A REMARK ON KAZHDAN’S THEOREM ON SEQUENCES OF BERGMAN METRICS

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Abstract. A proof is given to an assertion attributed to D. Kazhdan by D. Mumford. The argument works to prove a generalized assertion, which was due to Kazhdan according to S. T. Yau, under some additional assumptions.

0. Introduction

Let $M$ be a connected and paracompact complex manifold of dimension $n$ and let $\tilde{M} \to M$ be a covering space. In [M], Mumford states the following, attributing it to Kazhdan [K-1] (see also [K-2]).

(a) If $M$ is compact, $n = 1$, $\tilde{M} \simeq \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and $M = \mathbb{D} / \Gamma$, then, for any sequence $\Gamma_1 = \Gamma \supset \Gamma_2 \supset \Gamma_3 \supset \cdots \supset \Gamma_{\infty} = \{\text{id}\}$ of Fuchsian group $\Gamma_k$ satisfying $\bigcap_{k \in \mathbb{N}} \Gamma_k = \{\text{id}\}$, such that $\mathbb{D} / \Gamma_k$ (for $k \in \mathbb{N}$) are compact, the pullbacks $ds^2_k$ of the Bergman metrics of $\mathbb{D} / \Gamma_k$ to $\mathbb{D}$ satisfy

$$\lim_{k \to \infty} \lambda_k ds^2_k = ds^2_{\infty} = \frac{2 \, dz \, d\bar{z}}{(1 - |z|^2)^2}$$

with suitably chosen scalars $\lambda_k$.

However, neither this statement nor its proof is explicitly stated in [K-1, K-2].

In [Y], Yau asserts that Kazhdan proved the following.

(b) If $M = \tilde{M} / \Gamma$ and the manifold $\tilde{M}$ admits the Bergman metric, then for any sequence $\Gamma_1 = \Gamma \supset \Gamma_2 \supset \Gamma_3 \supset \cdots \supset \Gamma_{\infty} = \{\text{id}\}$ of properly discontinuous groups of automorphisms of $\tilde{M}$ with $[\Gamma_k, \Gamma_{k+1}] < \infty$ and $\bigcap_{k \in \mathbb{N}} \Gamma_k = \{\text{id}\}$, the pullbacks of the Bergman metrics $\tilde{M} / \Gamma_k$ converge on $\tilde{M}$ to the Bergman metric $ds^2_{\tilde{M}}$ of $\tilde{M}$.

Again, similarly as in [M], no literature is given for the statement and the proof in the reference of [Y].

In [R], Rhodes proved (a) under somewhat restrictive assumptions. The purpose of the present paper is to give a proof of (b) under some assumptions which are obviously satisfied if $\tilde{M} = \mathbb{D}$ and $M$ is compact. Accordingly, we shall obtain (a) with $\lambda_k = 1$ without any additional assumptions.

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1. Donnelly–Fefferman’s estimate

Let us recall an $L^2$ estimate for the $\bar{\partial}$-operator first due to Donnelly and Fefferman [D-F], in a generalized formulation after Gromov [G].

Let $(X, ds^2)$ be a complete Kähler manifold of pure dimension $n$ equipped with a $(0, 1)$-form $\gamma$ of class $C^1$ such that $\sqrt{-1} d\gamma$ is the fundamental form of $ds^2$.

Let $C^0_{p,q}(X)$ be the space of $C^\infty$ compactly supported $(p, q)$-forms on $X$ and let

$$\bar{\partial} : C^0_{p,q}(X) \rightarrow C^0_{p,q+1}(X)$$

be the complex exterior derivative of type $(0, 1)$.

For any $u, v \in C^0_{p,q}(X)$, let $\langle u, v \rangle$ denote the pointwise inner product of $u$ and $v$ with respect to $ds^2$, and put

$$(u, v) = \int_X \langle u, v \rangle \, dV,$$

where $dV$ denotes the volume form on $X$ with respect to $ds^2$. We put $|u| = \sqrt{(u, v)}$ and $\|u\| = \sqrt{(u, u)}$.

Let $L^p,q(X)$ denote the completion of $C^0_{p,q}(X)$ with respect to the norm $\|u\|$ and let $\bar{\partial}$ also denote the maximal extension of $\bar{\partial}$ as a closed operator from $L^p,q(X)$ to $L^p,q+1(X)$. The adjoint of $\bar{\partial} : L^p,q(X) \rightarrow L^p,q+1(X)$ is denoted by $\bar{\partial}^*$. The domains of $\bar{\partial}$ and $\bar{\partial}^*$ in $\bigcup_{p,q} L^p,q(X)$ will be denoted by $\text{Dom} \, \bar{\partial}$ and $\text{Dom} \, \bar{\partial}^*$, respectively.

**Theorem 1.1. (Cf. [D-F] and [G])** If $c := \sup_X |\gamma| < \infty$, then

$$c^{-1}|p + q - n| \|u\| \leq \|\bar{\partial} u\| + \|\bar{\partial}^* u\|$$

holds for any $u \in L^{p,q}(X) \cap \text{Dom} \, \bar{\partial} \cap \text{Dom} \, \bar{\partial}^*$.

**Corollary 1.1.** Under the above situation, let $u \in L^{p,q}(X) \cap \text{Ker} \, \bar{\partial}$ with $p + q \neq n$. Then there exists a $v \in L^{p,q-1}(X) \cap \text{Dom} \, \bar{\partial}$ satisfying $\bar{\partial} v = u$ and $\|v\| \leq c|p + q - n|^{-1}\|u\|$.

2. Assertion (b) with additional conditions

Let $M, \tilde{M}, \Gamma_k$ and $ds^2_M$ be as in (b), let $M_k = \tilde{M} / \Gamma_k$ and let $ds^2_k$ be the Bergman metric of $M_k$.

We recall at first a characterization of $ds^2_k$ as the solution to an extremal problem.

Let $\xi$ be a holomorphic tangent vector of $M_k$ at $p \in M_k$. By expressing any $\omega \in L^{n,0}(M_k) \cap \text{Ker} \, \bar{\partial}$ near $p$ as $\omega = \omega_0 \, dz_1 \wedge \cdots \wedge dz_n$ with respect to a fixed local coordinate $(z_1, \ldots, z_n)$ around $p$, we put

$$K(\xi) = \sup \{ |\xi \omega_0|^2 \mid \omega \in L^{n,0}(M_k) \cap \text{Ker} \, \bar{\partial}, \|\omega\| = 1, \omega_0(p) = 0 \}$$

and

$$L(p) = \sup \{ |\omega_0(p)|^2 \mid \omega \in L^{n,0}(M_k) \cap \text{Ker} \, \bar{\partial}, \|\omega\| = 1 \}. \quad (2.2)$$

Then the length $|\xi|_k$ of $\xi$ with respect to $ds^2_k$ satisfies

$$|\xi|_k^2 = K(\xi) / L(p). \quad (2.3)$$

From (2.3) we deduce the following criterion for the convergence of $ds^2_k$. 

PROPOSITION 2.1. Let \( \pi_k : \tilde{M} \to M_k \) be the projection. Suppose that the following two conditions are satisfied.

(i) For any \( \omega \in L^{n,0}(\tilde{M}) \cap \text{Ker} \, \tilde{\partial} \), for any \( \varepsilon > 0 \) and for any compact set \( Q \subset \tilde{M} \), there exists \( k_0 \in \mathbb{N} \) such that, for any \( k > k_0 \) one can find \( \omega_k \in L^{n,0}(M_k) \cap \text{Ker} \, \tilde{\partial} \) and an open set \( D_k \subset M \) containing \( Q \), such that \( \|\omega_k\| \leq \|\omega\| \) and \( \|\pi_k^* \omega_k - \omega\| \leq \varepsilon \).

(ii) For any compact set \( Q \subset M \) and for any \( \varepsilon > 0 \), there exists \( k_1 \in \mathbb{N} \) such that, for any \( k > k_1 \) and for any \( \omega_k \in L^{n,0}(M_k) \cap \text{Ker} \, \tilde{\partial} \), there exists \( \tilde{\omega}_k \in L^{n,0}(\tilde{M}) \cap \text{Ker} \, \tilde{\partial} \) such that \( \|\tilde{\omega}_k\| \leq \|\omega_k\| \) and \( \|\pi_k^* \omega_k - \tilde{\omega}_k\| \leq \varepsilon \).

Then \( \lim_{k \to \infty} \pi_k^* ds^2_k = ds^2_M \).

In fact, for any holomorphic tangent vector \( \xi \) of \( \tilde{M} \), from (i) we have \( \lim_{k \to \infty} K(\pi_k^* \xi) \geq K(\xi) \) and \( \lim_{k \to \infty} L(\pi_k(p)) \geq L(p) \), and from (ii) we have \( \lim_{k \to \infty} K(\pi_k^* \xi) \leq K(\xi) \) and \( \lim_{k \to \infty} L(\pi_k(p)) \leq L(p) \).

Combining Proposition 2.1 with Corollary 1.1, we shall prove the following.

THEOREM 2.1. In the situation of (b), assume moreover that the following are satisfied.

1. \( ds_M^2 \) is complete.
2. There exists a \((0, 1)\)-form \( \gamma \) of class \( C^1 \) on \( \tilde{M} \) such that \( \sup |\gamma| < \infty \) and that \( \sqrt{-1} \, d\gamma \) is the fundamental form of \( ds_M^2 \).
3. For any sequence \( Q_k \subset M_k \) (\( k = 1, 2, \ldots \)) of compact sets, there exists \( N \in \mathbb{N} \), a sequence of subsets \( A_k \subset M_k \) (\( k = 1, 2, \ldots \)) and a divergent sequence of positive real numbers \( r_k \) such that

\[
\sup \{\text{dist}(p, A_k) \mid p \in Q_k\} < r_k^2 - r_k
\]

holds, \( \pi_k^{-1} \) exists on \( \{p \mid \text{dist}(p, q) < r_k^2\} \) for all \( q \in A_k \), and

\[
\bigcap_{q \in A} \{p \mid \text{dist}(p, q) < r_k^2\} = \emptyset
\]

holds if \( A \subset A_k \) and \( \#A > N \). Here \( \text{dist}(\cdot, A_k) \) denotes the distance to \( A_k \) with respect to the metric induced by \( ds_M^2 \) and \( \#A \) is the cardinality of \( A \).

Then \( \lim_{k \to \infty} \pi_k^* ds_k^2 = ds_M^2 (= ds^2_\infty) \).

Proof. It suffices to verify the conditions (i) and (ii) of Proposition 2.1.

(i) Let \( \omega \in L^{n,0}(\tilde{M}) \cap \text{Ker} \, \tilde{\partial} \), let \( \varepsilon > 0 \) and let \( Q \subset \tilde{M} \) be any compact set. By \( \bigcap \Gamma_k = \{\text{id}\} \) and (3), there exists \( k_0 \in \mathbb{N} \) such that, for any \( k > k_0 \), one has a \( C^\infty \) partition of unity, say \( \{\chi_{k\mu}\}_{\mu=1}^{m_k} \), on \( M_k \) such that

there exist continuous maps \( \sigma_{k\mu} : \text{supp} \, \chi_{k\mu} \to \tilde{M} \)

such that \( \pi_k \circ \sigma_{k\mu} = \text{id} \) and \( \sigma_{k1}(\text{supp} \, \chi_{k1}) \supset Q \),

\[
\chi_{k1} \circ \pi_k | Q = 1,
\]

\[
\|\omega - (\chi_{k1} \circ \pi_k)\omega| \sigma_{k1}(\text{supp} \, \chi_{k1})\| < \varepsilon
\]

and

\[
\sum_{\mu} |\tilde{\partial}(\chi_{k\mu} \circ \pi_k)| < \varepsilon.
\]
We put
\[
\hat{\omega}_k = \begin{cases} 
\chi_{k1} \sigma_{k1}^* \omega & \text{on } \text{supp } \chi_{k1}, \\
0 & \text{on } M_k \setminus \text{supp } \chi_{k1}.
\end{cases}
\]
Then, by (1) and (2), it follows from Corollary 1.1 and (2.7) that there exist a constant \( C > 0 \) and \( C^\infty(n, 0) \)-forms \( v_{k\mu} \) on \( \text{supp } \chi_{k\mu} \) such that
\[
\bar{\partial} v_{k\mu} = \bar{\partial} \hat{\omega}_k | \text{supp } \chi_{k\mu}
\]
and
\[
\| v_{k\mu} \| \leq C \varepsilon \| \omega \|
\]
hold for all \( \mu \).
Then we put
\[
\omega_1 = \hat{\omega}_k - \sum_{\mu=1}^{m_k} \chi_{k\mu} v_{k\mu}.
\]
Here we extend \( \chi_{k\mu} v_{k\mu} \) trivially outside \( \text{supp } \chi_{k\mu} \).
Observe that we have
\[
\| \omega_1 - \hat{\omega}_k \| \leq C \varepsilon \| \omega \|
\]
and
\[
\| \bar{\partial} \omega_1 \| = \left\| \sum_{\mu} \bar{\partial} \chi_{k\mu} v_{k\mu} \right\| \leq C \varepsilon^2 \| \omega \|
\]
by (2.7)–(2.9). Here \( \| \bar{\partial} \omega_1 \| \) is with respect to \( \pi_k^* ds_M^2 \).
Hence, by a successive approximation one can find \( \tilde{\omega}_k \in L^{n, 0}(M_k) \cap \text{Ker } \bar{\partial} \) satisfying
\[
\| \omega_k - \hat{\omega}_k \| \leq 2C \varepsilon \| \omega \|
\]
if \( C \varepsilon \ll 1 \).
In view of (2.6) and (2.13), with sufficiently large \( Q \), it is easy to see the validity of (i).
(ii) Let \( \varepsilon > 0 \). Then, by (1), (2), \( \bigcap \Gamma_k = \{ \text{id} \} \) and by Corollary 1.1, it is easy to see that there exists \( k_1 \in \mathbb{N} \) such that, for any \( k > k_1 \), for any compact set \( Q_k \subset \tilde{M} \) such that \( \pi_k | Q_k \) is injective and for any \( \omega_k \in L^{n, 0}(M_k) \cap \text{Ker } \bar{\partial} \), one can find \( \tilde{\omega}_k \in L^{n, 0}(\tilde{M}) \cap \text{Ker } \bar{\partial} \) such that
\[
\| (\pi_k^* \omega_k - \tilde{\omega}_k) | Q_k \| < \varepsilon
\]
and
\[
\| \tilde{\omega}_k \| \leq \| \omega_k \| + \varepsilon
\]
hold. Thus we obtain (ii).

**Note added in proof.** It turned out that, in the situation of (a), the normality of \( \Gamma_k \) is needed for applying Theorem 2.1, so that the general case is still open.

**References**


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