THE GEOMETRY OF FROBENIOIDS II: POLY-FROBENIOIDS

Shinichi MOCHIZUKI

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Abstract. We develop the theory of Frobenioids associated to non-archimedean (mixed-characteristic) and archimedean local fields. In particular, we show that the resulting Frobenioids satisfy the properties necessary to apply the main results of the general theory of Frobenioids. Moreover, we show that the reciprocity map in the non-archimedean case, as well as a certain archimedean analogue of this reciprocity map, admit natural Frobenioid-theoretic translations, which are, moreover, purely category-theoretic, to a substantial extent (i.e., except for the extent to which this category-theoreticity is obstructed by certain ‘Frobenius endomorphisms’ of the relevant Frobenioids). Finally, we show that certain Frobenioids which naturally encode the global arithmetic of a number field may be ‘grafted’ (i.e., glued) onto the Frobenioids associated to non-archimedean and archimedean primes of the number field to obtain ‘poly-Frobenioids’. These poly-Frobenioids encode, in a purely category-theoretic fashion, most of the important aspects of the classical framework of the arithmetic geometry of number fields.

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Introduction

In this paper, we continue to develop the theory of Frobenioids, along the lines of [Mzk5]. By comparison with the theory of [Mzk5], however, here we introduce the following two new themes:

(A) the localization of the global arithmetic Frobenioids associated to number fields in [Mzk5, Example 6.3];

(B) the intertwining of the ‘étale-like’ and ‘Frobenius-like’ portions of the structure of the local Frobenioids introduced in the course of (A).

Both of these themes may be regarded as ‘special cases’ of the more general theme, discussed in [Mzk5, Section I2], of translating – via Frobenioids – into category-theoretic language the familiar scheme-theoretic framework of arithmetic geometry over number fields.

The first theme (A) is also motivated by the ultimate goal of the author, discussed in [Mzk5, Section I3], of developing a sort of ‘arithmetic Teichmüller theory for number fields’. That is to say, if one thinks of Teichmüller theory as a sort of ‘surgery operation’ on the (‘global’) objects under consideration, then it is natural to require that these ‘global objects’ be ‘dissectible into local objects’, so that one is free to perform such ‘local dissections’ in the course of a ‘surgery operation’. (Indeed, here it may be helpful to recall that the classical theory of deformations of proper curves proceeds precisely by ‘dissecting’ a proper curve into ‘local affine opens’ whose deformation theory is ‘trivial’ and then studying how these local affine opens may be glued together in a fashion that deforms the original tautological gluing. Alternatively, in classical complex Teichmüller theory, Teichmüller deformations of a Riemann surface are obtained by locally integrating the square root of a given (global) square differential so as to obtain local holomorphic coordinates, which one then proceeds to deform in a linear, real analytic fashion.)

With regard to the second theme (B), it is interesting to observe that this theme runs, in some sense, in the opposite direction to the theme discussed in [Mzk5, Section I4], of distinguishing or separating out from one another the ‘étale-like’ and ‘Frobenius-like’ portions of the structure of a Frobenioid. That is to say, the content of this theme (B) is that, for certain special types of Frobenioids (i.e., that arise in the localization theory of (A)), although the ‘étale-like’ and ‘Frobenius-like’ portions of the Frobenioid are distinguishable or separable from one another, they are, nonetheless, intertwined in a fairly essential way. In the case of the non-archimedean local Frobenioids that arise in (A), this intertwining may be observed by considering Frobenioid-theoretic versions of the Kummer and reciprocity maps that occur in the classical theory of non-archimedean (mixed-characteristic) local fields. In the case of the archimedean local Frobenioids that arise in (A), this intertwining may be observed by considering certain ‘circles’ – i.e., copies of $S^1$ – that occur (under suitable conditions) in the base category and units of the Frobenioid.

In Section 1, we observe that by applying the theory of [Mzk5], it is an ‘easy exercise’ to define Frobenioids associated to $p$-adic local fields – which we refer to as $p$-adic Frobenioids (cf. Example 1.1) – and to show that these Frobenioids satisfy the properties necessary to apply the main results of the theory of [Mzk5] (cf. Theorem 1.2(i) and Example 1.4(iii)). We also observe that the theory of temperoids in [Mzk2] furnishes (cf. Example 1.3) an example of a natural base category for these Frobenioids – especially, for instance, if one is interested in the arithmetic of hyperbolic curves (cf. Remark 5.6.2) – and discuss how base categories
of Frobenioids at localizations of a number field may be modified slightly so as to reflect the geometry of such localizations (cf. Example 1.4 and Proposition 1.5). One important aspect of the arithmetic of non-archimedean (mixed-characteristic) local fields is the existence of natural Kummer and reciprocity maps (cf. the discussion of theme (B) above). In Section 2 (cf. especially Theorem 2.4), we observe that the Frobenioid-theoretic versions of these maps are ‘category-theoretic’ to a substantial extent, but not completely (cf. Remark 2.4.2) – a situation that reflects the existence of certain ‘Frobenius (self-)equivalences’ in the case of Frobenioids.

In Section 3, we take up the study of Frobenioids associated to archimedean local fields – which we refer to as archimedean Frobenioids (cf. Example 3.3). Again, by applying the general framework of [Mzk5], it is a relatively easy matter to define such Frobenioids. Unlike the non-archimedean case, however, the archimedean case exhibits an additional ‘layer of complexity’ – namely, the existence of non-isotropic objects and non-co-angular morphisms, which reflect the ‘category-theoretic aspects’ of the geometry of the circle \( S^1 \subseteq \mathbb{C}^\times \) (cf. Lemma 3.2 and the Appendix). Indeed, it is precisely the example of archimedean Frobenioids that motivated both the terminology (e.g., ‘isotropic’, ‘co-angular’) and the results of this portion of the general theory of Frobenioids in [Mzk5]. After showing that the archimedean Frobenioids just defined satisfy the properties necessary to apply the main results of the theory of [Mzk5] (cf. Theorem 3.6(i), (ii) and Remark 4.2.1), we proceed in Section 4 to discuss the base categories – which we refer to as angloids (cf. Example 3.3) – obtained by ‘appending an extra copy of the circle to a given base category’ (cf. Proposition 4.1 and Corollary 4.2). This theory of angloids – in particular, the natural homeomorphisms that one obtains between the ‘circle in the base angloid’ and the ‘circle in the group of units of the Frobenioid’ – may, in some sense, be regarded as an archimedean analogue of the theory of the reciprocity map in the non-archimedean case (cf. the discussion of theme (B) above and Remark 4.2.1). Finally, we discuss (the ‘non-circular portion’ of) typical base categories that occur in the archimedean context – namely, temperoids (cf. Example 4.4) and categories naturally associated to hyperbolic Riemann surfaces (cf. Example 4.3).

In Section 5, we take up the issue of ‘grafting’ (i.e., gluing) the local Frobenioids introduced in Section 1 and Section 3 onto the global Frobenioids of [Mzk5, Example 6.3]. We begin by discussing this sort of ‘grafting operation’ for categories in substantial generality, and show, in particular, that, under suitable conditions, one may dissect such a grafted category into its local and global components (cf. Proposition 5.2(iv)). Next, we consider categories – which we refer to as poly-Frobenioids – obtained by grafting Frobenioids. By combining various generalities on ‘grafting’ with the extensive theory of [Mzk5], we obtain the result that ‘most natural aspects’ of the theory of poly-Frobenioids are category-theoretic (i.e., preserved by arbitrary equivalences of categories between poly-Frobenioids satisfying certain conditions – cf. Theorem 5.5). Finally, we discuss in some detail the poly-Frobenioids obtained by considering the localizations of number fields at non-archimedean and archimedean primes (cf. Example 5.6). This example is, in some sense, representative of the entire theory constituted by [Mzk5] and the present paper. In addition, it shows (cf. Remark 5.6.1) that the theory of poly-Frobenioids is strictly more general than the theory of Frobenioids (i.e., that ‘poly-Frobenioids are not just Frobenioids of a certain special type in disguise’).
0. Notation and conventions

In addition to the ‘Notation and conventions’ of [Mzk5, Section 0], we shall employ the following ‘Notation and conventions’ in the present paper.

Categories

If $C$ is a category and $\{\phi_i : A_i \to A\}_{i \in I}$ is a collection of arrows in $C$ all of which have codomain equal to $A$ and non-empty (i.e., non-initial) domains, then we shall say that this collection strongly (respectively, weakly) dissects $A$ if, for every pair of distinct elements $i, j \in I$, $C$ fails to contain a pair of arrows $\psi_i : B \to A_i, \psi_j : B \to A_j$ (respectively, a pair of arrows $\psi_i : B \to A_i, \psi_j : B \to A_j$ such that $\phi_i \circ \psi_i = \phi_j \circ \psi_j$), where $B$ is a non-empty object of $C$. We shall say that an object $A \in \text{Ob}(C)$ is strongly dissectible (respectively, weakly dissectible) if it admits a strongly (respectively, weakly) dissecting pair of arrows. If $A$ is not weakly (respectively, not strongly) dissectible, then we shall say that it is strongly indissectible (respectively, weakly indissectible). Thus, if $A$ is strongly dissectible (respectively, indissectible), then it is weakly dissectible (respectively, indissectible). If every object of $C$ is strongly dissectible (respectively, weakly dissectible; strongly indissectible; weakly indissectible), then we shall say that $C$ is of strongly dissectible (respectively, weakly dissectible; strongly indissectible; weakly indissectible) type.

Observe that by considering appropriate pre-steps and pull-back morphisms as in [Mzk5, Definition 1.3(i), (b), (c)], it follows immediately that (the underlying category of) any Frobenioid over a base category of weakly indissectible (respectively, strongly dissectible; weakly dissectible) type is itself of weakly indissectible (respectively, strongly dissectible; weakly dissectible) type.

Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a functor between categories $\mathcal{C}, \mathcal{D}$. Then we shall say that $\Phi$ is arrow-wise essentially surjective if it is surjective on abstract equivalence classes (cf. [Mzk5, Section 0]) of arrows. If, for every object $B \in \text{Ob}(\mathcal{D})$, there exists an object $A \in \text{Ob}(\mathcal{C})$, together with a morphism $\Phi(A) \to B$ in $\mathcal{D}$, then we shall say that $\Phi$ is relatively initial. If, for every object $A \in \text{Ob}(\mathcal{C})$, the object $\Phi(A)$ is non-initial, then we shall say that $\Phi$ is totally non-initial.

Let $\mathcal{C}$ be a category. If $\phi : A \to B$ is a morphism of $\mathcal{C}$, write

$$\mathcal{C}_\phi$$

for the category whose objects are factorizations

$$A \xrightarrow{\alpha} C \xrightarrow{\beta} B$$

of $\phi$, and whose morphisms

$$\{A \xrightarrow{\alpha_1} C_1 \xrightarrow{\beta_1} B\} \to \{A \xrightarrow{\alpha_2} C_2 \xrightarrow{\beta_2} B\}$$

are morphisms $\psi : C_1 \to C_2$ of $\mathcal{C}$ such that $\psi \circ \alpha_1 = \alpha_2, \beta_2 \circ \psi = \beta_1$. Also, we shall write

$$\mathcal{C}_\phi \subseteq \mathcal{C}$$

for the subcategory of monomorphisms of $\mathcal{C}$. We shall say that a morphism $\phi : A \to B$ of $\mathcal{C}$ (i.e., a monomorphism of $\mathcal{C}$) is totally ordered if the category of factorizations $\mathcal{C}_\phi^{-}$ of $\phi$ is
equivalent to a category of the form Order(E) (cf. [Mzk5, Section 0]), where E is a totally ordered set (so E may be recovered as the set of isomorphism classes of $C_\phi$ with the order relation determined by the arrows of $C_\phi$). If, in this situation, E satisfies the property that for any $a, b \in E$ such that $a < b$, there exists a $c \in E$ such that $a < c < b$, then we shall say that $\phi$ is continuously ordered. We shall say that a morphism of $C$ (i.e., a monomorphism of C) is quasi-totally ordered (respectively, quasi-continuously ordered) if it is an isomorphism or factors as a finite composite of totally ordered (respectively, continuously ordered) morphisms. If every totally ordered morphism (respectively, continuously ordered morphism) of $C$ is an isomorphism, then we shall say that $C$ is of strictly partially ordered (respectively, discontinuously ordered) type. If every morphism of $C$ is totally ordered (respectively, quasi-totally ordered; continuously ordered; quasi-continuously ordered), then we shall say that $C$ is of totally ordered (respectively, quasi-totally ordered; continuously ordered; quasi-continuously ordered) type.

Let $C, D, E$ be categories and let $\Phi : C \to E$ and $\Psi : D \to E$ be functors. Then we shall refer to as the category obtained by grafting $D$ onto $C$ (via $\Phi, \Psi$) the category

$$C \cdot E \cdot D$$

whose objects are objects of $C$ or objects of $D$, and whose morphisms are defined as follows. The morphisms $B \to A$, where $A, B \in \text{Ob}(C)$ (respectively, $A, B \in \text{Ob}(D)$), are simply the morphisms $B \to A$ in $C$ (respectively, $D$); such morphisms will be referred to as homogeneous. The morphisms $B \to A$, where $A \in \text{Ob}(C), B \in \text{Ob}(D)$, are the morphisms $\Psi(B) \to \Phi(A)$ in $E$; such morphisms will be referred to as heterogeneous. If $A \in \text{Ob}(C), B \in \text{Ob}(D)$, then there are no morphisms $A \to B$. Composition of morphisms is defined in the equivalent fashion. This completes the definition of $C \cdot E \cdot D$. Thus, there are natural full embeddings

$$C \hookrightarrow C \cdot E \cdot D, \quad D \hookrightarrow C \cdot E \cdot D,$$

so we may regard $C, D$ as full subcategories of $C \cdot E \cdot D$.

Let $I$ be a set. For each $i \in I$, let $C_i$ be a category. Then we shall write

$$\bigsqcup_{i \in I} C_i$$

for the category whose objects are objects of one of the $C_i$ and whose morphisms $A \to B$ are the morphisms of $C_i$ whenever $A, B \in \text{Ob}(C_i)$ (so there are no morphisms $A \to B$ whenever $A \in \text{Ob}(C_i), B \in \text{Ob}(C_j), i \neq j$). Thus, we have a natural functor

$$\bigsqcup_{i \in I} C_i \to \prod_{i \in I} C_i^\uparrow$$

(where ‘$\uparrow$’ is as in [Mzk5, Section 0]) given by sending an object $A \in \text{Ob}(C_i)$ to the object of the product category on the right whose component labeled $i$ is $A$, and whose other components are initial objects of the categories $C_j$, for $j \neq i$.

1. Non-archimedean primes

In this section, we define and study the basic properties of certain Frobenioids naturally associated to non-archimedean (mixed-characteristic) local fields.
Let \( p \in \text{Primes} \) be a prime number (cf. [Mzk5, Section 0]); write

\[ D_0 \]

for the full subcategory of connected objects of the Galois category of finite étale coverings of \( \text{Spec}(\mathbb{Q}_p) \) (cf. [Mzk5, Section 0]). Thus, \( D_0 \) is a connected, totally epimorphic category, which is of \( FSM \)-type, hence also of \( FSMFF \)-type (cf. [Mzk5, Section 0]).

**Example 1.1.** \((p\text{-adic Frobenioids})\)

(i) If \( \text{Spec}(K) \in \text{Ob}(D_0) \) (i.e., \( K \) is a finite extension of \( \mathbb{Q}_p \)), then write \( \mathcal{O}_K \subseteq K \) for the ring of integers of \( K \), \( \mathcal{O}_K^\times \subseteq \mathcal{O}_K \) for the group of units, and \( \mathcal{O}_K^{\leq} \subseteq \mathcal{O}_K \) for the multiplicative monoid of non-zero elements. Also, we shall use the following notation:

\[ \text{ord} \left( \mathcal{O}_K^{\leq} \right) \overset{\text{def}}{=} \mathcal{O}_K^{\leq} / \mathcal{O}_K^\times \subseteq \text{ord}(K^\times) \overset{\text{def}}{=} K^\times / \mathcal{O}_K^\times. \]

Thus, the assignment

\[ \text{Ob}(D_0) \ni \text{Spec}(K) \mapsto \text{ord}(\mathcal{O}_K^{\leq}) \overset{\text{rlf}}{=} R \geq 0 \in \text{Ob}(\mathcal{M}_{\text{Mon}}) \]

(where \( \mathcal{M}_{\text{Mon}} \) is as in [Mzk5, Section 0] and the superscript `rlf` is discussed in [Mzk5, Definition 2.4(i)]) determines a monoid \( \Phi_0 \) on \( D_0 \) (cf. [Mzk5, Definition 1.1(ii)]), which is easily verified to be non-dilating (cf. [Mzk5, Definition 1.1(i), (ii)]) and \( perf\)-factorial (cf. [Mzk5, Definition 2.4(i)]); the assignment

\[ \text{Ob}(D_0) \ni \text{Spec}(K) \mapsto K^\times \in \text{Ob}(\mathcal{M}_{\text{Mon}}) \]

determines a group-like (cf. [Mzk5, Definition 1.1(i), (ii)]) monoid \( \mathbb{B}_0 \) on \( D_0 \) together with a natural homomorphism of monoids

\[ \mathbb{B}_0 \rightarrow \Phi_0^{\text{gp}} \]

(i.e., by considering the natural surjection \( K^\times \rightarrow \text{ord}(K^\times) \)). Thus, by [Mzk5, Theorem 5.2(ii)], this data determines a (model) Frobenioid \( C_0 \)

which is easily verified to be of \emph{rationally standard} type (cf. [Mzk5, Theorem 5.2(iii)]) over a slim base category \( D_0 \) (cf. [Mzk1, Theorem 1.1.1(ii)] and [Mzk5, Theorem 6.2(iv) and Theorem 6.4(i)]). If \( \Lambda \) is a monoid type (cf. [Mzk5, Section 0]), then we define the notation \( C_0^\Lambda \) as

\[ C_0^\mathbb{Z} \overset{\text{def}}{=} C_0, \quad C_0^\mathbb{Q} \overset{\text{def}}{=} C_0^{\text{pf}}, \quad C_0^\mathbb{R} \overset{\text{def}}{=} C_0^{\text{rlf}} \]

(cf. [Mzk5, Propositions 3.2 and 5.3]). Thus, \( C_0^\mathbb{Q} \) (respectively, \( C_0^\mathbb{R} \)) is the model Frobenioid associated to the data

\[ \Phi_0^\mathbb{Q} \overset{\text{def}}{=} \Phi_0, \quad \mathbb{B}_0^\mathbb{Q} \overset{\text{def}}{=} \mathbb{B}_0^{\text{pf}} \overset{\Phi_0^{\text{gp}}}{} 
\]

(respectively, \( \Phi_0^\mathbb{R} \overset{\text{def}}{=} \Phi_0, \quad \mathbb{B}_0^\mathbb{R} \overset{\text{def}}{=} \mathbb{R} \cdot \Phi_0^{\text{birat}} \overset{}{\Phi_0^{\text{gp}}}) \)

(cf. [Mzk5, Proposition 5.5(iv)]). Note that the monoid \( \Phi_0^{\mathbb{Z}} \) is isomorphic to the \emph{constant monoid} determined by \( R \geq 0 \in \text{Ob}(\mathcal{M}_{\text{Mon}}) \) on \( D_0 \); in particular, the Frobenioid \( C_0^\mathbb{R} \) may be naturally identified with the \emph{elementary Frobenioid} determined by this constant monoid. Also,
we observe in passing that an object of \(C_0\) lying over \(\text{Spec}(K) \in \text{Ob}(D_0)\) may be thought of as a metrized line bundle on \(\text{Spec}(K)\) (i.e., a \(K\)-torsor \(V\) equipped with a ‘metric’, i.e., a choice of an element

\[
\mu_V \in \text{ord}(V)^{\text{rlf}}
\]

where we write \(\text{ord}(V) \overset{\text{def}}{=} V/\mathcal{O}_K^\times\) (\(\cong \mathbb{Z}\)), and the superscript ‘rlf’ denotes the result of changing the structure group of the \(\text{ord}(K^\times)\)-torsor \(\text{ord}(V)\) via the homomorphism \(\text{ord}(K^\times) \to \text{ord}(K^\times) \otimes_{\mathbb{Z}} \mathbb{R}\) – cf. \([\text{Mzk5}, \text{Theorem 5.2(i)}]\). Then a morphism in \(C_0\) from a metrized line bundle \((V, \mu_V)\) lying over \(\text{Spec}(K) \in \text{Ob}(D_0)\) to a metrized line bundle \((W, \mu_W)\) lying over \(\text{Spec}(L) \in \text{Ob}(D_0)\) consists of the following data: (a) a morphism \(\text{Spec}(K) \to \text{Spec}(L)\) in \(D_0\); (b) an element \(d \in \mathbb{N}_{\geq 1}\); (c) an isomorphism of \(K\)-torsors \(V^\otimes d \isom W|_K\) (where the ‘\(d\)th tensor power’ and ‘restriction from \(L\) to \(K\)’ are defined in the expected way) that is ‘integral’ with respect to the metrics \(\mu_{V^\otimes d}, \mu_{W|_K}\) induced on \(V^\otimes d, W|_K\) by \(\mu_V, \mu_W\), respectively (i.e., in the sense that \(\mu_{V^\otimes d} \in \text{ord}(V^\otimes d)^{\text{rlf}}\) maps to an element in the \(\text{ord}(\mathcal{O}_K^{\varphi_d})^{\text{rlf}}\)-orbit of \(\mu_{W|_K} \in \text{ord}(W|_K)^{\text{rlf}}\)).

(ii) Let \(D\) be a connected, totally epimorphic category, and let \(D \to D_0\) be a functor. Let

\[
\Phi \subseteq \Phi_0|_D (=\Phi_0|_D)
\]

be a monoprime (cf. \([\text{Mzk5}, \text{Section 0}]\); the convention of \([\text{Mzk5}, \text{Definition 1.1(ii)}]\)) subfunctor in monoids such that the image of the resulting homomorphism of group-like monoids on \(D\)

\[
\mathbb{B} \overset{\text{def}}{=} \mathbb{B}_0|_D \times (\Phi_0|_D)^{\text{gp}}|_D \Phi^{\text{gp}} \to \Phi^{\text{gp}}
\]

determines a subfunctor in non-zero monoids of \(\Phi^{\text{gp}}\) (i.e., for every \(A \in \text{Ob}(D)\), the homomorphism \(\mathbb{B}(A) \to \Phi^{\text{gp}}(A)\) is non-zero). We shall refer to a \(p\)-adic Frobenioid as the Frobenioid

\[
\mathcal{C}
\]

that arises as the model Frobenioid associated to this data \(\Phi, \mathbb{B} \to \Phi^{\text{gp}}\) (cf. \([\text{Mzk5}, \text{Theorem 5.2(ii)}]\)). If it holds that

\[
\Phi(K) \subseteq \Lambda - \text{ord}(\mathbb{Q}_p^\times) \quad (\subseteq \text{ord}(K^\times) \otimes_{\mathbb{Z}} \mathbb{R} = (\text{ord}(\mathcal{O}_K^{\varphi_d})^{\text{rlf}})^{\text{gp}})
\]

(respectively, \(\text{ord}(K^\times) \subseteq \Phi(K) \quad (\subseteq \text{ord}(K^\times) \otimes_{\mathbb{Z}} \mathbb{R} = (\text{ord}(\mathcal{O}_K^{\varphi_d})^{\text{rlf}})^{\text{gp}})\))

for every \(\text{Spec}(K) \in \text{Ob}(D_0)\), then we shall say that \(\Phi\) is absolutely primitive (respectively, fieldwise saturated).

**Theorem 1.2.** (Basic properties of \(p\)-adic Frobenioids) In the notation and terminology of Example 1.1 we have the following.

(i) If \(\Lambda = \mathbb{Z}\) (respectively, \(\Lambda = \mathbb{R}\)), then \(\mathcal{C}\) is of unit-profinite (respectively, unit-trivial) type. For arbitrary \(\Lambda\), the Frobenioid \(\mathcal{C}\) is of isotropic, model, \(\text{Aut}\)-ample, End-ample, and quasi-Frobenius-trivial type, but not of group-like type. If \(D\) is of FSMFF-type, then \(\mathcal{C}\) is of rationally standard type.

(ii) Let \(A \in \text{Ob}(\mathcal{C})\) and \(A_D \overset{\text{def}}{=} \text{Base}(A) \in \text{Ob}(D)\). Write \(A_0 \in \text{Ob}(D_0)\) for the image of \(A_D\) in \(D_0\). Then the natural action of \(\text{Aut}_D(A)\) on \(\mathcal{O}^{\varphi}(A), \mathcal{O}^\times(A)\) factors through \(\text{Aut}_{D_0}(A_0)\). If, moreover, \(\Lambda \in \{\mathbb{Z}, \mathbb{Q}\}\), then this factorization determines a faithful action of the image of \(\text{Aut}_C(A)\) in \(\text{Aut}_{D_0}(A_0)\) on \(\mathcal{O}^{\varphi}(A), \mathcal{O}^\times(A)\).
If \( D \) admits a terminal object, then \( C \) admits a pseudo-terminal object.

If \( D \) is slim, and \( \Lambda \in \{\mathbb{Z}, \mathbb{R}\} \), then \( C \) is also slim.

Suppose that \( \Phi \) is absolutely primitive. Then \( C \) is of base-trivial type. Moreover, the element \( p \in \mathbb{Q}_p^\times \) determines a characteristic splitting (cf. [Mzk5, Definition 2.3]) on \( C \).

**Proof.** First, we consider assertion (i). In light of the definition of \( C \) as a model Frobenioid, it follows from [Mzk5, Theorem 5.2(ii)], that \( C \) is of isotropic and model type. Since, for \( A_D \in \text{Ob}(D) \), the monoid \( \text{End}_D(A_D) \) acts trivially on \( \Phi(A_D) \), it follows immediately from the construction of a model Frobenioid (cf. [Mzk5, Theorem 5.2(i)]) that \( C \) is of \( \text{Aut-ample} \), \( \text{Aut}^{\text{sub-ample}} \), and \( \text{End-ample} \) type. Consideration of the kernel of the homomorphism of group-like monoids on \( D \)

\[
\mathcal{B} \rightarrow \Phi^{\text{gp}}
\]

reveals that if, moreover, \( \Lambda = \mathbb{Z} \) (respectively, \( \Lambda = \mathbb{R} \)), then \( C \) is of unit-profinite (respectively, unit-trivial) type; since the image monoids of this homomorphism are assumed to be non-zero, and \( \Phi \) is monoprime (cf. Example 1.1(ii)), it follows immediately that \( C \) is of quasi-Frobenius-trivial and (strictly) rational type, but not of group-like type. Thus, it follows immediately from [Mzk5, Theorem 5.2(iii)], that if \( D \) is of \( \text{FSMFF-type} \), then \( C \) is of rationally standard type. This completes the proof of assertion (i).

Assertions (ii) and (iii) follow immediately from the construction of a model Frobenioid (cf. Example 1.1(i) and (ii); [Mzk5, Theorem 5.2(i)]).

Assertion (iv) follows formally from [Mzk5, Proposition 1.13(iii)] (since, by assertion (i) of the present Theorem 1.2, ‘condition (b)’ of loc. cit. is always satisfied by objects of \( C \)).

Finally, assertion (v) follows by observing that if \( \Phi \) is absolutely primitive, then the homomorphism \( \mathcal{B} \rightarrow \Phi^{\text{gp}} \) considered above is surjective. Thus, it follows formally from the computation of isomorphism classes of \( C \) given in [Mzk5, Theorem 5.1(i)], that \( C \) is of base-trivial type; moreover, it is immediate that the image of \( p \in \mathbb{Q}_p^\times \) in \( K^\times \) (for \( \text{Spec}(K) \in \text{Ob}(D_0) \)) determines a characteristic splitting. This completes the proof of Theorem 1.2. \( \square \)

**Remark 1.2.1.** We observe in passing that if \( \Phi \) is absolutely primitive, then by Theorem 1.2(i) and (v), it follows that \( C \) admits unit-linear Frobenius functors as in [Mzk5, Proposition 2.5(iii)] (where one takes the ‘\( \Lambda \)’ of loc. cit. to be \( \mathbb{Z} \)), and unit-wise Frobenius functors as in [Mzk5, Corollary 2.6]. If, moreover, \( \Lambda = \mathbb{Z} \), then \( C \) admits unit-wise Frobenius functors as in [Mzk5, Proposition 2.9(ii)].

**Remark 1.2.2.** Consider the functors ‘\( \mathcal{O}^\triangleright (-) \)’, ‘\( \mathcal{O}^\times (-) \)’ on \( D \) determined by \( C \) (cf. [Mzk5, Proposition 2.2(ii) and (iii)]). Let

\[
u_D : (\text{Ob}(D) \ni A_D \mapsto u_{A_D} \in \mathcal{O}^\times(A_D))
\]

be a ‘section’ of the functor ‘\( \mathcal{O}^\times (-) \)’ (i.e., every arrow \( \phi : A_D \rightarrow B_D \) of \( D \) maps \( u_{B_D} \mapsto u_{A_D} \)). Suppose that \( \Phi \) is absolutely primitive and that \( \Lambda = \mathbb{Z} \) (so \( \mathcal{O}^\triangleright(A_D)^{\text{char}} = \mathcal{O}^\triangleright(A_D)/\mathcal{O}^\times(A_D) \cong \mathbb{Z}_{\geq 0} \)). Then if \( \tau \) is a characteristic splitting on \( C \), then it follows immediately that \( u_D \) determines a new characteristic splitting ‘\( u_D \cdot \tau \)’ by taking

\[
u_D \cdot \tau \equiv \left(\nu_{A_D} \cdot \eta\right)_{\geq 0} \subseteq \mathcal{O}^\triangleright(A_D)
\]

for \( \eta \in \tau(A_D) \subseteq \mathcal{O}^\triangleright(A_D) \) a generator of \( \tau(A_D) \). Moreover, one obtains a unique automorphism \( \mathcal{U} \) of the data

\[
(\Phi, \mathcal{B} \rightarrow \Phi^{\text{gp}})
\]
which is the identity on \( \mathcal{O}^\infty(\cdot) \) (regarded as a subfunctor of \( \mathcal{B} \)) and \( \Phi \), but which maps \( \tau \) (where we regard \( \mathcal{O}^\infty(\cdot) \) as a subfunctor of \( \mathcal{B} \) such that \( \mathcal{B} = \mathcal{O}^\infty(\cdot)_{\mathcal{B}} \)) to \( u_{\mathcal{D}} \cdot \tau \). Thus, \( \mathcal{U} \) induces a self-equivalence

\[
\Psi_{\mathcal{U}} : \mathcal{C} \to \mathcal{C}
\]

of the model Frobenioid \( \mathcal{C} \). In particular, \( \Psi_{\mathcal{U}} \) exhibits an example of a situation where \( p \in \mathbb{Q}^\times_p \) is mapped to some \( p \cdot u \in \mathbb{Q}^\times_p \), where \( u \in \mathbb{Z}^\times_p \) — a situation which, of course, never arises in conventional scheme theory (cf. the various Frobenius functors discussed in Remark 1.2.1 and [Mzk5, Introduction]).

**Example 1.3.** (Connected quasi-temperoids)

(i) If \( \Pi \) is a topological group, then we shall write

\[
\mathcal{B}^{\text{temp}}(\Pi)
\]

for the category whose objects are countable (i.e., of cardinality \( \leq \) the cardinality of the set of natural numbers), discrete sets equipped with a continuous \( \Pi \)-action and whose morphisms are morphisms of \( \Pi \)-sets (cf. [Mzk2, Section 3]). If \( \Pi \) may be written as an inverse limit of an inverse system of countable discrete topological groups, then we shall say that \( \Pi \) is tempered (cf. [Mzk2, Definition 3.1(i)]). Thus, if \( \Pi \) is profinite, then it is tempered, and there is a natural equivalence of categories \( \mathcal{B}^{\text{temp}}(\Pi)^0 \sim \mathcal{B}(\Pi)^0 \) (where the superscript ‘0’ is as in [Mzk5, Section 0]). If \( Z_\Pi(H) = \{1\} \) for every open subgroup \( H \subseteq \Pi \), then we shall say that \( \Pi \) is temp-slim (cf. [Mzk2, Definition 3.4(ii)]). Any category equivalent to a category of the form \( \mathcal{B}^{\text{temp}}(\Pi) \) for some tempered topological group \( \Pi \) will be referred to as a connected temperoid (cf. [Mzk2, Definition 3.1(ii)]). We recall that if \( \Pi \) is a tempered topological group, then \( \Pi \) is temp-slim if and only if the category \( \mathcal{B}^{\text{temp}}(\Pi) \) is slim (cf. [Mzk2, Corollary 3.3 and Remark 3.4.1]). In a similar vein, if \( \Pi \) is residually finite, then \( \mathcal{B}^{\text{temp}}(\Pi)^0 \) is Frobenius-slim (cf. [Mzk5, Remark 3.1.2]). If \( \Pi^0 \subseteq \Pi \) is an open subgroup, then we shall write

\[
\mathcal{B}^{\text{temp}}(\Pi, \Pi^0) \subseteq \mathcal{B}^{\text{temp}}(\Pi)
\]

for the full subcategory of objects that admit a morphism to the object \( \Pi/\Pi^0 \) (i.e., the set of cosets \( \Pi/\Pi^0 \) equipped with its natural \( \Pi \)-action from the left) of \( \mathcal{B}^{\text{temp}}(\Pi) \). Note that there is a natural equivalence of categories \( \mathcal{B}^{\text{temp}}(\Pi^0) \sim \mathcal{B}^{\text{temp}}(\Pi)/\Pi^0 \); in particular, we obtain natural functors \( \mathcal{B}^{\text{temp}}(\Pi^0) \to \mathcal{B}^{\text{temp}}(\Pi) \), \( \mathcal{B}^{\text{temp}}(\Pi^0)^0 \to \mathcal{B}^{\text{temp}}(\Pi)^0 \). A connected quasi-temperoid is a category that is equivalent to a category of the form \( \mathcal{B}^{\text{temp}}(\Pi, \Pi^0) \), where \( \Pi \) is a tempered topological group, and \( \Pi^0 \subseteq \Pi \) is an open subgroup (cf. [Mzk2, Definition A.1(i)])). One verifies immediately that if \( \mathcal{E} \) is a connected quasi-temperoid, then the category \( \mathcal{E}^0 (\subseteq \mathcal{E}) \) is connected, totally epimorphic, of strongly indissectible type (cf. Section 0), and of FSM-type (indeed, every monomorphism of \( \mathcal{E}^0 \) is an isomorphism), hence, in particular, of FSMFF-type (cf. [Mzk5, Section 0]).

(ii) Let \( \phi : \Pi_1 \to \Pi_2 \) be an open homomorphism of tempered topological groups (i.e., \( \phi \) is a continuous homomorphism, \( \phi(\Pi_1) \) is an open subgroup of \( \Pi_2 \), and \( \phi \) induces an isomorphism of topological groups \( \Pi_1/\ker(\phi) \to \phi(\Pi_1) \)). Then observe that \( \phi \) induces a natural functor

\[
\phi_* : \mathcal{B}^{\text{temp}}(\Pi_1)^0 \to \mathcal{B}^{\text{temp}}(\Pi_2)^0
\]
by composing the functor $B_{\text{temp}}(\phi(\Pi_1))^0 \rightarrow B_{\text{temp}}(\Pi_2)^0$ of (i) with the functor $B_{\text{temp}}(\Pi_1)^0 \rightarrow B_{\text{temp}}(\phi(\Pi_1))^0$ obtained by mapping a $\Pi_1$-set $E$ to the $\phi(\Pi_1)$-set $E' \overset{\text{def}}{=} E / \text{Ker}(\phi)$ given by taking the set of $\text{Ker}(\phi)$-orbits of $E$. When $\phi$ is surjective, one verifies immediately that the functor $\rho^*$ is left adjoint to the pull-back functor $B_{\text{temp}}(\Pi_2)^0 \rightarrow B_{\text{temp}}(\Pi_1)^0$ (i.e., obtained by composing the $\Pi_2$-action on a $\Pi_2$-set $F$ with $\phi$ so as to obtain a $\Pi_1$-set $F$).

(iii) Let $F = \mathbb{Q}_p$ or $\mathbb{R}$ and let $\tilde{F}$ be an algebraic closure of $F$. Then by (i) and (ii), any open homomorphism $\Pi \rightarrow Q$ from a tempered topological group $\Pi$, equipped with an open subgroup $\Pi^0 \subseteq \Pi$, to a quotient $G_F \rightarrow Q$ of the absolute Galois group $G_F \overset{\text{def}}{=} \text{Gal}(\tilde{F} / F)$ of $F$ determines a functor

$$B_{\text{temp}}(\Pi, \Pi^0)^0 \hookrightarrow B_{\text{temp}}(\Pi)^0 \rightarrow B_{\text{temp}}(Q)^0 \hookrightarrow B_{\text{temp}}(G_F)^0$$

(where the middle arrow ‘$\hookrightarrow$’ is a functor as in (ii)) between connected, totally epimorphic categories of FSM-type. In particular, when $F = \mathbb{Q}_p$, if we set $D \overset{\text{def}}{=} B_{\text{temp}}(\Pi, \Pi^0)^0$, then we obtain a functor $D \rightarrow D_0 = B_{\text{temp}}(G_{\mathbb{Q}_p})^0$ (cf. Example 1.1(ii)) which satisfies the hypotheses of Theorem 1.2(i). That is to say, in this case, the main results of the theory of [Mzk5] may be applied to the $p$-adic Frobenioids of Example 1.1(ii).

Example 1.4. (Localizations of number fields)
(i) Often it is of interest to relate finite extensions of a local field to finite extensions of number fields. This may be achieved as follows. Let $\tilde{F}$ be a Galois extension of a number field $F$ and let $v$ be a (not necessarily non-archimedean!) valuation of $F$. Write $F_v$ for the completion of $F$ at $v$, $\tilde{F}_v$ for the Galois extension of $F_v$ determined by $\tilde{F} / F$, $D_v \subseteq \text{Gal}(\tilde{F} / F)$ for the decomposition group of $v$ in $\text{Gal}(\tilde{F} / F)$ (which is well-defined up to conjugation),

$$\mathcal{Q}_0$$

for the full subcategory of the Galois category of finite étale coverings of $\text{Spec}(F)$ determined by objects of the form $\text{Spec}(F') \rightarrow \text{Spec}(F)$, where $F' \subseteq \tilde{F}$ is a finite field extension of $F$, and

$$\mathcal{P}_0$$

for the full subcategory of connected objects of the Galois category of finite étale coverings of $\text{Spec}(F_v)$ determined by objects of the form $\text{Spec}(F') \rightarrow \text{Spec}(F_v)$, where $F' \subseteq \tilde{F}_v$ is a finite field extension of $F_v$. Thus, restriction to $\text{Spec}(F_v)$ determines a natural functor

$$\rho : \mathcal{Q}_0 \rightarrow \mathcal{P}_0^\perp$$

(where the superscript ‘$\perp$’ is as in [Mzk5, Section 0]; thus, $\mathcal{P}_0^\perp$ is naturally equivalent to the full subcategory of the Galois category of finite étale coverings of $\text{Spec}(F_v)$ consisting of objects whose connected components form objects of $\mathcal{P}_0$). Write

$$\mathcal{E}_0$$

for the category whose objects are triples

$$(P, Q, \iota : P \rightarrow \rho(Q)),$$
where $P \in \text{Ob}(P_0)$, $Q \in \text{Ob}(Q_0)$, and $\iota$ is an isomorphism in $P_0^\perp$ of $P$ onto a connected component of $\rho(Q)$ -- and whose morphisms

$$(P, Q, \iota : P \to \rho(Q)) \to (P', Q', \iota' : P' \to \rho(Q'))$$

are pairs of morphisms $P \to P'$, $Q \to Q'$ in $P_0$, $Q_0$, respectively, that are compatible with $\iota$, $\iota'$. Thus, $E_0$ may be thought of as the category of `connected finite étale coverings of Spec($F_v$) equipped with a localization morphism (that satisfies certain properties) to a connected finite étale covering of Spec($F$) determined by a finite subextension of the extension $\overline{F}/F$'.

(ii) One verifies immediately that the category $E_0$ is connected and totally epimorphic. Also, we have natural functors $E_0 \to P_0$, $E_0 \to Q_0$. One verifies immediately that $E_0 \to P_0$ is faithful and arrow-wise essentially surjective (cf. Section 0), and that if $E_0 \in \text{Ob}(E_0)$ projects to an object $P_0 \in \text{Ob}(P_0)$, then the functor $E_0 \to P_0$ induces a bijection $\text{Aut}_{E_0}(E_0) \to \text{Aut}_{P_0}(P_0)$. Note that finite extensions $F_1 \subseteq F_2 \subseteq \overline{F}$ of $F$, together with a valuation $v_2$ of $F_2$ lying over a valuation $v_1$ of $F_1$ which, in turn, lies over $v$ such that the local extension $(F_2)_{v_2}/(F_1)_{v_1}$ is trivial yield examples of FSM-morphisms of $E_0$ which are not isomorphisms. Conversely, one verifies immediately that every FSM-morphism of $E_0$ which is not an isomorphism arises in this way. Thus, the category $E_0$ is not (in general) of FSM-type. On the other hand, one verifies immediately that $E_0$ is of FSMFF-type.

(iii) Suppose that $v$ is non-archimedean and lies over a rational prime $p$. Then one has an evident natural functor $P_0 \to D_0$. Moreover, by applying Proposition 1.5(ii) and (iii), below, given a connected, totally epimorphic category $P$ and a functor $P \to P_0$, one obtains a connected, totally epimorphic category $E \overset{\text{def}}{=} P \times_{P_0} E_0$ equipped with a functor $E \to P_0 \to D_0$ (cf. category `D', the functor `D $\to D_0$' of Example 1.1(iii)). If, moreover, $P$ is of FSM-type (cf., e.g., Example 1.3(i)), then $E$ is of FSMFF-type (cf. Proposition 1.5(viii)), hence, in particular, satisfies the hypotheses of Theorem 1.2(i). That is to say, in this case, the main results of the theory of [Mzk5] may be applied to the $p$-adic Frobenioids of Example 1.1(ii). Also, we observe that if $P$ is Frobenius-slim or slim (cf., e.g., Example 1.3(i)), then so is $E$ (cf. Proposition 1.5(iv)); if $P$ is of strongly indissectible type (cf., e.g., Example 1.3(i)), then so is $E$ (cf. Proposition 1.5(vii)).

**Proposition 1.5.** (Localizations of number fields) Let $P_0$, $E_0$ be as in Example 1.4, let $P$ be a category and let $P \to P_0$ be a functor. Denote by

$$E \overset{\text{def}}{=} P \times_{P_0} E_0$$

the categorical fiber product (cf. [Mzk5, Section 0]). (Thus, we have $1$-compatible natural projection functors $E \to P$, $E \to E_0$, $E \to P_0$.) We shall refer to a morphism of $E$ that projects to an isomorphism of $P$ as a $P$-isomorphism. Then we have the following.

(i) The natural projection functor $E \to P$ is faithful and arrow-wise essentially surjective. Moreover, if $E_1$, $E_2 \in \text{Ob}(E)$ project to $P_1$, $P_2 \in \text{Ob}(P)$, respectively, then the functor $E \to P$ induces a bijection

$$\lim_{E_3 \to E_1} \text{Hom}_E(E_3, E_2) \sim \text{Hom}_P(P_1, P_2),$$

where the inductive limit is over $P$-isomorphisms $E_3 \to E_1$. That is to say, `$P$ may be reconstructed from $E$ by inverting the $P$-isomorphisms'.
Let $P$.

**Proof.** First, we consider assertion (i). The faithfulness and arrow-wise essential surjectivity of $E \to P$ follow immediately from the corresponding properties of $E_0 \to P_0$ (cf. Example 1.4(ii)). The bijection of assertion (i) follows immediately from the definitions and the easily verified observation that such a bijection exists when $P \to P_0$ is the identity functor on $P_0$.

Next, we consider assertion (ii). Since $E \to P$ is essentially surjective (cf. assertion (i)), it is immediate that the connectedness of $E$ implies that of $P$. On the other hand, note that when $P$ is a one-morphism category, the connectedness of $E$ follows immediately from the bijection of assertion (i). If $P$ is an arbitrary connected category (equipped with a functor $P \to P_0$), then the connectedness of $E$ follows immediately from the arrow-wise essential surjectivity of $E \to P$ (cf. assertion (i)) by considering the ‘fibers’ of the functor $E \to P$ (which correspond to the case where $P$ is a one-morphism category).

Assertion (iii) follows immediately from the definitions, the total epimorphicity of $E_0$ and $P_0$, and the bijection of assertion (i).

Assertion (iv) follows immediately from the bijection of assertion (i) and the bijection $\{\text{Aut}_{P_0}(E_0) \to \text{Aut}_{P_0}(P_0)\}$ observed in Example 1.4(ii).

Assertions (v) and (vii) follow from the faithfulness of $E \to P$ (cf. assertion (i)), together with the bijection of assertion (i), by a routine verification.

Assertion (vi) follows formally from assertion (v). This completes the proof of assertions (i)--(vii).

Next, we consider assertion (viii). Since $P$ is totally epimorphic, it follows that if a composite of two morphisms $\alpha$, $\beta$ of $P$ is an isomorphism, then both $\alpha$ and $\beta$ are
isomorphisms (cf. the discussion of [Mzk5, Section 0]); a similar statement holds for \( P_0 \). Thus, by applying the equivalence of assertion (vi) both to the original \( P \) and to the case where \( P \to P_0 \) is the identity functor on \( P_0 \), we conclude that a morphism of \( E \) is an FSMI-morphism if and only if it is a \( P \)-isomorphism that projects to an FSMI-morphism of \( E_0 \). Thus, the fact that \( E \) is of \( FSMFF \)-type follows formally from the fact that \( E_0 \) is of \( FSMFF \)-type (cf. Example 1.4(ii)). This completes the proof of assertion (viii).

Finally, we observe that the portion of assertion (ix) concerning factorizations follows immediately from the fact that the assumption on \( D \) implies that if \( \alpha_{P_0} \circ \beta_{P_0} \) is any composite morphism of \( P_0 \), then either \( \alpha_{P_0} \) or \( \beta_{P_0} \) is an isomorphism. The portion of assertion (ix) concerning irreducibility then follows formally; the portion of assertion (ix) concerning totally ordered morphisms follows from the portion of assertion (ix) concerning factorizations, together with the portion of assertion (v) concerning monomorphisms (cf. Section 0).

2. The Kummer and reciprocity maps

The purpose of this section is to consider classical Kummer theory, as well as the reciprocity map of classical local class field theory, in the context of the ‘\( p \)-adic Frobenioids’ of Section 1.

We begin with the following general definition.

**Definition 2.1.** Let \( C \) be a Frobenioid (whose Frobenioid structure is given by a functor \( C \to F_{/\Phi^1} \), where \( \Phi^1 \) is a divisorial monoid on a connected, totally epimorphic category \( D \)) let \( A \in \text{Ob}(C) \), \( A_D \overset{\text{def}}{=} \text{Base}(A) \in \text{Ob}(D) \) and \( N \in \mathbb{N}_{\geq 1} \).

(i) Write
\[
\mu_N(A) \subseteq \mathcal{O}^\times(A)
\]
for the subgroup of elements of \( \mathcal{O}^\times(A) \) that are annihilated by \( N \). (Thus, if \( C \) is as in Example 1.1, with \( \Lambda = \mathbb{Z} \), then \( \mu_N(A) \) is isomorphic to a subgroup of \( \mathbb{Z}/N\mathbb{Z} \).) We shall refer to \( \mu_N(A) \) as the cyclotomic portion of \( \mathcal{O}^\times(A) \) (or \( \mathcal{O}^\psi(A) \)) of order \( N \). Write
\[
\text{Aut}_C/D(A) \overset{\text{def}}{=} \text{Im}(\text{Aut}_C(A)) \subseteq \text{Aut}_D(A_D)
\]
for the image of \( \text{Aut}_C(A) \) in \( \text{Aut}_D(A_D) \). (Thus, since \( \mathcal{O}^\psi(A) \) is abelian (cf. [Mzk5, Remark 1.3.1]), the natural action of \( \text{Aut}_C(A) \) on \( \mathcal{O}^\psi(A) \) factors through \( \text{Aut}_C/D(A) \).) Then we observe that the submonoid \( \mu_N(A) \subseteq \mathcal{O}^\times(A) \subseteq \mathcal{O}^\psi(A) \) is stabilized by the natural action of \( \text{Aut}_C/D(A) \) on \( \mathcal{O}^\psi(A) \). We shall say that \( A \) is \( \mu_N \)-saturated if the abstract group \( \mu_N(A) \) is isomorphic to \( \mathbb{Z}/N\mathbb{Z} \).

(ii) Suppose that \( A \) is \( \mu_N \)-saturated. Let \( H_A \subseteq \text{Aut}_C/D(A) \) be a subgroup, let
\[
f \in \mathcal{O}^\psi(A)^{H_A}
\]
(where the superscript \( H_A \) denotes the subset of elements fixed by the action of \( H_A \)) be an element that admits an \( N \)th root \( g \in \mathcal{O}^\psi(A) \) (i.e., \( g^N = f \)). Then since the action of \( H_A \) on the monoid \( \mathcal{O}^\psi(A) \) fixes \( f \), it follows that \( H_A \) stabilizes the subset \( \mu_N(A) \cdot g \subseteq \mathcal{O}^\psi(A) \) (i.e., the subset of \( N \)th roots of \( f \), cf. [Mzk5, Definition 1.3(vi)]), hence determines a cohomology
class\[\kappa_f \in H^1(H_A, \mu_N(A))\]

which is easily verified to be independent of the choice of \(g\), and which we shall refer to as the \textit{Kummer class of} \(f\). Here, we note that \(H_A\) acts \textit{trivially} on the cohomology module \(H^1(H_A, \mu_N(A))\). The assignment \(f \mapsto \kappa_f\) determines a homomorphism

\[
(O^>(A) \supseteq O^>(A)^{H_A} \cap O^>(A)^N \to H^1(H_A, \mu_N(A))
\]

which we shall refer to as the \textit{Kummer map}.

Now we return to our discussion of the \(p\)-adic Frobenoids \(C\) of Example 1.1. Let us assume that \(\Lambda = \mathbb{Z}, D\) is any category

\[
\mathcal{B}^{\text{temp}}(\Pi, \Pi^o)^0
\]

as in Example 1.3(iii), where \(\Pi \to Q, G_{\mathbb{Q}_p} \to Q\) (i.e., we take ‘\(F\)’ of \textit{loc. cit.} to be \(\mathbb{Q}_p\)), \(\Pi^o \subseteq \Pi\) are as in \textit{loc. cit.}, and we assume further that the surjection \(G_{\mathbb{Q}_p} \to Q\) is an \textit{isomorphism}. Write \(G \overset{\text{def}}{=} \text{Im}(\Pi) \subseteq Q, G^o \overset{\text{def}}{=} \text{Im}(\Pi^o) \subseteq Q, E \overset{\text{def}}{=} \mathcal{B}^{\text{temp}}(G, G^o)^0\) and \(K\) for the \textit{finite extension} of \(\mathbb{Q}_p\) determined by the open subgroup \(G \subseteq Q\). Thus, we have a \textit{natural functor}

\[
\mathcal{D} = \mathcal{B}^{\text{temp}}(\Pi, \Pi^o)^0 \to E = \mathcal{B}^{\text{temp}}(G, G^o)^0
\]

(cf. Example 1.3(ii) and (iii)).

\textbf{Definition 2.2.} Let \(H \subseteq G\) be a \textit{normal open subgroup}, let \(N \in \mathbb{N}_{\geq 1}\) and let \(A \in \text{Ob}(C)\). Write \(A_D \overset{\text{def}}{=} \text{Base}(A) \in \text{Ob}(\mathcal{D})\) and \(A_E \in \text{Ob}(\mathcal{E})\) for the projection of \(A\) to \(\mathcal{E}\).

(i) Note that the natural action by conjugation of \(\text{Aut}_C(A)\) on \(O^{\times}(A) \subseteq \text{Aut}_C(A))\) \textit{factors through the quotient}

\[
\text{Aut}_C(A) \to G_A \overset{\text{def}}{=} \text{Aut}_C(A)/(\text{Ker}(\text{Aut}_C(A) \to \text{Aut}_E(A_E)))
\]

induced by the functor \(C \to \mathcal{E}\). Moreover, the functor \(C \to \mathcal{E}\) induces a natural inclusion \(G_A \hookrightarrow \text{Aut}_E(A_E)\), which is an \textit{isomorphism} if, for instance, \(A_D\) is \textit{Galois} (where we recall that \(C\) is \textit{Aut-ample}, cf. Theorem 1.2(i)). If \(A_D\) is Galois, then we have a \textit{natural surjective outer homomorphism}

\[
G \twoheadrightarrow \text{Aut}_E(A_E) \twoheadrightarrow G_A
\]

which thus determines a surjective outer homomorphism

\[
H \to H_A \overset{\text{def}}{=} \text{Im}(H) \subseteq G_A,
\]

which, in light of the fact that \(H\) is \textit{normal} in \(G\), is well-defined, up to \textit{composition with conjugation by an element of} \(G\), despite the fact that the homomorphism \(G \to G_A\) is only determined up to composition with an inner automorphism.

(ii) We shall say that \(A\) is \((N, H)\)-\textit{saturated} if the following conditions are satisfied:

(a) \(A\) is \(\mu_N\)-saturated;

(b) \(A_D\) is Galois;
(c) the natural surjective homomorphism $H \twoheadrightarrow H_A$ induces isomorphisms on first cohomology modules

\[ H^1(H_A, \mu_N(A)) \sim H^1(H, \mu_N(A)), \quad H^1(H_A, \mathbb{Z}/N\mathbb{Z}) \sim H^1(H, \mathbb{Z}/N\mathbb{Z}) \]

and a surjection on second cohomology modules $H^2(H_A, \mu_N(A)) \rightarrow H^2(H, \mu_N(A))$ (conditions which are unaffected by composition with conjugation by an element of $G$).

In this situation, we shall write

\[ \mathcal{F}_N(A) \overset{\text{def}}{=} H^2(H_A, \mu_N(A))/\text{Ker}(H^2(H_A, \mu_N(A)) \rightarrow H^2(H, \mu_N(A))), \]

so $\mathcal{F}_N(A) \cong H^2(H, \mu_N(A)) \cong \mathbb{Z}/N\mathbb{Z}$ (cf., e.g., [NSW, Chapter 7, Theorem 7.2.6]).

(iii) Define

\[ \mathcal{O}^{\square}(-) \overset{\text{def}}{=} \mathcal{O}^{\square}(A) \]

if $\Phi$ is fieldwise saturated (cf. Example 1.1(ii)), and

\[ \mathcal{O}^{\square}(-) \overset{\text{def}}{=} \mathcal{O}^{\times}(-) \]

if $\Phi$ is not fieldwise saturated (e.g., if $\Phi$ is absolutely primitive).

Remark 2.2.1. Thus, it follows immediately from the definitions, together with the finiteness of the cohomology modules $H^1(H, \mu_N(A))$, $H^2(H, \mu_N(A))$ (cf., e.g., [NSW, Chapter 7, Theorem 7.2.6]), that given an $A' \in \text{Ob}(C)$, it follows that, for any $N$, $H$ as in Definition 2.2, there exists a pull-back morphism $A'' \rightarrow A'$ in $C$ such that $A''$ is $(N, H)$-saturated.

In the notation of Definition 2.2, suppose that $A$ is $(N, H)$-saturated. Write $A_{\mathcal{E}} = \text{Spec}(L)$ (so $L$ is a finite extension of $K$). Note that $H_A$ acts naturally on $L$, $\mathcal{O}^{\square}(A)$. Moreover, we have a natural isomorphism

\[ (\mathcal{O}^{\square}(A) \supseteq \mathcal{O}^{\square}(A)^{H} \sim \mathcal{O}^{\square}(L^{H}) \]

(where the superscript ‘$H$’ denotes the submonoids/subfields of elements on which $H_A$ acts trivially). Observe that, in light of the definition of the notation ‘$\mathcal{O}^{\square}(-)$’ in Definition 2.2(iii), the (first cohomology module portion of the) Galois-theoretic condition of Definition 2.2(ii)(c), implies (upon translation into ‘extension field-theoretic language’) that any element

\[ f \in \mathcal{O}^{\square}(A)^{H} \]

admits an $N$th root $g \in \mathcal{O}^{\square}(A)$, i.e., $g^N = f$. Thus, the Kummer map (cf. Definition 2.1(ii)) is defined on all of $\mathcal{O}^{\square}(A)^{H}$.

Next, let us recall that, by the well known duality theory of non-archimedean (mixed-characteristic) local fields (cf., e.g., [NSW, Chapter 7, Theorem 7.2.6]), the cup product on group cohomology determines an isomorphism

\[ H^1(H, \mu_N(A)) \sim H^{ab} \otimes H^2(H, \mu_N(A)) \]

and hence (in light of the condition of Definition 2.2(ii)(c)) an isomorphism (induced by the cup product on group cohomology)

\[ H^1(H_A, \mu_N(A)) \sim H^{ab}_A \otimes \mathcal{F}_N(A) \]

which is independent of the choice of the homomorphism $H \twoheadrightarrow H_A$ among its various $G$-conjugates and compatible with the natural actions of $G_A/H_A$ on its domain and codomain. Write $\eta_f \in H^{ab}_A \otimes \mathcal{F}_N(A)$ for the image of $\kappa_f$ via this last isomorphism.
Definition 2.3. The assignments \( f \mapsto \kappa_f, f \mapsto \eta_f \) determine homomorphisms
\[
O^\square(A)^H \to H^1(H_A, \mu_N(A)), \quad O^\square(A)^H \to H^1_{ab}(\pi N(A))
\]
which we shall refer to as the Kummer and reciprocity maps (associated to the \((N, H)\)-saturated object \(A\)), respectively.

**Theorem 2.4.** (Category-theoreticity of the Kummer and reciprocity maps) For \( i = 1, 2 \), let \( p_i \in \mathfrak{Primes} \) and \( C_i \) a \( p_i \)-adic Frobenioid (cf. Example 1.1(ii)) whose monoid type (i.e., the ‘\( A \)’ of Example 1.1) is \( \mathbb{Z} \), whose defining divisor monoid (i.e., the ‘\( \Phi \)’ of Example 1.1) we denote by \( \Phi_i \), whose defining rational function monoid (i.e., the ‘\( \mathbb{B} \)’ of Example 1.1) we denote by \( \mathbb{B}_i \), and whose base category \( D_i \) satisfies
\[
D_i = \mathcal{B}_{\text{temp}}(\Pi_i, \Pi_i^0)\]
(cf. Example 1.3(iii)), where \( \Pi_i \to \mathbb{Q}_p \) is a closed homomorphism of temp-slim tempered topological groups (cf. [Mzk1, Theorem 1.1.1(ii)]), \( \Pi_i^0 \subseteq \Pi_i \) is an open subgroup, \( G_i \) is an open normal subgroup, \( G_i^0 = \text{Im}(\Pi_i) \subseteq Q_i \), \( E_i = \mathcal{B}_{\text{temp}}(G_i, G_i^0) \). (Thus, we have a natural functor \( D_i \to E_i \).) Suppose further that we have been given open normal subgroups \( H_1 \subseteq G_1, H_2 \subseteq G_2 \), together with an equivalence of categories
\[
\Psi : C_1 \sim C_2,
\]
which (cf. Theorem 1.2(i), Example 1.3(i) and [Mzk5, Theorem 3.4(v)]) necessarily induces a 1-compatible equivalence of categories \( \Psi_{\text{Base}} : D_1 \sim D_2 \), hence an outer isomorphism of topological groups
\[
\Pi_1 \sim \Pi_2
\]
(cf. [Mzk2, Proposition 3.2 and Theorem A.4]) that lies over an outer isomorphism of topological groups
\[
G_1 \sim G_2
\]
(cf. Theorem 1.2(ii)). Assume that this isomorphism \( G_1 \sim G_2 \) maps \( H_1 \) onto \( H_2 \). Then (relative to the notation introduced in Definition 2.2) we have the following.

(i) \( \Phi_i \) is fieldwise saturated if and only if \( \Phi_2 \) is. Moreover, \( p_1 = p_2 \) and \( \Psi \) maps \((N, H_1)\)-saturated objects to \((N, H_2)\)-saturated objects and induces isomorphisms of monoids/modules
\[
O^\square(A_1)^H_1 \sim O^\square(A_2)^H_2, \quad (H_1)_{A_1} \sim (H_2)_{A_2}, \quad (G_1)_{A_1} \sim (G_2)_{A_2}
\]
\[
H^1((H_1)_{A_1}, \mu_N(A_1)) \sim H^1((H_2)_{A_2}, \mu_N(A_2)), \quad \mathfrak{S}_N(A_1) \sim \mathfrak{S}_N(A_2)
\]
(where \( \Psi(A_1) = A_2 \) and for \( i = 1, 2, A_i \in \text{Ob}(C_i) \) is \((N, H_i)\)-saturated) which are compatible with the respective Kummer and reciprocity maps
\[
O^\square(A_i)^H_i \to H^1((H_i)_{A_i}, \mu_N(A_i)), \quad O^\square(A_i)^H_i \to (H_i)_{A_i}^{ab} \otimes \mathfrak{S}_N(A_i)
\]
(where \( i = 1, 2 \)) as well as with the various natural actions of \((G_1)_{A_1}/(H_1)_{A_1}, (G_2)_{A_2}/(H_2)_{A_2}\).
(ii) If the \( \Phi_i \) are fieldwise saturated, then the isomorphism \( \mathfrak{s}_N(A_1) \xrightarrow{\sim} \mathfrak{s}_N(A_2) \) of (i) is compatible with the natural isomorphisms

\[
\mathfrak{s}_N(A_1) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}
\]

(cf., e.g., [NSW, Chapter 7, Theorem 7.2.6]).

Proof. First, we consider assertion (i). We begin by observing the (easily verified) fact that \( \Phi_i \) is fieldwise saturated if and only if the following two conditions hold.

(a) The inductive limit monoid

\[
\lim \Phi_i(B)
\]

(where \( B \) ranges over the objects of \( \mathcal{D}_i \) ) is divisible.

(b) For every pull-back morphism \( \phi : B \rightarrow C \) of \( \mathcal{C}_i \) that projects to a Galois covering \( \phi_D : B_D \rightarrow C_D \) of \( \mathcal{D}_i \), the injection \( \mathcal{O}^\triangleright(C) \hookrightarrow \mathcal{O}^\triangleright(B) \) induced by \( \phi \) (cf. [Mzk5, Proposition 1.11(iv)]) determines a bijection

\[
\mathcal{O}^\triangleright(C) \xrightarrow{\sim} \mathcal{O}^\triangleright(B)^{\text{Gal}(B/C)}
\]

(where the superscript ‘\( \text{Gal}(B/C) \)’ denotes the submonoid of \( \text{Gal}(B/C) \)-invariants; \( \text{Gal}(B/C) \equiv \text{Gal}(B_D/C_D) \)).

Moreover, since \( \mathcal{C}_i \) is a Frobenioid of \textit{rationally standard} type (cf. Theorem 1.2(i) and Example 1.3(ii)) over a \textit{slim} base category (cf. Example 1.3(i)), it follows from [Mzk5, Corollaries 4.10 and 4.11(ii), (iii)], that \( \Psi \) induces a 1-compatible equivalence of categories \( \Psi^{\text{Base}} : \mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2 \), as well as compatible isomorphisms of functors \( \Phi_1 \xrightarrow{\sim} \Phi_2 \), \( \mathbb{B}_1 \xrightarrow{\sim} \mathbb{B}_2 \) – where we regard ‘\( \mathcal{O}^\triangleright(-) \)’ as a subfunctor of \( \mathbb{B}_i \) (such that \( \mathbb{B}_i \) is the groupification of ‘\( \mathcal{O}^\triangleright(-) \)’) which is preserved by the isomorphism \( \mathbb{B}_1 \xrightarrow{\sim} \mathbb{B}_2 \) – hence that these conditions (a) and (b) are preserved by \( \Psi \). Thus, \( \Phi_1 \) is fieldwise saturated if and only if \( \Phi_2 \) is. The fact that \( p_1 = p_2 \) follows from the existence of the isomorphism \( G_1 \xrightarrow{\sim} G_2 \) (cf. [Mzk1, Proposition 1.2.1(i)])). The remainder of assertion (i) follows immediately from the fact (just observed) that \( \Psi \) preserves ‘\( \mathcal{O}^\triangleright(-) \)’, together with the ‘manifestly group-/category-theoretic’ construction given above of the Kummer and reciprocity maps. This completes the proof of assertion (i).

Next, we consider assertion (ii). For \( i = 1, 2 \), write \( K_i \) for the finite extension of \( \mathbb{Q}_p \) (where \( p \equiv p_1 = p_2 \) ) determined by the open subgroup \( G_i \leq Q_i \), and \( \overline{K}_i \) for the algebraic closure of \( K_i \) used to define the Galois group \( Q_i \) (so \( G_i = \text{Gal}(\overline{K}_i/K_i) \)). Then since \( \Phi_i \) is fieldwise saturated, it follows – by varying the objects \( A_i \) (that correspond via \( \Psi \) ) and reconstructing the multiplicative group associated to the field determined by the image of \( A_i \) in \( \mathcal{B}(G_i, G_i^\triangleright) \) as the groupification of the monoid \( \mathcal{O}^\triangleright(A_i) = \mathcal{O}^\triangleright_0(A_i) \) – that \( \Psi \) induces a pair of compatible isomorphisms

\[
G_1 \xrightarrow{\sim} G_2, \quad \overline{K}_1^\times \xrightarrow{\sim} \overline{K}_2^\times,
\]

where this pair is well-defined up to composition with automorphisms of the pair \( (G_2, \overline{K}_2^\times) \) 

induced by elements of \( G_2 \). For \( i = 1, 2 \), denote by \( \mu(\overline{K}_i^\times) \subseteq \overline{K}_i^\times \) the torsion subgroup of \( \overline{K}_i^\times \).

Now let us recall (cf. the theory of the Brauer group of a non-archimedean (mixed-characteristic) local field, as exposed, for instance, in [Serre, Section 1] and [Mzk1, proof of
Proposition 1.2.1(vii)) that the natural isomorphism

\[ H^2(G_i, \mu(\overline{K}_i^\times)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z} \]

may be constructed as the composite of the natural isomorphism

\[ H^2(G_i, \mu(\overline{K}_i^\times)) \xrightarrow{\sim} H^2(G_i, \overline{K}_i^\times) \]

(induced by the natural inclusion \( \mu(\overline{K}_i^\times) \hookrightarrow \overline{K}_i^\times \)) with the inverse of the natural isomorphism

\[ H^2(G_i, (K_i^\text{unr})^\times) \xrightarrow{\sim} H^2(G_i, \overline{K}_i^\times) \]

(induced by the inclusion \((K_i^\text{unr})^\times \hookrightarrow \overline{K}_i^\times\), where \(K_i^\text{unr} \subseteq \overline{K}_i\) denotes the maximal unramified extension of \(K_i\) and \(G_i^\text{unr} \overset{\text{def}}{=} \text{Gal}(K_i^\text{unr} / K_i)\), followed by the natural isomorphism

\[ H^2(G_i^\text{unr}, (K_i^\text{unr})^\times) \xrightarrow{\sim} H^2(G_i^\text{unr}, \mathbb{Z}) \xrightarrow{\sim} H^2(\mathbb{Q}, \mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \]

(where the first arrow is induced by the valuation map \((K_i^\text{unr})^\times \rightarrow \mathbb{Z}\) and the isomorphism \(G_i^\text{unr} \xrightarrow{\sim} \hat{\mathbb{Z}}\) is induced by the Frobenius element). Since the isomorphism \(G_1 \xrightarrow{\sim} G_2\) is automatically compatible with the quotients \(G_i \xrightarrow{\sim} G_i^\text{unr}\), as well as with the Frobenius elements \(\in G_i^\text{unr}\) (cf., e.g., [Mzk1, Proposition 1.2.1(ii) and (iv)]), we thus conclude that the natural isomorphisms

\[ H^2(G_i, \mu(\overline{K}_i^\times)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z} \]

are compatible with the isomorphism

\[ H^2(G_1, \mu(\overline{K}_1^\times)) \xrightarrow{\sim} H^2(G_2, \mu(\overline{K}_2^\times)) \]

induced by \(\Psi\). Since, moreover, the natural isomorphisms

\[ \mathfrak{f}_N(A_i) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z} \]

in question are easily verified to be ‘isomorphisms induced on subquotients’ by the natural isomorphisms \(H^2(G_i, \mu(\overline{K}_i^\times)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}\) (cf., e.g., [NSW, Chapter 7, Corollary 7.1.4]), it thus follows that the natural isomorphisms \(\mathfrak{f}_N(A_i) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}\) are compatible with the isomorphism \(\mathfrak{f}_N(A_1) \xrightarrow{\sim} \mathfrak{f}_N(A_2)\) induced by \(\Psi\), as desired. This completes the proof of assertion (ii).

\[ \square \]

Remark 2.4.1. By allowing \(N \in \mathbb{N}_{\geq 1}\) to vary, it is a routine exercise to show that:

One obtains a profinite version of Theorem 2.4(i), so long as one allows for a \((G_i)_{A_i}/(H_i)_{A_i}\)-indeterminacy (where \(i = 1, 2\)) in the resulting profinite Kummer and reciprocity maps.

Indeed, when one allows \(N\) – hence also the objects \(A_i\), where \(i = 1, 2\) – to vary, one must exercise care with respect to the coefficients ‘\(\mu_N(A_i)\)’, as one allows \(A_i\) to vary. The fact that the coefficients then vary does not cause a problem – so long as one allows for a \((G_i)_{A_i}/(H_i)_{A_i}\)-indeterminacy – since (cf. the discussion preceding Definition 2.3) \((H_i)_{A_i}\) acts trivially on \(H^1((H_i)_{A_i}, \mu_N(A_i))\) and \(\mathfrak{f}_N(A_i)\). (Here, we note that we implicitly used the well-known – (cf., e.g., [Mzk1, Proposition 1.2.1(vii)]) fact that \((G_i)_{A_i}\) acts trivially on \(\mathfrak{f}_N(A_i)\) in the proof of Theorem 2.4(ii).)
Remark 2.4.2. Note (relative to Theorem 2.4(ii)) that if the $\Phi_i$ are not fieldwise saturated, then the natural isomorphisms
\[ \mathfrak{F}_N(A_1) \sim \mathbb{Z}/N\mathbb{Z} \]
are not, in general, compatible with the isomorphism $\mathfrak{F}_N(A_1) \sim \mathfrak{F}_N(A_2)$ induced by $\Psi$. Indeed, when the $\Phi_i$ are absolutely primitive (cf. Remark 1.2.1), an example of such a $\Psi$ is provided by the unit-wise Frobenius functor of [Mzk5, Proposition 2.9(ii)], which acts on $\mathfrak{F}_N(A_i)$ (relative to the natural isomorphisms $\mathfrak{F}_N(A_i) \sim \mathbb{Z}/N\mathbb{Z}$) by ‘raising to the $\xi$th power’ (cf. ‘the compatibility with the reciprocity map’ asserted in Theorem 2.4(i) and [Mzk5, Proposition 2.9(ii)(a) and (c)]).

3. Archimedean primes

In this section, we define and study the basic properties of certain Frobenioids naturally associated to archimedean local fields. In particular, we construct our first example (i.e., among the various examples presented thus far in [Mzk5] and the present paper) of a Frobenioid which is not of isotropic type. In fact, it is this example that motivated the theory (developed in [Mzk5]) of Frobenioids that are not necessarily of isotropic type; in particular, this example is also the source of the terminology ‘isotropic’ and ‘co-angular’ in the theory of [Mzk5].

Write $\mathcal{D}_0$ for the subcategory of connected objects of the Galois category of finite étale coverings of $\text{Spec}(\mathbb{R})$ (cf. [Mzk5, Section 0]). Thus, $\mathcal{D}_0$ is a connected, totally epimorphic category, which is of FSM-type, hence also of FSMFF-type (cf. [Mzk5, Section 0]).

We begin by introducing some terminology.

Definition 3.1.
(i) An archimedean local field is defined to be a topological field that is isomorphic (as a topological field) to either the (topological) field of real numbers $\mathbb{R}$ – in which case we shall refer to the archimedean local field as being real – or the (topological) field of complex numbers $\mathbb{C}$ – in which case we shall refer to the archimedean local field as being complex. Recall that the topological field $\mathbb{R}$ has no non-trivial automorphisms, while the unique non-trivial automorphism of the topological field $\mathbb{C}$ is given by complex conjugation. If $K, L$ are archimedean local fields, then any inclusion of topological rings
\[ \iota : K \hookrightarrow L \]
is either an isomorphism or satisfies the property that there exist isomorphisms of topological fields $\alpha_K : K \sim \mathbb{R}, \alpha_L : L \sim \mathbb{C}$ such that $\alpha_L \circ \iota \circ \alpha_K^{-1} : \mathbb{R} \hookrightarrow \mathbb{C}$ is the natural inclusion.

(ii) If $K$ is an archimedean local field, then we shall denote by
\[ \mathcal{O}_K^\times \subseteq K^\times \]
the topological subgroup of $K^\times$ of elements of norm 1 and refer to it as the group of units of $K$. Let us write
\[ \mathbb{S}^1 \overset{\text{def}}{=} \mathcal{O}_\mathbb{C}^\times \subseteq \mathbb{C}^\times, \quad \text{ord}(K^\times) \overset{\text{def}}{=} K^\times / \mathcal{O}_K^\times \]
and observe that the usual absolute value $| - |$ on $K$ determines a *natural isomorphism* $\text{ord}(K^\times) \sim \mathbb{R}_{>0}$. Also, note that we have a *canonical decomposition*

$$O_K^\times \times \text{ord}(K^\times) \sim K^\times$$

(i.e., given by mapping a pair $(z, \lambda)$, where $z \in O_K^\times, \lambda \in \mathbb{R}_{>0} \equiv \text{ord}(K^\times)$, to $z \cdot \lambda \in K^\times$).

(iii) Let $K$ be an archimedean local field. Then we shall refer to as an *angular region*

$$\mathbb{A} \subseteq K^\times$$

any subset $\mathbb{A}$ of the form $B \times C$ (i.e., relative to the canonical decomposition of (ii)), where $B \subseteq O_K^\times$ is an open subset whose intersection with each connected component of $O_K^\times$ is (non-empty and) *connected* (which implies that $B = O_K^\times$ whenever $K$ is real), and $C \subseteq \text{ord}(K^\times) \equiv \mathbb{R}_{>0}$ is an interval of the form $(0, \lambda]$, where $\lambda \in \mathbb{R}_{>0}$. In this situation, we shall refer to $\lambda$ as the *tip* of the angular region $\mathbb{A}$ and to the subset

$$\partial \mathbb{A} \overset{\text{def}}{=} \{ a \in \mathbb{A} \mid |a| = \lambda \} \subseteq \mathbb{A}$$

as the *boundary* of $\mathbb{A}$ (so $\partial \mathbb{A}$ maps bijectively via the projection $K^\times \rightarrow O_K^\times$ to $B$). If $B = O_K^\times$, then we shall refer to the angular region $\mathbb{A}$ as *isotropic*. If $\mathbb{A}' \subseteq K^\times$ is another angular region, then we shall refer to $\mathbb{A}, \mathbb{A}'$ as *co-angular* if the projections of $\mathbb{A}, \mathbb{A}'$ to $O_K^\times$ coincide. We shall refer to a subset of $K$ as an *angular region* if it is the union with the subset $\{0\}$ of an angular region of $K^\times$. Note that if $V$ is any *one-dimensional vector space* over $K$, then it makes sense to speak of angular regions, tips of angular regions (i.e., objects that project to an object of $\mathbb{D}_0$), boundaries of angular regions, isotropic angular regions, and co-angular angular regions of $V$, $V^\times$. Also, if, for $i = 1, 2$, $\mathbb{A}_i \subseteq V_i$ is an angular region of a one-dimensional vector space $V_i$ over $K$, then one verifies immediately that the subset

$$\mathbb{A}_1 \otimes_K \mathbb{A}_2 \overset{\text{def}}{=} \{ a_1 \otimes_K a_2 \mid a_1 \in \mathbb{A}_1, a_2 \in \mathbb{A}_2 \} \subseteq V_1 \otimes_K V_2$$

is an angular region of $V_1 \otimes_K V_2$.

(iv) If

$$\iota: K \hookrightarrow L$$

is an inclusion of archimedean local fields, and $\mathbb{A}_K \subseteq V_K$ is an angular region of a one-dimensional vector space $V_K$ over $K$, then the pair $(V_K, \mathbb{A}_K)$ determines a pair

$$(V_K, \mathbb{A}_K)|_L \overset{\text{def}}{=} (V_L, \mathbb{A}_L),$$

where $V_L \overset{\text{def}}{=} V_K \otimes_K L$, and $\mathbb{A}_L \subseteq V_L$ is the angular region of $V_L$ defined as follows. If $\iota$ is an *isomorphism*, then we take $\mathbb{A}_L \subseteq V_L$ to be the image of $\mathbb{A}_K$ via the natural bijection $V_K \sim V_L$. If $\iota$ is *not* an isomorphism, then we take $\mathbb{A}_L \subseteq V_L$ to be the (necessarily isotropic) angular region of $V_L$ given by the $O_L^\times$-orbit of the image of $\mathbb{A}_K$ via the natural inclusion $V_K \hookrightarrow V_L$.

(v) If $\mathcal{F}$ is a category equipped with a functor $\mathcal{F} \rightarrow \mathcal{D}_0$, then we shall write $\mathcal{F}[\mathbb{R}]$ (respectively, $\mathcal{F}[\mathbb{C}]$) for the full subcategory of *real* (respectively, *complex*) objects – i.e., objects that project to an object of $\mathcal{D}_0$ determined by a real (respectively, complex)
archimedean local field. If \( \mathcal{F} = \mathcal{F}([\mathbb{R}]) \) (respectively, \( \mathcal{F} = \mathcal{F}([\mathbb{C}]) \)), then we shall say that \( \mathcal{F} \) is real (respectively, complex). If, for every \( A \in \text{Ob}(\mathcal{F}([\mathbb{R}])) \), there exists a morphism \( B \to A \) in \( \mathcal{F} \), where \( B \in \text{Ob}(\mathcal{F}([\mathbb{C}])) \), together with an automorphism of \( \beta \in \text{Aut}_{\mathcal{F}}(B) \) that projects to the unique non-trivial automorphism of \( \mathcal{T}_0 \), then we shall say that \( \mathcal{F} \) is complexifiable. If \( \mathcal{F} \) is connected, and, moreover, the categories \( \mathcal{F}([\mathbb{R}]), \mathcal{F}([\mathbb{C}]) \) are either empty or connected, then we shall say that \( \mathcal{F} \) is \( \mathcal{R} \mathcal{C} \)-connected. Suppose that \( \mathcal{F} \) is \( \mathcal{T} \mathcal{O} \mathcal{L} \mathcal{O} \mathcal{M} \mathcal{P} \mathcal{I} \mathcal{C} \mathcal{I} \mathcal{F} \mathcal{I} \mathcal{R} \) epimorphic. Then we shall say that an object \( A \in \text{Ob}(\mathcal{F}) \) is an \( \mathcal{R} \mathcal{C} \)-anchor if \( A \) determines an anchor (cf. [Mzk5, Section 0]) of \( \mathcal{F}([\mathbb{C}]) \). We shall say that an object \( A \in \text{Ob}(\mathcal{F}) \) is an \( \mathcal{R} \mathcal{C} \)-subanchor if there exists a morphism \( A \to B \) in \( \mathcal{F} \), where \( B \) is an \( \mathcal{R} \mathcal{C} \)-anchor. We shall say that an object \( A \in \text{Ob}(\mathcal{F}) \) is an \( \mathcal{R} \mathcal{C} \)-iso-subanchor if there exist an \( \mathcal{R} \mathcal{C} \)-subanchor \( B \in \text{Ob}(\mathcal{F}) \), a subgroup \( G \subseteq \text{Aut}_{\mathcal{F}}(B) \), and a morphism \( B \to A \) in \( \mathcal{F} \) which is a mono-minimal categorical quotient in \( \mathcal{F} \) (cf. [Mzk5, Section 0]) of \( B \) by \( G \). If every object of (the totally epimorphic category) \( \mathcal{F} \) is an \( \mathcal{R} \mathcal{C} \)-subanchor, then we shall say that \( \mathcal{F} \) is of \( \mathcal{R} \mathcal{C} \)-iso-subanchor type. We shall say that (the totally epimorphic category) \( \mathcal{F} \) is of \( \mathcal{R} \mathcal{C} \)-standard type if \( \mathcal{F} \) is:

(a) \( \mathcal{R} \mathcal{C} \)-connected;
(b) complexifiable;
(c) of \( \mathcal{F} \mathcal{S} \mathcal{M} \mathcal{F} \mathcal{F} \mathcal{F} \)-type;
(d) of \( \mathcal{R} \mathcal{C} \)-iso-subanchor type.

Before proceeding, we note the following elementary result concerning the geometry of the circle \( \mathbb{S}^1 \).

**Lemma 3.2.** (Connected open subsets of the circle) Let \( A, B \subseteq \mathbb{S}^1 \) be (non-empty) connected open subsets such that \( A \subseteq B \). We shall refer to a connected open subset \( C \subseteq B \) that contains \( A \) as an \( (A, B) \)-subset. If \( n \in \mathbb{Z} \), write \( \phi_n : \mathbb{S}^1 \to \mathbb{S}^1 \) for the map given by \( \mathbb{S}^1 \ni z \mapsto z^n \).

(i) \( A = B \) if and only if every \( (A, \mathbb{S}^1) \)-subset \( C \) is, in fact, equal to \( A \). In particular, \( A = \mathbb{S}^1 \) if and only if every \( (A, \mathbb{S}^1) \)-subset \( C \) is, in fact, equal to \( A \).

(ii) There exists a \( w \in \mathbb{S}^1 \) such that the translated open subset \( w \cdot A \) satisfies \( \phi_{-1}(w \cdot A) = w \cdot A \) (i.e., \( w \cdot A \) is invariant with respect to complex conjugation). For \( n \) a positive integer, there exists a (non-empty) connected open subset \( A' \subseteq A \) such that \( \phi_{-1}(A') = w \cdot A' \), where \( w^n = 1 \), if and only if \( \phi_n(A) \cap \{1, -1\} \neq \emptyset \).

(iii) The complement \( \mathbb{S}^1 \setminus A \) is of cardinality \( \leq 1 \) if and only if there does not exist an \( (A, \mathbb{S}^1) \)-subset \( A' \) such that \( A' \neq A, \mathbb{S}^1 \).

(iv) Suppose that for some \( 0 \neq n \in \mathbb{Z} \), \( \phi_n(A) \subseteq A \). Then either \( A = \mathbb{S}^1 \) or \( |n| = 1 \). Moreover, (in either of these cases) \( \phi_n(A) = A \).

(v) There exists a finite subset \( E \subseteq \mathbb{Z} \) such that for any \( n \in \mathbb{Z} \setminus E \), and any \( (A, \mathbb{S}^1) \)-subset \( A' \), the restriction of \( \phi_n \) to \( A' \) is surjective, but not injective. In particular, for \( n \in \mathbb{Z} \setminus E \), \( \phi_n(A) = \phi_n(B) = \mathbb{S}^1 \).

(vi) Suppose that \( A \neq B \). Consider the following conditions on \( A, B \):

(a) For any two \( (A, B) \)-subsets \( A_1, A_2 \), it holds that either \( A_1 \subseteq A_2 \) or \( A_2 \subseteq A_1 \).

(b) For any two \( (A, B) \)-subsets \( A_1, A_2 \) such that \( A_1 \subseteq A_2, A_1 \neq A_2 \), there exists an \( (A_1, A_2) \)-subset \( A_3 \) such that \( A_3 \neq A_1, A_2 \).

(c) The complement \( B \setminus A \) is connected.

(d) If \( B = \mathbb{S}^1 \), then \( B \setminus A \) is of cardinality \( \leq 1 \).

(e) \( B \neq \mathbb{S}^1 \).
Then condition (a) holds if and only if both conditions (c) and (d) hold; both conditions (a) and (b) hold if and only if both conditions (c) and (e) hold. If the pair \((A, B)\) satisfies conditions (a) and (b), then we shall say that the pair \((A, B)\) is continuously ordered.

(vii) Suppose that \(A \neq B, B \neq S^1\). Then there exists an \((A, B)\)-subset \(C\) such that the pairs \((A, C)\) and \((C, B)\) are continuously ordered.

(viii) Suppose that \(B \setminus A\) is of cardinality \(\geq 1\). Then there exist \((A, B)\)-subsets \(A_1, A_2\) such that \(B = A_1 \cup A_2, A_1 \neq S^1, A_2 \neq S^1\).

(ix) Suppose that \(A \neq B\), and that there does not exist a connected open subset \(D \subseteq B\) such that \(A \cap D = \emptyset\). Then \(B = S^1\), and \(B \setminus A\) is of cardinality \(\leq 1\).

(x) If \(A \neq S^1\), then \(A\) is not homeomorphic to \(S^1\).

(xi) Every automorphism of the topological group \(S^1\) is equal to either \(\phi_1\) or \(\phi_{-1}\).

(xii) Write

\[
\text{Homeo}(S^1), \quad \text{Trans}(S^1) \subseteq \text{Homeo}(S^1), \quad \text{Refl}(S^1) \subseteq \text{Homeo}(S^1)
\]

for the group of self-homeomorphisms of \(S^1\), the subgroup (naturally isomorphic to \(S^1\)) determined by the translations by elements of \(S^1\), and the subgroup generated by \(\text{Trans}(S^1), \phi_{-1}\), respectively. Then \(\text{Refl}(S^1) \cong S^1 \rtimes (\mathbb{Z}/2\mathbb{Z}), \text{Trans}(S^1) \subseteq \text{Refl}(S^1)\) is equal to the subgroup of infinitely divisible elements, the normalizer in \(\text{Homeo}(S^1)\) of either \(\text{Trans}(S^1)\) or \(\text{Refl}(S^1)\) is equal to \(\text{Refl}(S^1)\) and the centralizer in \(\text{Homeo}(S^1)\) of \(\text{Trans}(S^1)\) is equal to \(\text{Trans}(S^1)\).

**Proof.** Assertions (i), (ii), (iii), (vi), (vii), (viii) are immediate from the well-known structure of \(S^1\) (and, in the case of assertion (vii), the equivalence, given in assertion (vi), of conditions (a) and (b) with conditions (c) and (e)).

To verify assertion (iv), note that, relative to the standard metric on the tangent bundle of \(S^1\) (i.e., the metric induced by the standard euclidean metric on \(\mathbb{C}\) via the natural inclusion \(S^1 \hookrightarrow \mathbb{C}\), \(|d\phi_n| = |n| \geq 1\), i.e., \(\phi_n\) multiplies lengths by \(|n|\). Suppose that \(A \neq S^1, |n| \geq 2 > 1\). Then \(|d\phi_n| > 1\), so it follows that the length of \(\phi_n(A)\) is strictly greater than the length of \(A\), in contradiction to the inclusion \(\phi_n(A) \subseteq A\). Thus, either \(A = S^1\) or \(|n| = 1\). The fact that \(\phi_n(A) = A\) then follows immediately. This completes the proof of assertion (iv).

Assertion (v) follows by taking \(E\) to be the (manifestly finite) set of \(n\) such that \(|n|\) times the length of \(A\) fails to exceed (i.e., is \(\leq\)) the length of \(S^1\).

To verify assertion (ix), observe that our assumption on \((A, B)\) implies that there does not exist an \((A, B)\)-subset \(C\) such that \((A, C), (C, B)\) are continuously ordered; thus, by assertion (vii), it follows that \(B = S^1\), and (again by our assumption on \((A, B)\)) that \(B \setminus A\) is of cardinality \(\leq 1\).

Assertion (x) follows, for instance, by observing that the fundamental group of \(S^1\) is non-trivial, whereas the fundamental group of \(A \neq S^1\) is trivial.

Assertion (xi) follows, for instance, by thinking of \(S^1\) as a quotient \(\mathbb{R}/\mathbb{Z}\) and applying the easily verified fact that every automorphism of the (additive) topological group \(\mathbb{R}\) is given by multiplication by a non-zero element of \(\mathbb{R}\).

As for assertion (xii), the isomorphism \(\text{Refl}(S^1) \cong S^1 \rtimes (\mathbb{Z}/2\mathbb{Z})\), as well as the characterization of the subgroup \(\text{Trans}(S^1) \subseteq \text{Refl}(S^1)\), are immediate from the definitions. In light of this characterization, it follows that the normalizer of \(\text{Refl}(S^1)\) in \(\text{Homeo}(S^1)\)
is contained in the normalizer of \( \text{Trans}(S^1) \) in \( \text{Homeo}(S^1) \). Since the topology of \( \text{Trans}(S^1) (\cong S^1) \) may be recovered from the action of \( \text{Trans}(S^1) \) on the topological space \( S^1 \), it follows immediately that any element of the normalizer of \( \text{Trans}(S^1) \) in \( \text{Homeo}(S^1) \) determines, by conjugation, an automorphism of the topological group \( \text{Trans}(S^1) \cong S^1 \), hence, by assertion (xi), lies in the subgroup of \( \text{Homeo}(S^1) \) generated by \( \text{Refl}(S^1) \) and the centralizer of \( \text{Trans}(S^1) \) in \( \text{Homeo}(S^1) \). Thus, to complete the proof of assertion (xii), it suffices to show that this centralizer is equal to \( \text{Trans}(S^1) \), or, equivalently (since \( \text{Trans}(S^1) \) acts transitively on \( S^1 \)), that every element \( \alpha \) of this centralizer that fixes 1 in \( S^1 \) is equal to the identity. But this follows by considering, for an arbitrary \( z \in S^1 \), the corresponding translation \( \tau_z \in \text{Trans}(S^1) \), which yields the desired relation \( z = \tau_z(1) = \tau_z(\alpha(1)) = \alpha(\tau_z(1)) = \alpha(z) \). □

**Example 3.3.** (Archimedean Frobenioids, angular Frobenioids and angloids)

(i) Now we define a category

\[ \mathcal{C}_0 \]

as follows. The objects of \( \mathcal{C}_0 \) are triples \((\text{Spec}(K), V_K, \mathbb{A}_K)\), where \( \text{Spec}(K) \in \text{Ob}(\mathcal{D}_0) \), \( V_K \) is a one-dimensional \( K \)-vector space and \( \mathbb{A}_K \subseteq V_K \) is an angular region. We shall say that the object \((\text{Spec}(K), V_K, \mathbb{A}_K)\) is naively isotropic if \( \mathbb{A}_K \) is isotropic. The morphisms of \( \mathcal{C}_0 \)

\[ \phi : (\text{Spec}(L), V_L, \mathbb{A}_L) \to (\text{Spec}(K), V_K, \mathbb{A}_K) \]

consist of data as follows:

(a) a morphism \( \text{Base}(\phi) : \text{Spec}(L) \to \text{Spec}(K) \) of \( \mathcal{D}_0 \);

(b) an element \( \deg_{\mathbb{F}_L}(\phi) \overset{\text{def}}{=} d \in \mathbb{N}_{\geq 1} \); and

(c) an isomorphism of \( L \)-vector spaces \( V_L^\otimes \overset{\sim}{\to} V_K|_L \) that maps \( \mathbb{A}_L^\otimes \) into \( \mathbb{A}_K|_L \).

One verifies immediately that this category \( \mathcal{C}_0 \) is connected and totally epimorphic. If, whenever \( L \) is complex, the image \( \text{Im}(\mathbb{A}_L^\otimes) \subseteq V_K|_L \) is co-angular with \( \mathbb{A}_K|_L \), then we shall say that \( \phi \) is naively co-angular. (Thus, if \( L \) is real, then \( \phi \) is always naively co-angular.) Also, we shall write \( \text{Div}(\phi) \overset{\text{def}}{=} \log(\lambda) \in \mathbb{R}_{\geq 0} \), for the largest \( \lambda \in \mathbb{R}_{>0} \) such that \( \lambda \cdot \text{Im}(\mathbb{A}_L^\otimes) \subseteq \mathbb{A}_K|_L \). If we denote by

\[ \Phi_0 : \mathcal{D}_0 \to \text{Mon} \]

the functor determined by the assignment

\[ (\text{Ob}(\mathcal{D}_0) \ni \text{Spec}(K)) \mapsto \text{ord}(K) \cong \mathbb{R}_{>0} \cong \mathbb{R}_{\geq 0} \]

(where the isomorphism \( \mathbb{R}_{>0} \overset{\sim}{\to} \mathbb{R}_{\geq 0} \) is given by the natural logarithm), then the triple ‘\((\text{Base}(\cdot), \text{Div}(\cdot), \deg_{\mathbb{F}_L}(\cdot))\)’ determines a pre-Frobenioid structure (cf. [Mzk5, Definition 1.1(iv)])

\[ \mathcal{C}_0 \to \mathbb{F}_{\Phi_0} \]

on \( \mathcal{C}_0 \). Similarly, if \( \mathcal{D} \) is any connected, totally epimorphic category, and \( \mathcal{D} \to \mathcal{D}_0 \) is a functor, then by setting

\[ \mathcal{C} \overset{\text{def}}{=} \mathcal{C}_0 \times_{\mathcal{D}_0} \mathcal{D} , \quad \Phi \overset{\text{def}}{=} \Phi_0|_{\mathcal{D}} \]

we obtain a pre-Frobenioid structure

\[ \mathcal{C} \to \mathbb{F}_{\Phi} \]

on \( \mathcal{C} \).
(ii) By Lemma 3.2(i), it follows immediately that an object of \( C \) is isotropic (in the sense of [Mzk5, Definition 1.2(iv)]) if and only if it is naively isotropic, and that a morphism of \( C \) is co-angular (in the sense of [Mzk5, Definition 1.2(iii)]) if and only if it is naively co-angular. Also, we note that, by Lemma 3.2(iv), every endomorphism of a non-isotropic object of \( C \) is linear and co-angular and every endomorphism of an isotropic object of \( C \) is co-angular. Moreover, the following conditions on an object of \( C \) are equivalent: (a) the object is isotropic; (b) the object is Frobenius-trivial; and (c) the object is Frobenius-ample. Also, we observe that every object of \( C \) is metrically trivial. Now a routine verification reveals that \( C \) satisfies the conditions of [Mzk5, Definition 1.3], hence that \( C \) is a Frobenioid. Moreover, by Lemma 3.2(v), it follows that \( C \) is of Frobenius-isotropic type, so it makes sense to speak of the perfection \( C^{\text{pf}} \) of \( C \) (cf. [Mzk5, Definition 3.1(iii)]). If \( \Lambda \) is a monoid type, then we define \( C^\Lambda \) as

\[
C^\mathbb{Z} \defeq C, \quad C^\mathbb{Q} \defeq C^{\text{pf}}, \quad C^\mathbb{R} \defeq C^{\text{rlf}}
\]

(cf. [Mzk5, Proposition 5.3]). We shall refer to a Frobenioid \( C^\Lambda \) as an archimedean Frobenioid.

(iii) Write

\[
A_0 \subseteq C_0, \quad A \subseteq C
\]

for the respective subcategories determined by the isometries. Thus, we have a natural equivalence of categories \( A \overset{\sim}{\to} A_0 \times D_0 \). Moreover, a routine verification reveals that \( A \) satisfies the conditions of [Mzk5, Definition 1.3], hence that \( A \) is a Frobenioid over the base category \( D \), which is, in fact, of group-like type. Also, we observe that the isotropic objects of \( A \) are precisely the isotropic objects of \( C \), and that the co-angular morphisms of \( A \) are precisely the co-angular isometries of \( C \). We shall refer to \( A \) as an angular Frobenioid. Write

\[
N^\text{lin} \defeq A^{\text{lin}} \subseteq A, \quad N_0^\text{lin} \defeq A_0^{\text{lin}} \subseteq A_0
\]

(i.e., the respective subcategories determined by the linear morphisms – cf. [Mzk5, Definition 1.2(iv)]). We shall refer to a category \( N \) as a non-rigidified angloid (over the base category \( D \)). Note that there is a natural equivalence of categories \( N \overset{\sim}{\to} N_0 \times D_0 \).

(iv) Observe that all real objects of \( N_0 \) are isomorphic. Thus, it makes sense to define

\[
R_0 \defeq (N_0)_A, \quad R \defeq R_0 \times D_0 \quad \text{for } A \in \text{Ob}(N_0[\mathbb{R}]).
\]

We shall refer to a category \( R \) as a rigidified angloid (over the base category \( D \)). If \( D \) is RC-connected, then one verifies immediately that \( C^\Lambda, A, N, R \) are RC-connected (hence, in particular, connected) and totally epimorphic. Although \( N, R \) are not equipped with (pre-)Frobenioid structures, we shall often apply ‘Frobenioid-theoretic terminology’ to objects and morphisms of \( N, R \); such terminology is to be interpreted as referring to the images of the objects and morphisms in question in \( C \) via the natural functors \( N \to C, R \to C \); also, we shall apply the notation ‘\( \mathcal{O}(\mathcal{O}^{\text{pf}}(\mathcal{O}^{\times}(\mathcal{O}^{\times}(\mathcal{O}^{\times}(\mathcal{O}^{\text{pf}}(\mathcal{O}^{\times}(-)))))))) \)’ to objects of \( N, R \) to denote the monoids of base-identity endomorphisms (i.e., as objects of \( N, R \)), automorphisms (i.e., as objects of \( N, R \)), respectively.

(v) Note that if \( \mathcal{F} \) is one of the categories \( A, N, R \), then any morphism of \( \mathcal{F} \) that is obtained as the isotropic hull of an object of \( \mathcal{F} \) whose angular region is determined by the complement
in $S^1$ of a single element of $S^1$ is an FSMI-morphism of $\mathcal{F}$ (cf. Lemma 3.2(iii)). We shall refer to such a morphism as a slit morphism of $\mathcal{F}$. Note that the existence of slit morphisms implies that $\mathcal{F}$ is not of FSM-type.

Remark 3.3.1. Note that it follows immediately from the simple, explicit structure of the categories $C_0$, $A_0$, $N_0$, $R_0$ introduced in Example 3.3 that in any of the categories $\mathcal{C}$, $\mathcal{A}$, $\mathcal{N}$, or $\mathcal{R}$, a base-isomorphism between complex objects is a monomorphism if and only if it induces an injection on underlying angular regions.

PROPOSITION 3.4. (FSM-morphisms) In the notation and terminology of Example 3.3, let $\mathcal{F}$ be one of the following categories: $\mathcal{A}$, $\mathcal{N}$, $\mathcal{R}$. If $\mathcal{F} = \mathcal{A}$ (respectively, $\mathcal{F} = \mathcal{N}$, $\mathcal{F} = \mathcal{R}$), then set $\mathcal{F}_0 = A_0$ (respectively, $\mathcal{F}_0 = N_0$, $\mathcal{F}_0 = R_0$). Then we have the following.

(i) Fiberwise-surjective morphisms of $\mathcal{F}$ project to fiberwise-surjective morphisms of $\mathcal{D}$.

(ii) Let $\phi : A \to B$ be a monomorphism of $\mathcal{F}$ such that (at least) one of the following two conditions is satisfied (cf. Remark 3.4.1 below): (a) $\phi$ projects to an isomorphism of $\mathcal{D}$; (b) $\phi$ admits a factorization $A \to A' \to B$ as a composite of a morphism of Frobenius type $A \to A'$ and a linear morphism $A' \to B$ such that any isotropic hull $A' \to A''$ of $A'$ is either an isomorphism or a slit morphism (cf. Example 3.3(v)). Then $\phi$ projects to a monomorphism $\phi_D$ of $\mathcal{D}$.

(iii) FSM-morphisms of $\mathcal{F}$ project to FSM-morphisms of $\mathcal{D}$.

(iv) Suppose that $\mathcal{D}$ is complexifiable. Then any FSM-morphism (respectively, FSMI-morphism) of $\mathcal{F}$ that projects to an isomorphism of $\mathcal{D}$ projects to an FSM-morphism (respectively, FSMI-morphism) of $\mathcal{F}_0$.

(v) Any morphism of $\mathcal{F}$ that projects to an irreducible morphism (respectively, isomorphism, irreducible morphism) of $\mathcal{D}$, to an isomorphism (respectively, irreducible morphism, irreducible morphism) of $\mathcal{F}_0$, and to an isomorphism (respectively, isomorphism, non-isomorphism) of $\mathcal{D}_0$ is an irreducible morphism of $\mathcal{F}$.

(vi) If $\phi$ is a morphism of $\mathcal{F}$ such that $\phi_D \overset{\text{def}}{=} \text{Base}(\phi)$ admits a factorization $\phi_D = \alpha_D \circ \beta_D$ in $\mathcal{D}$, then there exist morphisms $\alpha$, $\beta$ of $\mathcal{F}$ lifting $\alpha_D$, $\beta_D$, respectively, such that $\phi = \alpha \circ \beta$ in $\mathcal{F}$. In particular, irreducible morphisms of $\mathcal{F}$ project to either isomorphisms or irreducible morphisms of $\mathcal{D}$; FSM-morphisms of $\mathcal{F}$ project to either isomorphisms or FSMI-morphisms of $\mathcal{D}$.

(vii) Let $\phi$ be a morphism of $\mathcal{F}$ that projects to a pull-back morphism of $\mathcal{F}_0$ and to an FSM-morphism (respectively, FSMI-morphism) of $\mathcal{D}$. Then $\phi$ is an FSM-morphism (respectively, FSMI-morphism) of $\mathcal{F}$.

(viii) Suppose that $\mathcal{D}$ is complexifiable, RC-connected and of FSMFF-type. Then $\mathcal{F}$ is complexifiable, RC-connected (hence, in particular, connected), totally epimorphic, and of FSMFF-type.

Proof. Assertion (i) follows immediately from the observation that if $A \in \text{Ob}(\mathcal{F})$ projects to an object $A_D \in \text{Ob}(\mathcal{D})$, then any morphism $B_D \to A_D$ of $\mathcal{D}$ lifts to a morphism $B \to A$ of $\mathcal{F}$, for some $B \in \text{Ob}(\mathcal{F})$.

To verify assertion (ii), let $\alpha_D, \beta_D : C_D \to A_D \overset{\text{def}}{=} \text{Base}(A)$ be morphisms of $\mathcal{F}$ such that $\phi_D \circ \alpha_D = \phi_D \circ \beta_D$, where $\phi_D \overset{\text{def}}{=} \text{Base}(\phi)$. Then to complete the proof of assertion (ii), it suffices to show that $\alpha_D = \beta_D$. Write $\phi_0 : A_0 \to B_0$ for the projection of $\phi$ to $\mathcal{F}_0$ and $\alpha_{D_0}, \beta_{D_0} : C_{D_0} \to A_{D_0}$, $\phi_{D_0} : A_{D_0} \to B_{D_0}$ for the respective projections of $\alpha$, $\beta$, $\phi$ to $\mathcal{D}_0$.
(so $\phi_D \circ \alpha_D = \phi_D \circ \beta_D$). Now I claim that there exist linear isometries $\alpha_0$, $\beta_0 : C_0 \to A_0$ that lift $\alpha_D$, $\beta_D$, respectively, and, moreover, satisfy the relation $\phi_0 \circ \alpha_0 = \phi_0 \circ \beta_0$. Indeed, this follows immediately if $\alpha_D = \beta_D$ (which is the case whenever condition (a) is satisfied, i.e., whenever $\phi_D$ is an isomorphism), by taking $\alpha_0 = \beta_0$ to be a pull-back morphism.

On the other hand, if $\alpha_D \neq \beta_D$, but condition (b) is satisfied, then it follows from (the latter portion of) Lemma 3.2(ii), that there exist $\alpha_0$, $\beta_0$ as desired, where $\alpha_0$ is related to $\beta_0$ by an automorphism of $C_0$ that lies over the complex conjugation automorphism of $C_D$. This completes the proof of the claim. Thus, by applying the natural equivalence of categories $\mathcal{F} \approx \mathcal{F}_0 \times_D \mathcal{D}$, we conclude that there exist linear isometries $\alpha$, $\beta : C \to A$ that simultaneously lift $\alpha_0$, $\beta_0$ and $\alpha_D$, $\beta_D$, respectively, and, moreover, satisfy the relation $\phi \circ \alpha = \phi \circ \beta$. Since $\phi$ is a monomorphism of $\mathcal{F}$, we thus conclude that $\alpha = \beta$, hence that $\alpha_D = \beta_D$, as desired. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertions (i) and (ii), together with the observation that if $\phi$ is a fiberwise-surjective morphism of $\mathcal{F}$ that does not satisfy condition (a) of assertion (ii), then the fiberwise-surjectivity of $\phi$ implies (cf. Lemma 3.2(ix)) that $\phi$ necessarily satisfies condition (b) of assertion (ii).

Next, we consider assertion (iv). Let $\phi : A \to B$ be an FSM-morphism of $\mathcal{F}$ that projects to an isomorphism of $\mathcal{D}$. Thus, the projection $\phi_0 : A_0 \to B_0$ of $\phi$ to $\mathcal{F}_0$ projects to an isomorphism $\phi_D$ of $D_0$. Now since $\mathcal{D}$ is complexifiable, it follows immediately that one can lift arbitrary pairs of morphisms $\alpha_0$, $\beta_0 : C_0 \to A_0$ of $\mathcal{F}_0$ such that $\phi_0 \circ \alpha_0 = \phi_0 \circ \beta_0$ (which, since $\phi_D$ is an isomorphism, implies that the respective projections $\alpha_D$, $\beta_D$ of $\alpha_0$, $\beta_0$ to $D_0$ coincide) to morphisms $\alpha$, $\beta : C \to A$ of $\mathcal{F}$ such that $\phi \circ \alpha = \phi \circ \beta$, hence that $\alpha = \beta$ (since $\phi$ is a monomorphism), so $\alpha_0 = \beta_0$, i.e., $\phi_0$ is a monomorphism. Similarly, (since $\mathcal{D}$ is complexifiable) any morphism $\gamma_0 : C_0 \to B_0$ of $\mathcal{F}_0$ lifts to a morphism $\gamma : C \to B$ of $\mathcal{F}$; thus, the fiberwise-surjectivity of $\phi$ implies that of $\phi_0$. If $\phi$ is irreducible, then it follows immediately from the fact that $\phi$ projects to an isomorphism of $\mathcal{D}$ that $\phi_0$ is also irreducible. This completes the proof of assertion (iv).

Assertion (v) follows immediately from the fact that since $\mathcal{F}_0$, $\mathcal{D}$ are total epimorphic, if a composite morphism $\alpha \circ \beta$ of $\mathcal{F}_0$ or $\mathcal{D}$ is an isomorphism, then so are $\alpha$, $\beta$.

Assertion (vi) follows immediately from the fact that if $\alpha_D \circ \beta_D$ is any composite morphism of $D_0$, then either $\alpha_D$ or $\beta_D$ is an isomorphism (cf. also assertion (iii)).

Next, we consider assertion (vii). Since every non-iso morphism of $D_0$ (hence also every pull-back morphism of $\mathcal{F}_0$ which is not an isomorphism – cf. [Mzk5, Proposition 1.7(v)], for pull-back morphisms is irreducible, it follows from assertion (v) that it suffices to show that $\phi$ is an FSM-morphism. Since $\phi : A \to B$ projects to a pull-back morphism $\phi_0 : A_0 \to B_0$ of $\mathcal{F}_0$, it follows that given any morphism $\psi_0 : C_0 \to B_0$, it holds that any factorization $\psi_D = \phi_D \circ \zeta_D$ (for some morphism $\zeta_D : C_D \to A_D$ of $D_0$), where $\phi_D : A_D \to B_D$, $\psi_D : C_D \to B_D$ are the respective projections of $\phi_0$, $\psi_0$ to $D_0$, lifts to a factorization $\psi_0 = \phi_0 \circ \zeta_0$ for some $\zeta_0 : C_0 \to A_0$ (cf. the definition of a ‘pull-back morphism’ in [Mzk5, Definition 1.2(ii)]). Thus, (cf. also the observation made in the proof of assertion (i)) that if $E \in \text{Ob}(\mathcal{F})$ projects to an object $E_D \in \text{Ob}(\mathcal{D})$, then any morphism $F_D \to E_D$ in $D$ lifts to a morphism $F \to E$ of $\mathcal{F}$ the fiberwise-surjectivity of $\phi$ follows immediately from that of the projection $\phi_D$ of $\phi$ to $D$. To show that $\phi$ is a monomorphism, let $\alpha$, $\beta : C \to A$ be morphisms of $\mathcal{F}$ such that $\phi \circ \alpha = \phi \circ \beta$. Since $\phi_D$ is a monomorphism, it follows that the respective projections $\alpha_D$, $\beta_D$ of $\alpha$, $\beta$ to $D$ coincide, hence that the respective projections $\alpha_D$, $\beta_D$ of $\alpha$, $\beta$ to $D$ coincide.
of $\alpha, \beta$ to $D_0$ coincide. Thus, again by the definition of a ‘pull-back morphism’ (cf. [Mzk5, Definition 1.2(ii)]), we conclude that the respective projections $\alpha_0, \beta_0$ of $\alpha, \beta$ to $F_0$ coincide, hence that $\alpha = \beta$, as desired. This completes the proof of assertion (vii).

Finally, we consider assertion (viii). It is immediate from the definition of $F$ that $F$ is complexifiable, RC-connected (hence, in particular, connected) and totally epimorphic.

Next, I claim that $F_0$ is of FSMFF-type. Indeed, let $\phi : A \to B$ be an FSM-morphism of $F_0$. Then by assertion (iii), it follows that $\phi$ projects to an isomorphism of $D_0$. Suppose that $A, B$ are real. Now if $F_0 = N_0$ or $F_0 = R_0$, then one concludes immediately that $\phi$ is an LB-invertible pre-step, hence an isomorphism (cf. [Mzk5, Proposition 1.4(iii)]). If $F_0 = A_0$, then there exists a complex isotropic object $C$ of $F_0$ and a linear morphism $C \to A$; thus, the existence of torsion elements in $O^\times(C)$ of arbitrary order when $F_0 = A_0$ implies (since $\phi$ is a monomorphism) that the LB-invertible base-isomorphism (cf. the definition of $F_0$!) $\phi$ is linear, hence an isomorphism. In particular, it follows that every FSMI-morphism of $F_0$ has complex domain and codomain.

Now suppose that $A, B$ are complex. Then observe that if the isometric base-isomorphism $\phi$ is linear, then it is an isometric pre-step, hence (cf. the fiberwise-surjectivity of $\phi$, Lemma 3.2(ix) and Example 3.3(v)) either an isomorphism or a slit morphism. Moreover, even if $\phi$ is not linear, by Lemma 3.2(iv) (which may be interpreted as asserting that ‘angular regions never shrink’) and (v), there exists an integer $N$ such that given any composite

$$\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_2 \circ \phi_1$$

of (not necessarily FSMI-) morphisms $\phi_1, \ldots, \phi_n$ such that the domain of $\phi_1$ is equal to $A$, it holds that the cardinality of the set of $j$ (where $j = 1, \ldots, n$) such that $\phi_j$ is an FSMI-morphism (which implies that $\phi_j$ is a monomorphism which is a base-isomorphism between complex objects – cf. Remark 3.3.1) and non-linear is $\leq N$. In this situation, any linear FSMI-morphism $\phi_j$ (where $j = 1, \ldots, n$) is necessarily a slit morphism (as observed above) – which implies that the codomain of $\phi_j$ is isotropic, so no $\phi_j'$, where $j' > j$, can be a slit morphism; we thus conclude that the cardinality of the set of $j \in \{1, \ldots, n\}$ such that $\phi_j$ is an FSMI-morphism is $\leq N + 1$.

Next, observe that (by [Mzk5, Proposition 1.7(ii)], applied to the Frobenioi $A_0$) $\phi$ admits a factorization $\phi = \alpha \circ \beta$, where $\beta$ is a morphism of Frobenius type, and $\alpha$ is an isometric pre-step (hence, in particular, a monomorphism – cf., e.g., [Mzk5, Definition 1.3(v)(a)]). Since $\phi$ is fiberwise-surjective, it follows formally that $\alpha$ is an FSM-morphism, hence that $\alpha$ is either a slit morphism (hence, in particular, an FSMI-morphism – cf. Example 3.3(v)) or an isomorphism. On the other hand, $\beta$ may be written as a composite $\beta = \beta_b \circ \cdots \circ \beta_1$ of prime-Frobenius morphisms $\beta_1, \ldots, \beta_b$. Since each of the $\beta_j$ (where $j = 1, \ldots, b$) is co-angular, and $\phi$ (hence also $\beta$) is a monomorphism, it follows immediately by induction on $j$ that each of the $\beta_j$ (where $j = 1, \ldots, b$) is a monomorphism (i.e., more concretely: induces an injection on underlying angular regions – cf. Remark 3.3.1), hence (by a formal argument, since $\phi$ is fiberwise-surjective, and $\alpha$ is a monomorphism) that each of the $\beta_j$ is an FSM-morphism. Moreover, since $\beta_j$ is a prime-Frobenius morphism, it follows immediately from the fact that $\beta_j$ induces a bijection on underlying angular regions that $\beta_j$ is irreducible, hence an FSMI-morphism. Thus, we conclude that $\phi$ factors as a composite of finitely many FSMI-morphisms. This completes the proof of the claim that $F_0$ is of FSMFF-type.
To prove that $\mathcal{F}$ is of FSFF-type, we reason as follows. First, since FSMI-morphisms of $\mathcal{F}$ project to either isomorphisms or FSMI-morphisms of $\mathcal{D}$ (cf. assertion (vi)), and $\mathcal{D}$ is of FSFF-type, it follows that for every $A \in \text{Ob}(\mathcal{F})$, there exists an integer $N$ such that given any composite

$$\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_2 \circ \phi_1$$

of FSMI-morphisms $\phi_1, \ldots, \phi_n$ such that the domain of $\phi_1$ is equal to $A$, it holds that the cardinality of the set of $j$ (where $j = 1, \ldots, n$) such that $\phi_j$ fails to project to an isomorphism of $\mathcal{D}$ is $\leq N$. If $\phi_j$ projects to an isomorphism of $\mathcal{D}$, then by assertion (iv), it follows that $\phi_j$ projects to an FSMI-morphism of $\mathcal{F}_0$. Thus, by our discussion of composites of (not necessarily FSMI-) morphisms in $\mathcal{F}_0$, it follows that there exists an $N'$ (depending only on $A!$) such that the cardinality of the set of $j \in \{1, \ldots, n\}$ for which $\phi_j$ projects to an isomorphism of $\mathcal{D}$ is $\leq N'$. But this implies that $n \leq N + N'$.

Next, let $\phi$ be an arbitrary FSM-morphism of $\mathcal{F}$. Observe that (by [Mzk5, Definition 1.3(iv)(a)] applied to the Frobenioid $A_0$) the projection $\phi_0$ of $\phi$ to $\mathcal{F}_0$ admits a factorization $\phi_0 = \alpha_0 \circ \beta_0$, where $\alpha_0$ is a pull-back morphism, and $\beta_0$ is a base-isomorphism. Since the projection of $\beta_0$ to $\mathcal{D}_0$ is an isomorphism, it thus follows that there exists a factorization $\phi = \alpha \circ \beta$ in $\mathcal{F}$, where $\alpha$, $\beta$ lift $\alpha_0$, $\beta_0$ respectively, and $\beta_0 \overset{\text{def}}{=} \text{Base}(\beta) \in \text{Arr}(\mathcal{D})$ is an isomorphism. Since $\phi_0 \overset{\text{def}}{=} \text{Base}(\phi) \in \text{Arr}(\mathcal{D})$ is an FSM-morphism (cf. assertion (iii)), it thus follows that $\alpha_0 \overset{\text{def}}{=} \text{Base}(\alpha) = \phi_0 \circ \beta_0^{-1} \in \text{Arr}(\mathcal{D})$ is an FSM-morphism, hence (by assertion (vii)) that $\alpha$ is an FSM-morphism. Moreover, since $\alpha$ is a monomorphism, it follows formally from the fact that $\phi$ is an FSM-morphism that $\beta$ is an FSM-morphism, hence (by assertion (iv), since $\beta_0$ is an isomorphism) that $\beta_0$ is an FSM-morphism of $\mathcal{F}_0$. Thus, by factoring $\beta_0$ as a composite of FSMI-morphisms of $\mathcal{F}_0$ (since we have already shown that $\mathcal{F}_0$ is of FSFF-type), we conclude that we may write $\beta = \beta_b \circ \cdots \circ \beta_1$, where the $\beta_j$ (for $j = 1, \ldots, b$) are morphisms of $\mathcal{F}$ that project to FSMI-morphisms of $\mathcal{F}_0$ and to isomorphisms of $\mathcal{D}$. Now it follows formally that the $\beta_j$ are monomorphisms, hence (again by a formal argument, since $\beta$ is an FSM-morphism) that the $\beta_j$ are FSM-morphisms, hence (cf. assertion (v)) FSMI-morphisms. Moreover, by factoring $\alpha_0$ as a composite of FSMI-morphisms of $\mathcal{D}$ (since $\mathcal{D}$ is, by assumption, of FSFF-type) and applying assertion (vii), we conclude that $\alpha$, $\beta$, hence also $\phi$ admits a factorization as a composite of FSMI-morphisms of $\mathcal{F}$. This completes the proof of assertion (viii). □

**Remark 3.4.1.** Note that there exist monomorphisms of $\mathcal{G}$ that do not satisfy either of the conditions of Proposition 3.3(ii), and which fail to project to monomorphisms of $\mathcal{D}$. Indeed, by using angular regions whose projections to $\mathbb{S}^1$ fail to intersect $\{1, -1\}$ (cf. Lemma 3.2(ii)), one may construct examples of linear monomorphisms $\phi : A \to B$ of $\mathcal{G}_0$, where $A$ is complex, and $B$ is real (so the projection of $\phi$ to $\mathcal{D}_0$ is not a monomorphism).

**Proposition 3.5. (Iso-subanchors)** In the notation and terminology of Example 3.3, let $\mathcal{F}$ be one of the categories $\mathcal{C}$, $\mathcal{A}$; $\mathcal{G}$ one of the categories $\mathcal{N}$, $\mathcal{R}$; $\mathcal{H}$ one of the categories $\mathcal{F}$, $\mathcal{G}$. If $\mathcal{F} = \mathcal{C}$ (respectively, $\mathcal{F} = \mathcal{A}$; $\mathcal{G} = \mathcal{N}$; $\mathcal{G} = \mathcal{R}$; $\mathcal{H} = \mathcal{F}$; $\mathcal{H} = \mathcal{G}$), then set $\mathcal{F}_0 = \mathcal{C}_0$ (respectively, $\mathcal{F}_0 = \mathcal{A}_0$; $\mathcal{G}_0 = \mathcal{N}_0$; $\mathcal{G}_0 = \mathcal{R}_0$; $\mathcal{H}_0 = \mathcal{F}_0$; $\mathcal{H}_0 = \mathcal{G}_0$). Suppose further that $\mathcal{D}$ is of RC-iso-subanchor type. Then we have the following.

(i) Let $A \in \text{Ob}(\mathcal{H})$ and suppose that $B_D \to A_D \overset{\text{def}}{=} \text{Base}(A)$ is a monomorphic categorical quotient of $B_D$ by a group $G_D \subseteq \text{Aut}_D(B_D)$ in $\mathcal{D}$. Then there exists a
pull-back morphism \( B \to A \) that lifts \( B_\mathcal{D} \to A_\mathcal{D} \) and a group \( G \subseteq \text{Aut}_\mathcal{H}(B) \) that maps isomorphically to \( G_\mathcal{D} \) such that \( B \to A \) is a mono-minimal categorical quotient of \( B \) by \( G \) in \( \mathcal{H} \).

(ii) The Frobenioid \( \mathcal{F} \) is quasi-isotropic, i.e., an object of \( \mathcal{F} \) is non-isotropic if and only if it is an iso-subanchor of \( \mathcal{F} \).

(iii) \( \mathcal{G} \) is of \( \text{RC-iso-subanchor type} \).

(iv) \( \mathcal{F} \) is not of \( \text{RC-iso-subanchor type} \).

Proof. First, we consider assertion (i). By [Mzk5, Definition 1.3(i)(c)], there exists a pull-back morphism \( B \to A \) of \( \mathcal{H} \) that lifts \( B_\mathcal{D} \to A_\mathcal{D} \). Write \( B_0 \to A_0, B_{D_0} \to A_{D_0} \) for the respective projections of \( B \to A \) to \( \mathcal{H}_0, \mathcal{D}_0 \). Now observe that it follows immediately from the simple, explicit structure of \( \mathcal{H}_0 \) (cf. Example 3.3(i)–(iv)) that the natural surjection

\[
\text{Aut}(\mathcal{H}_0)_{A_0}(B_0) \to \text{Aut}(\mathcal{D}_0)_{A_{D_0}}(B_{D_0})
\]
splits, hence that there exists a group \( G \subseteq \text{Aut}_\mathcal{H}(B) \) that maps isomorphically to \( G_\mathcal{D} \). Since \( B_\mathcal{D} \to A_\mathcal{D} \) is a categorical quotient of \( B_\mathcal{D} \) by \( G_\mathcal{D} \) in \( \mathcal{D} \), it follows again from the simple, explicit structure of \( \mathcal{H}_0 \) (cf. Example 3.3(i)–(iv)) that \( B \to A \) is a categorical quotient of \( B \) by \( G \) in \( \mathcal{H} \). If \( B \to B' \) is a monomorphism that violates the mono-minimality of \( B \to A \), then it follows from the fact that \( B \to A \) is a pull-back morphism, hence, in particular, an isometry, that \( B \to B' \) is an isometry (which implies that even if \( \mathcal{H} = \mathcal{C} \), the arrow \( B \to B' \) lies in \( \mathcal{A} \)), hence, by Proposition 3.4(ii), that the projection \( B_{D_0} \to B'_{D_0} \) of \( B \to B' \) to \( \mathcal{D} \) is a monomorphism that violates the mono-minimality of \( B_{D_0} \to A_{D_0} \). (Here, we observe that if \( B \to B' \) fails to project to an isomorphism of \( \mathcal{D}_0 \) (i.e., fails to satisfy condition (a) of Proposition 3.4(ii)), then the pull-back morphism \( B \to A \) also fails to project to an isomorphism of \( \mathcal{D}_0 \); but this implies that \( B \) is isotropic, so the morphism \( B \to B' \) necessarily satisfies condition (b) of Proposition 3.4(ii).) Thus, we conclude that \( B_{D_0} \to B'_{D_0} \) is an isomorphism. In particular, the pull-back morphism \( B \to A \) admits a factorization \( B \to B' \to A \), where \( B \to B' \) is a base-isomorphism which (cf. [Mzk5, Proposition 1.7(v)]) is also a pull-back morphism. But this implies (cf. [Mzk5, Remark 1.2.1]) that \( B \to B' \) is an isomorphism, hence that \( B \to A \) is mono-minimal, as desired. This completes the proof of assertion (i).

Next, we consider assertion (ii). By [Mzk5, Remark 3.1.1], it suffices to show that every non-isotropic (hence necessarily complex!) \( A \in \text{Ob}(\mathcal{F}) \) is an iso-subanchor of \( \mathcal{F} \). Since \( A_\mathcal{D} \overset{\text{def}}{=} \text{Base}(A) \in \text{Ob}(\mathcal{D}[\mathcal{C}]) \) is an RC-iso-subanchor of \( \mathcal{D} \), it follows that there exists a mono-minimal categorical quotient (in \( \mathcal{D} \)) \( B_\mathcal{D} \to A_\mathcal{D} \) of \( B_\mathcal{D} \) by a group \( G_\mathcal{D} \subseteq \text{Aut}_\mathcal{D}(B_\mathcal{D}) \), where \( B_\mathcal{D} \) is an RC-subanchon of \( \mathcal{D} \). Thus, by assertion (i), we obtain a pull-back morphism \( B \to A \) of \( \mathcal{F} \) that lifts \( B_\mathcal{D} \to A_\mathcal{D} \) and which is a mono-minimal categorical quotient of \( B \) by some group \( G \subseteq \text{Aut}_\mathcal{F}(B) \). Here, we note that since \( A \) is non-isotropic, it follows (cf. [Mzk5, Definition 1.3(vii)(b)]) that \( B \) is non-isotropic. Since \( B_\mathcal{D} \) is an RC-subanchon of \( \mathcal{D} \), it follows that there exists a morphism \( B_\mathcal{D} \to C_\mathcal{D} \) in \( \mathcal{D}[\mathcal{C}] \), where \( C_\mathcal{D} \) is an anchor of \( \mathcal{D}[\mathcal{C}] \); note, moreover, that since every morphism of \( \mathcal{D}_0[\mathcal{C}] \) is an isomorphism, we may assume that this morphism \( B_\mathcal{D} \to C_\mathcal{D} \) is the projection to \( \mathcal{D} \) of a pull-back morphism \( B \to C \) of \( \mathcal{F} \) (so \( C \) is non-isotropic). Thus, to show that \( A \) is an iso-subanchor of \( \mathcal{F} \), it suffices to show that \( C \) is an anchor of \( \mathcal{F} \).
Thus, let \( \phi : C \to C' \) be an irreducible morphism of \( \mathcal{F} \). Since \( \phi \) is irreducible, it follows (cf. the factorizations of \([\text{Mzk5}, \text{Definition 1.3(iv)}\) and \((v))\) that \( \phi \) is either a pull-back morphism, an isometric pre-step, a co-angular pre-step with irreducible zero divisor, or a prime-Frobenius morphism. Since \( \mathbb{R}_{\geq 0} \) has no irreducible elements (cf. \([\text{Mzk5}, \text{Section 0}]\)), it follows that \( \phi \) is not a co-angular pre-step. If \( \phi \) is a pull-back morphism or a prime-Frobenius morphism (hence a non-pre-step – cf. \([\text{Mzk5}, \text{Remark 1.2.1}]\)), then since \( C \) is non-isotropic, it follows (by considering the factorization of \( \phi \) through an isotropic hull of \( C \), together with the irreducibility of \( \phi \)) that \( C' \) is non-isotropic, hence, in particular, complex. Thus, the finiteness of the collection of isomorphism classes of \( C \mathcal{F} \) arising from (irreducible) \( \phi \) which are pull-back morphisms (respectively, prime-Frobenius morphisms) then follows immediately from the fact that \( C_D \) is an anchor of \( \mathcal{D}[C] \) (respectively, from Lemma 3.2(v)). Finally, (again since \( C \) is non-isotropic!) the finiteness of the collection of isomorphism classes of \( C \mathcal{F} \) arising from (irreducible) \( \phi \) which are isometric pre-steps follows formally from Lemma 3.2(iii) and (vii). This completes the proof of assertion (ii).

Next, we consider assertion (iii). Just as in the case of assertion (ii), we may apply assertion (i) to conclude that to complete the proof of assertion (iii), it suffices to show that any \( C \in \text{Ob}(\mathcal{G}[C]) \) such that \( C_D \overset{\text{def}}{=} \text{Base}(C) \in \text{Ob}(\mathcal{D}[C]) \) is an anchor of \( \mathcal{D}[C] \) is itself an anchor of \( \mathcal{G}[C] \). Thus, let \( \phi : C \to C' \) be an irreducible morphism of \( \mathcal{G}[C] \). Just as in the argument used to prove assertion (ii), since \( \phi \) is irreducible, it follows (cf. the factorizations of \([\text{Mzk5}, \text{Definition 1.3(iv)}\) and \((v))\) that \( \phi \) is either a pull-back morphism or an isometric pre-step (since all morphisms of \( \mathcal{G} \) are linear isometries). Now (cf. the argument used to prove assertion (ii)) the finiteness of the collection of isomorphism classes of \( C \mathcal{F} \) arising from (irreducible) \( \phi \) which are pull-back morphisms (respectively, isometric pre-steps) follows formally from the fact that \( C_D \) is an anchor of \( \mathcal{D}[C] \) (respectively, from Lemma 3.2(iii) and (vii)). This completes the proof of assertion (iii).

Finally, we consider assertion (iv). Suppose that \( \mathcal{F} \) is of \( \text{RC-iso-subanchor} \) type. Then it follows formally that \( \mathcal{F} \) admits a complex object, hence, in particular, a complex isotropic object \( A \in \text{Ob}(\mathcal{F}) \). Since \( A \) is an \( \text{RC-iso-subanchor} \), it follows immediately, by taking isotropic hulls (cf. the argument of \([\text{Mzk5}, \text{Remark 3.1.1}]\)), that \( \mathcal{F} \) admits a complex isotropic \( \text{RC-subanchor} B \), hence (cf. \([\text{Mzk5}, \text{Definition 1.3(vii)(b)}]\)) that \( \mathcal{F} \) admits a complex isotropic \( \text{RC-anchor} C \), i.e., that \( \mathcal{F}[C] \) admits a complex isotropic anchor \( C \). But this contradicts the existence of prime-Frobenius morphisms, of arbitrary prime degree, with domain equal to \( C \) (cf. \([\text{Mzk5}, \text{Proposition 1.10(iv)}]\)). This completes the proof of assertion (iv). \( \square \)

Remark 3.5.1. In the context of Proposition 3.5, we observe that it is important to consider not just anchors, but also subanchors since, for instance in the case of the base categories associated to connected temperoids (cf. Example 1.3), it is not difficult to construct examples of base categories that appear naturally in arithmetic geometry and which have objects that are subanchors, but not anchors. (Indeed, this phenomenon occurs in the case of the temperoid associated to a tempered group (such as a non-abelian discrete finitely generated free group) which admits a topological subquotient isomorphic to an infinite direct sum

\[
\bigoplus_{\mathbb{Z}/p\mathbb{Z}}
\]

of copies of \( \mathbb{Z}/p\mathbb{Z} \), equipped with the discrete topology.) Moreover, it is important to consider not just anchors and subanchors, but also \( \text{RC-anchors and RC-subanchors} \) – i.e., in short,
complex anchors and subanchors – since the arguments applied in the proof of Proposition 3.5 require one to work with non-isotropic (hence complex) objects that map to anchors of the base category. Thus, one is obliged to work with RC-iso-subanchors in order to accommodate real objects.

Our main result concerning the Frobenioids of Example 3.3 is the following.

**Theorem 3.6. (Basic properties of Archimedean, angular Frobenioids)** In the notation and terminology of Example 3.3, let \( \mathcal{F} \) be one of the following Frobenioids: \( C^A \) or \( \mathcal{A} \) (where, when \( \mathcal{F} = \mathcal{A} \), we take \( \Lambda = \mathbb{Z} \)). Also, denote by \( \Phi^\wedge \) the restriction \( \Phi^\wedge |_D \) to \( D \) of the functor \( \Phi^\circ : D_0 \to \text{Mon} \) determined by the assignment

\[
\text{(Ob}(D_0) \ni A \mapsto \text{Spec}(K) \mapsto \mathcal{O}^\times_K
\]

and by \( \Phi^{\text{fd}} \) the functor \( \mathcal{D} \to \text{Mon} \) given by \( \text{(Ob}(D) \ni D \mapsto \Phi^{\text{SP}}(D) \times \Phi^\wedge(D) \) (so projecting to the two factors of the product monoid yields natural transformations \( \Phi^{\text{fd}} \mapsto \Phi^\wedge \) and \( \Phi^{\text{fd}} \mapsto \Phi^{\text{SP}} \). If \( \Lambda = \mathbb{Z} \) (respectively, \( \Lambda = \mathbb{Q} \), \( \Lambda = \mathbb{R} \)), then write \( (\Phi^{\text{fd}})^\Lambda \overset{\text{def}}{=} \Phi^{\text{fd}} \) (respectively, \( (\Phi^{\text{fd}})^\Lambda \overset{\text{def}}{=} (\Phi^{\text{fd}})^\wedge \), \( (\Phi^{\text{fd}})^\Lambda \overset{\text{def}}{=} \Phi^{\text{SP}} \)). Then we have the following:

(i) The Frobenioid \( (C^A)^\text{istr} \) is of isotropic, base-trivial, and model type, with rational function monoid naturally isomorphic to \( (\Phi^{\text{fd}})^\Lambda \) and the canonical decomposition of Definition 3.1(ii) determines a characteristic splitting (cf. [Mzk5, Definition 2.3]) on \( C^A \). If \( \Lambda \geq \mathbb{Q} \), then \( (C^A)^\text{istr} = C^A \). For arbitrary \( \Lambda \), there is a natural equivalence of categories \( (C^A)^\text{istr} \sim C^A \), compatible with the Frobenioid structures; moreover, the Frobenioid \( C^A \) is of Aut-ample, Aut\_sub-ample, End-ample, and metrically trivial type, but not of group-like type. If, moreover, \( D \) is of FSMFF- and RC-iso-subanchor type, then \( C^A \) is of rationally standard type.

(ii) The Frobenioid \( \mathcal{A}^{\text{istr}} \) is of base-trivial and model type, with rational function monoid naturally isomorphic to \( \Phi^\wedge \). The Frobenioid \( \mathcal{A} \) is of Aut-ample, Aut\_sub-ample, End-ample, group-like, and metrically trivial type. If, moreover, \( D \) is of FSMFF- and RC-iso-subanchor type, then \( \mathcal{A} \) is of standard (but not of rationally standard) type.

(iii) Every morphism \( \phi : B \to A \) of \( C \) factors uniquely as a composite \( \phi = \beta \circ \alpha \), where \( \alpha \) is an isometry (hence belongs to \( \mathcal{A} \), and \( \beta \in \text{Ob}(\mathcal{D}) \subseteq \Phi^{\text{fd}}(A) \) belongs to the submonoid \( \Phi^{\text{SP}}(A) \times \{1\} \subseteq \Phi^{\text{fd}}(A) \) (cf. the characteristic splitting of (i)). Moreover, the assignment \( \phi \mapsto \alpha \) determines an ‘isometrization functor’

\[
\mathcal{C} \to \mathcal{A}
\]

which, together with the isotropification functors \( \mathcal{C} \to \mathcal{C}^{\text{istr}} \), \( \mathcal{A} \to \mathcal{A}^{\text{istr}} \) (cf. [Mzk5, Proposition 1.9(v)]), determines an equivalence of categories

\[
\mathcal{C} \sim \mathcal{A} \times_{\mathcal{A}^{\text{istr}}} \mathcal{C}^{\text{istr}}
\]

that is 1-compatible with the natural functors to \( \mathcal{F}_\Phi \) on both sides.

(iv) Let \( A \in \text{Ob}(\mathcal{F}) \) and \( A_D \overset{\text{def}}{=} \text{Base}(A) \in \text{Ob}(\mathcal{D}) \). Write \( A_0 \in \text{Ob}(D_0) \) for the image of \( A_D \) in \( D_0 \). Then the natural action of \( \text{Aut}_\mathcal{F}(A) \) on \( \mathcal{O}^\times(A) \), \( \mathcal{O}^\times(A) \) factors through \( \text{Aut}_{D_0}(A_0) \). If, moreover, \( \Lambda \in \{\mathbb{Z}, \mathbb{Q}\} \), then this factorization determines a faithful action of the image of \( \text{Aut}_\mathcal{F}(A) \) in \( \text{Aut}_{D_0}(A_0) \) on \( \mathcal{O}^\times(A) \), \( \mathcal{O}^\times(A) \).
Let $A \in \text{Ob}(\mathcal{F})$. Then the group $\mathcal{O}^\times(A)$ is trivial if and only if one of the following holds: (a) $\Lambda = \mathbb{Z}$ and $A$ is complex non-isotropic; (b) $\Lambda = \mathbb{Q}$ and $A$ is real; and (c) $\Lambda = \mathbb{R}$. The group $\mathcal{O}^\times(A)$ is nontrivial and torsion free (and in fact isomorphic to $\mathbb{S}^1 \otimes \mathbb{Z} \mathbb{Q}$) if and only if $\Lambda = \mathbb{Q}$ and $A$ is complex. The group $\mathcal{O}^\times(A)$ is of order two if and only if $\Lambda = \mathbb{Z}$ and $A$ is real. The group $\mathcal{O}^\times(A)$ has infinitely many torsion elements (and is in fact isomorphic to $\mathbb{S}^1$) if and only if $\Lambda = \mathbb{Z}$ and $A$ is complex isotropic.

If $\mathcal{D}$ admits a pseudo-terminal object, then $\mathcal{F}$ admits a pseudo-terminal object.

Suppose that $\Lambda = \mathbb{Z}$. Let $A \in \text{Ob}(\mathcal{F})$ be complex. Then the assignment that maps an isometric pre-step $B \rightarrow A$ of $\mathcal{F}$ to the image of the boundary $\partial \mathcal{A}_B$ of the angular region $\mathcal{A}_B$ of $B$ in the boundary $\partial \mathcal{A}_A$ of the angular region $\mathcal{A}_A$ of $A$ determines an equivalence of categories — which is functorial (cf. [Mzk5, Proposition 1.9(ii) and (iii)]) — in $A$

$$\mathcal{F}_A^{\text{imtr-pre}} \sim \text{Open}^0(\partial \mathcal{A}_A),$$

where $\mathcal{F}_A^{\text{imtr-pre}} \subseteq \mathcal{F}$ denotes the full subcategory determined by the arrows which are isometric pre-steps (cf. [Mzk5, Proposition 1.9]), $\mathcal{F}_A^{\text{imtr-pre}} \overset{\text{def}}{=} (\mathcal{F}_A^{\text{imtr-pre}})_A$ and $\text{Open}^0(\partial \mathcal{A}_V)$ is the category of connected open subsets of $\partial \mathcal{A}_V$ (cf. the Appendix).

In particular (cf. Theorem A.2(vi)), the topological space $\partial \mathcal{A}_A$ may be recovered functorially from the category $\mathcal{F}_A^{\text{imtr-pre}}$. Finally, if $A$ is isotropic, then the action of $\mathcal{O}^\times(A)$ on $\partial \mathcal{A}_A$ determines on $\partial \mathcal{A}_A$ a structure of torsor over this group.

If $\Lambda = \mathbb{Z}$, then $\mathcal{F}[\mathbb{C}]$ is of weakly dissectible type (cf. Section 0).

Suppose that $\mathcal{D}$ is of strongly indissectible type. If $\mathcal{D}$ is not complexifiable, then we assume further that $\Lambda \neq \mathbb{Z}$. Then $\mathcal{F}^{\text{istr}}$ is of strongly indissectible type.

If $\mathcal{D}$ is slim and $\Lambda \in \{\mathbb{Z}, \mathbb{R}\}$, then $\mathcal{F}$ is also slim.

**Proof:** First, we consider assertion (i). By (the earlier portion of) Lemma 3.2(ii), it is immediate from the construction of $\mathcal{C}^\Lambda$ that $\mathcal{C}^\Lambda$ is of Aut-ample, Aut$^{\text{sub-ample}}$, and End-ample type. Also, it is immediate from the construction of $\mathcal{C}^\Lambda$ that $\mathcal{C}^\Lambda$ is of metrically trivial, (strictly) rational, and birationally Frobenius-normalized type, but not of group-like type, and from the construction of $(\mathcal{C}^\Lambda)^{\text{istr}}$ that $(\mathcal{C}^\Lambda)^{\text{istr}}$ is of base-trivial and isotropic type. Since $\mathcal{D}_0$ admits a terminal object, it thus follows from [Mzk5, Proposition 2.9(ii)], that $(\mathcal{C}_0^\Lambda)^{\text{istr}}$, hence also $(\mathcal{C}^\Lambda)^{\text{istr}}$ (which may be constructed as a categorical fiber product $(\mathcal{C}_0^\Lambda)^{\text{istr}} \times_{\mathcal{D}_0} \mathcal{D}$), is of model type. Moreover, it is immediate from the construction of $(\mathcal{C}^\Lambda)^{\text{istr}}$ that the rational function monoid of $(\mathcal{C}^\Lambda)^{\text{istr}}$ is naturally isomorphic to $O^{\text{chi}}$ (cf. also the canonical decomposition of Definition 3.1(ii)); in a similar vein, it is immediate from the definitions that the canonical decomposition of Definition 3.1(ii) determines a characteristic splitting on $\mathcal{C}^\Lambda$. The fact that if $\Lambda \geq \mathbb{Q}$, then $(\mathcal{C}^\Lambda)^{\text{istr}} = \mathcal{C}^\Lambda$ is immediate from the definitions. The existence of a natural equivalence of categories $(\mathcal{C}^\Lambda)^{\text{un-trl}} \sim \mathcal{C}^\mathbb{R}$, compatible with the Frobenioid structures, follows immediately from the construction of $\mathcal{C}^\Lambda$. Also, it is immediate from the construction of $\mathcal{C}^\mathbb{R}$ that every object of $((\mathcal{C}^\Lambda)^{\text{un-trl}})^{\text{birat}} = (\mathcal{C}^\mathbb{R})^{\text{birat}}$ is Frobenius-compact. Thus, if, moreover, $\mathcal{D}$ is of FSMFF- and RC-iso-subanchor type, then (since $\Phi$ is manifestly non-dilating), to complete the proof of assertion (i), it suffices to observe that by Proposition 3.5(ii), $\mathcal{C}$ is of quasi-isotropic type. This completes the proof of assertion (i).
Next, we consider assertion (ii). The fact that the Frobenioid $\mathcal{A}$ is of Aut-ample, Aut\textsuperscript{ab}-ample, End-ample, group-like, and metrically trivial type, as well as the fact that the Frobenioid $\mathcal{A}^{\text{str}}$ is of base-trivial and model type, together with the description of the rational function monoid of the Frobenioid $\mathcal{A}^{\text{str}}$, follow immediately from the corresponding facts for $\mathcal{C}$ and $\mathcal{C}^{\text{str}}$ proven in assertion (i) (together with the construction of $\mathcal{A}$ and $\mathcal{A}^{\text{str}}$). If, moreover, $\mathcal{D}$ is of FSMFF- and RC-iso-subanchor type (which implies, in particular, that $\mathcal{D}$ admits complex objects), then one verifies immediately that the complex isotropic objects of $\mathcal{A}$ are Frobenius-compact; moreover, the quasi-isotropicity of $\mathcal{A}$ follows from Proposition 3.5(ii), while the Frobenius-isotropy, and Frobenius-normalizedness of $\mathcal{A}$ follows immediately from the corresponding properties for $\mathcal{C}$ (together with the construction of $\mathcal{A}$ and $\mathcal{C}$); thus, we conclude that $\mathcal{A}$ is of standard type. On the other hand, since, as is easily verified, $(\mathcal{A}^{\text{un-str}})^{\text{birat}}$ is of unit-trivial type, it follows that $\mathcal{A}$ is not of rationally standard type. This completes the proof of assertion (ii).

Next, we consider assertion (iii). The existence and uniqueness of the factorization asserted in the statement of assertion (iii) follows immediately from the construction of $\mathcal{C}$; also, it is immediate that this factorization yields an ‘isometrization functor’ $\mathcal{C} \to \mathcal{A}$. The resulting functor

$$\mathcal{C} \xrightarrow{\sim} \mathcal{A} \times_{\mathcal{A}^{\text{str}}} \mathcal{C}^{\text{str}}$$

is then manifestly essentially surjective and 1-compatible with the natural functors to $F_\phi$ on both sides. Now the fully faithfulness of this functor is immediate from the factorization just discussed, together with the 1-compatibility with the natural functors to $F_\phi$ on both sides. This completes the proof of assertion (iii).

Assertions (iv), (v) and (vi) follow immediately from the definitions.

Assertion (vii) follows immediately from the definitions and Theorem A.2(vi) of the Appendix (since $S^1$ is clearly sober and locally connected).

Assertion (viii) follows by considering disjoint angular regions.

Assertion (ix) follows, in light of the strong indissectibility assumption on $\mathcal{D}$, by reducing (cf. our assumption concerning the case when $\mathcal{D}$ is not complexifiable) to the easily verified fact that the categories $(\mathcal{C}^{\Lambda}_{\mathcal{A}})^{\text{str}}[\mathbb{R}]$ (when $\Lambda \neq \mathbb{Z}$) and $(\mathcal{C}^{\Lambda}_{\mathcal{A}})^{\text{str}}[\mathcal{C}]$ (for arbitrary $\Lambda$) are of strongly indissectible type.

Assertion (x) follows formally from [Mzk5, Proposition 1.13(iii)] (since, by assertions (v) and (vii) of the present Theorem 3.6, either ‘condition (a)’ or ‘condition (b)’ of loc. cit. is always satisfied by objects of $\mathcal{F}$).

Remark 3.6.1. Note that the topology of $O^x(\mathcal{A})$ (\(\cong S^1\)), for complex isotropic $\mathcal{A} \in \text{Ob}(\mathcal{C})$, may be recovered from the category-theoretic structure of $\mathcal{C}$ (cf. Theorem 3.6(vii)) precisely because of the existence of the non-isotropic objects. This is the principal reason for the inclusion of non-isotropic objects in the theory of Frobenioids.

4. Angloids as base categories

In this section, we show (cf. Corollary 4.2(v) and Remark 4.2.1) that the categories called ‘angloids’, which were constructed in Example 3.3(iii) and (iv), satisfy those properties required of a base category of an archimedean or angular Frobenioid (cf. Theorem 3.6(i) and (ii)), in order for the main results of the general theory of Frobenioids developed
in [Mzk5] to apply. In addition, we study various other basic properties of angloids, certain of which may be regarded (cf. Remark 4.2.1) as a sort of archimedean analogue of the category-theoreticity of the (non-archimedean) local reciprocity map, as discussed in Theorem 2.4(i) and (ii). Finally, we discuss the main motivating examples (cf. Examples 4.3 and 4.4) from arithmetic geometry of the theory developed thus far for archimedean primes.

**Proposition 4.1.** (Basic properties of angloids) In the notation and terminology of Example 3.3, let \( G \) be one of the categories \( \mathcal{N} \) or \( \mathcal{R} \); if \( G = \mathcal{N} \) (respectively, \( G = \mathcal{R} \)), then write \( G_0 \equiv N_0 \) (respectively, \( G_0 \equiv R_0 \)). Then we have the following.

(i) Let \( A \in \text{Ob}(G) \) be complex. Then the assignment that maps a base-isomorphism (or, equivalently, an isometric pre-step – cf. the definition of \( G \) in Example 3.3(iii) and (iv))

\[ B \to A \]

of \( G \) to the image of the boundary \( \partial \mathcal{A}_B \) of the angular region \( \mathcal{A}_B \) of \( B \) in the boundary \( \partial \mathcal{A}_A \) of the angular region \( \mathcal{A}_A \) of \( A \) determines an equivalence of categories – which is functorial (cf. [Mzk5, Proposition 1.9(ii) and (iii)]) in \( \mathcal{A} - G \imtr\text{-pre} \sim \text{Open}^0(\partial \mathcal{A}_A) \), where \( G \imtr\text{-pre} \subseteq G \) denotes the full subcategory determined by the arrows which are isometric pre-steps (cf. [Mzk5, Proposition 1.9]); \( G_A \imtr\text{-pre} \equiv (G \imtr\text{-pre})_A ; \text{Open}^0(\partial \mathcal{A}_V) \) is the category of connected open subsets of \( \partial \mathcal{A}_V \) (cf. the Appendix). In particular (cf. Theorem A.2(vi)), the topological space \( \partial \mathcal{A}_A \) may be recovered functorially from the category \( G_A \imtr\text{-pre} \). Finally, if \( A \) is isotropic, and \( G = \mathcal{N} \), then the action of \( \mathcal{O}^\times(A) \) on \( \partial \mathcal{A}_A \) determines on \( \partial \mathcal{A}_A \) a structure of torsor over this group.

(ii) A morphism of \( G \) with real domain or codomain is an isomorphism if and only if it is a base-isomorphism.

(iii) Suppose that \( \mathcal{D} \) is of discontinuously ordered type. Then every continuously ordered (cf. Section 0) morphism of \( G \) is a base-isomorphism (i.e., an isometric pre-step – cf. the definition of \( G \) in Example 3.3(iii) and (iv)).

(iv) Suppose that \( \mathcal{D} \) is of discontinuously ordered type. Then a morphism \( \phi : A \to B \) of \( G \) is a base-isomorphism (or, equivalently, an isometric pre-step – cf. the definition of \( G \) in Example 3.3(iii) and (iv)) if and only if there exists a morphism \( \psi : C \to A \) of \( G \) and quasi-continuously ordered (cf. Section 0) morphisms \( C \to C_1, C \to C_2 \) of \( G \) such that the object of \( cG \) constituted by the composite arrow \( \phi \circ \psi : C \to B \) is an inductive limit of the diagram

\[ C_2 \leftarrow C \to C_1 \]

in \( cG \).

(v) If \( \mathcal{D} \) is of strongly dissectible type, then so is \( G \). If \( \mathcal{D} \) is of weakly indissectible type, then so is \( G \).

**Proof.** Assertion (i) follows immediately from the definitions and Theorem A.2(vi) of the Appendix (cf. the proof of Theorem 3.6(vii)).

Assertion (ii) follows immediately from the definition of \( G \) in Example 3.3(iii) and (iv).

Next, we consider assertion (iii). Let \( \phi : A \to B \) be a continuously ordered (hence, in particular, totally ordered) morphism of \( G \) and \( \phi_\mathcal{D} \equiv \text{Base}(\phi) \in \text{Arr}(\mathcal{D}) \). Let \( \phi = \beta \circ \alpha \) be a
factorization of \( \phi \), where \( \alpha, \beta \) are monomorphisms which are not isomorphisms. Thus, \( \alpha, \beta \) are continuously ordered (hence, in particular, totally ordered). Then I claim that neither of the projections \( \alpha_0, \beta_0 \) of \( \alpha, \beta \) to \( \mathcal{G}_0 \) is an isomorphism. Indeed, suppose that the projection \( \gamma_0 \) of \( \gamma \in \{ \alpha, \beta \} \) to \( \mathcal{G}_0 \) is an isomorphism. Then one verifies immediately that the natural projection functor determines an equivalence of categories \( \mathcal{G}_\gamma \too \mathcal{D}_{\mathcal{Y} \mathcal{D}} \) (where we write \( \gamma_\mathcal{D} \eqdef \text{Base}(\gamma) \)). Since \( \gamma \) is continuously ordered, we thus conclude that \( \gamma_\mathcal{D} \) is continuously ordered. On the other hand, since \( \mathcal{D} \) is of discontinuously ordered type, this implies that \( \gamma_\mathcal{D} \) is an isomorphism, hence that \( \gamma \) is an isomorphism, a contradiction. This completes the proof of the claim. Since every morphism between real objects of \( \mathcal{G}_0 \) is an isomorphism (cf. assertion (ii)), we thus conclude that the codomain of \( \alpha \) is complex.

Next, I claim that \( \alpha \) is a base-isomorphism. Indeed, suppose that \( \alpha_\mathcal{D} \eqdef \text{Base}(\alpha) \) is not an isomorphism. Since \( \alpha_0 \) is not an isomorphism, it thus follows that by lifting \( \alpha_\mathcal{D} \) to a morphism of \( \mathcal{G} \) that projects to an isomorphism of \( \mathcal{G}_0 \) and lifting \( \alpha_0 \) to a morphism of \( \mathcal{G} \) that projects to an isomorphism of \( \mathcal{D} \), we thus obtain two factorizations of \( \alpha \), i.e., two objects of \( \mathcal{G}^\mathcal{D} \), neither of which ‘dominates’ the other (cf. the fact that \( \mathcal{D}, \mathcal{G}_0 \) are totally epimorphic!) – a contradiction (since \( \alpha \) is totally ordered). This completes the proof of the claim. Thus, in summary, \( \alpha \) is an isometric pre-step. More concretely, the morphism \( \alpha \) may be thought of as an ‘enlargement of the angular region that defines \( A \)’. In particular, by taking the union of the angular regions that arise from such enlargements, one obtains a factorization \( \phi = \beta^\infty \circ \alpha^\infty \), where \( \alpha^\infty : A \to A^\infty \) is an isometric pre-step (hence, in particular, a base-isomorphism), such that every \( \alpha \) appearing in a factorization \( \phi = \beta \circ \alpha \) as above determines a factorization \( \alpha^\infty = \alpha' \circ \alpha \).

Next, I claim that \( \beta^\infty \) is an isomorphism. Indeed, suppose that \( \beta^\infty \) is not an isomorphism. Now if \( \beta^\infty \) is a monomorphism, then \( \alpha^\infty \) and \( \beta^\infty \) determine an object of \( \mathcal{G}^\mathcal{D} \) which (by the definition of \( \alpha^\infty \), our assumption that \( \beta^\infty \) is not an isomorphism) contradicts the fact that \( \phi \) is continuously ordered. Thus, we conclude that \( \beta^\infty : A^\infty \to B \) is not a monomorphism, i.e., that there exist distinct morphisms \( \gamma' \), \( \gamma'' \) : \( C \to A^\infty \) in \( \mathcal{G} \) such that \( \beta^\infty \circ \gamma' = \beta^\infty \circ \gamma'' \). On the other hand, by the definition of \( \alpha^\infty \) in terms of angular regions, it follows immediately that there exists a factorization \( \phi = \beta \circ \alpha \), where we write \( \alpha : A \to E \) and \( \beta : E \to B \), that forms an object of \( \mathcal{G}^\mathcal{D} \), together with morphisms \( \delta : D \to C \) and \( \epsilon' \), \( \epsilon'' : D \to E \) of \( \mathcal{G} \) such that \( \gamma' \circ \delta = \xi \circ \epsilon' \) and \( \gamma'' \circ \delta = \xi \circ \epsilon'' \), where \( \xi : E \to A^\infty \) is the morphism arising from the definition of \( \alpha^\infty \) (so \( \beta^\infty \circ \xi = \beta \)), and \( \delta \) is an isometric pre-step. Thus, we conclude that \( \beta^\infty \circ \xi \circ \epsilon' = \beta^\infty \circ \gamma' \circ \delta = \beta^\infty \circ \gamma' \circ \delta = \beta^\infty \circ \xi \circ \epsilon'' \), i.e., that \( \beta \circ \epsilon' = \beta \circ \epsilon'' \). But since \( \beta \) is a monomorphism, this implies that \( \epsilon' = \epsilon'' \), hence that \( \gamma' \circ \delta = \gamma'' \circ \delta \), i.e., (since \( \delta \) is an epimorphism!) that \( \gamma' = \gamma'' \) – a contradiction. This completes the proof of the claim. Since \( \phi = \beta^\infty \circ \alpha^\infty \), and \( \alpha^\infty \) is a base-isomorphism, we thus conclude that \( \phi \) is also a base-isomorphism, as desired. This completes the proof of assertion (iii).

Next, we consider assertion (iv). To verify the sufficiency of the condition in the statement of assertion (iv), observe that by assertion (iii), the morphisms \( C \to C_1 \) and \( C \to C_2 \) are base-isomorphisms (i.e., isometric pre-steps); thus, any isotropic hull \( C \to C' \) of \( C \) factors through \( C \to B \) (cf. [Mzk5, Proposition 1.9(vii)]), which, by the total epimorphy of \( \mathcal{D} \), implies that \( C \to B \), hence also \( A \to B \), is a base-isomorphism, as desired. Next, we consider necessity. In the real case, this necessity follows immediately from assertion (ii). In the complex case, the necessity in question follows from Lemma 3.2(viii) (which we apply to
construct the morphisms $C \rightarrow C_1$ and $C \rightarrow C_2$ (for an appropriate isometric pre-step $C \rightarrow A$) and Lemma 3.2(vii) (which we apply to show that these morphisms $C \rightarrow C_1$ and $C \rightarrow C_2$ are quasi-continuously ordered isometric pre-steps). (Here, we note that ‘continuously ordered pairs’ in the context of Lemma 3.2 give rise to ‘continuously ordered isometric pre-steps’.) This completes the proof of assertion (iv).

Finally, we observe that assertion (v) follows by considering appropriate isometric pre-steps and pull-back morphisms to relate ‘dissections in $\mathcal{G}$’ to ‘dissections in $\mathcal{D}$’ (cf. the discussion of Section 0).

**Corollary 4.2. (Category-theoreticity of factors of angloids)** For $i = 1, 2$, let $\mathcal{D}_i$ be a connected, totally epimorphic category of discontinuously ordered type, equipped with a functor $\mathcal{D}_i \rightarrow \mathcal{D}_0$, with respect to which $\mathcal{D}_i$ is complexifiable let $\mathcal{G}_i$ be one of the categories $\mathcal{N}_i \overset{\text{def}}{=} \mathcal{N}_0 \times_{\mathcal{D}_0} \mathcal{D}_i$, $\mathcal{R}_i \overset{\text{def}}{=} \mathcal{R}_0 \times_{\mathcal{D}_0} \mathcal{D}_i$ (cf. Example 3.3(iii) and (iv)), and let $\psi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$

be an equivalence of categories. If $\mathcal{G}_i = \mathcal{N}_i$ (respectively, $\mathcal{G}_i = \mathcal{R}_i$), then set $\mathcal{G}_{i,0} \overset{\text{def}}{=} \mathcal{N}_0$ (respectively, $\mathcal{G}_{i,0} \overset{\text{def}}{=} \mathcal{R}_0$). Then we have the following.

(i) If $\mathcal{G}_{1,0} = \mathcal{N}_0$ (respectively, $\mathcal{G}_{1,0} = \mathcal{R}_0$), then $\mathcal{G}_{2,0} = \mathcal{N}_0$ (respectively, $\mathcal{G}_{2,0} = \mathcal{R}_0$). In particular, it makes sense to write $\mathcal{G}_0 \overset{\text{def}}{=} \mathcal{G}_{1,0} = \mathcal{G}_{2,0}$.

(ii) The category of $\mathcal{G}_0$ is slim. If $\mathcal{G}_0 = \mathcal{N}_0$ (respectively, $\mathcal{G}_0 = \mathcal{R}_0$), then there is a natural outer isomorphism between the group of isomorphism classes of self-equivalences of the category $\mathcal{G}_0$ and the group of translations and reflections $\text{Refl}(\mathbb{S}^1)$ of Lemma 3.2(xii) (respectively, the group of self-homeomorphisms of $\mathbb{S}^1$ that commute with the complex conjugation automorphism $\iota : \mathbb{S}^1 \rightarrow \mathbb{S}^1$).

(iii) Suppose that $\mathcal{D}_i$ is RC-connected. Then there exists a 1-unique 1-commutative diagram

$$
\begin{array}{ccc}
\mathcal{G}_1 & \xrightarrow{\psi} & \mathcal{G}_2 \\
\downarrow & & \downarrow \\
\mathcal{G}_0 & = & \mathcal{G}_0
\end{array}
$$

where the vertical arrows are the natural functors and the horizontal arrows are equivalences of categories.

(iv) If $\mathcal{D}_1$, $\mathcal{D}_2$ are Frobenius-slim, then so are $\mathcal{G}_1$, $\mathcal{G}_2$. Suppose that $\mathcal{D}_1$, $\mathcal{D}_2$ are slim. Then $\mathcal{G}_1$, $\mathcal{G}_2$ are slim, and there exists a 1-unique 1-commutative diagram

$$
\begin{array}{ccc}
\mathcal{G}_1 & \xrightarrow{\psi} & \mathcal{G}_2 \\
\downarrow & \, & \downarrow \\
\mathcal{D}_1 & \xrightarrow{\psi^{\text{Base}}} & \mathcal{D}_2
\end{array}
$$

where the vertical arrows are the natural functors and the horizontal arrows are equivalences of categories; both composite functors are rigid.
If (for \( i = 1, 2 \)) \( D_i \) is of RC-standard type, then so is \( G_i \). In particular, in this case, \( G_i \) satisfies all of the hypotheses on ‘\( D \)’ in Theorem 3.6(i) and (ii).

**Proof.** First, we observe that \( \Psi \) preserves base-isomorphisms (cf. Proposition 4.1(iv)). Since the complex objects of \( G_i \) are precisely the objects that appear as domains or codomains of base-isomorphisms which are not isomorphisms (cf. Proposition 4.1(i) and (ii)), it thus follows that \( \Psi \) preserves real and complex objects. For \( i = 1, 2 \), suppose that \( A_i \in \text{Ob}(G_i) \) is complex, and that \( A_2 = \Psi(A_1) \). (Here, we note that since \( D_1 \) is complexifiable, it follows that such \( A_1, A_2 \) always exist.) Then \( \Psi \) induces a homeomorphism

\[
\partial H_{A_1} \sim \partial H_{A_2}
\]

(cf. Proposition 4.1(i) and Theorem A.2(ii)), which is functorial in \( A_1, A_2 \). In particular, it follows (cf. Lemma 3.2(x)) that \( \Psi \) preserves isotropic objects and isotropic hulls.

Next, we consider assertion (i). Observe (cf. Theorem 3.6(v)) that \( O^\times(A_i) = \{1\} \) if and only if either of the following conditions holds: (a) \( G_i = R_i \); (b) \( G_i = N_i \), and \( A_i \) is non-isotropic. Moreover, one verifies immediately that there exist non-isotropic \( A_i \) such that the natural homomorphism \( \text{Aut}_{G_i}(A_i) \to \text{Aut}_{D_i}(\text{Base}(A_i)) \) (whose kernel is equal to \( O^\times(A_i) \)) is surjective (cf. the Aut-ampleness of Theorem 3.6(ii)). In particular, we conclude that \( G_i = R_i \) if and only if there exist non-isotropic \( A_i \) such that if \( A_i \to A_i^{\text{istr}} \) is an isotropic hull, then the natural homomorphism \( \text{Aut}_{G_i}(A_i) \to \text{Aut}_{G_i}(A_i^{\text{istr}}) \) (induced by the defining property of the isotropic hull) is an isomorphism. This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us observe that the existence of the functorial homeomorphism \( \partial H_{A_1} \sim \partial H_{A_2} \) discussed above (when \( D_i = D_0 \)) implies that \( G_0 \) is slim. Now let us consider this functorial homeomorphism for complex isotropic \( A_i \). Note that since all complex isotropic objects of \( G_0 \) are isomorphic, we may assume (for the remainder of the proof of the present assertion (ii)), without loss of generality, that \( A_1 = A_2 \). Write \( I \subseteq \text{Homeo}(S^1) \) for the image in \( \text{Homeo}(S^1) \) (cf. Lemma 3.2(xii)) of \( \text{Aut}_{G_0}(A_i) \). Observe that this image is given (for a suitable identification of \( \partial H_{A_i} \) with \( S^1 \)) by the subgroup \( \text{Refl}(S^1) \) (cf. Lemma 3.2(xiii)), in the case \( G_0 = N_0 \), and by the subgroup (of order 2) generated by \( i \) in the case \( G_0 = R_0 \). In particular, the subset of \( I \) consisting of elements that reverse the orientation of \( S^1 \) – i.e., of elements that determine the structure of real objects of \( G_0 \) as categorical quotients of complex isotropic objects – is preserved by conjugation by arbitrary elements of the normalizer of \( I \) in \( \text{Homeo}(S^1) \) (cf. Lemma 3.2(xii)). Thus, by thinking of the functorial homeomorphism \( \partial H_{A_1} \sim \partial H_{A_2} \) as an element of \( \text{Homeo}(S^1) \), we conclude that the isomorphism classes of self-equivalences of \( G_0 \) may be identified with the normalizer of \( I \) in \( \text{Homeo}(S^1) \) (which may be computed when \( G_0 = N_0 \) by applying Lemma 3.2(xiii)). This completes the proof of assertion (ii).

Next, we consider assertion (iii). First, let us observe that since \( D_i \) is complexifiable, it follows that we may choose \( A_i \) to be (complex) isotropic and such that the natural homomorphism \( \text{Aut}_{G_i}(A_i) \to \text{Aut}_{D_i}(\text{Base}(A_i)) \) is surjective (a condition that may be phrased in category-theoretic terms as the condition that \( A_i \) admit an automorphism (in \( G_i \)) that reverses the orientation of \( \partial H_{A_i} \)). Let \( B_i \) be a complex isotropic object of \( G_i \) and write \( \text{Aut}_{G}(B_i) \) for the group of automorphisms of \( \partial H_{B_i} \) induced by automorphisms of \( B_i \) (in \( G_i \)). Since \( D_i \) is RC-connected, we obtain a homeomorphism \( \partial(A_i, B_i) : \partial H_{A_i} \sim \partial H_{B_i} \) by applying [Mzk5, Proposition 1.9(ii) and (iii)], to various morphisms of \( G_i(\mathbb{C}) \) to ‘connect’
Moreover, by projecting to $G_0$, one verifies immediately that the $\text{Aut}_{\partial}(A_i)$-orbit – which we denote by $\partial(A_i, B_i)$ – of this homeomorphism $\partial(A_i, B_i)$ depends only on $B_i$ (i.e., not on the choice of morphisms used to ‘connect’ $A_i$ to $B_i$). In a similar vein, by projecting to $G_0$, one verifies immediately that conjugation by any homeomorphism $\in \partial(A_i, B_i)$ maps $\text{Aut}_{\partial}(B_i)$ into $\text{Aut}_{\partial}(A_i)$. In particular, by applying the complexity of $\mathcal{D}_i$ in the case of real objects of $\mathcal{G}_i$, we conclude as in the proof of assertion (ii) that this functorial homeomorphism determines a 1-unique 1-commutative diagram as in the statement of assertion (iii).

Next, we consider assertion (iv). First, let us observe that the portion of assertion (iv) concerning the Frobenius-slimness and slimness of $\mathcal{G}_1, \mathcal{G}_2$ follows immediately from Proposition 4.1(i) (cf. the use of condition (a) in the proof of [Mzk5, Proposition 1.13(iii)]), together with the assumption that $\mathcal{D}_1, \mathcal{D}_2$ are Frobenius-slim or slim; similarly, the rigidity of the composite functors in the diagram appearing in the statement of assertion (iv) (if such a diagram exists) follows immediately from the slimness of $\mathcal{D}_1, \mathcal{D}_2$ (cf. [Mzk5, Proposition 1.13(i)]). Now assume that $\mathcal{D}_1, \mathcal{D}_2$ are slim. Next, let us observe that the functor

$$\mathcal{G}_i \to \mathcal{G}_i^{\text{istr}}$$

(where $\mathcal{G}_i^{\text{istr}} \subseteq \mathcal{G}_i$ is the full subcategory determined by the isotropic objects) induced by the isotropification functor of [Mzk5, Proposition 1.9(v)], may be reconstructed, up to isomorphism, as the left adjoint to the natural inclusion functor $\mathcal{G}_i^{\text{istr}} \hookrightarrow \mathcal{G}_i$ (cf. the analogous fact for Frobenioids, discussed in [Mzk5, Proposition 1.9(v)]). Moreover, one verifies immediately that the natural projection functor $\mathcal{G}_i^{\text{istr}} \to \mathcal{D}_i$ is full and essentially surjective, and that if $A, B \in \text{Ob}(\mathcal{G}_i^{\text{istr}})$, then

$$\text{Hom}_{\mathcal{D}_i}(\text{Base}(A), \text{Base}(B))$$

may be reconstructed as the quotient of $\text{Hom}_{\mathcal{G}_i^{\text{istr}}}(A, B) = \text{Hom}_{\mathcal{G}_i}(A, B)$ by the natural action of $O^\times(A)$ (cf. the analogous fact for Frobenioids, i.e., the faithfulness discussed in [Mzk5, Proposition 3.3(iv)]). Since we have already seen that $\Psi$ preserves isotropic objects (hence, in particular, the subcategory $\mathcal{G}_i^{\text{istr}} \subseteq \mathcal{G}_i$), to complete the proof of assertion (iv), it suffices to show that $\Psi$ preserves $O^\times(-)$.

Let $A \in \text{Ob}(\mathcal{G}_i^{\text{istr}})$. Then the projection functor $\mathcal{G}_i \to \mathcal{D}_i$ induces an equivalence of categories

$$(\mathcal{G}_i)^{\text{istr}}_A \sim (\mathcal{D}_i)_{\text{Base}(A)};$$

cf. [Mzk5, Definition 1.3(i)(c)] and the fact (cf. [Mzk5, Proposition 1.4(ii)]) that a morphism of a Frobenoid is LB-invertible and linear if and only if it is a pull-back morphism. Thus, since $\mathcal{D}_i$ is slim, the group

$$\text{Aut}((\mathcal{G}_i)^{\text{istr}}_A \to \mathcal{D}_i) \cong \text{Aut}((\mathcal{D}_i)_{\text{Base}(A)} \to \mathcal{D}_i)$$

is trivial. In particular (composition with the projection functor $\mathcal{G}_i \to \mathcal{D}_i$ reveals that), any

$$\alpha \in \text{Aut}((\mathcal{G}_i)^{\text{istr}}_A \to \mathcal{G}_i)$$
consists of \textit{base-identity} automorphisms, i.e., that by considering the automorphism of \( A \) determined by \( \alpha \), we obtain a \textit{natural isomorphism} 

\[ O^\times (A) \xrightarrow{\sim} \text{Aut}((\mathcal{G}_t)^\text{istr}_A \rightarrow \mathcal{G}_t) \]

(cf. the analogue of this argument for Frobenioids given in the proof of \([\text{Mzk5}, \text{Corollary 4.11(i)}]\)). In particular, we conclude that \( \Psi \) preserves ‘\( O^\times (\cdot) \)' as desired. This completes the proof of assertion (iv).

Finally, we observe that assertion (v) follows formally from Proposition 3.4(viii) and 3.5(iii).

\begin{remark}
(i) Thus, by Corollary 4.2(v), if \( D \) is any category of \textit{discontinuously ordered} type, equipped with a functor \( D \rightarrow D_0 \) such that \( D \) is \textit{RC-standard} type, then the resulting categories \( \mathcal{N} \) and \( \mathcal{R} \) form ‘\textit{suitable base categories}', from the point of view of Theorem 3.6(i) and (ii). That is to say:

\textit{The main results of the theory of} \([\text{Mzk5}]\) \textit{may be applied to archimedean or angloid Frobenioids over the base categories} \( \mathcal{N}, \mathcal{R} \).

Moreover, in the case of \( \mathcal{N} \), if \( A \) is a complex isotropic object of such an \textit{archimedean} or \textit{angloid Frobenioid} that projects to a complex isotropic object \( A_N \) of the base category \( \mathcal{N} \), then one obtains natural \textit{category-theoretic} isomorphisms of the \textit{topological group structure} of the ‘\( S^1 \)’ (i.e., the ‘\( O^\times (\cdot) \)’) of \( A \) with the ‘\( S^1 \)’ of \( A_N \) via Theorem 3.6(vii), Proposition 4.1(i) and Corollary 4.2(ii) and (iii). Furthermore:

Such category-theoretic isomorphisms between the ‘\textit{Frobenioid-theoretic}’ and ‘\textit{base category-theoretic}’ copies of ‘\( S^1 \)’ may be regarded as an \textit{archimedean analogue} of the category-theoreticity of the \textit{(non-archimedean) local reciprocity map}, as discussed in Theorem 2.4(i) and (ii).

Indeed, this sort of ‘archimedean analogue of the local reciprocity map’ constituted one of the \textit{main goals} of the author in the development of the (technically somewhat cumbersome – cf., especially, Propositions 3.4 and 3.5) theory of Section 3.

(ii) It is perhaps of interest to note that in the ‘archimedean analogue of the category-theoreticity of the local reciprocity map’ discussed in (i), if one forgets the vast collection of terminology and ‘general nonsense’ developed so far in the theory of Section 3 and the present Section 4, then the ‘\textit{main substantive ingredients}’ of the theory appear to be the following:

(a) the \textit{reconstruction} of \( S^1 \) from the associated category of open connected subsets \( \text{Open}^0(S^1) \) (cf. the theory of the Appendix);

(b) the fact that \( \text{Ref}(S^1) \) is equal to its own \textit{normalizer} in \( \text{Homeo}(S^1) \) (cf. Lemma 3.2(xii)).

Here, we recall that (a) was applied in the proof of Theorem 3.6(vii) and Proposition 4.1(i), while (b) was applied in the proof of Corollary 4.2(ii).

(iii) Write \( G_K \) for the \textit{absolute Galois group} of a \textit{non-archimedean \textit{(mixed-characteristic) local field} \( K \)}. Then relative to the point of view discussed in (i), in which \( \mathcal{G}_0 \) (especially \( N_0 \)) plays the role of a sort of \textit{archimedean version} of \( G_K \), the \textit{slimness} of \( \mathcal{G}_0 \) (cf. Corollary 4.2(ii))
may be thought of as a sort of *archimedean analogue* of the slinness of $G_K$ (cf., e.g., [Mzk1, Theorem 1.1.1(ii)]). In a similar vein, the ‘category-theoreticity of the projection to $G_0$’ given in Corollary 4.2(iii) may be thought of as a sort of *archimedean version of the group-theoreticity of the projection to $G_K$* given in [Mzk1, Lemma 1.3.8].

Finally, we discuss some examples from *arithmetic geometry at archimedean primes* of categories ‘$D$’ which satisfy the hypotheses of Corollary 4.2(v).

**Example 4.3.** (Categories associated to hyperbolic Riemann surfaces)

(i) In the notation and terminology of [Mzk3, Section 2], let $(X^*, \phi^*)$ be an *RC-Teichmüller pair* (roughly speaking; a hyperbolic Riemann surface considered up to complex conjugation and equipped with a non-zero *square differential*) let $\pi_1(X^*) \to \Pi$ (where $\pi_1(X^*)$ is the topological fundamental group of $X^*$) be a *dense morphism* of *tempered topological groups* such that $\Pi$ is ‘*totally ramified at infinity*’ and ‘*stack-resolving*’ and let $\Box \in \{\mathfrak{P}, \mathfrak{R}, \mathfrak{S}\}$. Then associated to this data, one may define (cf. [Mzk3, the discussion preceding Proposition 2.2]) a category

$$D \overset{\text{def}}{=} \text{Loc}_\Pi^\Box(X^*, \phi^*)$$

of parallelogram, rectangle, or square localizations (depending on the choice of $\Box$) of $(X^*, \phi^*)$. Roughly speaking, this category consists of two (mutually exclusive) types of objects: (a) *complete objects*, which are coverings of the hyperbolic Riemann surface in question (as prescribed by $\Pi$); (b) *parallelogram objects*, which are (pre-compact) parallelograms, rectangles, or squares (depending on the choice of $\Box$) on the (universal covering of the non-critical locus of the) Riemann surface in question, relative to the natural parameters (in the sense of Teichmüller theory) determined by the given square differential. The main result proven concerning this category $D$ in [Mzk3] is that if $\Box = \mathfrak{R}, \mathfrak{S}$ (respectively, $\Box = \mathfrak{P}$), then one may reconstruct the given Riemann surface, together with its *conformal structure* (respectively, *quasiconformal structure*) – up to possible confusion with the corresponding complex conjugate structures – in an entirely *category-theoretic* fashion from the category $D$ (cf. [Mzk3, Theorem 2.3]). It is immediate from the definition of $D$ that $D$ is equipped with a *natural functor* $D \to D_0$ (that maps ‘objects of complex type’ of $D$ (cf. [Mzk3, Proposition 1.5(i)]) to complex objects of $D_0$ and ‘objects of real type’ of $D$ (cf. [Mzk3, Proposition 1.5(ii)]) to real objects of $D_0$).

(ii) Here, we wish to observe that:

$D$ (equipped with the natural functor $D \to D_0$) is a *slim* category of *RC-standard, FSM-*, and *strictly partially ordered* (hence, a fortiori, of *discontinuously ordered*) type (cf. Remark 4.2.1).

Indeed, $D$ is *slim* (cf. [Mzk3, Theorem 2.3(i)]), *totally epimorphic* (cf. [Mzk3, Proposition 2.2(v)]), *RC-connected* (cf. the definition of $D$ in loc. cit.), and *complexifiable* (cf. our assumption that $\Pi$ is ‘stack-resolving’). Since the complete objects of $D$ that arise from *finite Galois coverings* of $X^*$ of *complex type* (where we note that such objects always exist, by our assumption that $\Pi$ is ‘stack-resolving’) form *RC-anchors* of $D$, it follows immediately from the definition of $D$ in loc. cit. that $D$ is of *RC-iso-subanchor* type. Next, let us observe that it follows immediately from the description of *monomorphisms* of $D$
given in [Mzk3, Proposition 2.2(ix)] that \( \mathcal{D} \) is of \( \text{FSM} \)- (hence, \textit{a fortiori}, of \( \text{FSMFF} \)-) type. Since, moreover, any parallelogram \( P_1 \) (of type ‘\( \square \)’) contained in (a compact subset of) a parallelogram \( P_2 \) (of type ‘\( \square \)’) of the Euclidean plane admits ‘intermediary parallelograms’ (of type ‘\( \square \)’) \( P_3 \) (i.e., \( P_1 \subseteq P_3 \subseteq P_2 \), \( P_3 \neq P_1, P_2 \)) in two dimensions (hence, in particular, intermediary \( P_3, P_4 \) (of type ‘\( \square \)’) neither of which is contained in the other), it follows from the description of monomorphisms of \( \mathcal{D} \) given in [Mzk3, Proposition 2.2(ix)], that \( \mathcal{D} \) is of \textit{strictly partially ordered} type.

(iii) Now, suppose that, in the notation of Example 1.4, we take ‘\( \text{strictly partially ordered} \) \( \mathcal{D} \) the description of monomorphisms of \( \text{Loc} \) (of type ‘\( \square \)’) allows one to rule the case where \( \text{amply quadratic} \) \( P \) (i.e., \( P_1 \subseteq P_3 \subseteq P_2 \), \( P_3 \neq P_1, P_2 \)) in two dimensions (hence, in particular, intermediary \( P_3, P_4 \) (of type ‘\( \square \)’) neither of which is contained in the other), it follows from the description of monomorphisms of \( \mathcal{D} \) given in [Mzk3, Proposition 2.2(ix)], that \( \mathcal{D} \) is of \textit{strictly partially ordered} type.

Thus, \( \mathcal{E} \) may be identified with \( \mathcal{D} \times \mathcal{D}_0 \mathcal{E}_0 \), and we note that, so long as \( \widetilde{F}/F \) is \textit{amply quadratic}, even if \( \mathcal{P}_0 \to \mathcal{D}_0 \) \textit{fails} to be an equivalence of categories, the difference between \( \mathcal{D} \) and \( \mathcal{P} \) amounts, in effect, to forming ‘\( \mathcal{D} \)’ as above, but with the original ‘\( X^* \)’ replaced by the result of ‘tensoring \( X^* \) over \( \mathbb{R} \) with \( \mathbb{C} \)’. (Put another way, the assumption that \( \widetilde{F}/F \) is \textit{amply quadratic} allows one to rule the case where \textit{every} object in the image of \( \mathcal{P}_0 \) in \( \mathcal{D}_0 \) is \textit{real}.) Then:

Suppose that \( \widetilde{F}/F \) is \textit{amply quadratic}. Then \( \mathcal{E} \) (equipped with the natural functor \( \mathcal{E} \to \mathcal{D}_0 \)) is a \textit{slim} category of \textit{RC-standard} and \textit{discontinuously ordered} type (cf. Remark 4.2.1).

Indeed, the \textit{slimness} of \( \mathcal{E} \) follows immediately from the slimness of \( \mathcal{D} \) (cf. (ii) above) and Proposition 1.5(iv); the \textit{total epimorphicity} of \( \mathcal{E} \) follows immediately from the total epimorphicity of \( \mathcal{D} \) (cf. (ii) above) and Proposition 1.5(iii); the \textit{RC-connectedness} follows immediately from the RC-connectedness of \( \mathcal{D} \) (cf. (ii) above) and Proposition 1.5(ii); the \textit{complexifiable} of \( \mathcal{E} \) follows immediately from the complexifiability of \( \mathcal{D} \) (cf. (ii) above) and our assumption that \( \widetilde{F}/F \) is \textit{amply quadratic}; the fact that \( \mathcal{E} \) is of \textit{FSMFF-type} follows from the fact that \( \mathcal{D} \) is \textit{totally epimorphic} and of \textit{FSM-type} (cf. (ii) above) and Proposition 1.5(viii); the fact that \( \mathcal{E} \) is of \textit{RC-iso-subanchor} type follows from the corresponding fact for \( \mathcal{D} \) (cf. (ii) above) and our assumption that \( \widetilde{F}/F \) is \textit{amply quadratic}. Finally, by Proposition 1.5(ix), it follows that \textit{totally ordered} morphisms of \( \mathcal{E} \) project to \textit{totally ordered} morphisms of \( \mathcal{D} \), which are \textit{isomorphisms} (since \( \mathcal{D} \) is of \textit{strictly partially ordered} type – cf. (ii) above); thus, it follows immediately from the ‘discrete structure’ of \( \mathcal{E}_0 \) that \( \mathcal{E} \) is of \textit{discontinuously ordered} type, as desired.

(iv) Finally, we observe that by considering parallelogram objects whose images in \( X^*(\mathbb{C}) \) are \textit{sufficiently small} and \textit{disjoint}, it follows immediately that the categories \( \mathcal{D}, \mathcal{E} \) are of \textit{strongly dissectible} type (cf. Proposition 1.5(viii)).

Remark 4.3.1. Note that, by contrast to the categories of Riemann surfaces discussed in Example 4.3, the categories of Riemann surfaces ‘\( \text{Loc}_{\Pi}(X^*) \)’ of [Mzk3, Section 1] – which are intended to embody the \textit{upper half-plane uniformization} of a hyperbolic Riemann surface – \textit{fail} to be of \textit{FSMFF-type}. Indeed, if \( \mathcal{D} \) is the \textit{open unit disc} in the complex plane, and \( V \subseteq D, U \subseteq V \) \textit{are connected open subsets}, then the inclusion \( U \leftrightarrow V \) determines a monomorphism of this category ‘\( \text{Loc}_{\Pi}(X^*) \)’. Thus, if, for instance, \( U \) is the \textit{complement}
of an (infinite) compact line segment in $D$, then $U \to D$ determines an FSM-morphism (which is not an isomorphism) such that monomorphism factorizations of this morphism correspond to connected open subsets $W \subseteq D$ containing $U$. In particular, it follows immediately that this morphism $U \to D$ fails to factor as a composite of finitely many FSMI-morphisms. Also, we observe that if $U$ is the complement of an infinite discrete subset of $D$, then ‘filling in successive points of the complement $D \setminus U$’ (cf. the slit morphisms of Example 3.3(v)) yields an infinite sequence of FSMI-morphisms $U \to U_1 \to U_2 \to \cdots$. Thus, in summary, the category $\text{Loc}/\Pi_1(X^\ast)$ does not satisfy either of the two conditions in the definition of a ‘category of FSMFF-type’ (cf. [Mzk5, Section 0]). Moreover, we observe that if, for instance, $X^\ast$ is non-compact, then its upper half-plane uniformization constitutes a morphism from an ‘object which is not of finite type’ (in the terminology of [Mzk3, Section 1]) to an ‘object of finite type’ which does not have pre-compact image. Thus, it does not appear that this situation may be remedied by imposing a simple ‘pre-compact image condition’ of the sort that occurs in the definition of the categories of Example 4.3 (cf. [Mzk3, Section 2]).

**Example 4.4. (Archimedean quasi-temperoids)**

(i) Let

$$\mathcal{D} \overset{\text{def}}{=} B^{\text{temp}}(\Pi, \Pi^\circ)^0$$

be any category as in Example 1.3(iii), where $\Pi \to Q$, $G_\mathbb{R} \to Q$ (i.e., we take ‘$F$’ of loc. cit. to be $\mathbb{R}$), $\Pi^\circ \subseteq \Pi$ are as in loc. cit., and we assume further that the surjection $G_\mathbb{R} \to Q$ is an isomorphism. Thus, we have a natural functor $\mathcal{D} = B^{\text{temp}}(\Pi, \Pi^\circ)^0 \to \mathcal{D}_0$ (cf. Example 1.3(ii) and (iii)).

(ii) Now we observe that:

$\mathcal{D}$ (equipped with the natural functor $\mathcal{D} \to \mathcal{D}_0$) is a category of RC-standard, FSM-, and strictly partially ordered (hence, a fortiori, of discontinuously ordered) type (cf. Remark 4.2.1). If, moreover, $\Pi$ is temp-slim, then $\mathcal{D}$ is slim.

Indeed, this follows via a similar (but somewhat easier) argument to the argument applied in Example 4.3(ii) (cf. also Example 1.3(i)).

(iii) Now, suppose that, in the notation of Example 1.4, we take ‘$v$’ to be an archimedean valuation. Observe that we have a natural functor $\mathcal{P}_0 \to \mathcal{D}_0$. Set $\mathcal{P} \overset{\text{def}}{=} \mathcal{D} \times_{\mathcal{D}_0} \mathcal{P}_0$, $\mathcal{E} \overset{\text{def}}{=} \mathcal{P} \times_{\mathcal{P}_0} \mathcal{E}_0 \to \mathcal{D} \times_{\mathcal{D}_0} \mathcal{E}_0$ (cf. Example 4.3(iii)), and we note that, so long as $\tilde{F}/F$ is amply quadratic, even if $\mathcal{P}_0 \to \mathcal{D}_0$ fails to be an equivalence of categories, the difference between $\mathcal{D}$ and $\mathcal{P}$ amounts, in effect, to forming ‘$\mathcal{D}$’ as above, but with the original ‘$\Pi$’, ‘$\Pi^\circ$’ replaced by the kernel of the given morphism $\Pi \to Q \cong G_\mathbb{R}$ and its intersection with $\Pi^\circ$, respectively. Then:

If $\tilde{F}/F$ is amply quadratic, then $\mathcal{E}$ (equipped with the natural functor $\mathcal{E} \to \mathcal{D}_0$) is a category of RC-standard and discontinuously ordered type (cf. Remark 4.2.1).

If, moreover, $\Pi$ is temp-slim, then $\mathcal{E}$ is slim.

Indeed, this follows via a similar (but somewhat easier) argument to the argument applied in Example 4.3(iii).

(iv) Finally, we observe that the categories $\mathcal{D}$, $\mathcal{E}$ are of strongly indissectible type (cf. Example 1.3(i) and Proposition 1.5(vii)).
5. Poly-Frobenioids

In this section, we develop the 'general nonsense' necessary to construct categories obtained by ‘gluing together’ various Frobenioids in a certain fashion. We shall refer to the categories obtained in this way as poly-Frobenioids. This ‘gluing problem’ is motivated by the goal of ‘gluing together’ the Frobenioids of [Mzk5, Example 6.3] (which embody the global arithmetic of number fields) to the p-adic and archimedean Frobenioids of Examples 1.1 and 3.3 (which embody the arithmetic of local fields) in a fashion that reflects, at the level of category theory, the way in which localizations of number fields are typically treated in arithmetic geometry.

Definition 5.1.
(i) We shall refer to a collection of grafting data as any collection of data

\[ \mathfrak{A} \overset{\text{def}}{=} \left( A_\otimes, \{ A_i \}_{i \in I}, \{ A_\bot \}_{i \in I}, \{ \zeta_i : A_\otimes \to A_\bot \}_{i \in I}, \{ \eta_i : A_i \to A_\bot \}_{i \in I} \right), \]

where \( I \) is a set; the \( A_\otimes, A_i, A_\bot \) are connected, totally epimorphic categories; ‘\( \bot \)’ is as in [Mzk5, Section 0]; the arrows \( \zeta_i \) are functors, which we shall refer to as global contact functors; the functors \( \eta_i \), which we shall refer to as local contact functors, are relatively initial (cf. Section 0). We shall refer to \( A_\otimes \) as the global component of \( \mathfrak{A} \) and to the \( A_i \) as the local components of \( \mathfrak{A} \). The grafted category associated to \( \mathfrak{A} \) is defined to be the category

\[ \mathcal{G} \overset{\text{def}}{=} A_\otimes \prec \left( \prod_{i \in I} A_i \right) \left( \bigcoprod_{i \in I} A_\bot \right), \]

where ‘\( \prec \)’ is defined relative to the functors

\[ A_\otimes \to \prod_{i \in I} A_i \overset{\zeta_i}{\to} \prod_{i \in I} A_\bot \]

(cf. Section 0); the arrow in the first line is the functor determined by the \( \zeta_i \); the first arrow in the second line is the natural functor (cf. Section 0); the second arrow in the second line is the functor determined by the \( \eta_i \). Thus, (cf. Section 0) we may think of the global component, as well as the local components, of \( \mathfrak{A} \) as full subcategories of \( \mathcal{G} \). An object or arrow that belongs to the global component (respectively, a local component \( i \in I \)) of \( \mathcal{G} \) will be referred to as global (respectively, \( i \)-local; local) objects. A morphism of \( \mathcal{G} \) from an \( i \)-local object to a global object will be referred to as \( i \)-heterogeneous.

(ii) We shall say that \( \mathfrak{A} \) is of uniformly dissectible type if the following conditions are satisfied:

(a) there exist at least two elements \( i \in I \) for which \( \zeta_i \) is totally non-initial (cf. Section 0);
(b) for each \( i \in I \), \( A_i \) is either of weakly indissectible type or of strongly dissectible type;
(c) there exists an \( i \in I \) such that \( A_i \) is of weakly indissectible type, and \( \zeta_i \) is totally non-initial.
If, for \( \iota \in I, \eta_\iota \) is an equivalence of categories, then we shall say that \( \mathfrak{A} \) is of \( \iota \)-tactile type; if \( \mathfrak{A} \) is of \( \iota \)-tactile type for all \( \iota \in I \), then we shall say that \( \mathfrak{A} \) is of totally tactile type. A morphism of \( \mathcal{G} \) whose domain is \( (\iota-) \)-local (for some \( \iota \in I \)), whose codomain is global, and which is minimal-adjoint (cf. \([Mzk5\text{, Section 0]}\)) to the morphisms with local codomain, will be referred to as \( (\iota-) \)-pre-tactile. A morphism \( A \to B \) of \( \mathcal{G} \) whose domain is \( (\iota-) \)-local (for some \( \iota \in I \)), whose codomain is global, and which corresponds (cf. the definition of \( \mathcal{G} \)) to an isomorphism of \( \eta_\iota(A) \) onto a connected component of \( \zeta_\iota(B) \), will be referred to as \( (\iota-) \)-tactile.

If \( \eta \) is \( (\iota-) \)-heterogeneous morphism \( \phi: A \to B \) of \( \mathcal{G} \) factors as a composite \( A \to A' \to B \), where \( A' \to B \) is \( (\iota-) \)-tactile, then we shall refer to this factorization as \( \eta(\iota-) \)-tactile factorization of \( \phi \). If, for some \( \iota \in I \), \( \{A_j \to B\}_{j \in J} \) forms a collection of \( \iota \)-tactile morphisms of \( \mathcal{G} \) that corresponds (cf. the definition of \( \mathcal{G} \)) to an isomorphism \( \prod_{j \in J} \eta_\iota(A_j) \sim \zeta_\iota(B) \), then we shall refer to the collection of morphisms \( \{A_j \to B\}_{j \in J} \) as a complete \( (\iota-) \)-tactile collection (for \( B \)).

(iii) If

\[
\mathfrak{A}' \equiv \left( A'_\circ, \{A'_i\}_{i \in I'}, \{A'_i\}_{i \in I'}, \{\zeta'_i: A'_i \to A'_i\}_{i \in I'}, \{\eta'_i: A'_i \to A'_i\}_{i \in I'} \right)
\]

is another collection of grafting data, then we shall refer to as an equivalence of collections of grafting data

\[
\Psi: \mathfrak{A} \sim \mathfrak{A}'
\]

a bijection \( \Psi_I: I \sim I' \), together with equivalences of categories

\[
\Psi_\circ: A_\circ \sim A'_\circ
\]

and

\[
\Psi_i: A_i \sim A'_i; \quad \Psi_j: A_j \sim A'_j
\]

for \( \iota \in I \), \( \iota' \equiv \Psi_I(\iota) \), which are \( 1 \)-compatible with the various functors \( \zeta_i, \zeta'_i, \eta_i, \eta'_i \). (Thus, the equivalences of collections of grafting data \( \mathfrak{A} \sim \mathfrak{A}' \), together with isomorphisms (in the evident sense) between such equivalences, form a category.) If \( \mathcal{G}' \) is the grafted category associated to \( \mathfrak{A}' \), then it is immediate that \( \Psi \) determines an equivalence of categories \( \mathcal{G} \sim \mathcal{G}' \).

Remark 5.1.1. Since the local components of \( \mathcal{G} \) are assumed to be connected (cf. Definition 5.1(i)), it follows immediately that the \( \mathcal{G}_\iota \) are precisely the connected components (cf. \([Mzk5\text{, Section 0]}\)) of the category \( \mathcal{G}_\iota \), and that the category \( \mathcal{G}_\iota \) is naturally equivalent to the coproduct (cf. Section 0) of the categories \( \mathcal{G}_\iota \).

Remark 5.1.2. Let \( \iota \in I \). Then the following assertions follow immediately from the definitions and the total epimorphicity assumption in Definition 5.1(i). Every \( \iota \)-tactile morphism of \( \mathcal{G} \) is \( \iota \)-pre-tactile. Suppose further that \( \mathfrak{A} \) is of \( \iota \)-tactile type. Then any two \( \iota \)-tactile factorizations of an \( \iota \)-heterogeneous morphism \( \phi: A \to B \) of \( \mathcal{G} \) determine isomorphic objects of the category of factorizations \( \mathcal{G}_\phi \) (cf. Section 0); every global object \( B \) of \( \mathcal{G} \) admits a complete \( \iota \)-tactile collection; if \( \{A_j \to B\}_{j \in J} \) is any collection of \( \iota \)-tactile morphisms of \( \mathcal{G} \), then this collection is a complete \( \iota \)-tactile collection for \( B \) if and only if every \( \iota \)-heterogeneous morphism \( C \to B \) factors through precisely one of the \( A_j \to B \); finally, a morphism of \( \mathcal{G} \) is \( \iota \)-tactile if and only if it is \( \iota \)-pre-tactile.
Proposition 5.2. (Dissection of grafted categories) Let \( \mathfrak{A}, \mathfrak{A}' \) be collections of grafting data of uniformly dissectible type, whose associated categories we denote by \( \mathcal{G}, \mathcal{G}' \), respectively. Then we have the following.

(i) The global objects of \( \mathcal{G} \) are strongly dissectible.

(ii) Let \( i \in I \). Suppose that \( \mathcal{A}_i \) is of weakly indissectible (respectively, strongly dissectible) type. Then the \( i \)-local objects of \( \mathcal{G} \) are weakly indissectible (respectively, strongly dissectible).

(iii) An object of \( \mathcal{G} \) is global if and only if it is a strongly dissectible object that appears as the codomain of a morphism whose domain is weakly indissectible.

(iv) Every equivalence of categories \( \Xi : \mathcal{G} \to \mathcal{G}' \) induces an equivalence of categories \( \mathcal{G}_{\otimes} \to \mathcal{G}'_{\otimes} \), a bijection \( \Xi_I : I \to I' \), and, for each \( I \ni i \mapsto i' \in I' \), an equivalence of categories \( \mathcal{G}_i \to \mathcal{G}'_i \). In particular, \( \Xi \) preserves global and local objects.

(v) Suppose that \( \mathfrak{A}, \mathfrak{A}' \) are of totally tactile type. Then there is a natural equivalence of categories between the category of equivalences \( \mathfrak{A} \sim \mathfrak{A}' \) and the category of equivalences \( \mathcal{G} \sim \mathcal{G}' \).

Proof. To verify assertion (i), let \( A \in \text{Ob}(\mathcal{A}_{\otimes}) \). Since (cf. Definition 5.1(ii)(a) and Definition 5.1(i)) there exist distinct \( i_1, i_2 \in I \) such that (for \( j = 1, 2 \)) \( \zeta_{ij} \) is totally non-initial, and \( \eta_{ij} \) is relatively initial, it follows that there exists (for \( j = 1, 2 \)) a morphism \( A_j \to A \), where \( A_j \in \text{Ob}(\mathcal{G}) \) is \( j \)-local. On the other hand, since no object of \( \mathcal{G} \) admits a morphism to both \( A_1 \) and \( A_2 \), it follows that \( A_1 \to A, A_2 \to A \) form a strongly dissecting pair of arrows, as desired.

Assertion (ii) follows immediately from the definitions (together with the fact that the domain of any morphism of \( \mathcal{G} \) whose codomain is \( i \)-local is itself \( i \)-local).

Assertion (iii) follows immediately from assertions (i) and (ii) (cf. also Definition 5.1(ii)(b)), together with the existence (cf. Definition 5.1(ii)(c) and Definition 5.1(i)) of an \( i \in I \) such that \( \mathcal{A}_i \) is of weakly indissectible type, \( \zeta_i \) is totally non-initial, and \( \eta_i \) is relatively initial.

Assertion (iv) follows immediately from assertion (iii) and Remark 5.1.1.

Finally, assertion (v) follows immediately from assertion (iv), by considering complete tactile collections of morphisms (cf. Remark 5.1.2).

Observe that our discussion so far has nothing to do with Frobenioids (at least in an explicit sense). We now return to discussing Frobenioids.

Definition 5.3. For \( i = 1, 2 \), let \( \mathcal{C}_i \) be a Frobenioid over a base category \( \mathcal{D}_i \), with perfect-factorial divisor monoid \( \Phi_i \). Note that \( \Phi_i \) extends naturally to a functor \( \Phi_i^\top \) on \( \mathcal{D}_i^\top \) by assigning to a coproduct of objects \( \{ A_j \}_{j \in J} \) of \( \mathcal{D}_i \) the direct sum of the monoids \( \Phi_i(A_j) \).

(i) Suppose that \( \mathcal{C}_1 \) is of isotropic type. Then we shall refer to a functor

\[ \Psi : \mathcal{C}_1 \to \mathcal{C}_2^\top \]

as GC-admissible (i.e., ‘global contact-admissible’) if the following conditions are satisfied:

(a) there exists a functor \( \Psi^{\text{Base}} : \mathcal{D}_1 \to \mathcal{D}_2^\top \) which is 1-compatible with \( \Psi \) (relative to the natural projection functors \( \mathcal{C}_i \to \mathcal{D}_i \), for \( i = 1, 2 \));

(b) \( \Psi \) maps an arrow of Frobenius degree \( d \in \mathbb{N}_{\geq 1} \) of \( \mathcal{C}_1 \) to an arrow of \( \mathcal{C}_2^\top \) each of whose component arrows (\( \in \text{Arr}(\mathcal{C}_2) \)) is of Frobenius degree \( d \);
(c) there exists a natural transformation \( \Psi^\Phi : \Phi_1 \rightarrow \Phi_1^{rlf}|_{C_i} \) that is compatible with \( \Psi \) relative to the functors \( C_i \rightarrow \mathbb{F}_{\Phi_i} \) (for \( i = 1, 2 \)) that define the Frobenioid structures of the \( C_i \) (in other words: \( \Psi^\Phi \) is compatible with \( \Psi \) relative to the operation of taking the zero divisor);

(d) every component object \( \in \text{Ob}(C_2) \) of an object in the image of \( \Psi \) is isotropic.

(ii) We shall refer to a functor
\[
\Psi : C_1 \rightarrow C_2
\]
as LC-admissible (i.e., ‘local contact-admissible’) if there exists a subfunctor of group-like monoids
\[
\Phi^{\text{cnst}} \subseteq (\Phi_1^{rlf})^{\text{birat}}
\]
(where ‘rlf’ is as in [Mzk5, Definition 2.4(i)]; cf. also [Mzk5, Proposition 5.3]) on \( D_1 \) of ‘constant realified rational functions’ such that the following conditions are satisfied:

(a) there exists an equivalence of categories \( \Psi^{\text{Base}} : D_1 \stackrel{\sim}{\rightarrow} D_2 \) which is 1-compatible with \( \Psi \) (relative to the natural projection functors \( C_i \rightarrow D_i \), for \( i = 1, 2 \));

(b) \( \Psi \) is compatible with Frobenius degrees;

(c) the natural inclusion \( \Phi_1 \hookrightarrow \Phi_1^{rlf} \) factors as a composite of inclusions
\[
\Phi_1 \hookrightarrow \Phi_2|_{D_1} \hookrightarrow \Phi_1^{rlf}
\]
of functors of monoids on \( D_1 \) that are compatible with \( \Psi \) relative to the functors \( C_i \rightarrow \mathbb{F}_{\Phi_i} \) (for \( i = 1, 2 \)) that define the Frobenioid structures of the \( C_i \) (in other words: relative to the operation of taking the zero divisor);

(d) applying the operation of ‘groupification’ to the monoids of (c) induces inclusions
\[
\Phi_1^{\text{birat}} \hookrightarrow \Phi_2^{\text{birat}}|_{D_1} \hookrightarrow (\Phi_1^{rlf})^{\text{birat}} = \mathbb{R} \cdot \Phi_1^{\text{birat}}
\]
(where we recall that \( \Phi_i^{\text{birat}} \subseteq \Phi_i^{gp} \), for \( i = 1, 2 \) — cf. [Mzk5, Proposition 4.4(iii)]) of functors of monoids on \( D_1 \);

(e) the image of the inclusion \( \Phi_2^{\text{birat}}|_{D_1} \hookrightarrow (\Phi_1^{rlf})^{\text{birat}} \) of (d) is equal to the subfunctor of (group-like) monoids \( \Phi_2^{\text{birat}} \cap \Phi_1^{\text{cnst}} \subseteq (\Phi_1^{rlf})^{\text{birat}} \);

(f) the functor \( \Phi^{\text{cnst}} \) is ‘constant’ on \( D_1 \) in the sense that \( \Phi^{\text{cnst}} \) maps every arrow of \( D_1 \) to an isomorphism;

(g) \( \Psi \) preserves isotropic objects and co-angular morphisms.

We shall say that \( \Psi \) is LC-unit-admissible if \( \Psi \) is LC-admissible and, moreover, the following condition is satisfied:

(h) if \( A_2 = \Psi(A_1) \in \text{Ob}(C_2) \), where \( A_1 \in \text{Ob}(C_1) \), then the induced homomorphism of abelian groups \( \mathcal{O}^\times(A_1) \rightarrow \mathcal{O}^\times(A_2) \) induces a surjection on perfections \( \mathcal{O}^\times(A_1)^{pf} \rightarrow \mathcal{O}^\times(A_2)^{pf} \).

(iii) We shall refer to as a collection of Frobenioid-theoretic grafting data any collection of grafting data
\[
\mathcal{C} \overset{\text{def}}{=} \left( C_\otimes, \{ C_i \}_{i \in I}, \{ \zeta_i : C_\otimes \rightarrow C_i \} \right)_{i \in I}, \{ \eta_i : C_i \rightarrow C_i \} \}_{i \in I}
\]
such that the following conditions are satisfied: (a) the categories \( C_\otimes, C_i, C_i \) (for \( i \in I \)) are equipped with Frobenioid structures, the base categories of which we denote by \( D_\otimes, D_i, \)
and, moreover, for every factorization $A$ (respectively, $B$) the divisor monoids $\Phi_\circ, \Phi_\circ, \Phi_\circ$ (for $\iota \in I$) are all perf-factorial; (c) the global contact functors $\zeta_\iota$ (for $\iota \in I$) are GC-admissible (which implies, in particular, that $C_\circ$ is of isotropic type); (d) the local contact functors $\eta_\iota$ (for $\iota \in I$) are LC-admissible. A category obtained as the grafted category associated to such data $C$ will be referred to as a poly-Frobenioid $\mathcal{C}$. If all of the $\eta_\iota$ (for $\iota \in I$) are LC-unit-admissible, then we shall say that $C$ or $\mathcal{C}$ is of LC-unit-admissible type. Observe that $C$ determines a collection of base-category grafting data $\mathcal{D}$, which is of totally tactile type (cf. (ii)(a)); the associated grafted category $\mathcal{D}$ will be referred to as the poly-base category of the poly-Frobenioid $\mathcal{C}$. (Here, we observe that, by a (harmless!) abuse of notation, $\mathcal{C}_\circ$ denotes both the category $\mathcal{C}_\circ$ and the category $\mathcal{G}_\circ$ of Definition 5.1(i); a similar remark holds for $\mathcal{D}_\circ$, $\mathcal{D}_I$, $\mathcal{D}_I$ (see (iv)). Thus, we have a natural projection functor $\mathcal{C} \to \mathcal{D}$. If $\mathcal{P}$ is a property of Frobenioids (respectively, base categories of Frobenioids), then we shall say that $C$ or $\mathcal{C}$ (respectively, $\mathcal{D}$ or $\mathcal{D}$) satisfies this property $\mathcal{P}$ if the Frobenioids $C_\circ, C_I, C_I$ (for $\iota \in I$) (respectively, the base categories $\mathcal{D}_\circ, \mathcal{D}_I, \mathcal{D}_I$ (for $\iota \in I$)) all satisfy this property $\mathcal{P}$. (In particular, one must be careful to distinguish between the assertion that $\mathcal{D}$ satisfies $\mathcal{P}$ as an abstract category and the assertion that $\mathcal{D}$ satisfies $\mathcal{P}$ as a poly-base category of a poly-Frobenioid.)

(iv) Let $\phi: A \to B$ be a morphism of the poly-Frobenioid $\mathcal{C}$. We shall denote the projection of $\phi$ to $\mathcal{D}$ by $\text{Base}(\phi)$. Observe that by conditions (i)(b) and (ii)(b), above, it makes sense to speak of the Frobenius degree $\deg_{\text{Fr}}(\phi) \in \mathbb{N}_{\geq 1}$ of $\phi$ if $\deg_{\text{Fr}}(\phi) = 1$, then we shall say that $\phi$ is linear. If, for $\iota \in I$, $\phi$ is $\iota$-local (respectively, $\iota$-heterogeneous), then it makes sense to speak of the zero divisor $\text{Div}(\phi) \in \Phi_\iota(A)$ (respectively, $\text{Div}(\phi) \in \Phi_\iota(A)$); if $\phi$ is global, then it makes sense to speak of the zero divisor $\text{Div}(\phi) \in \Phi_\iota(A)$. We shall say that $\phi$ satisfies a property $\mathcal{P}$ of arrows of Frobenioids if $\phi$ is homogeneous, and, moreover, satisfies the property $\mathcal{P}$ when regarded as an arrow of one of the local or global component Frobenioids of $\mathcal{C}$; a similar convention will be applied to objects of $\mathcal{C}$. We shall say that $\phi$ is base-($\iota$)-tactile if it projects to a (n) ($\iota$)-tactile morphism of $\mathcal{D}$ (for $\iota \in I$). We shall say that $\phi$ is birationally ($\iota$)-tactile if it is linear and base-($\iota$)-tactile (for $\iota \in I$), with isotropic domain. We shall say that $\phi$ is base-($\iota$)-pre-tactile if $\phi$ is ($\iota$)-heterogeneous, and, moreover, for every factorization $A \to A' \to B$ of $\phi$, where $A'$ is ($\iota$)-local, it holds that $A \to A'$ is a base-isomorphism (for $\iota \in I$). We shall say that $\phi$ is birationally ($\iota$)-pre-tactile if $\phi$ is ($\iota$)-heterogeneous, and, moreover, for every factorization $A \to A' \to B$ of $\phi$, where $A'$ is ($\iota$)-local, it holds that $A \to A'$ is a co-angular pre-step (for $\iota \in I$). If $\phi$ factors as a composite $A \to A' \to B$, where $A'$ is base($\iota$)-tactile (respectively, birationally ($\iota$)-tactile) (for some $\iota \in I$), then we shall refer to this factorization as a base-($\iota$)-tactile (respectively, birationally ($\iota$)-tactile) factorization of $\phi$. If, for some $\iota \in I$, $\{A_j \to B\}_{j \in J}$ forms a collection of birationally $\iota$-tactile morphisms of $\mathcal{C}$ that projects to a complete $\iota$-tactile collection of $\text{Base}(B)$, then we shall refer to the collection of morphisms $\{A_j \to B\}_{j \in J}$ as a complete birationally ($\iota$)-tactile collection (for $B$).

(v) A routine check reveals (cf. Remarks 5.3.1 and 5.3.3, below and [Mzk5, Definition 2.4(i), Proposition 3.2(ii), (iii), and Proposition 4.4(iv)]) that applying the operations

\[ 'pf', 'un-tr', 'rfl', 'birat' or 'ISTR' \]

(where 'pf' is defined whenever $C$ is of Frobenius-isotropic type) to each of the Frobenioids $C_\circ, C_I, C_I$ (for $\iota \in I$) yields a new collection of Frobenioid-theoretic grafting data $C^{\text{pf}},$
of the component arrows of $\Psi_1(\phi)$

We observe in passing that, in the notation of Definition 5.3(i), the functor Remark 5.3.2.

Proposition 1.4(ii)

linear morphism functor $\Psi_1$.

Observe that if a morphism Remark 5.3.1.

Proposition 1.4(ii)]

a morphism of Frobenius type is a

Remark 5.3.3. Observe that (in the notation of Definition 5.3(ii)) any LC-admissible functor $\Psi : \mathcal{C}_1 \to \mathcal{C}_2$ induces an equivalence of categories $\mathcal{C}_1^{\text{clf}} \sim \mathcal{C}_2^{\text{clf}}$ between the associated realifications. Indeed, this follows immediately from Definition 5.3(ii)(a)–(d) (cf. also the definition of the ‘realification’ in [Mzk5, Proposition 5.3]). In particular (cf. also Definition 5.3(ii)(g) and [Mzk5, Proposition 1.4(ii)]), $\Psi$ preserves morphisms of Frobenius type, pre-steps, pull-back morphisms, linear morphisms, isometries, isotropic hulls, co-angular morphisms, base-isomorphisms, and base-identity endomorphisms.

Remark 5.3.4. It is immediate from the definitions that ‘birationally pre-tactile’ implies ‘base-pre-tactile’, that ‘birationally tactile’ implies ‘base-tactile’, that ‘birationally tactile’ implies ‘birationally pre-tactile’ and that ‘base-tactile’ implies ‘base-pre-tactile’ (cf. Remark 5.1.2).

**PROPOSITION 5.4. (Birationally tactile factorizations)** In the notation of Definition 5.3(iii), suppose that the poly-Frobenioid $\mathcal{C}$ is of LC-unit-admissible, Frobenius-normalized, and perfect type. Let $\phi : A \to B$ be an $\iota$-heterogeneous morphism, where $\iota \in I$, of $\mathcal{C}$, whose image in the birationalization (cf. Definition 5.3(v)) $\mathcal{C}^{\text{birat}}$ of $\mathcal{C}$ we denote by $\phi^{\text{birat}} : A^{\text{birat}} \to B^{\text{birat}}$. Then we have the following.

(i) $\phi^{\text{birat}}$ admits a tactile factorization.

(ii) $\phi^{\text{birat}}$ is tactile if and only if it is pre-tactile.
(iii) There exists a co-angular pre-step $\psi : C \to A$ such that $\phi \circ \psi$ admits a birationally tactile factorization.
(iv) There exists a co-angular pre-step $\psi : C \to A$ such that $\phi \circ \psi$ is base-tactile (respectively, birationally tactile) if and only if $\phi \circ \psi$ is base-pre-tactile (respectively, birationally pre-tactile).

Proof. First, we consider assertion (i). Since every connected component of an object in the essential image of the global contact functor labeled $\iota$ is isotropic (cf. Definition 5.3(ii)(d)), it follows that $\phi^{\text{birat}}$ factors through any isotropic hull of $A^{\text{birat}}$, thus, we may assume without loss of generality that $A$, $A^{\text{birat}}$ (cf. [Mzk5, Proposition 4.4(iv)]) are isotropic. Also, by factoring $\phi$ through a morphism of Frobenius type with domain $A$ (cf. [Mzk5, Definition 1.3(ii) and (iv)]), we may assume without loss of generality that $\phi$ is linear.

Since the isomorphism class of an isotropic (hence Frobenius-trivial) $\delta$ (iii) there exists a co-angular pre-step $C$ exists an object $\phi$ which is of Frobenius-normalized (cf. assertion (i)) and the total epimorphicity of $\Phi_1$, Proposition 4.4(i), Proposition 1.11(vii) and the inclusions 'birationalization' (cf. [Mzk5, Proposition 4.4(i), Proposition 1.11(vii)] and the inclusions ‘birationalization’ (cf. [Mzk5, Proposition 5.5(i)]) are perfect type; [Mzk5, Proposition 5.5(i)]]), it follows that we may assume that $\phi$ is of LC-unit-admissible, Frobenius-normalized, and perfect type; [Mzk5, Proposition 5.5(i)]), it follows that we may assert $\phi$ admits a factorization

$$\eta_i(A)^{\text{birat}} \xrightarrow{a} \eta_i(D)^{\text{birat}} \xrightarrow{\delta} \xi_i(B)^{\text{birat}}$$

(where $\delta$ determines a tactile morphism of $C^{\text{birat}}$; we use the superscript ‘birat’ to denote the images of objects or arrows of Frobenioids or poly-Frobenioids in their respective birationalizations). Since $A$, $A^{\text{birat}}$, hence also $\eta_i(A)^{\text{birat}}$ (cf. Definition 5.3(ii)(g)), are isotropic, it follows that the first arrow $\alpha : \eta_i(A)^{\text{birat}} \to \eta_i(D)^{\text{birat}}$ of this factorization is linear, co-angular and (cf. the definition of ‘birat’!) isometric. It thus follows that $\alpha$ is a pull-back morphism (cf. [Mzk5, Proposition 1.4(ii)]) hence (cf. [Mzk5, Definition 1.3(ii)(c)] and the definition of a ‘pull-back morphism’ in [Mzk5, Definition 1.2(ii)]) that this arrow differs from the image in $\mathcal{C}_i^{\text{birat}}$ of a pull-back morphism $\beta : A^{\text{birat}} \to D^{\text{birat}}$ in $\mathcal{C}_i^{\text{birat}}$ by composition with a unit of $\mathcal{C}_i^{\text{birat}}$, i.e., an element $u \in \mathcal{O}^{\times}(\eta_i(A)^{\text{birat}})$ (so $\alpha = \eta_i(D)^{\text{birat}}$ o $u$). Moreover, if we apply the isomorphism ‘$\Phi_1^{\text{birat}}[\mathcal{D}] \to \Phi_1^{\text{birat}} + \mathcal{C}_i^{\text{birat}}$‘ of Definition 5.3(ii)(e), it follows that we may write $u = u_1 \cdot u_{\text{const}}$, where $u_1 = \eta_i(\psi)$ for some $\psi \in \mathcal{O}^{\times}(A^{\text{birat}})$, and the image of $u_{\text{const}} \in \mathcal{O}^{\times}(\eta_i(A)^{\text{birat}})$ in $\Phi_1^{\text{birat}}$ lies in $\Phi_1^{\text{const}}$; by applying Definition 5.3(ii)(f) and (h) (cf. also our assumption that the poly-Frobenioid $C$ is of LC-unit-admissible, Frobenius-normalized, and perfect type; [Mzk5, Proposition 5.5(i)]), it follows that we may write

$$u_{\text{const}} = v_{\text{const}} \cdot \eta_i^{\text{birat}}(w_{\text{const}}),$$

where $v_{\text{const}} \in \mathcal{O}^{\times}(\eta_i(D)^{\text{birat}})$, $w_{\text{const}} \in \mathcal{O}^{\times}(A^{\text{birat}})$. Thus, by replacing $\alpha$ by $v_{\text{const}}^{-1} \circ \alpha$ and $\beta$ by $\beta \circ u_{\text{const}} \circ v_1$, we conclude that we may assume that $\alpha = \eta_i^{\text{birat}}(\beta)$. Then $\beta$, $\delta$ determine a tactile factorization of $\phi^{\text{birat}}$, as desired. This completes the proof of assertion (i).

Now assertion (ii) follows immediately from the existence of tactile factorizations (cf. assertion (i)) and the total epimorphicity of $\mathcal{C}_i^{\text{birat}}$ (cf. Remark 5.1.2). Assertion (iii) follows immediately from assertion (i); the definition of the ‘birationalization’ (cf. [Mzk5, Proposition 4.4(i), Proposition 1.11(vii)] and the inclusions ‘birationalization’ (cf. [Mzk5, Proposition 5.5(i)]), which imply that pre-steps of the underlined local components may be dominated by pre-steps of the non-underlined local components).
Finally, assertion (iv) follows immediately from assertion (iii) and the total epimorphicity of $D_i$ (cf. also Remark 5.3.4, [Mzk5, Proposition 1.9(iv)] and the derivation of assertion (ii) from assertion (i)). □

**THEOREM 5.5. (Category-theoreticity of poly-frobenioids)** For $i = 1, 2$, let $C_i$ be a poly-Frobenioid of poly-standard type and let

$$\Psi : C_1 \sim\rightarrow C_2$$

be an equivalence of categories. Then we have the following.

(i) $\Psi$ preserves global and local objects and induces a compatible bijection $I_1 \sim\rightarrow I_2$ between the sets $I_1, I_2$ of local components of $C_1, C_2$. Moreover, $C_1$ is of poly-non-group-like type if and only if $C_2$ is.

(ii) $\Psi$ preserves the full subcategories $C_i^{\text{str}} \subseteq C_i$.

(iii) Let $\square \in \{\text{pf, un-tr, rlf, birat}\}$. If $\square = \text{un-tr}$, then we assume further that the poly-base category $D_i$ is Frobenius-slim. If $\square = \text{rlf}$, then we assume further that $C_i$ is of polynatorially standard type, and that the poly-base category $D_i$ is slim. If $\square \neq \text{birat}$, then we assume further that $C_i$ is of poly-non-group-like type. Then there exists a 1-unique 1-commutative diagram

$\begin{array}{ccc}
C_1 & \xrightarrow{\Psi} & C_2 \\
\downarrow & & \downarrow \\
C_1^{\square} & \xrightarrow{\Psi^{\square}} & C_2^{\square}
\end{array}$

(where the vertical arrows are the natural functors and the horizontal arrows are equivalences of categories).

(iv) Suppose that the poly-base category $D_i$ is slim, and that $C_i$ is of poly-non-group-like type. Then there exists a 1-unique 1-commutative diagram

$\begin{array}{ccc}
C_1 & \xrightarrow{\Psi} & C_2 \\
\downarrow & & \downarrow \\
D_1 & \xrightarrow{\Psi^{\text{Base}}} & D_2
\end{array}$

(where the vertical arrows are the natural functors and the horizontal arrows are equivalences of categories).

(v) Suppose that $C_i$ is of polynatorially standard and poly-non-group-like type, and that the poly-base category $D_i$ is slim. Then $\Psi$ induces isomorphisms between the divisor monoids of corresponding (cf. (i)) local or global components of $C_1, C_2$. Moreover, these isomorphisms are compatible with the local and global contact functors that determine the poly-Frobenioid structures of $C_1, C_2$ (a statement that makes sense in light of the inclusions ‘$\Phi_2|_{D_1} \hookrightarrow \Phi^{\text{rlf}}_1$’ of Definition 5.3(ii)(c)).

**Proof.** Assertion (i) follows formally from Proposition 5.2(iv) and [Mzk5, Theorem 3.4(ii)]. In light of assertion (i), assertion (ii) follows from [Mzk5, Theorem 3.4(i)]. Next, let us observe that $C_i^{\text{pf}}$ may be constructed from $C_i$ by considering appropriate inductive limits.
involving \textit{pairs of morphisms of Frobenius type}, just as in the case of Frobenioids (cf. \cite[Definition 3.1(ii) and (iii)]{Mzk5}). Since, whenever \( \square \neq \text{birat} \) (so \( C_i \) is of \textit{poly-non-group-like} type), \( \Psi \) preserves \textit{pairs of morphisms of Frobenius type of the same Frobenius degree} (cf. assertion (i) and \cite[Theorem 3.4(iii)]{Mzk5}), we thus conclude that assertion (iii) holds when \( \square = \text{pf} \). In a similar vein, it follows from the inclusions \( ‘\Phi_2|_{D_i} \hookrightarrow \Phi_i^{\text{fr}}’ \) of Definition 5.3(ii)(c) (which imply that pre-steps of the \textit{underlined} local components may be dominated by pre-steps of the \textit{non-underlined} local components) that \( C_i^{\text{birat}} \) may be constructed from \( C_i \) by ‘\text{inverting the co-angular pre-steps}’, just as in the case of Frobenioids (cf. \cite[Proposition 4.4(i)]{Mzk5}). Since \( \Psi \) preserves \textit{co-angular pre-steps} (cf. assertion (i) and \cite[Theorem 3.4(ii) and Corollary 4.10]{Mzk5}), we thus conclude that assertion (iii) holds when \( \square = \text{birat} \). Thus, in the following, we assume that \( \square \neq \text{birat} \) (so \( C_i \) is of \textit{poly-non-group-like} type).

Next, observe that by \cite[Proposition 5.5(iii)]{Mzk5}, it follows that \( C_i^{\text{pf}} \) is of \textit{standard} type (i.e., each of its local and global component Frobenioids is of standard type). Thus, by ‘\text{applying the last two arguments in succession}’, it follows that we obtain a \textit{1-unique} \textit{1-commutative diagram}

\begin{equation}
\begin{array}{ccc}
C_1 & \xrightarrow{\Psi} & C_2 \\
\downarrow & & \downarrow \\
(C_1^{\text{pf}})^{\text{birat}} & \xrightarrow{\Psi_{\text{birat}}} & (C_2^{\text{pf}})^{\text{birat}}
\end{array}
\end{equation}

(where the vertical arrows are the natural functors and the horizontal arrows are equivalences of categories). (Note that here, it is not clear that the \textit{uniform dissectibility} – hence, \textit{a fortiori}, the property of being of \textit{poly-standard} type – of \( C_i \) implies that of \( C_i^{\text{pf}} \). Thus, one cannot apply to ‘\( C_i^{\text{pf}} \)’ the portion of Theorem 5.5 already proven for ‘\( C_i \)’. Nevertheless, this does not cause any problems since uniform dissectibility is only used to ‘\text{dissect}’ the poly-Frobenioid in question; thus, the necessary ‘dissection’ of \( C_i^{\text{pf}} \) follows immediately from the dissection already discussed of \( C_i \).)

Next, let us observe that since, for \( A \in \text{Ob}(C_i^{\text{pf}}) \) with image \( A^{\text{pf}} \in \text{Ob}(C_i^{\text{pf}}) \), we have a natural isomorphism \( O^\times(A)^{\text{pf}} \cong O^\times(A) \) (cf. \cite[Proposition 5.5(i)]{Mzk5}), it follows that the \textit{LC-unit-admissibility} of \( C_i \) implies that of \( C_i^{\text{pf}} \) (cf. Definition 5.3(v)). Thus, \( C_i^{\text{pf}} \) is of \textit{standard, perfect} (cf. \cite[Proposition 3.2(iii)]{Mzk5}), and \textit{LC-unit-admissible} type. In particular, we may apply Proposition 5.4 to \( C_i^{\text{pf}} \). Thus, it follows that global objects of \( (C_i^{\text{pf}})^{\text{birat}} \) admit \textit{complete tactile collections} (cf. Proposition 5.4(i)), which are, moreover, \textit{preserved by} \( \Psi \) (cf. Proposition 5.4(ii) and the fact that, in light of assertion (i), the definition of ‘pre-tactile’ is \textit{manifestly category-theoretic}). (Here, we note in passing that if one wishes to restrict oneself to operating in \( C_i^{\text{pf}} \), then instead of working with complete tactile collections in \( (C_i^{\text{pf}})^{\text{birat}} \), one can instead work with \textit{complete birationally tactile collections} in \( C_i^{\text{pf}} \) – cf. Proposition 5.4(iii) and (iv)).

Now let us consider assertion (iv). Since the poly-base category \( D_i \) is assumed to be \textit{slim}, it follows from \cite[Theorem 3.4(v)]{Mzk5} (cf. also assertion (i) and our assumption that \( C_i \) is of \textit{poly-non-group-like} type) that \( \Psi \) induces equivalences of categories between the various local and global components of \( D_i \) which are \textit{compatible} with the equivalences of categories induced by \( \Psi \) between the various local and global components of \( C_i \).
Moreover, by considering complete tactile collections in \((C_i^{\text{pf}})^{\text{birat}}\), it follows immediately that these equivalences of categories between the various local and global components of \(D_i\) are compatible with the local and global contact functors of \(D_i\). Thus, we obtain a 1-unique 1-commutative diagram as in assertion (iv).

Next, we consider unit-trivializations (i.e., the case ‘\(\Box = \text{un-tr}'\)). By the Frobenius-slimness assumption, it follows from [Mzk5, Theorem 3.4(iv)], that \(\Psi\) induces equivalences of categories between the various local and global components of \(C_i^{\text{un-tr}}\) which are compatible with the equivalences of categories induced by \(\Psi\) between the various local and global components of \(C_i\). Thus, to obtain a diagram as in assertion (iii), it suffices to show that \(\Psi\) preserves unit-equivalent pairs of co-objective heterogeneous morphisms of \(C_i^{\text{str}}\) (i.e., pairs that map to the same arrow in \(C_i^{\text{un-tr}}\)). By the faithfulness portion of [Mzk5, Proposition 3.3(iv)] (applied to the underlined local components of the \(C_i\)!), it follows that it suffices to show that \(\Psi\) maps a unit-equivalent pair of heterogeneous morphisms \(\phi_1, \psi_1 : A \to B\) of \(C_1\) to a pair of morphisms \(\phi_2 \equiv \Psi(\phi_1), \psi_2 \equiv \Psi(\psi_1)\) of \(C_2\) such that \(\degFr(\phi_2) = \degFr(\psi_2), \Div(\phi_2) = \Div(\psi_2)\) and \(\text{Base}(\phi_2) = \text{Base}(\psi_2)\).

To do this, it suffices to show that \(\degFr(\phi_2) = \Div(\phi_2) = \Div(\psi_2)\) may be ‘computed' entirely in terms of homogeneous morphisms (whose behavior with respect to \(\Psi\) and the operation of taking the unit-trivialization is already well-understood). Thus, it suffices to observe the following fact: the Frobenius degree may be computed by considering the existence of factorizations \(A \to A' \to B\), where \(A \to A'\) is a (necessarily homogeneous!) morphism of Frobenius type (cf. [Mzk5, Definition 1.3(iv)(a)]). The property of metric equivalence (i.e., ‘\(\Div(\phi_2) = \Div(\psi_2)\)') of a pair of linear co-objective morphisms \(A \to B\) may be described (cf. the inclusions ‘\(\Phi_2|_{D_i} \hookrightarrow \Phi_1^{\text{pf}}\)' of Definition 5.3(ii)(c)) by projecting these morphisms to morphisms \(A^{\text{pf}} \to B^{\text{pf}}\) of \(C_i^{\text{pf}}\) and considering the condition that one of these morphisms \(A^{\text{pf}} \to B^{\text{pf}}\) factors through an arbitrary given co-angular pre-step \(A^{\text{pf}} \to (A')^{\text{pf}}\) (cf. [Mzk5, Definition 1.3(iii)(d) and Definition 1.3(iv)(a)]) if and only if the other does. (Here, we use the fact that \(\Psi^{\text{pf}}\) preserves co-angular pre-steps of \(C_i^{\text{pf}}\) – cf. [Mzk5, Theorem 3.4(ii)].) Finally, the property of base-equivalence (i.e., ‘\(\text{Base}(\phi_2) = \text{Base}(\psi_2)\)') of a pair of linear, metrically equivalent, co-objective morphisms \(A \to B\) may be described by projecting these morphisms to morphisms \(A^{\text{pf}} \to B^{\text{pf}}\) of \(C_i^{\text{pf}}\) and considering the condition (cf. [Mzk5, Definition 1.3(iii)(c), Definition 1.3(iv)(a) and Definition 1.3(vi)]) that these morphisms may be obtained from one another by composition with a (manifestly homogeneous!) element \(\in O^\times(A^{\text{pf}}) \cong O^\times(A)^{\text{pf}}\). (Note that here, we must apply the \(\text{LC-unit-admissibility}\) of \(C_i\).) This completes the proof of assertion (iii) in the case ‘\(\Box = \text{un-tr}'\'.

Next, we consider divisor monoids (i.e., assertion (v)). By [Mzk5, Corollary 4.11(iii)], it follows that \(\Psi\) induces isomorphisms between the divisor monoids of corresponding local or global components of \(C_1, C_2\). The asserted compatibility then follows by considering global co-angular pre-steps \(B^{\text{pf}} \to C^{\text{pf}}\) of \(C_i^{\text{pf}}\) (cf. [Mzk5, Definition 1.3(iii)(d)]), together with birationally tactile morphisms \(A^{\text{pf}} \to B^{\text{pf}}\) (whose existence follows from the fact that \(D_i\) is of totally tactile type, [Mzk5, Definition 1.3(i)(b)] and the inclusions ‘\(\Phi_2|_{D_i} \hookrightarrow \Phi_1^{\text{pf}}\)' of Definition 5.3(ii)(c)), which allow one to compute the image of \(\Div(B^{\text{pf}} \to C^{\text{pf}})\) in (the value of the appropriate divisor monoid at) \(A^{\text{pf}}\) as the difference

\[
\Div(A^{\text{pf}} \to B^{\text{pf}} \to C^{\text{pf}}) - \Div(A^{\text{pf}} \to B^{\text{pf}}),
\]
where we note that each of the terms in this difference may be computed as the supremum of the $\text{Div}(A^{\text{pf}} \to (A')^{\text{pf}})$, where $A^{\text{pf}} \to (A')^{\text{pf}}$ is a (necessarily homogeneous!) co-angular pre-step through which the heterogeneous morphism in question (i.e., $A^{\text{pf}} \to B^{\text{pf}} \to C^{\text{pf}}$ or $A^{\text{pf}} \to B^{\text{pf}}$) factors. This completes the proof of assertion (v).

Finally, assertion (iii) in the case $\Box = \text{rlf}$ follows by applying \cite{Mzk5, Corollary 5.4}, to the various local and global components of $C_i$, in light of the compatibility of assertion (v). $\Box$

We are now ready to discuss the main motivating example of the theory of \cite{Mzk5} and the present paper, an example to which most of the main results of this theory may be applied. The example arises, in effect, by ‘grafting’ the global example of \cite{Mzk5, Example 6.3}, onto the local examples of Examples 1.1 and 3.3 of the present paper.

**Example 5.6.** (Poly-Frobenioids associated to number fields)

(i) Let $\widetilde{F}/F$ be a Galois extension of a number field $F$ and write $G_\otimes \overset{\text{def}}{=} \text{Gal}(\widetilde{F}/F)$, $\mathcal{V}(F)$ for the set of valuations on $F$ (where we identify complex archimedean valuations with their complex conjugates – cf. \cite{Mzk5, Example 6.3}). For $v \in \mathcal{V}(F)$, write $F_v$ for the completion of $F$ at $v$, $\widetilde{F}_v$, for the Galois extension of $F_v$ determined by $\widetilde{F}$, $\mathcal{D}_v \overset{\text{def}}{=} \text{Gal}(\widetilde{F}_v/F_v) \subseteq G_\otimes$ (so $\mathcal{D}_v$ is well-defined up to conjugation in $G_\otimes$). Also, we assume that we have been given monoid types

$$\Lambda_\otimes, \quad \{\Lambda_v\}_{v \in \mathcal{V}(F)}, \quad \{\Lambda_v\}_{v \in \mathcal{V}(F)}$$

satisfying $\Lambda_v \supseteq \Lambda_\otimes, \Lambda_v \supseteq \Lambda_v$, for all $v \in \mathcal{V}(F)$. Note that (if we take the ‘$\widetilde{F}/F$’, ‘$v$’ of Example 1.4 to be the ‘$\widetilde{F}/F$’, ‘$v$’ of the present discussion, then) the category ‘$\mathcal{P}_0$’ of Example 1.4 may be identified with the category $\mathcal{P}_v \overset{\text{def}}{=} \mathcal{B}(\mathcal{D}_v)^0$. Thus, the functor ‘$\mathcal{E}_0 \to \mathcal{P}_0$’ of Example 1.4 determines a functor which we denote by $\mathcal{E}_v \to \mathcal{P}_v$.

(ii) Let

$$\Pi_\otimes \to G_\otimes$$

be a surjection of residually finite tempered groups. Then the category

$$\mathcal{D}_\otimes \overset{\text{def}}{=} \mathcal{B}(\Pi_\otimes)^0$$

is connected, totally epimorphic, Frobenius-slim and of FSM- and strongly indissectible (hence, in particular, weakly indissectible) type – cf. Example 1.3(i). Thus, if ‘$\mathcal{C}_{\widetilde{F}/F}$’ is as in \cite{Mzk5, Example 6.3}, then

$$C^{\mathcal{Z}}_\otimes \overset{\text{def}}{=} C_\otimes \overset{\text{def}}{=} C_{\widetilde{F}/F} \times \mathcal{B}(G_\otimes) \mathcal{D}_\otimes, \quad C^{\otimes}_\otimes \overset{\text{def}}{=} C^{\text{pf}}, \quad C^{\mathcal{R}}_\otimes \overset{\text{def}}{=} C^{\text{rlf}}_\otimes$$

determines a Frobenioid $\mathcal{C}^{\Lambda_\otimes}$ which is of rationally standard type over a Frobenius-slim base category (cf. \cite{Mzk5, Theorem 6.4(i)}), for the case $\Pi_\otimes = G_\otimes$; the case of arbitrary $\Pi_\otimes$ is entirely similar). If, moreover, $\Pi_\otimes$ is temp-slim, then $\mathcal{D}_\otimes$ is slim (cf. Example 1.3(i)).

(iii) Let $v \in \mathcal{V}(F)$ be non-archimedean, let $\widetilde{K}_v/F_v$ be a Galois extension containing $\widetilde{F}_v$ and $G_v \overset{\text{def}}{=} \text{Gal}(\widetilde{K}_v/F_v)$. Thus, we have a natural surjection $G_v \to \mathcal{D}_v \subseteq G_\otimes$. Suppose further that we have been given a commutative diagram

\[
\begin{array}{ccc}
\Pi_\otimes & \longrightarrow & G_\otimes \\
\downarrow & & \downarrow \\
\Pi_v & \longrightarrow & G_v
\end{array}
\]
of residually finite tempered groups (where the lower horizontal surjection and the vertical homomorphism on the right are the morphisms that were given previously). Let

$$\mathcal{D}_v$$

be one of the categories $\mathcal{B}(\Pi_v)^0$, $\mathcal{B}(\Pi_v)^0 \times \mathcal{P}_v \mathcal{E}_v$ (where the functor $\mathcal{B}(\Pi_v)^0 \to \mathcal{P}_v$ is the functor determined by the surjection $\Pi_v \to G_v \to \mathfrak{D}_v$ – cf. Example 1.3(ii)). Then the category $\mathcal{D}_v$ is connected, totally epimorphic, Frobenius-slim, and of FSMFF- and strongly indissectible (hence, in particular, weakly indissectible) type (cf. Example 1.3(i) and Example 1.4(iii)). If, moreover, $\Pi_v$ is temp-slim, then $\mathcal{D}_v$ is slim (cf. Example 1.3(i) and Example 1.4(iii)). Write

$$\mathcal{C}^{\Lambda_v}_v, \mathcal{C}^{\Delta_v}_v$$

for the respective categories ‘$\mathcal{C}$’ of Example 1.1(ii), obtained by taking the ‘$\mathcal{D}$’ of loc. cit. to be $\mathcal{D}_v$, the ‘$\Lambda$’ of loc. cit. to be $\Lambda_v$, $\Lambda_v$, and the ‘$\Phi$’ of loc. cit. to be functors $\Phi_v, \Phi_v : \mathcal{D}_v \to \mathfrak{M}_\text{Mon}$ as in loc. cit. such that the following conditions are satisfied:

(a) $\Phi_v \subseteq \Phi_v$;

(b) the resulting functors $\Phi_v^{\text{birat}}, \Phi_v^{\text{birat}}$ satisfy $\Phi_v^{\text{birat}} = \Phi_v^{\text{birat}} + \Phi_v^{\text{cnst}}$ for some group-like functor $\mathcal{D}_v \to \mathfrak{M}_\text{Mon}$ which is constant (i.e., maps all arrows of $\mathcal{D}_v$ to isomorphisms of $\mathfrak{M}_\text{Mon}$);

(c) $\Phi_v$ is fieldwise saturated (cf. Example 1.1(iii)) and supported by $\Lambda_v$ (cf. [Mzk5, Definition 2.4(ii)]).

Thus, $\mathcal{C}^{\Lambda_v}_v, \mathcal{C}^{\Delta_v}_v$ are Frobenioids which are of rationally standard type over a Frobenius-slim base category (cf. Theorem 1.2(i)) and the underlying category of $\mathcal{C}^{\Lambda_v}_v, \mathcal{C}^{\Delta_v}_v$ is of weakly indissectible type (cf. the discussion of Section 0). Moreover (since $\Lambda_v \geq \Lambda_\otimes, \Lambda_v \geq \Lambda_v$, for all $v \in \mathcal{V}(F)$), it follows immediately from the definitions (by restricting arithmetic divisors on number fields to divisors on their localizations and non-zero elements of number fields to elements of their completions) that we obtain functors

$$\mathcal{C}^{\Lambda_\otimes}_v \to (\mathcal{C}^{\Delta_v}_v)^\top, \mathcal{C}^{\Lambda_v}_v \to \mathcal{C}^{\Delta_v}_v$$

the first of which is $G\text{-admissible}$ and $1$-compatible with the natural localization functor $\mathcal{D}_\otimes \to \mathcal{D}_v^\top$ (arising from the homomorphism ‘$\Pi_v \to \Pi_\otimes$’ – cf. the ‘pull-back functor’ of Example 1.3(ii)), and the second of which is $LC$-unit-admissible (cf. [Mzk5, Proposition 5.5(i)]). Also, we note that the ‘restriction of arithmetic divisors’ portion of the first of these functors is possible precisely because of condition (c).

(iv) Let $v \in \mathcal{V}(F)$ be archimedean let $\tilde{K}_v/F_v$ be a Galois extension containing $\tilde{F}_v$ such that $\tilde{K}_v$ is complex and let $G_v \overset{\text{def}}{=} \text{Gal}(\tilde{K}_v/F_v)$. Thus, we have a natural surjection $G_v \twoheadrightarrow \mathfrak{D}_v \subseteq G_\otimes$. Suppose further that we have been given a commutative diagram

$$\begin{array}{ccc}
\Pi_v & \longrightarrow & G_v \\
\downarrow & & \downarrow \\
\Pi_\otimes & \longrightarrow & G_\otimes
\end{array}$$

of residually finite tempered groups (where the lower horizontal surjection and the vertical homomorphism on the right are the morphisms that were given previously). Let

$$\mathcal{D}_v$$
be one of the following categories:

(a) \( \mathcal{B}(\Pi_v)^0, \mathcal{B}(\Pi_v)^0 \times \mathcal{P}_v \mathcal{E}_v \) (where the functor \( \mathcal{B}(\Pi_v)^0 \to \mathcal{D}_v \) is the functor determined by the surjection \( \Pi_v \to G_v \to \mathcal{D}_v \) – cf. Example 1.3(ii));

(b) one of the categories of hyperbolic Riemann surfaces ‘\( \mathcal{D} \)’ or ‘\( \mathcal{E} \)’ of Example 4.3(i) – (iii), where the tempered group ‘\( \Pi \)’ of loc. cit. is equal to \( \Pi_v \) (in a fashion compatible with the surjection \( \Pi_v \to G_v \));

(c) a non-rigidified or rigidified angloid (i.e., ‘\( \mathcal{N} \)’ or ‘\( \mathcal{R} \)’) over a base category as in (a) above (cf. Remark 4.2.1);

(d) a non-rigidified or rigidified angloid (i.e., ‘\( \mathcal{N} \)’ or ‘\( \mathcal{R} \)’) over a base category as in (b) above (cf. Remark 4.2.1).

If \( \mathcal{D}_v \) is taken to be one of the categories that involves \( \mathcal{E}_v \), then let us assume further that the extension \( \tilde{F}/F \) is {	extit{amply quadratic}} (cf. Example 4.3(iii)). Then (cf. Example 1.3(i), Proposition 1.5(iv), Proposition 4.1(v), Corollary 4.2(iv), (v), Example 4.3(ii)–(iv) and Proposition 5.5(i)

the category \( \mathcal{D}_v \) is Frobenius-slim and of RC-standard (hence, in particular, connected, totally epimorphic and of FSMFF-type) and either strongly dissectible or weakly indissectible type; moreover, in cases (b) and (d), \( \mathcal{D}_v \) is always slim, while in cases (a) and (c), \( \mathcal{D}_v \) is slim whenever \( \Pi_v \) is temp-slim. Write

\[
\mathcal{C}^\Lambda_v, \quad \mathcal{C}^\Lambda_v
\]

for the respective categories ‘\( \mathcal{C}^\Lambda \)' of Example 3.3(ii) obtained by taking the ‘\( \mathcal{D} \)’ of loc. cit. to be \( \mathcal{D}_v \) and the ‘\( \Lambda \)' of loc. cit. to be \( \Lambda_v, \Lambda_v \). Thus, \( \mathcal{C}^\Lambda_v, \mathcal{C}^\Lambda_v \) are Frobenioids which are of rationally standard type over a Frobenius-slim base category (cf. Theorem 3.6(i)); the underlying category of \( \mathcal{C}^\Lambda_v, \mathcal{C}^\Lambda_v \) is of either strongly dissectible or weakly indissectible type (cf. the discussion of Section 0). Moreover, (since \( \Lambda_v, \Lambda_v, \Lambda_v \geq \Lambda_v \geq v \), for all \( v \in \mathcal{V}(F) \)), it follows immediately from the definitions (by restricting arithmetic divisors on number fields to divisors on their localizations and non-zero elements of number fields to elements of their completions) that we obtain functors

\[
\mathcal{C}^\Lambda_v \to (\mathcal{C}^\Lambda_v)^\top, \quad \mathcal{C}^\Lambda_v \to (\mathcal{C}^\Lambda_v)^\top
\]

the first of which is GC-admissible and 1-compatible with the natural localization functor \( \mathcal{D}_v \to \mathcal{D}_v^\top \) (arising from the homomorphism ‘\( \Pi_v \to \Pi_v \)’ – cf. the ‘pull-back functor’ of Example 1.3(ii)), and the second of which is LC-unit-admissible (cf. [Mzk5, Proposition 5.5(i)]).

(v) Thus, the data

\[
\mathcal{C} \defn \left( \mathcal{C}^\Lambda, \{\mathcal{C}^\Lambda_v\}_{v \in \mathcal{V}(F)}, \{\mathcal{C}^\Lambda_v\}_{v \in \mathcal{V}(F)}, \{\mathcal{C}^\Lambda_v \to (\mathcal{C}^\Lambda_v)^\top\}_{v \in \mathcal{V}(F)}, \{\mathcal{C}^\Lambda_v \to (\mathcal{C}^\Lambda_v)^\top\}_{v \in \mathcal{V}(F)} \right)
\]

determines a poly-Frobenioid \( \mathcal{C} \) of poly-rationally standard and poly-non-group-like type over a Frobenius-slim poly-base category \( \mathcal{D} \). In particular, all but the portion requiring ‘slimness’ of Theorem 5.5 applies to \( \mathcal{C} \); the conditions on the data of the present Example 5.6 necessary for the poly-base category to be slim are as discussed above.

Remark 5.6.1. Observe that unlike the case with Frobenoids, poly-Frobenioids are not necessarily totally epimorphic! In a similar vein (cf. the use of the injectivity condition of [Mzk5, Definition 1.1(ii)(a)], in the proof of the total epimorphicity portion
of [Mzk5, Proposition 1.5]), we observe that (again, unlike the case with Frobenioids) the map induced by a heterogeneous morphism between the values of the divisor monoids at the domain and codomain is not necessarily injective! Indeed, these phenomena occur, for instance, if one considers the poly-Frobenioids of Example 5.6, when \( \Lambda_\odot = \mathbb{R} \).

**Remark 5.6.2.** Just as in the case of archimedean \( v \), it is natural to consider \( \Pi_\odot, \Pi_v \) (for non-archimedean \( v \)) arising from the arithmetic fundamental group of a hyperbolic curve. One may then apply to such \( \Pi_\odot, \Pi_v \) various results from the absolute anabelian geometry of hyperbolic curves (cf., e.g., [Mzk4]).

**Remark 5.6.3.** One verifies immediately that, when considering equivalences of categories between poly-Frobenioids of the sort discussed in Example 5.6(v), the induced equivalences of categories between the respective global components determine a ‘degree’ \( \in \mathbb{R}_{>0} \), as in [Mzk5, Theorem 6.4(ii)]. Moreover, just as in [Mzk5, Theorem 6.4(iii)], this degree is \( \in \mathbb{Q}_{>0} \) (even if \( \Lambda_\odot = \mathbb{R} \! \)!) whenever it holds that the \( \Lambda_v \leq \mathbb{Q} \) for all \( v \in \mathbb{V}(F) \). Similarly, just as in [Mzk5, Theorem 6.4(iv)], this degree = 1 (even if \( \Lambda_\odot = \mathbb{R} \! \)!) whenever it holds that the \( \Lambda_v = \mathbb{Z} \) for all \( v \in \mathbb{V}(F) \).

**Appendix. Categorical representation of topological spaces**

In this appendix, we discuss certain classical results concerning how a topological space may be represented by means of a category.

Let \( X \) be a topological space. Then we shall write

\[
\text{Subset}(X)
\]

for the category whose objects are the subsets of \( X \) (including \( X \), the empty set), and whose morphisms are the inclusions of subsets of \( X \). Write

\[
\text{Open}(X) \subseteq \text{Subset}(X), \quad \text{Closed}(X) \subseteq \text{Subset}(X)
\]

for the full subcategories determined by the open and closed subsets, respectively. Also, we shall denote by

\[
\text{Open}^0(X) \subseteq \text{Open}(X)
\]

the full subcategory determined by the (non-empty) connected open subsets and by

\[
\text{Shv}(X)
\]

the category of sheaves (valued in sets, relative to some universe fixed throughout the discussion) on \( X \).

**Definition A.1.**

(i) We shall say that \( X \) is locally connected if, for every open subset \( U \subseteq X \) and every point \( x \in U \), there exists a connected open subset \( V \subseteq U \) such that \( x \in V \). We shall say that \( X \) is sober if, for every irreducible closed subset \( F \subseteq X \), there exists a unique point \( x \in F \) such that \( F \) is equal to the closure of the set \( \{ x \} \) in \( X \) (cf. [John, p. 230]).

(ii) We shall refer to a collection \( \{ A_i \}_{i \in I} \) of distinct objects of \( \text{Open}^0(X) \) as a collection of disjoint objects if, for any pair of distinct elements \( i, j \) of \( I \), there does not exist an object \( C \in \text{Ob}(\text{Open}^0(X)) \) that admits a morphism in \( \text{Open}^0(X) \) to both \( A_i \) and \( A_j \).
(iii) Denote by \( \text{Disjt}(\text{Open}^0(X)) \)

the category defined as follows: an object of this category is a collection of disjoint objects \( \{U_i\}_{i \in I} \) (where \( I \) is a (possibly empty) set). A morphism of this category

\[
\{U_i\}_{i \in I} \rightarrow \{V_j\}_{j \in J}
\]

consists of a function \( f : I \rightarrow J \) and a collection of morphisms (in \( \text{Open}^0(X) \)) \( U_i \rightarrow V_{f(i)} \) (where \( i \) ranges over the elements of \( I \)). Thus, by assigning to an object of \( \text{Open}^0(X) \) the collection of objects of \( \text{Open}^0(X) \) consisting of this single object, and to a collection \( \{U_i\}_{i \in I} \) the object of \( \text{Open}(X) \) constituted by the union of the connected open sets determined by the \( U_i \), we obtain natural functors

\[
\text{Open}^0(X) \rightarrow \text{Disjt}(\text{Open}^0(X)), \quad \text{Disjt}(\text{Open}^0(X)) \rightarrow \text{Open}(X)
\]

the first of which is easily verified to be fully faithful.

Remark A.1.1. Thus, if \( X \) is locally connected, then one verifies immediately that, relative to the notation of Definition A.1(ii), the \( A_i \) are disjoint if and only if the open subsets of \( X \) to which the \( A_i \) correspond are pair-wise mutually ‘disjoint’ in the usual sense.

**Theorem A.2. (Categorical representation of topological spaces)** Let \( X, Y \) be topological spaces. Then we have the following.

(i) The category \( \text{Open}(X) \) is equivalent to the opposite category to \( \text{Closed}(X) \).

(ii) The categories \( \text{Subset}(X), \text{Open}(X), \text{Open}^0(X) \) and \( \text{Closed}(X) \) are slim.

(iii) By assigning to an object of \( \text{Open}(X) \) the sheaf on \( \text{Open}(X) \) represented by the given object, we obtain a natural functor

\[
\text{Open}(X) \rightarrow \text{Shv}(X)
\]

which is fully faithful.

(iv) Suppose that \( X, Y \) are sober. Then passing to the induced equivalence on the categories ‘\( \text{Open}(\_\_\_) \)’ determines a bijection between the equivalences of categories

\[
\text{Open}(X) \sim \text{Open}(Y)
\]

(considered up to isomorphism) and the homeomorphisms \( X \sim Y \).

(v) Suppose that \( X \) is locally connected. Then the natural functor

\[
\text{Disjt}(\text{Open}^0(X)) \rightarrow \text{Open}(X)
\]

is an equivalence of categories.

(vi) Suppose that \( X, Y \) are sober and locally connected. Then passing to the induced equivalence on the categories ‘\( \text{Open}^0(\_\_\_) \)’ determines a bijection between the equivalences of categories

\[
\text{Open}^0(X) \sim \text{Open}^0(Y)
\]

(considered up to isomorphism) and the homeomorphisms \( X \sim Y \).
Proof. The equivalence of assertion (i) is obtained by associating to an open set of $X$ the closed set of $X$ given by its complement. Assertion (ii) follows immediately from the fact that, by definition, the categories in question have no non-trivial automorphisms. Assertion (iii) follows immediately from the definitions and ‘Yoneda’s Lemma’. To verify assertion (iv), we recall (cf. [John, Theorem 7.24]) that passing to the induced equivalence on the categories ‘Shv(−)’ determines a bijection between the equivalences of categories

$$\text{Shv}(X) \overset{\sim}{\rightarrow} \text{Shv}(Y)$$

(considered up to isomorphism) and the homeomorphisms $X \sim Y$. Note, moreover, that an open $V$ of $X$ is a union of opens $\{U_\alpha\}_{\alpha \in I}$ of $X$ if and only if $V$ forms (i.e., in $\text{Open}(X)$) an inductive limit (a purely category-theoretic notion!) of the system constituted by the $U_\alpha$. Thus, to complete the proof of assertion (iv), it suffices, in light of the natural embedding of assertion (iii), to observe that $\text{Shv}(X)$ may be reconstructed (by the definition of a ‘sheaf’!) directly from the category $\text{Open}(X)$, in a fashion that is compatible with the natural embedding of assertion (iii). This completes the proof of assertion (iv).

Next, we consider assertion (v). First, I claim that every open $V$ of $X$ is a disjoint union of connected opens of $X$. Indeed, consider the equivalence relation on objects of $\text{Open}^0(V)$ generated by the pre-equivalence relation that two distinct objects $A, B$ of $\text{Open}^0(V)$ are ‘pre-equivalent’ if the pair of objects $A, B$ fails to form a collection of disjoint objects of $\text{Open}^0(V)$ (i.e., the connected opens corresponding to $A, B$ intersect). Denote by $I$ the set of equivalence classes of objects of $\text{Open}^0(V)$, relative to this equivalence relation. For $i \in I$, write $U_i \in \text{Ob}(\text{Open}(X))$ for the union of the connected opens that lie in the class $i$. Then it follows immediately from the definitions that $U_i$ is a connected open, hence forms an object of $\text{Open}^0(X)$. Since, moreover, $X$ is locally connected, it follows that the union of $U_i$ (as $i$ ranges over the elements of $I$) is equal to $V$. This completes the proof of the claim. Now assertion (v) follows formally.

Finally, we observe that assertion (vi) follows immediately from assertions (iv) and (v), together with the easily verified observation that the category $\text{Disjt}(\text{Open}^0(−))$ may be reconstructed directly from the category $\text{Open}^0(−)$, in a fashion that is compatible with the natural embedding $\text{Open}^0(X) \hookrightarrow \text{Disjt}(\text{Open}^0(X))$.

Remark A.2.1. Note that neither of the two conditions of ‘soberness’ and ‘local connectedness’ implies the other. Indeed, suppose that the underlying set of $X$ is countably infinite, and that the proper (i.e., $\neq X$) closed subsets of $X$ are precisely the finite subsets of $X$. (Consider, for instance, the Zariski topology on the set of closed points of the affine line over a countable algebraically closed field.) Then observe that $X$ is irreducible, and that any non-empty open subset $Y \subseteq X$ is homeomorphic to $X$, hence, in particular, irreducible. Since irreducible topological spaces are connected, it thus follows that $X$ is locally connected. On the other hand, since $X$ is irreducible, but clearly fails to admit a generic point, it follows that $X$ is not sober. In the ‘opposite direction’, any infinite ‘profinite set’ (i.e., a projective limit of finite sets – e.g., the underlying topological space of a profinite group) is Hausdorff, hence sober, but satisfies the property that every open subset is totally disconnected, hence fails to be locally connected.
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Shinichi Mochizuki
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
(E-mail: motizuki@kurims.kyoto-u.ac.jp)