_1 \cdots N_\ell} \]

\[ = \exp\left(\int_0^1 \cdots \int_0^1 \log S_{r+\ell,k}(x + t \cdot \alpha, (\omega, \alpha)) \, dt\right). \tag{4.1b} \]

**Proof.** As the proof is the same, we only show (4.1b). From (3.1b) we have

\[ \lim_{N_1, \ldots, N_\ell \to \infty} \frac{1}{N_1 \cdots N_\ell} \log S_{r+\ell,k}(x, \left(\omega_1, \ldots, \omega_\ell, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_\ell} \alpha_\ell\right)) \]

\[ = \lim_{N_1, \ldots, N_\ell \to \infty} \frac{1}{N_1 \cdots N_\ell} \sum_{h_j=0, j=1}^{N_j-1} \log S_{r+\ell,k}(x + \frac{h_1}{N_1} \alpha_1 + \cdots + \frac{h_\ell}{N_\ell} \alpha_\ell, (\omega, \alpha)) \]

\[ = \int_0^1 \cdots \int_0^1 \log S_{r+\ell,k}(x + t_1 \alpha_1 + \cdots + t_\ell \alpha_\ell, (\omega, \alpha)) \, dt_1 \cdots dt_\ell. \]

This proves Theorem 4.1.

\[ \square \]

5. Generalized Raabe’s formulas

We establish generalized Raabe’s formulas in several versions.
THEOREM 5.1. We have the following.

(i) Zeta–Raabe. For all $s \in \mathbb{C}$

$$\int_0^1 \cdots \int_0^1 \zeta_{r+\ell}(s, x + t \cdot \alpha, (\omega, \alpha)) \, dt$$

$$= \frac{1}{\alpha_1 \cdots \alpha_\ell \cdot (s-1) \cdots (s-\ell)} \zeta_r(s-\ell, x, \omega).$$  \hspace{1cm} (5.1a)

(ii) Gamma-Raabe:

$$\int_0^1 \cdots \int_0^1 \log \Gamma_{r+\ell,k}(x + t \cdot \alpha, (\omega, \alpha)) \, dt$$

$$= \frac{1}{\alpha_1 \cdots \alpha_\ell} (-1)^{\ell} \log \Gamma_{r,\ell+k}(x, \omega)$$

$$+ \frac{1}{\alpha_1 \cdots \alpha_\ell (k+\ell)!} \left( \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{k+\ell} \right) \zeta_r(-\ell, x, \omega).$$ \hspace{1cm} (5.1b)

(iii) Sine-Raabe:

$$\int_0^1 \cdots \int_0^1 \log S_{r+\ell,k}(x + t \cdot \alpha, (\omega, \alpha)) \, dt = \frac{1}{\alpha_1 \cdots \alpha_\ell} (-1)^{\ell} \log S_{r,\ell+k}(x, \omega).$$ \hspace{1cm} (5.1c)

Proof. To prove (5.1a) it is sufficient to show the formula when $\text{Re} \, s > r + \ell$ because both sides can be meromorphically extended to the entire plane $\mathbb{C}$. Now we prove

$$\int_0^{\alpha_1} \cdots \int_0^{\alpha_\ell} \zeta_{r+\ell}(s, x + u_1 + \cdots + u_\ell, (\omega, \alpha)) \, du_1 \cdots du_\ell$$

$$= \frac{1}{(s-1) \cdots (s-\ell)} \zeta_r(s-\ell, x, \omega)$$

for $\text{Re} \, s > r + \ell$. This iterated integration is performed as follows. We first show that

$$\int_0^{\alpha_\ell} \zeta_{r+\ell}(s, x + u_1 + \cdots + u_\ell, (\omega, \alpha)) \, du_\ell$$

$$= \frac{1}{s-1} \zeta_{r+\ell-1}(s-1, x + u_1 + \cdots + u_{\ell-1}, (\omega, \alpha_1, \ldots, \alpha_{\ell-1})).$$

Then, it is easy to see that the inductive procedure gives the desired result. Now,

$$\int_0^{\alpha_\ell} \zeta_{r+\ell}(s, x + u_1 + \cdots + u_\ell, (\omega, \alpha)) \, du_\ell$$

$$= \int_{x+u_1+\cdots+u_{\ell-1}+\alpha_\ell}^{x+u_1+\cdots+u_{\ell-1}} \zeta_{r+\ell}(s, y, (\omega, \alpha)) \, dy.$$
Here, the periodicity (in other words, the ladder structure of the multiple Hurwitz zeta function)

\[ \zeta_{r+\ell}(s, y, (\omega, \alpha)) = \zeta_{r+\ell}(s, y, (\omega, \alpha)) - \zeta_{r+\ell-1}(s, y, (\omega, \alpha_1, \ldots, \alpha_{\ell-1})) \]

displays actually that

\[ \int_0^\alpha \zeta_{r+\ell}(s, x + u_1 + \cdots + u_\ell, (\omega, \alpha)) \, du_\ell \]

\[ = \int_0^\alpha \zeta_{r+\ell-1}(s, y, (\omega, \alpha_1, \ldots, \alpha_{\ell-1})) \, dy \]

\[ = \sum_{n_1, \ldots, n_r \geq 0, m_1, \ldots, m_l \geq 0} \int_0^\alpha (x + u_1 + \cdots + u_{\ell-1} + n_1 \omega_1 + \cdots + n_r \omega_r + m_1 \alpha_1 + \cdots + m_{\ell-1} \alpha_{\ell-1})^{-s} \, dy \]

\[ = \frac{1}{s-1} \zeta_{r+\ell-1}(s-1, x + u_1 + \cdots + u_{\ell-1}, (\omega, \alpha_1, \ldots, \alpha_{\ell-1})). \]

Thus, we obtain the assertion (5.1a).

Next we prove (5.1b). Differentiating the formula (5.1a) at \( s = -k \), we obtain Raabe’s formula for the generalized gamma function as

\[ \int_0^1 \cdots \int_0^1 \log \Gamma_{r+\ell,k}(x + t \cdot \alpha, (\omega, \alpha)) \, dt \]

\[ = \frac{1}{\alpha_1 \cdots \alpha_\ell} \frac{(-1)^\ell k!}{(k + \ell)!} \times \left[ \log \Gamma_{r+\ell+k}(x, \omega) + \sum_{k' = k+1}^{k+\ell} \frac{1}{k'} \right]. \]

This proves (5.1b). Using this, we prove (5.1c) as follows:

\[ \int_0^1 \cdots \int_0^1 \log S_{r+\ell,k}(x + t \cdot \alpha, (\omega, \alpha)) \, dt \]

\[ = -\int_0^1 \cdots \int_0^1 \log \Gamma_{r+\ell,k}(x + t \cdot \alpha, (\omega, \alpha)) \, dt \]

\[ + (-1)^{r+\ell+k} \int_0^1 \cdots \int_0^1 \log \Gamma_{r+\ell,k}(|\alpha| + |\alpha| - x - t \cdot \alpha, (\omega, \alpha)) \, dt \]
= - \int_0^1 \cdots \int_0^1 \log \Gamma_{r+\ell,k}(x + t \cdot \alpha, (\omega, \varrho)) \, dt \\
+ (-1)^{r+\ell+k} \int_0^1 \cdots \int_0^1 \log \Gamma_{r+\ell,k}(|\omega| - x + t \cdot \alpha, (\omega, \varrho)) \, dt \\
= \frac{1}{\alpha_1 \cdots \alpha_\ell} \frac{(-1)^{\ell k}}{(k + \ell)!} \left[ (-\log \Gamma_{r,\ell+k}(x, \omega) + (-1)^{r+\ell+k} \log \Gamma_{r,\ell+k}(|\omega| - x, \omega)) \\
+ \left( \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{k+\ell} \right) \right] \times \{-\zeta_r(-k - \ell, x, \omega) + (-1)^{r+\ell+k} \zeta_r(-k - \ell, |\omega| - x, \omega)\} \\
= \frac{1}{\alpha_1 \cdots \alpha_\ell} \frac{(-1)^{\ell k}}{(k + \ell)!} \log S_{r,\ell+k}(x, \omega),

where we have used the following lemma, which claims the vanishing property of \(\zeta_r(s, x, \omega)\), in the last stage of the calculation.

**Lemma 5.2.** For each integer \(m \geq 0\)

\[-\zeta_r(-m, x, \omega) + (-1)^{r+m} \zeta_r(-m, |\omega| - x, \omega) = 0.\]

**Proof.** Let

\[F_{r,m}(t, x, \omega) := \frac{-e^{-tx} + (-1)^{r+m} e^{-t(|\omega|-x)}}{(1 - e^{-t\omega_1}) \cdots (1 - e^{-t\omega_r})} = \sum_{k \geq -r} c_k(x, \omega) t^k\]

be the Laurent expansion around \(t = 0\) with polynomials \(c_k(x, \omega)\) in \(x\). Then we have

\[-\zeta_r(s, x, \omega) + (-1)^{r+m} \zeta_r(s, |\omega| - x, \omega) = \frac{1}{\Gamma(s)} \int_0^\infty F_{r,m}(t, x, \omega) t^{s-1} \, dt\]

in \(\text{Re } s > r + 1\), and the standard procedure for obtaining analytic continuation shows that

\[-\zeta_r(-m, x, \omega) + (-1)^{r+m} \zeta_r(-m, |\omega| - x, \omega) = \frac{1}{\Gamma(s)} \int_0^1 F_{r,m}(t, x, \omega) t^{s-1} \, dt \bigg|_{s = -m} = (-1)^m m! c_m(x, \omega).\]

Moreover, the asymmetry

\[F_{r,m}(-t, x, \omega) = (-1)^{m-1} F_{r,m}(t, x, \omega)\]

asserts

\[(-1)^k c_k(x, \omega) = (-1)^{m-1} c_k(x, \omega).\]

This, in particular, shows \(c_m(x, \omega) = 0\). Thus, we have the lemma.

This completes the proof of Theorem 5.1.
6. Examples

Our main theorem would give many examples by taking period variously. In this section, we report a few restricted examples. We write $S_r(x) = S_r(x, (1, \ldots, 1))$ and $S_{r,k}(x) = S_{r,k}(x, (1, \ldots, 1))$ for simplicity.

**Example 6.1.** We have the following examples:

$$\lim_{N \to \infty} S_{r+1}\left(x, \left(1, \ldots, 1, \frac{1}{N}\right)\right)^{1/N} = S_{r+1}(x)^{-r} S_r(x)^{-x}, \tag{6.1a}$$

$$\lim_{N_1, N_2 \to \infty} S_{r+2}\left(x, \left(1, \ldots, 1, \frac{1}{N_1}, \frac{1}{N_2}\right)\right)^{1/(N_1 N_2)} = S_{r+2}(x)^{(r+1)/2} S_r(x)^{-(r-1)/2} S_r(x)^{(r-x)^2/2}, \tag{6.1b}$$

$$\lim_{N_1, N_2, N_3 \to \infty} S_{r+3}\left(x, \left(1, \ldots, 1, \frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3}\right)\right)^{1/(N_1 N_2 N_3)} = S_{r+3}(x)^{-r(r+1)/6} S_{r+2}(x)^{-2(r+1)/3} S_r(x)^{(r-x)^3/3}, \tag{6.1c}$$

$S_{r,1}(x) = S_{r+1}(x)^{-r} S_r(x)^{-x}, \tag{6.1d}$

$S_{r,2}(x) = S_{r+2}(x)^{(r+1)} S_{r+1}(x)^{(2x-2r-1)} S_r(x)^{(r-x)^2}, \tag{6.1e}$

$S_{r,3}(x) = S_{r+3}(x)^{(r+1)(r+2)} S_{r+2}(x)^{3r(r+1)(x-r-1)} S_{r+1}(x)^{(3x^2-3r(2r+1)x+(3r^3+3r^2+r))} S_r(x)^{(r-x)^3}. \tag{6.1f}$

From Theorem 1.1, we first note that

$$\lim_{N_1, \ldots, N_{\ell} \to \infty} S_{r+\ell}\left(x, \left(1, \ldots, 1, \frac{1}{N_1} \alpha_1, \ldots, \frac{1}{N_{\ell}} \alpha_{\ell}\right)\right)^{1/N_1 \cdots N_{\ell}} = S_{r,\ell}(x)^{(-1)^{\ell}/\ell!}. \tag{6.1g}$$

Hence, (6.1a), (6.1b) and (6.1c) follow immediately from (6.1d), (6.1e) and (6.1f), respectively. Therefore, we show (6.1d), (6.1e) and (6.1f) as follows.

We write $\zeta_r(s, x) = \zeta_r(s, x, (1, \ldots, 1))$ simply. An easily verified formula

$$\zeta_r(s, x) = \sum_{n_1, \ldots, n_r \geq 0} (n_1 + \cdots + n_r + x)^{-s} = \sum_{n=0}^{\infty} \frac{(n + r - 1)}{(r - 1)} (n + x)^{-s}$$

shows that

$$\zeta_r(s - 1, x) = r \zeta_{r+1}(s, x) + (x - r) \zeta_r(s, x),$$

$$\zeta_r(s - 2, x) = r(r + 1) \zeta_{r+2}(s, x) + r(2x - 2r - 1) \zeta_{r+1}(s, x) + (x - r)^2 \zeta_r(s, x),$$

and

$$\zeta_r(s - 3, x) = r(r + 1)(r + 2) \zeta_{r+3}(s, x) + 3r(r + 1)(x - r - 1) \zeta_{r+2}(s, x) + (3r x^2 - 3r(2r + 1)x + 3r^3 + 3r^2 + r) \zeta_{r+1}(s, x) + (x - r)^3 \zeta_r(s, x).$$
In fact, these are respectively equivalent to the following equalities which can be checked directly:

\[
(n + x) \binom{n + r - 1}{r - 1} = r \binom{n + r}{r} + (x - r) \binom{n + r - 1}{r - 1},
\]

\[
(n + x)^2 \binom{n + r - 1}{r - 1} = r(r + 1) \binom{n + r + 1}{r + 1} + r(2x - 2r - 1) \binom{n + r}{r}
+ (x - r)^2 \binom{n + r - 1}{r - 1},
\]

and

\[
(n + x)^3 \binom{n + r - 1}{r - 1} = r(r + 1)(r + 2) \binom{n + r + 2}{r + 2} + 3r(r + 1)(x - r - 1) \binom{n + r + 1}{r + 1}
+ (3rx^2 - 3r(2r + 1)x + 3r^3 + 3r^2 + r) \binom{n + r}{r}
+ (x - r)^3 \binom{n + r - 1}{r - 1}.
\]

Hence, differentiating the above three equations at \( s = 0 \), respectively, we have

\[
\Gamma_{r,1}(x) = \Gamma_{r+1}(x)^{r} \Gamma_{r}(x)^{x-r},
\]

\[
\Gamma_{r,2}(x) = \Gamma_{r+2}(x)^{(r+1)} \Gamma_{r+1}(x)^{(2x-2r-1)} \Gamma_{r}(x)^{(x-r)^2},
\]

and

\[
\Gamma_{r,3}(x) = \Gamma_{r+3}(x)^{(r+1)(r+2)} \Gamma_{r+2}(x)^{3r(r+1)(x-r-1)}
\times \Gamma_{r+1}(x)^{3rx^2 - 3r(2r+1)x + (3r^3 + 3r^2 + r)} \Gamma_{r}(x)^{(x-r)^3},
\]

where the period \((1, \ldots, 1)\) is omitted. It remains to calculate

\[
S_{r,1}(x) = \Gamma_{r,1}(x)^{-1} \Gamma_{r,1}(r - x)^{(-1)^{r+1}},
\]

\[
S_{r,2}(x) = \Gamma_{r,2}(x)^{-1} \Gamma_{r,2}(r - x)^{(-1)^{r+2}},
\]

and

\[
S_{r,3}(x) = \Gamma_{r,3}(x)^{-1} \Gamma_{r,3}(r - x)^{(-1)^{r+3}}
\]

from the above formulas for \( \Gamma_{r,\ell}(x) = \Gamma_{r,\ell}(x, (1, \ldots, 1)) \). We note that the periodicity of \( \zeta_r(s, x) \) shows that

\[
\Gamma_{r+1}(r - x) = \Gamma_{r+1}(r + 1 - x) \Gamma_{r}(r - x),
\]

\[
\Gamma_{r+2}(r - x) = \Gamma_{r+2}(r + 1 - x) \Gamma_{r+1}(r - x)
= \Gamma_{r+2}(r + 2 - x) \Gamma_{r+1}(r + 1 - x)^2 \Gamma_{r}(r - x),
\]
and

\[ \Gamma_{r+3}(r-x) = \Gamma_{r+3}(r+1-x) \Gamma_{r+2}(r-x) \]
\[ = \Gamma_{r+3}(r+2-x) \Gamma_{r+2}(r+1-x) \cdot \Gamma_{r+2}(r-x) \]
\[ = \Gamma_{r+3}(r+3-x) \Gamma_{r+2}(r+2-x)^2 \Gamma_{r+1}(r+1-x) \cdot \Gamma_{r+2}(r-x) \]
\[ = \Gamma_{r+3}(r+3-x) \Gamma_{r+2}(r+2-x)^3 \Gamma_{r+1}(r+1-x)^3 \Gamma_r(r-x). \]

Thus, we have

\[
S_{r,1}(x) = \left( \Gamma_{r+1}(x)^r \Gamma_r(x)^{x-r} \right)^{(-1)^{r+1}} \times \left( \Gamma_{r+1}(r-x)^r \Gamma_r(r-x)^{-x} \right)^{(-1)^{r+1}} \\
= (\Gamma_{r+1}(x)^r \Gamma_r(x)^{x-r})^{(-1)^{r+1}} \times (\Gamma_{r+1}(r-x)^r \Gamma_r(r-x)^{-x})^{(-1)^{r+1}} \\
= S_{r+1}(x)^r \Gamma_r(x)^{x-r},
\]

\[
S_{r,2}(x) = (\Gamma_{r+2}(x)^{(r+1)} \Gamma_{r+1}(r-x)^{(2x-2r-1)} \Gamma_r(x)^{(x-r)^2})^{(-1)} \\
\times \left( \Gamma_{r+2}(r-x)^{(r+1)} \Gamma_{r+1}(r-x)^{(2x-2r-1)} \Gamma_r(r-x)^{(x-r)^2} \right)^{(-1)^{r+2}} \\
= \left( \Gamma_{r+2}(x)^{(r+1)} \Gamma_{r+1}(r-x)^{(2x-2r-1)} \Gamma_r(x)^{(x-r)^2} \right)^{(-1)^{r+2}} \\
\times \left( \Gamma_{r+2}(r-x)^{(r+1)} \Gamma_{r+1}(r-x)^{(2x-2r-1)} \Gamma_r(r-x)^{(x-r)^2} \right)^{(-1)^{r+2}} \\
= S_{r+2}(x)^{(r+1)} S_{r+1}(x)^{(2x-2r-1)} \Gamma_r(x)^{(x-r)^2},
\]

and

\[
S_{r,3}(x) = (\Gamma_{r+3}(x)^{(r+1)} \Gamma_{r+2}(x)^{(r+2)} \Gamma_{r+1}(r-x)^{(3r+1)(x-r-1)}) \\
\times \left( \Gamma_{r+3}(x)^{3r^2-3r(2r+1)x+(3r^3+3r^2+r)} \Gamma_r(x)^{(x-r)^3} \right)^{(-1)^{r+3}} \\
\times \left( \Gamma_{r+3}(r-x)^{(r+1)(r+2)} \Gamma_{r+2}(r-x)^{(3r+1)(-x-1)} \Gamma_{r+1}(r-x)^{(3x^2+3x+1)} \Gamma_r(r-x)^{-x^3} \right)^{(-1)^{r+3}} \\
= \left( \Gamma_{r+3}(x)^{(r+1)(r+2)} \Gamma_{r+2}(x)^{(3r+1)(x-r-1)} \Gamma_{r+1}(r-x)^{(3x^2+3x+1)} \Gamma_r(r-x)^{-x^3} \right)^{(-1)^{r+3}} \\
\times \left( \Gamma_{r+3}(r+3-x) \Gamma_{r+2}(r+2-x)^3 \Gamma_{r+1}(r+1-x)^3 \Gamma_r(r-x)^{(r+1)(r+2)} \Gamma_{r+2}(r-x)^{(3r+1)(-x-1)} \Gamma_{r+1}(r-x)^{(3x^2+3x+1)} \Gamma_r(r-x)^{-x^3} \right)^{(-1)^{r+3}}
\]
\[(\Gamma_{r+3}(x))^{-1}\Gamma_{r+3}(r + 3 - x)^{(-1)^{r+3}}r(r+1)(r+2)\]
\[\times (\Gamma_{r+2}(x))^{-1}\Gamma_{r+2}(r + 2 - x)^{(-1)^{r+2}}3r(r+1)(x-r-1)\]
\[\times (\Gamma_{r+1}(x))^{-1}\Gamma_{r+1}(r + 1 - x)^{(-1)^{r+1}}3x^2-3r(2r+1)x+(3r^3+3r^2+r)\]
\[\times (\Gamma_r(x))^{-1}\Gamma_r(r - x)^{(-1)^r}(x-r)^3\]
\[= S_{r+3}(x)^{r(r+1)(r+2)}S_{r+2}(x)^{3r(r+1)(x-r-1)}\]
\[\times S_{r+1}(x)^{3rx^2-3r(2r+1)x+(3r^3+3r^2+r)}S_r(x)(x-r)^3.\]

This shows (6.1d), (6.1e) and (6.1f).

\[\square\]

Remark. When the period \(\omega\) is equal to \((1, \ldots, 1)\), the generalized sine functions \(S_{r,k}(x)\) \((k = 1, 2, 3)\) can be expressed as a product of several power functions of the multiple sine functions \(S_\ell(x)\) \((\ell \geq r)\) as we have seen explicitly in the examples above. Actually, the proof indicates that, for any \(r\) and \(k\), \(\Gamma_{r,k}(x)\) and \(S_{r,k}(x)\) can be expressed in the same manner. See Theorem 1.2 and 1.7 in [KOW] for explicit formulas when \(r = 1\). It is not, however, necessarily true when the period \(\omega\) is generic.

Acknowledgements. This work was partially supported by Grant-in-Aid for Scientific Research (B) N. 15340012.

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