HYPERCONVEXITY AND NEIGHBORHOOD BASIS OF REINHARDT DOMAINS

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Abstract. In this paper, we prove that every bounded Reinhardt domain in $\mathbb{C}^n$ admitting a weak Stein neighborhood basis is hyperconvex. In addition, we show that every unbounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$ satisfying the Fu condition admits a Stein neighborhood basis.

1. Introduction

For a bounded domain $D$ in $\mathbb{C}^n$, a weak Stein neighborhood basis of $D$ is a family of pseudoconvex domains $\{\Omega_k\}_{k \geq 1}$ such that $\overline{D} \subset \Omega_k$, $k \in \mathbb{N}$ and $D = \text{int} \bigcap_{k \in \mathbb{N}} \Omega_k$. Generally, for a bounded fat pseudoconvex Reinhardt domain $D$ in $\mathbb{C}^n$, there may not exist a weak Stein neighborhood basis of $D$. But any bounded hyperconvex Reinhardt domain $D$ has a Stein neighborhood basis which means that there is a family of pseudoconvex domains $\{\Omega_k\}_{k \geq 1}$ such that $\Omega_k$ is a domain of holomorphy and $\overline{D} \subset \Omega_k$, $k \in \mathbb{N}$ and $\overline{D} = \bigcap_{k \in \mathbb{N}} \Omega_k$ (see [9]). Note that every fat domain having a Stein neighborhood basis always has a weak Stein neighborhood basis. For example, the Hartogs triangle is a bounded fat pseudoconvex Reinhardt domain in $\mathbb{C}^2$ but it does not allow a weak Stein neighborhood basis.

It is worth noting that every pseudoconvex domain containing the closure of the Hartogs triangle also contains the closure of the unit bidisc [4]. Note that the unit bidisk is hyperconvex, while the Hartogs triangle is not so. By observing this phenomenon, in [9] (or [5]), the authors showed that if any bounded fat Reinhardt domain $D$ in $\mathbb{C}^n$ has a Stein neighborhood basis, then $D$ is hyperconvex. In this paper, we shall show that the same result still holds under a weaker condition. Namely, every bounded Reinhardt domain in $\mathbb{C}^n$ having a weak Stein neighborhood basis is hyperconvex.

On the other hand, it is natural to ask whether an unbounded Reinhardt domain having a Stein neighborhood basis can be hyperconvex. Unfortunately, it is impossible because any hyperconvex Reinhardt domain is bounded [13]. Now we are interested in the following: Under what condition will an unbounded Reinhardt domain admit a Stein neighborhood basis? As a partial answer to that question, we shall show that there exists a class of unbounded pseudoconvex Reinhardt domains admitting a Stein neighborhood basis.

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2. Preliminaries

A set $D$ in $\mathbb{C}^n$ is said to be Reinhardt whenever

$$(z_1, \ldots, z_n) \in D \implies (e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) \in D, \quad \theta_i \in \mathbb{R}, \ 1 \leq i \leq n.$$ 

For $j = 1, \ldots, n$, we set $V_j := \{z \in \mathbb{C}^n : z_j = 0\}$, $V := \bigcup_{1 \leq j \leq n} V_j$, and $D_\ast := D \setminus V$. Denote $V_I := \{z \in \mathbb{C}^n : z_I = 0\}$ where $I = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$. Given a Reinhardt subset $D$ of $\mathbb{C}^n$, we denote by $\log(D_\ast)$ its logarithmic image, i.e.

$$\log(D_\ast) := \{(\log |z_1|, \ldots, \log |z_n|) \in \mathbb{R}^n : (z_1, \ldots, z_n) \in D_\ast\}.$$ 

For a domain $D$ in $\mathbb{C}^n$, we denote by $E(D)$ the envelope of holomorphy of $D$, i.e.

$$E(D) := \text{int} \bigcap_{D \subset U \subset \mathbb{C}^n} U,$$

where $U$ is a pseudoconvex domain.

Observe that $E(D)$ is the smallest domain of holomorphy containing $D$ and if $D$ is a Reinhardt domain, then $E(D)$ is also a Reinhardt domain in $\mathbb{C}^n$ (see [2]). Therefore, the envelope of holomorphy of any Reinhardt domain in $\mathbb{C}^n$ is a pseudoconvex Reinhardt domain. Note that if $D \subset G$, then $E(D) \subset E(G)$ (see [12]). For more information regarding envelopes of holomorphy we refer the reader to [6, 7].

We denote by $\mathcal{PSH}(D)$ the set of all plurisubharmonic functions on $D$, and by $P(z, \varepsilon)$ the open polydisc in $\mathbb{C}^n$ of multiradius $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n), \varepsilon_j > 0$, centered at $z \in \mathbb{C}^n$.

**Definition 2.1.** A domain $D$ in $\mathbb{C}^n$ is hyperconvex if there exists a negative continuous plurisubharmonic exhaustion function $u$ on $D$, i.e. $u \in (\mathcal{C} \cap \mathcal{PSH})(D), u < 0$ on $D$, and $\lim_{z \to \partial D} u(z) = 0$ (here and below $\infty \in \partial D$ if $D$ is unbounded). Such a function $u$ is sometimes called a continuous plurisubharmonic barrier on $D$.

The notion of hyperconvexity is an important concept in pluripotential theory. For hyperconvexity, it is enough to have a weak plurisubharmonic barrier for every boundary point, as follows.

**Proposition 2.2.** [1] A domain $D$ in $\mathbb{C}^n$ is hyperconvex if and only if there is a weak plurisubharmonic barrier at every point $\xi \in \partial D$, i.e. there exists a negative plurisubharmonic function $\psi$ on $D$ such that $\lim_{z \to \xi} \psi(z) = 0$.

This notion is perhaps most clearly stated by Błocki [1]. The point $\xi$ as in Proposition 2.2 is called a barrier point for $D$.

Let us recall the following geometrical criterion for a Reinhardt domain in $\mathbb{C}^n$ to be pseudoconvex.

**Proposition 2.3.** [6, 12] Let $D$ be a Reinhardt domain in $\mathbb{C}^n$. Then the following conditions are equivalent:

1. $D$ is pseudoconvex;
2. $\log(D_\ast)$ is convex, and if $D \cap V_j \neq \emptyset$ for some $j \in \{1, \ldots, n\}$, then

$$(z_1, \ldots, z_j, \ldots, z_n) \in D \implies (z_1, \ldots, \lambda z_j, \ldots, z_n) \in D, \quad |\lambda| \leq 1.$$
Let us mention that the previous criterion plays an important role in the study of the pseudoconvexity of Reinhardt domains.

For an index set $I = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$, we define the canonical projection $\pi_I : \mathbb{C}^n \to \mathbb{C}^{n-k}$ by $\pi_I(z) := (z_{m_1}, \ldots, z_{m_{n-k}})$, where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\{m_1, \ldots, m_{n-k}\} := \{1, \ldots, n\} \setminus I$. The following result is easily obtained.

**Proposition 2.4.** [13] Let $D$ be a pseudoconvex Reinhardt domain in $\mathbb{C}^n$ and $D \cap V_I \neq \emptyset$ where $I = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$. Then $\pi_I(D)$ is also a pseudoconvex Reinhardt domain in $\mathbb{C}^{n-k}$.

**Definition 2.5.** For a bounded domain $D \subset \mathbb{C}^n$ we say that:

- $D$ has a weak Stein neighborhood basis if $D = \operatorname{int} \bigcap_{\overline{D} \subset U \subset \mathbb{C}^n} U$, where $U$ is a domain of holomorphy.

- $D$ admits a Stein neighborhood basis if $\overline{D} = \bigcap_{\overline{D} \subset U \subset \mathbb{C}^n} U$, where $U$ is a domain of holomorphy.

Notice that any domain in $\mathbb{C}^n$ admitting a (weak) Stein neighborhood basis is a domain of holomorphy. Obviously, the second notion from the above definitions plus a fatness condition implies the first one. For details, we refer to e.g. [7]. Furthermore, there is a bounded pseudoconvex domain that has a weak Stein neighborhood basis, but not a Stein neighborhood basis (see e.g. [11]).

3. Hyperconvexity of bounded Reinhardt domains

Before stating our main result let us make some preparations.

**Proposition 3.1.** (Cf. [3]) Let $D$ be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then, for every point $\xi \in (\partial D) \setminus V$, there exists a weak plurisubharmonic barrier $u$ on $D$ at $\xi$.

Combining Proposition 2.2 and Proposition 3.1, we obtain the following result.

**Corollary 3.2.** Let $D$ be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. $D$ is hyperconvex if and only if, for every point $\xi \in (\partial D) \cap V$, there exists a weak plurisubharmonic barrier $u$ on $D$ at $\xi$.

In the following we shall often use the following geometrical characterization of hyperconvex Reinhardt domains due to P. Pflug and W. Zwonek (see [13]).

**Theorem 3.3.** [13] Let $D$ be a pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then the following conditions are equivalent:

1. $D$ is hyperconvex;
2. $D$ is bounded and if $\overline{D} \cap V_j \neq \emptyset$ then $D \cap V_j \neq \emptyset$ for $1 \leq j \leq n$. 

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The second condition in (2) is called the ‘Fu-condition’. It is an immediate consequence of Theorem 3.3 that every bounded pseudoconvex Reinhardt domain $D$ with $\overline{D} \cap V = \emptyset$ is hyperconvex.

To verify our main theorem, we need the following results.

**Lemma 3.4.** Let $D$ be a bounded domain in $\mathbb{C}^n$ having a weak Stein neighborhood basis $\{U_\alpha\}_{\alpha \geq 1}$. Then $D$ is fat.

**Lemma 3.5.** If $D$ is a bounded domain in $\mathbb{C}^n$ having a weak Stein neighborhood basis, then $D$ has a weak Stein neighborhood basis $\{W_\alpha\}_{\alpha \geq 1}$ such that $\overline{W_{\alpha+1}} \subset W_\alpha$ for $\alpha \in \mathbb{N}$.

**Lemma 3.6.** Let $D$ be a bounded Reinhardt domain in $\mathbb{C}^n$ having a weak Stein neighborhood basis. Then there exists a weak Reinhardt Stein neighborhood basis $\{W_j\}_{j \geq 1}$ of $D$ such that $\overline{W_{j+1}} \subset W_j$ for $j \in \mathbb{N}$.

**Proof.** By Lemma 3.5, we may assume that $D$ has a weak Stein neighborhood basis $\{U_j\}_{j \geq 1}$ such that $\overline{U_{j+1}} \subset U_j$ for $j \in \mathbb{N}$ and each $U_j$ is bounded. Since $D$ is a bounded Reinhardt domain and $\overline{D} \subset U_j$ for $j \in \mathbb{N}$, we can choose a strictly decreasing sequence $\{r_j\}_{j \geq 1}$ of positive numbers with $P(z, (r_j, \ldots, r_j)) \subset U_j$ for any $z \in \partial D$, and for each $z = (z_1, z_2, \ldots, z_n) \in \partial D$ and $j \geq 1$, put $A_z(j) := A_{z_1}(j) \times \cdots \times A_{z_n}(j)$, where

$$A_{z_s}(j) := \begin{cases} \{z \in \mathbb{C} : \max(0, |z_s| - r_j) < |z| < |z_s| + r_j\} (z_s \neq 0) \\ \{z \in \mathbb{C} : |z| < r_j\} (z_s = 0) \end{cases} (1 \leq s \leq n).$$

Then each $A_z(j)$ ($z \in \partial D$) is a Reinhardt domain with $A_z(j) \subset U_j$, and also $M_j := D \cup (\bigcup_{z \in \partial D} A_z(j))$ is a Reinhardt domain contained in $U_j$. Moreover,

$$D \Subset M_{j+1} \Subset M_j \subset U_j \quad (j \geq 1), \quad D = \operatorname{int} \bigcap_{j \geq 1} U_j = \operatorname{int} \bigcap_{j \geq 1} M_j.$$

We proceed by setting $W_j := E(M_j)$ for $j \geq 1$, and then we get $\overline{W_{j+1}} \subset W_j$ by virtue of [2] and [11].

Now we are going to state and prove our main result.

**Theorem 3.7.** A bounded Reinhardt domain $D$ in $\mathbb{C}^n$ having a weak Stein neighborhood basis is hyperconvex.

**Proof.** By virtue of Lemma 3.6, we can take a family of bounded pseudoconvex Reinhardt domains $\{U_k\}_{k \geq 1}$ such that $\overline{U_{k+1}} \subset U_k$ for $k \geq 1$ and

$$D = \operatorname{int} \bigcap_{k \geq 1} U_k.$$

Let $a \in (\partial D) \cap V$. We shall first show that $a \neq 0$. Suppose that the origin 0 is contained in $\partial D$, and take a point $p = (p_1, \ldots, p_n) \in D_s$. Then 0 and $p$ are contained in $U_k$ for any $k \geq 1$. Since $p \notin V$, one has $p_i \neq 0$ for $i = 1, \ldots, n$. Put

$$E := \{(z_1, \ldots, z_n) : |z_i| \leq |p_i|, i = 1, \ldots, n\}.$$

By Proposition 2.3, one has $E \subset U_k$ for any $k \geq 1$, and so $E \subset \bigcap_{k \geq 1} U_k$. Because the origin 0 is an interior point of $E$, one has

$$0 \in \operatorname{int} E \subset \operatorname{int} \bigcap_{k \geq 1} U_k = D.$$
which is a contradiction to our assumption. Therefore, we have $a \neq 0$. Now, suppose that $a = (a_1, \ldots, a_{n-j}, 0, \ldots, 0)$ and $a_1 \cdots a_{n-j} \neq 0$ for some $0 \leq j < n$. We are going to show that $\pi(a) \in \partial \pi(D)$, where $\pi := \pi_{\{n-j+1, \ldots, n\}}$. Notice that $\pi(a) \in \pi(D) \subset \pi(D)$. Assume that $\pi(a) \in \pi(D)$. Then there is a point $a'' \in C^j$ such that $a := (\pi(a), a'') \in D$. Moreover, we can choose a Reinhardt open neighborhood $N = N(a'')$ in $D$. Fix $m \geq 1$. Observe that $a \in \partial D \subset U_m$. Hence we have $U_m \cap V_k \neq \emptyset$ for every $n-j+1 \leq k \leq n$. But since each $U_m$ is a pseudoconvex Reinhardt domain, it follows from Proposition 2.3 that

$$W := \{(z, \lambda_1 w_1, \ldots, \lambda_j w_j) \in C^{n-j} \times C^j \mid (z, w_1, \ldots, w_j) \in N, |\lambda_k| \leq 1, k = 1, \ldots, j \} \subset U_m.$$  

In particular, $a \in W$. But since $m \geq 1$ is arbitrary, we get that

$$a \in W = \text{int} W \subset \text{int} \bigcap_{m \geq 1} U_m = D,$$

which is a contradiction to the fact that $a \in \partial D$. Consequently, it must be $\pi(a) \notin \pi(D)$. Hence $\pi(a) \in \partial \pi(D)$. On the other hand, since $\pi(D)$ is a bounded pseudoconvex Reinhardt domain, Lemma 3.1 implies that there exists a weak plurisubharmonic barrier $u$ at $\pi(a)$ in $\pi(D)$. Therefore, the function $u \circ \pi$ is a weak plurisubharmonic barrier at $a$ in $D$, and so we are done by Corollary 3.2. \hfill \Box

Now we are going to discuss related examples. To do this, for every $\alpha > 0$ we define the (possibly multivalued) function

$$\lambda^\alpha := \begin{cases} e^{\alpha (\ln |\lambda| + i \arg \lambda)} & (\lambda \in \mathbb{C}_\ast), \\ 0 & (\lambda = 0), \end{cases}$$

and for every set $D$ in $\mathbb{C}^n$ we denote

$$D^\alpha := \{(z_1^\alpha, \ldots, z_n^\alpha) : (z_1, \ldots, z_n) \in D\}.$$

To proceed with our discussion we need the following auxiliary result.

**Proposition 3.8.** [9] Let $D$ be a bounded hyperconvex Reinhardt domain in $\mathbb{C}^n$. Then $\{D^\alpha\}_{\alpha > 1}$ is a Stein neighborhood basis of $D$.

In fact, its proof in [9] essentially guarantees that the previous statement without the assumption of boundedness is still true. For details, we would like to give the proof in the next section.

**Example 3.9.** (a) Since every pseudoconvex domain containing $\overline{\Delta_H}$ also contains the unit bidisk, $\Delta_H$ does not have a weak Stein neighborhood basis. Note that $0 \in \partial \Delta_H$.

(b) The domain $G := \{(z, w) \in \mathbb{D} : 0 < |w| - |z| < 1/2\}$ is not pseudoconvex. Moreover, $\mathcal{E}(G) = \Delta_H$ does not have a weak Stein neighborhood basis. Observe that the origin is a boundary point of $\mathcal{E}(G)$.

Consequently, we obtain the following characterization of hyperconvexity in Reinhardt domains.

**Theorem 3.10.** Let $D$ be a bounded Reinhardt domain in $\mathbb{C}^n$. Then the following conditions are equivalent.
Let us close this section with the following statement.

**Corollary 3.12.** Any bounded Reinhardt domain in \( \mathbb{C}^n \) admitting a weak Stein neighborhood basis always has a Stein neighborhood basis.

**4. Remarks on unbounded Reinhardt domains**

In this section we ask whether an unbounded Reinhardt domain has a Stein neighborhood basis? From now on, we are going to give a class of pseudoconvex Reinhardt domains for which the answer to the previous question is always positive. To see this, we first need the following lemma.

**Lemma 4.1.** Any pseudoconvex Reinhardt domain in \( \mathbb{C}^n \) satisfying the Fu condition is fat.

**Proof.** Suppose that \( D \subset \mathbb{C}^n \) is a pseudoconvex Reinhardt domain but not fat. Then there is a point \( z^0 = (z_1^0, \ldots, z_n^0) \in (\text{int} D) \setminus D \). First, assume that \( z^0 \notin V \). Observe that

\[
\log |z^0| \in \log((\text{int} D)_a) = \log(D_a).
\]

Here, in the above equality, we have used the fact that the convexity of \( \log(D_a) \) implies that \( \log((\text{int} D)_a) \subset \log(D_a) \). Hence we have \( z^0 \in D \), which is a contradiction. Next, suppose that \( z^0 \in V_\mu \) for some \( \mu \in \{1, 2, \ldots, n\} \). Since \( D \) is a pseudoconvex Reinhardt domain, there are only two cases:

(i) if there does not exist a point \( (z_1, \ldots, z_n) \in D \) such that \( z_\mu \neq 0 \) and \( z_v = z^0_v \) for \( v \neq \mu \), then \( D \) is not connected, which is a contradiction;

(ii) if there does not exist a point \( z \in D \) with \( z \in V_\mu \), that is, \( D \cap V_\mu = \emptyset \), the domain \( D \) can no longer satisfy the Fu condition. \( \square \)

From now on, similar to Definition 2.5, we say that an unbounded pseudoconvex Reinhardt domain \( D \) in \( \mathbb{C}^n \) has a Stein neighborhood basis if \( D \) is fat and

\[
\overline{D} = \bigcap_{\overline{D} \subset U \subset \mathbb{C}^n} U,
\]

where \( U \) is a domain of holomorphy.
Theorem 4.2. If $D \subset \mathbb{C}^n$ is an unbounded pseudoconvex Reinhardt domain satisfying the Fu condition, then $D$ has a Stein neighborhood basis.

Proof. Assume by changing coordinates if necessary that $(1, \ldots, 1) \in D$. It suffices to prove that $\{D^\alpha\}_{\alpha > 1}$ is a Stein neighborhood basis for $D$. We first prove that for $\alpha > 1$ the domain $D^\alpha$ is pseudoconvex Reinhardt. Fix $\alpha > 1$. If $(z_1, \ldots, z_n) \in D$ and $\lambda_i \in \mathbb{C}$ with $|\lambda_i| = 1$ where $i \in \{1, \ldots, n\}$, then $(\lambda_1^{1/\alpha} z_1, \ldots, \lambda_n^{1/\alpha} z_n) \in D$ and also $(\lambda_1 z_1^0, \ldots, \lambda_n z_n^0) \in D^\alpha$. Hence $D^\alpha$ is Reinhardt. Observe that $\log(D^\alpha)$ is convex and

$$\log[(D^\alpha)_a] = \alpha \log(D_a),$$

so the set $\log[(D^\alpha)_a]$ is also convex. Assume now that $D^\alpha \cap V_j \neq \emptyset$ for some $j \in \{1, \ldots, n\}$. Then it is clear that $D \cap V_j \neq \emptyset$, and because of the pseudoconvexity of $D$ one has

$$(z_1, \ldots, z_j, \ldots, z_n) \in D \implies (z_1, \ldots, \lambda_1^{1/\alpha} z_j, \ldots, z_n) \in D, \quad |\lambda| \leq 1.$$ 

So, by Proposition 2.3, we get the pseudoconvexity of $D^\alpha$.

Next, to show that $D \subset \bigcap_{\alpha > 1} D^\alpha$, let $a = (a_1, \ldots, a_n) \in \overline{D}$ and $\alpha > 1$. We shall discuss the following three cases.

(c1) The case $a \notin V$. Because of the convexity of $\log(D_a)$ one has $\overline{\log(D_a)} \subset \alpha \log(D_a)$. Hence, by (†), $(\log |a_1|, \ldots, \log |a_n|) \in \log(D^\alpha)_a$. But since $D^\alpha$ is Reinhardt, we have $a \in D^\alpha$.

(c2) The case $a = (0, \ldots, 0)$. Since $\overline{D} \cap V_j \neq \emptyset$ for any $j \in \{1, \ldots, n\}$, the Fu condition for $D$ implies that $D \cap V_j \neq \emptyset$ for any $j \in \{1, \ldots, n\}$. By Proposition 2.3, we get $a \in D \subset D^\alpha$.

(c3) The case $a \in V \setminus \{0\}$. Without loss of generality, we may assume that $a = (0, \ldots, 0, a_{k+1}, \ldots, a_n)$, where $a_{k+1} \cdots a_n \neq 0, 1 \leq k < n$. Denote $\pi := \pi_{\{1, \ldots, k\}}$. Then $\pi(D)$ is a pseudoconvex Reinhardt domain in $\mathbb{C}^{n-k}$ and

$$\pi(a) \not\subset \bigcup_{j=k+1}^n \pi(V_j), \quad \pi(a) \in \pi(D) \subset \pi(D).$$

This is a similar situation as in the case (c1). Hence, we obtain that $\pi(a) \in [\pi(D)]^\alpha = \pi(D^\alpha)$. On the other hand, because of the Fu condition for $D$, for each $j \in \{1, \ldots, k\}$ one has $D \cap V_j \neq \emptyset$ and so $D^\alpha \cap V_j \neq \emptyset$. Thus it follows from Proposition 2.3 that $a \in D^\alpha$, as desired.

Finally, it remains to see that $\bigcap_{\alpha > 1} D^\alpha \subset \overline{D}$. For this, let $a \in \bigcap_{\alpha > 1} D^\alpha$. Notice that the convexity of $\log(D_a)$ implies that

$$\bigcap_{\alpha > 1} \log[(D^\alpha)_a] = \bigcap_{\alpha > 1} \alpha \log(D_a) = \log(D_a).$$

Now, we are going to discuss the following three cases.

(k1) The case $a \notin V$. Note that $(\log |a_1|, \ldots, \log |a_n|) \in \log[(D^\alpha)_a]$ for any $\alpha > 1$. By (‡), it follows that $\log |a| \in \log(D_a) = \log(\overline{D})_a$ and also $a \in \overline{D}$.

(k2) The case $a = (0, \ldots, 0)$. Since $0 \in D^\alpha$ for any $\alpha > 1$, one has $0 \in D$ and so $0 \in \overline{D}$. 

(k3) The case \( a \in V \setminus \{0\} \). We may assume without loss of generality that \( a = (0, \ldots, 0, a_{k+1}, \ldots, a_n) \), where \( a_{k+1} \cdots a_n \neq 0 \), \( 1 \leq k < n \). By the case (k1) one has \( \pi(a) \in \pi(D) \). Observe the following.

(i) The case \( \pi(a) \in \pi(D) \). Let \( j \in \{k+1, \ldots, n\} \). Since \( D^a \cap V_j \neq \emptyset \), one has \( D \cap V_j \neq \emptyset \). By Theorem 3.10, we get \( a \in D \).

(ii) The case \( \pi(a) \in \partial(\pi(D)) \). Suppose that \( a / \in D \), then there is an open polydisc

\[
P(a) := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| < \varepsilon_1, \ldots, |z_k| < \varepsilon_k, |z_{k+1} - a_{k+1}| < \varepsilon_{k+1}, \ldots, |z_n - a_n| < \varepsilon_n \right\}
\]

such that \( P(a) \subset \mathbb{C}^n \setminus \overline{D} \). Put \( P(\pi(a)) := \pi(P(a)) \). Because \( P(\pi(a)) \cap \pi(D) = \emptyset \), we obtain \( \pi(a) \notin \partial(\pi(D)) \), which is a contradiction. Thus, \( a \in D \).

\[\Box\]

Remark 4.3. Theorem 3.3 shows that every hyperconvex Reinhardt domain is bounded. So, we would like to mention that any unbounded pseudoconvex Reinhardt domain cannot be hyperconvex although it has a Stein neighborhood basis.

Finally, we shall give such an example.

Example 4.4. Let \( E := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < C\} \). Clearly, \( E \) is a unbounded pseudoconvex Reinhardt domain satisfying the Fu condition. Now, if we set \( U_n := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < C + 1/n\} \) for \( n \in \mathbb{N} \), then \( \{U_n\}_{n \geq 1} \) is a Stein neighborhood basis of \( E \), but \( E \) is not a hyperconvex.

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