DECAY PROPERTY OF REGULARITY-LOSS TYPE
AND NONLINEAR EFFECTS FOR
SOME HYPERBOLIC-ELLIPTIC SYSTEM

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Abstract. We study the decay property of a certain nonlinear hyperbolic-elliptic system with 2mth-order elliptic part, which is a modified version of the simplest radiating gas model. It is proved that, for \( m \geq 2 \), the system verifies a decay property of the regularity-loss type that is characterized by the parameter \( m \). This dissipative property is very weak in the high-frequency region and causes a difficulty in showing the global existence of solutions to the nonlinear problem. By employing the time-weighted energy method together with the optimal decay for lower-order derivatives of solutions, we overcome this difficulty and establish a global existence and asymptotic decay result. Furthermore, we show that the solution approaches the nonlinear diffusion wave described by the self-similar solution of the Burgers equation as time tends to infinity.

1. Introduction

In this paper we consider the following hyperbolic-elliptic system:

\[
\begin{align*}
    u_t + f(u)_x + q_x &= 0, \\
    (-1)^m \partial_x^{2m} q + q + u_x &= 0,
\end{align*}
\]

with the initial data

\[ u(x, 0) = u_0(x), \]

where \( m \) is an integer with \( m \geq 2 \) and \( f(u) \) is a given smooth function of \( u \). We note that if \( m = 1 \) and \( f(u) = u^2/2 \), then (1.1) becomes

\[
\begin{align*}
    u_t + (u^2/2)_x + q_x &= 0, \\
    -q_{xx} + q + u_x &= 0,
\end{align*}
\]

which is the simplest model system of a radiating gas (see [1, 11]). A general class of hyperbolic-elliptic system including (1.3) (but not the system (1.1) with \( m \geq 2 \)) was studied in [7–9] for the initial value problem. For such general hyperbolic-elliptic systems, the global existence of solutions has been proved in [7–9] for small initial data in \( H^s \) with \( s \geq 2 \).

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Moreover, it is shown in [7] that if the initial data are small in $H^s \cap L^1$ with $s \geq 3$, then the solutions verify the decay property
\[ \| \partial_x^k u(t) \|_{L^2} \leq C (1 + t)^{-1/4 - k/2} \] (1.4)
for $0 \leq k \leq s - 1$ and approach the corresponding diffusion waves as $t \to \infty$ (cf. [8, 9]). Also, the model system (1.3) was investigated in detail in [10, 12, 14] on the asymptotic stability of nonlinear waves: see [8] and [9] for shock waves and [12] for rarefaction waves.

On the other hand, the system (1.1) with $m = 2$ has been studied very recently in [2, 4]. These papers proved the global solvability and asymptotic behavior of solutions for small initial data in $H^s \cap L^1$ with $s \geq 7$. These solutions satisfy the optimal decay estimate (1.4) only for lower-order derivatives, i.e. for $0 \leq k \leq (s + 1)/2 - 2$.

The main purpose of this paper is to extend the above-mentioned global existence and stability results in [2, 4] to the system (1.1) for any $m \geq 2$. As was observed in [2, 4], the dissipative property of the system (1.1) with $m = 2$ is weak and of the regularity-loss type. To see this, we apply the Fourier transform to the linearized system of (1.1) and find that the corresponding eigenvalue is
\[ \lambda_m(i \xi) = -\alpha i \xi - \eta_m(\xi), \]
where $\alpha = f'(0)$ and $\eta_m(\xi) = \xi^2/(1 + \xi^{2m})$. We see that the dissipativity of the system (1.1) with $m \geq 2$ is weak in the high-frequency region in the sense that, for $m \geq 2$, Re $\lambda_m(i \xi) = -\eta_m(\xi) \to 0$ as $|\xi| \to \infty$.

Let $e^{tA}$ be the corresponding semigroup:
\[ (e^{tA} \phi)(x) := \mathcal{F}^{-1}[e^{\lambda_m(i \xi)t} \hat{\phi}(\xi)](x). \]
When $m = 1$, this semigroup verifies the standard decay property
\[ \| \partial_x^k e^{tA} \phi \|_{L^2} \leq C (1 + t)^{-1/4 - k/2} \| \phi \|_{L^1} + C e^{-ct} \| \partial_x^k \phi \|_{L^2}. \] (1.5)
On the other hand, when $m \geq 2$, we will show in Lemma 3.1 that the semigroup $e^{tA}$ satisfies the following decay property of the regularity-loss type:
\[ \| \partial_x^k e^{tA} \phi \|_{L^2} \leq C (1 + t)^{-1/4 - k/2} \| \phi \|_{L^1} + C (1 + t)^{-l/2(m-1)} \| \partial_x^{k+l} \phi \|_{L^2}. \] (1.6)
This is a straightforward generalization of the decay result obtained in [2, 4] for $m = 2$. The second term on the right-hand side of (1.6), which comes from the high-frequency part, shows that we can get the quantitative decay rate $t^{-l/2(m-1)}$ only by assuming the additional $l$th-order regularity on the initial data. We note that a similar decay property of the regularity-loss type was also observed in [13] and [5] for the dissipative Timoshenko system. Such a regularity-loss property causes a difficulty in deriving the decay estimates of solutions to the nonlinear problem (1.1), (1.2). This difficulty can be resolved by combining suitably weaker decay estimates for higher-order derivatives of solutions which are obtained by the time-weighted energy method. The details will be explained in Section 4 (see also [2, 4] and [6]).

The effect on the parameter $m \geq 2$ also appears in the regularity assumption on the initial data when we construct global solutions to the nonlinear problem (1.1), (1.2). In fact, as mentioned above, we need to require the regularity $u_0 \in H^s$ with $s \geq 2$ for $m = 1$. On the other hand, the regularity assumption needed for $m = 2$ in [2, 4] was $u_0 \in H^s \cap L^1$
Decay property of regularity-loss type

with \( s \geq 7 \). When \( m \geq 2 \), this regularity assumption can be generalized as \( u_0 \in H^s \cap L^1 \) with \( s \geq 4m - 1 \). Also, it will be proved that the corresponding global solutions satisfy the optimal decay estimate (1.4) for \( 0 \leq k \leq [(s + 1)/m] - 2 \). For details, see Theorem 4.1.

The requirement \( s \geq 4m - 1 \) on the regularity might be technical but it seems necessary in our proof. In fact, in constructing global solutions, it is crucial to show the desired \textit{a priori} estimates of solutions and this will be done by employing the time-weighted energy method. To complete these energy estimates, we need to use the following optimal decay estimate for the \( L^\infty \) norm of the first-order derivative \( u_x \):

\[
\|u_x(t)\|_{L^\infty} \leq C(1 + t)^{-1}.
\]

This estimate requires the validity of the \( L^2 \) decay estimate (1.4) for \( 1 \leq k \leq 2 \). Consequently, we need to put the hypothesis \([(s + 1)/m] - 2 \geq 2 \), namely, \( s \geq 4m - 1 \). For the details of the discussion, see Section 4.

In order to study more detailed large-time behavior of solutions to (1.1), (1.2), we consider the following approximation of (1.1):

\[
\begin{align*}
vt + f(v)_x + q_x &= 0, \\
q + v_x &= 0,
\end{align*}
\]

(1.7)

that is obtained from (1.1) by neglecting the highest-order term \((-1)^m \partial_x^{2m} q\). By eliminating \( q \), this approximation (1.7) can be reduced to the viscous conservation law

\[
v_t + f(v)_x = v_{xx},
\]

which may be further approximated by the simpler viscous conservation law

\[
v_t + \left( \alpha v + \frac{\beta}{2} v^2 \right)_x = v_{xx},
\]

(1.8)

where \( \alpha = f'(0) \) and \( \beta = f''(0) \). When \( \beta = f''(0) > 0 \), this simpler equation (1.8) is finally reduced to the Burgers equation

\[
w_t + (w^2/2)_y = w_{yy},
\]

(1.9)

where we have changed the variables as \( x = y + \alpha t \) and \( w = \beta v \).

It is known that the asymptotic profile for \( t \to \infty \) of the solution \( w(y, t) \) to the Burgers equation (1.9) is described by its self-similar solution \( W(y, t; M) \) which carries the same mass \( M \) of the initial data \( w_0(y) \), that is,

\[
\int_{\mathbb{R}} W(y, t; M) \, dy = M = \int_{\mathbb{R}} w_0(y) \, dy.
\]

Consequently, we expect that the solution \( u(x, t) \) to (1.1), (1.2) is well approximated for \( t \to \infty \) by the nonlinear diffusion wave \( v(x, t) \) that is defined in terms of the self-similar solution \( W(y, t; M) \) of the Burgers equation as

\[
v(x, t) = \beta^{-1} W(x - \alpha t, t; \beta M),
\]

(1.10)

where the mass \( M \) is determined by \( M = \int_{\mathbb{R}} u_0(x) \, dx \), so that we have \( \int_{\mathbb{R}} v(x, t) \, dx = \int_{\mathbb{R}} u_0(x) \, dx \). Indeed, under the additional condition \( u_0 \in L^1_1 \), we can show that the solution
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1.1. Notation

Let $F[f]$ denote the Fourier transform of $f$ defined by

$$F[f](ξ) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ixξ} \, dx,$$

and let $F^{-1}[f]$ be the Fourier inverse transform of $f$:

$$F^{-1}[f](x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(ξ)e^{ixξ} \, dξ.$$

For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{R})$ denotes the usual Lebesgue space on $\mathbb{R}$ with the norm $\|\cdot\|_{L^p}$.

For $s \in \mathbb{R}$, $H^s = H^s(\mathbb{R})$ denotes the $L^2$ sense Sobolev space, equipped with the norm

$$\|f\|_{H^s} := \|(1 + |ξ|^2)^{s/2} \hat{f}(ξ)\|_{L^2_ξ} = \left(\int_{\mathbb{R}} (1 + |ξ|^2)^s |\hat{f}(ξ)|^2 \, dξ\right)^{1/2}.$$

For $α \in \mathbb{R}$, let $L^1_α = L^1_α(\mathbb{R})$ denote the weighted $L^1$ space with the norm

$$\|f\|_{L^1_α} := \int_{\mathbb{R}} (1 + |x|^α)|f(x)| \, dx.$$

Let $k$ be a non-negative integer and let $I$ be an interval contained in $[0, \infty)$. Then $C^k(I; H^s)$ denotes the space of $k$-times continuously differentiable functions on $I$ with values in $H^s = H^s(\mathbb{R})$.

In this paper, we denote various generic positive constants by $C$ or $c$.

2. Preliminaries

In this section we give several preliminary inequalities used in this paper.

**Lemma 2.1.** [4] Let $1 \leq p, q, r \leq \infty$ with $1/p = 1/q + 1/r$. Then we have

$$\|∂^k_x(uv)\|_{L^p} \leq C(\|u\|_{L^q}\|∂^k_xv\|_{L^r} + \|v\|_{L^q}\|∂^k_xu\|_{L^r}), \quad k \geq 0, \quad (2.1)$$

and

$$\|[∂^k_x, u]v_x\|_{L^p} \leq C(\|ux\|_{L^q}\|∂^k_xv\|_{L^r} + \|v_x\|_{L^q}\|∂^k_xu\|_{L^r}), \quad k \geq 1, \quad (2.2)$$

where $[∂^k_x, A]B = ∂^k_x(AB) - A ∂^k_xB$ denotes the commutator.
These estimates (2.1) and (2.2) are easy consequences of the following inequality in [4]. For non-negative integers $k_1$, $k_2$ and $k$ with $k = k_1 + k_2$, we have

$$\|\partial_x^{k_1} u_1 \partial_x^{k_2} u_2\|_{L^p} \leq C(\|u_1\|_{L^q} \|\partial_x^{k_1} u_2\|_{L^r} + \|u_2\|_{L^q} \|\partial_x^{k_2} u_1\|_{L^r}),$$

(2.3)

where $p$, $q$ and $r$ are the same as in Lemma 2.1. The following is a variant of the estimate (2.3) and is found in [7] and [3]. Let $N \geq 2$, and let $k_1, \ldots, k_N$ and $k$ be non-negative integers with $k = k_1 + \cdots + k_N$. Then we have

$$\left\| \prod_{j=1}^N \partial_x^{k_j} u \right\|_{L^p} \leq C \|u\|_{L^\infty}^{N-2} \|u\|_{L^q} \|\partial_x^k u\|_{L^r},$$

(2.4)

where $p$, $q$ and $r$ are also the same as in Lemma 2.1.

**Lemma 2.2.** Let $1 \leq p, q, r \leq \infty$ with $1/p = 1/q + 1/r$, and let $k$ be a non-negative integer. Suppose that $a(u)$ is a smooth function of $u$ satisfying $a(u) = O(1)$. Then we have

$$\|\partial_x^k (a(u) u)\|_{L^p} \leq K (1 + \|u\|_{L^\infty})^k \|\partial_x^k u\|_{L^p},$$

(2.5)

$$\|\partial_x^k (a(u) u^N)\|_{L^p} \leq K (1 + \|u\|_{L^\infty})^k \|u\|_{L^\infty}^{N-2} \|u\|_{L^q} \|\partial_x^k u\|_{L^r},$$

(2.6)

where $N$ is an integer with $N \geq 2$. Here $K = K(\|u\|_{L^\infty})$ is a quantity depending on $\|u\|_{L^\infty}$ and is regarded as an increasing function of $\|u\|_{L^\infty}$.

**Proof.** First we prove (2.5). The estimate (2.5) is trivial for $k = 0$. When $k \geq 1$, we have

$$\partial_x^k (a(u) u) = a(u) \partial_x^k u + [\partial_x^k, a(u)] u.$$  

(2.7)

Obviously, we have $\|a(u) \partial_x^k u\|_{L^p} \leq K \|\partial_x^k u\|_{L^p}$ for some $K = K(\|u\|_{L^\infty})$. On the other hand, $[\partial_x^k, a(u)] u$ consists of the terms of the form

$$b(u) \prod_{j=1}^J \partial_x^{k_j} u,$$

where $2 \leq J \leq k + 1$, $k = k_1 + \cdots + k_J$, and $b(u) = O(1)$. By applying (2.4) with $q = \infty$ and $r = p$, we can estimate this term as

$$\left\| b(u) \prod_{j=1}^J \partial_x^{k_j} u \right\|_{L^p} \leq K \|u\|_{L^\infty}^{J-1} \|\partial_x^k u\|_{L^p},$$

where $K = K(\|u\|_{L^\infty})$. Therefore, we obtain

$$\|\partial_x^k (a(u) u)\|_{L^p} \leq K \|\partial_x^k u\|_{L^p} + \sum_{J=2}^{k+1} K \|u\|_{L^\infty}^{J-1} \|\partial_x^k u\|_{L^p}$$

$$\leq K (1 + \|u\|_{L^\infty})^k \|\partial_x^k u\|_{L^p}.$$  

(2.8)

Next we show (2.6) for $N = 2$. By applying (2.1) and (2.5), we get

$$\|\partial_x^k (a(u)^2)\|_{L^p} \leq C(\|\partial_x^k (a(u) u)\|_{L^r} \|u\|_{L^s} + \|a(u) u\|_{L^q} \|\partial_x^k u\|_{L^r})$$

$$\leq K ((1 + \|u\|_{L^\infty})^k \|\partial_x^k u\|_{L^r} \|u\|_{L^s} + \|u\|_{L^q} \|\partial_x^k u\|_{L^r})$$

$$\leq K (1 + \|u\|_{L^\infty})^k \|u\|_{L^q} \|\partial_x^k u\|_{L^r},$$

where $K = K(\|u\|_{L^\infty})$. Therefore, we have

$$\|\partial_x^k (a(u) u)^2\|_{L^p} \leq C(\|\partial_x^k (a(u)^2)\|_{L^r} \|u\|_{L^s} + \|a(u)^2\|_{L^q} \|\partial_x^k u\|_{L^r})$$

$$\leq K ((1 + \|u\|_{L^\infty})^k \|\partial_x^k u\|_{L^r} \|u\|_{L^s} + \|u\|_{L^q} \|\partial_x^k u\|_{L^r})$$

$$\leq K (1 + \|u\|_{L^\infty})^k \|u\|_{L^q} \|\partial_x^k u\|_{L^r}.$$
which proves (2.6) for \( N = 2 \). The general case can be proved by the induction argument in \( N \) so that we omit the details. The proof of Lemma 2.2 is complete.

Finally in this section, we consider the linear elliptic equation

\[
(-1)^m \partial_x^{2m} q + q = h,
\]  

(2.9)

where \( m \) is a positive integer and \( h \) is a given function defined on \( \mathbb{R} \). We derive the estimate for the solution \( q \) to (2.9) in terms of \( h \).

**Lemma 2.3.** Let \( l \) be an integer which is not necessarily non-negative. Then the solution \( q \) to (2.9) verifies the elliptic estimate

\[
\|q\|_{H^{l+2m}} \leq C \|h\|_{H^l}.
\]  

(2.10)

**Proof.** Taking the Fourier transform of (2.9), we find that \( q \) verifies

\[
\hat{q}(\xi) = (1 + \xi^{2m})^{-1} \hat{h}(\xi).
\]

Therefore, we have

\[
\|q\|_{H^{l+2m}}^2 = \int_{\mathbb{R}} (1 + \xi^{2})^{l+2m-2} |\hat{h}(\xi)|^2 d\xi
\]

\[
\leq C \int_{\mathbb{R}} (1 + \xi^{2})^{l} |\hat{h}(\xi)|^2 d\xi = C \|h\|_{H^l}^2.
\]

This completes the proof of Lemma 2.3.

\[ \square \]

3. **Decay property of regularity-loss type**

We study the decay property of the linearized system of (1.1):

\[
\begin{align*}
\frac{\partial u}{\partial t} + \alpha u_x + q_x &= 0, \\
(-1)^m \partial_x^{2m} q + q + u_x &= 0,
\end{align*}
\]  

(3.1)

where \( \alpha = f'(0) \) and \( m \) is an integer with \( m \geq 2 \). We take the Fourier transform of (3.1) to get

\[
\begin{align*}
\hat{u}_t + \alpha i \xi \hat{u} + i \xi \hat{q} &= 0, \\
(1 + \xi^{2m}) \hat{q} + i \xi \hat{u} &= 0.
\end{align*}
\]  

(3.2)

Eliminating \( \hat{q} \) in (3.2), we have the single ordinary differential equation

\[
\hat{u}_t + (\alpha i \xi + \eta_m(\xi)) \hat{u} = 0,
\]

where

\[
\eta_m(\xi) = \xi^2 / (1 + \xi^{2m}).
\]  

(3.3)

Solving this ordinary differential equation with the initial condition \( \hat{u}(\xi, 0) = \hat{u}_0(\xi) \), we have

\[
\hat{u}(\xi, t) = e^{-(\alpha i \xi + \eta_m(\xi))t} \hat{u}_0(\xi).
\]  

(3.4)

Now we define the semigroup \( e^{tA} \) associated with the linearized system (3.1) by

\[
(e^{tA} \phi)(x) := \mathcal{F}^{-1} [e^{-(\alpha i \xi + \eta_m(\xi))t} \hat{\phi}(\xi)](x).
\]  

(3.5)
Note that the solution to the linearized problem (3.1) with the initial data $u(x, 0) = u_0(x)$ is given as $u(x, t) = e^{tA}u_0(x)$. Note also that our original problem (1.1), (1.2) is transformed into the integral equation
\begin{equation}
    u(t) = e^{tA}u_0 - \int_0^t e^{(t-\tau)A} \partial_x g(u)(\tau) d\tau,
\end{equation}
where we put $g(u) = f(u) - f(0) - f'(0)u = O(|u|^2)$.

As in [4] and [5], we show that the semigroup $e^{tA}$ verifies the decay property of the regularity-loss type.

**Lemma 3.1.** Let $m \geq 2$ be an integer and let $e^{tA}$ be the semigroup associated with the linearized system (3.1) defined in (3.5). Then we have
\begin{equation}
    \| \partial_x e^{tA} \phi \|_{L^2} \leq C (1 + t)^{-1/4 - k/2} \| \phi \|_{L^1} + C (1 + t)^{-l/(m-1)} \| \partial_x^{k+l} \phi \|_{L^2},
\end{equation}
where $k$ and $l$ are non-negative integers.

**Proof.** Applying the Plancherel theorem, we have
\begin{align}
    \| \partial_x e^{tA} \phi \|_{L^2}^2 &= \| (i \xi)^k e^{-(\alpha i \xi + \eta_m(\xi))t} \hat{\phi}(\xi) \|_{L^2_{\xi}}^2 \\
    &= \int_{|\xi| \leq 1} \xi^{2k} e^{-2\eta_m(\xi)t} |\hat{\phi}(\xi)|^2 d\xi + \int_{|\xi| \geq 1} \xi^{2k} e^{-2\eta_m(\xi)t} |\hat{\phi}(\xi)|^2 d\xi \\
    &=: I_1 + I_2.
\end{align}

In the low-frequency part $I_1$, we see that $\eta_m(\xi) = \xi^2/(1 + \xi^{2m}) \geq \xi^2/2$. Therefore, we obtain
\begin{equation}
    I_1 \leq \int_{|\xi| \leq 1} \xi^{2k} e^{-\xi^2 t} |\hat{\phi}(\xi)|^2 d\xi \\
    \leq \sup_{|\xi| \leq 1} |\hat{\phi}(\xi)|^2 \int_{|\xi| \leq 1} \xi^{2k} e^{-\xi^2 t} d\xi \leq C (1 + t)^{-1/2 - k} \| \phi \|_{L^1}.
\end{equation}

On the other hand, in the high-frequency part $I_2$, we see that $\eta_m(\xi) = \xi^2/(1 + \xi^{2m}) \geq 1/2 \xi^{2(m-1)}$. Hence we obtain
\begin{align}
    I_2 &\leq \int_{|\xi| \geq 1} \xi^{2k} e^{-t/\xi^{2(m-1)}} |\hat{\phi}(\xi)|^2 d\xi \\
    &\leq \sup_{|\xi| \geq 1} \{ (1/\xi^2) e^{-t/\xi^{2(m-1)}} \} \int_{|\xi| \geq 1} \xi^{2(k+l)} |\hat{\phi}(\xi)|^2 d\xi \\
    &\leq C (1 + t)^{-l/(m-1)} \| \partial_x^{k+l} \phi \|_{L^2}.
\end{align}

Substituting these estimates into (3.8) gives the desired decay estimate (3.7). This completes the proof of Lemma 3.1.

Next we consider an approximation of the linearized system (3.1). We neglect the highest-order term $(-1)^m \partial_x^{2m} q$ in (3.1). Then, eliminating $q$ from the resulting system, we get a single equation
\begin{equation}
    v_t + \alpha v_x = v_{xx},
\end{equation}
where $v(x, t) = v(t)$. Note that the solution to the linearized problem (3.1) with the initial data $v(x, 0) = v_0(x)$ is given as
where $\alpha = f'(0)$. The associated semigroup $e^{tA_0}$ is defined as
\[
(e^{tA_0}\phi)(x) := \mathcal{F}^{-1}\left[e^{-(\alpha t \xi + \xi^2)t} \hat{\phi}(\xi)\right](x),
\] (3.12)
which can be expressed in terms of the heat kernel $G(x, t) = (4\pi t)^{-1/2}e^{-x^2/4t}$ as
\[
(e^{tA_0}\phi)(x) = (G(\cdot, t) \ast \phi)(x - \alpha t).
\] (3.13)

Concerning this simple semigroup $e^{tA_0}$, we have the following decay estimates.

**Lemma 3.2.** [4] Let $e^{tA_0}$ be the semigroup defined in (3.12), and let $k$ and $l$ be integers with $0 \leq l \leq k$. Then we have
\[
\|\partial_x^k e^{tA_0} \phi\|_{L^2} \leq Ct^{-1/4-k/2}\|\phi\|_{L^1},
\] (3.14)
\[
\|\partial_x^k e^{tA_0} \phi\|_{L^2} \leq C(1 + t)^{-1/4-k/2}\|\phi\|_{L^1} + Ce^{-ct} t^{-(k-l)/2}\|\partial_x^l \phi\|_{L^2}.
\] (3.15)

Moreover, if $\int_{\mathbb{R}} \phi(x) \, dx = 0$, then we have the following improved decay estimates:
\[
\|\partial_x^k e^{tA_0} \phi\|_{L^2} \leq Ct^{-3/4-k/2}\|\phi\|_{L^1},
\] (3.16)
\[
\|\partial_x^k e^{tA_0} \phi\|_{L^2} \leq C(1 + t)^{-3/4-k/2}\|\phi\|_{L^1} + Ce^{-ct} t^{-(k-l)/2}\|\partial_x^l \phi\|_{L^2}.
\] (3.17)

Next we show that the semigroup $e^{tA}$ in (3.5) can be approximated by the semigroup $e^{tA_0}$ in (3.12).

**Lemma 3.3.** Let $m \geq 2$ be an integer. Let $e^{tA}$ and $e^{tA_0}$ be the semigroups defined in (3.5) and (3.12), respectively. Then we have
\[
\|\partial_x^k (e^{tA} - e^{tA_0}) \phi\|_{L^2} \leq C(1 + t)^{-1/4-m-k/2}\|\phi\|_{L^1} + C(1 + t)^{-l/2(m-1)}\|\partial_x^l \phi\|_{L^2},
\] (3.18)
where $k$ and $l$ are non-negative integers.

**Remark 3.4.** The decay rate $t^{-1/4-m-k/2}$ of the first term on the right-hand side of (3.18) is better than the usually expected rate $t^{-3/4-k/2}$. We note that the usual decay rate $t^{-3/4-k/2}$ is sufficient in the proof of Theorem 5.1.

**Proof of Lemma 3.3.** By applying the Plancherel theorem, we have
\[
\|\partial_x^k (e^{tA} - e^{tA_0}) \phi\|_{L^2}^2 = \int_{|\xi| \leq 1/2} \xi^{2k} |e^{-\eta_m(\xi)t} - e^{-\xi^2t}|^2 |\hat{\phi}(\xi)|^2 \, d\xi
\]
\[
+ \int_{|\xi| \geq 1/2} \xi^{2k} |e^{-\eta_m(\xi)t} - e^{-\xi^2t}|^2 |\hat{\phi}(\xi)|^2 \, d\xi
\]
\[
=: I_1 + I_2.
\] (3.19)

In the low-frequency region $|\xi| \leq 1/2$, we calculate
\[
e^{-\eta_m(\xi)t} - e^{-\xi^2t} = \int_0^1 \frac{d}{d\theta} e^{-\xi^2t + \theta(\eta_m(\xi) - \xi^2)t} d\theta
\]
\[
= - (\eta_m(\xi) - \xi^2) t e^{-\xi^2t} \int_0^1 e^{-\theta(\eta_m(\xi) - \xi^2)t} d\theta.
\]
Here we see that $|\eta_m(\xi) - \xi^2| = \xi^{2(m+1)}/(1 + \xi^{2m}) \leq \xi^{2(m+1)}$. Hence we have
\[
|e^{-\eta_m(\xi)t} - e^{-\xi^2t}| \leq \xi^{2(m+1)}te^{-\xi^2t}e^{\xi^{2(m+1)t}} 
\leq \xi^{2(m+1)}te^{-c\xi^2t} \leq C\xi^{2m}e^{-c\xi^2t}
\]
for $|\xi| \leq 1/2$, where $c$ is a positive constant. Hence we have
\[
|e^{-\eta_m(\xi)t} - e^{-\xi^2t}| \leq \xi^{2(m+1)}te^{-c\xi^2t} \leq C\xi^{2m}e^{-c\xi^2t}
\]

for $|\xi| \leq 1/2$, where $c$ is a positive constant. Consequently, we can estimate the low-frequency part $I_1$ as
\[
I_1 \leq C\int_{|\xi| \leq 1/2} \xi^{2k+4m}e^{-c\xi^2t} |\hat{\phi}(\xi)|^2 d\xi 
\leq C\|\hat{\phi}\|_{L^\infty}^2 \int_{|\xi| \leq 1/2} \xi^{2k+4m}e^{-c\xi^2t} d\xi \leq C(1 + t)^{-1/2 - 2m-k}\|\phi\|^2_{L^1}. 
(3.20)
\]

On the other hand, for the high-frequency part $I_2$, we have
\[
I_2 \leq 2\int_{|\xi| \geq 1/2} \xi^{2k}e^{-\eta_m(\xi)t} |\hat{\phi}(\xi)|^2 d\xi + 2\int_{|\xi| \geq 1/2} \xi^{2k}e^{-2\xi^2t} |\hat{\phi}(\xi)|^2 d\xi 
=: I_{21} + I_{22}. 
(3.21)
\]

Here the term $I_{22}$ is easily estimated as
\[
I_{22} \leq C\int_{|\xi| \geq 1/2} \xi^{2(k+l)}e^{-c\xi^2t} |\hat{\phi}(\xi)|^2 d\xi \leq Ce^{-ct}\|\partial^{k+l}\phi\|^2_{L^2}. 
\]

Also, we can estimate the term $I_{21}$ just in the same way as in (3.10) and obtain $I_{21} \leq C(1 + t)^{-1/(m-1)}\|\partial^{k+l}\phi\|^2_{L^2}$. These observations give the desired estimate (3.18). This completes the proof of Lemma 3.3. \(\square\)

4. Global existence and asymptotic decay

The aim of this section is to prove the following theorem concerning the global existence and asymptotic decay of solutions to the problem (1.1), (1.2).

**Theorem 4.1. (Global existence and asymptotic decay)** Let $m \geq 2$ and $s \geq 4m - 1$. Assume that $u_0 \in H^s \cap L^1$ and put $E_1 = \|u_0\|_{H^s} + \|u_0\|_{L^1}$. Then there is a positive constant $\delta_0$ such that if $E_1 \leq \delta_0$, then the initial value problem (1.1), (1.2) has a unique global solution $(u, q)(x, t)$ with
\[
\begin{align*}
&u \in C^0([0, \infty); H^s) \cap C^1([0, \infty); H^{s-1}), \\
&q \in C^0([0, \infty); H^{s+2m-1}).
\end{align*}
\]

The solution verifies the following optimal decay estimates:
\[
\|\partial_x^ku(t)\|_{L^2} \leq CE_1(1 + t)^{-1/4-k/2} \quad (4.1)
\]
for $k$ with $0 \leq k \leq \lfloor (s + 1)/m \rfloor - 2$ and
\[
\|\partial_x^kq(t)\|_{H^{2m}} \leq CE_1(1 + t)^{-3/4-k/2} \quad (4.2)
\]
for $k$ with $0 \leq k \leq \lfloor (s + 1)/m \rfloor - 3$. 

The key to the proof of this theorem is to derive the desired \textit{a priori} estimates of solutions. We derive these \textit{a priori} estimates by employing the time-weighted energy method used in [4] which is combined with the optimal decay estimates for lower-order derivatives of solutions.

4.1. Time-weighted energy estimate

In this subsection, we derive a suitable energy inequality for the problem (1.1), (1.2) by employing the time-weighted energy method. For this purpose, we introduce a time-weighted energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ by

$$E(t)^2 := \sum_{k=0}^{\lfloor s/m \rfloor} \sup_{0 \leq \tau \leq t} (1 + \tau)^{k-1/2} \| \partial_x^k u(\tau) \|^2_{H^{s-km}},$$ (4.3)

$$D(t)^2 := \sum_{k=0}^{\lfloor s/m \rfloor} \int_0^t (1 + \tau)^{k-3/2} \| \partial_x^k u(\tau) \|^2_{H^{s-km}} d\tau.$$ (4.4)

To get the desired estimates for $E(t)$ and $D(t)$, we need to use the following time-weighted norm $M_k(t)$ that measures the optimal decay for the $L^\infty$ norm of the $k$th-order derivative of solutions:

$$M_k(t) := \sup_{0 \leq \tau \leq t} (1 + \tau)^{(k+1)/2} \| \partial_x^k u(\tau) \|_{L^\infty}.$$ (4.5)

Our time-weighted energy inequality is then given as follows.

**Proposition 4.2.** Let $s \geq 2$ and suppose that $u_0 \in H^s$. Let $(u, q)(x, t)$ be a solution to the problem (1.1), (1.2). Then we have the following time-weighted energy inequality:

$$E(t)^2 + D(t)^2 \leq CE_0^2 + K(M(t))M_1(t)D(t)^2,$$ (4.6)

where $E_0 = \| u_0 \|_{H^s}$, and $K(M(t))$ denotes a quantity depending on $M(t)$ and is regarded as an increasing function of $M(t)$.

**Proof.** We first show the standard energy inequalities. We multiply (1.1a) and (1.1b) by $u$ and $q$, respectively, and add the resulting two equations. This yields

$$\frac{1}{2} u_t^2 + q^2 + (\partial_x^m q)^2 + F_x = 0,$$

where $F = \int_0^u f'(s) s \, ds + uq + (-1)^m \sum_{j=0}^{m-1} (-1)^j \partial_x^j q \partial_x^{2m-j-1} q$. Integrating this equality with respect to $x$, we get

$$\frac{d}{dt}\| u \|^2_{L^2} + c \| q \|^2_{H^m} \leq 0.$$ (4.7)

Next, to show similar energy inequalities for derivatives, we apply $\partial_x^k$ to (1.1) and obtain

$$\partial_x^k u_t + f'(u)\partial_x^k u_x + \partial_x^k q_x = -[\partial_x^k, f'(u)] u_x,$$ (4.8a)

$$(-1)^m \partial_x^{2m+k} q + \partial_x^k q + \partial_x^k u_x = 0,$$ (4.8b)
where \([\cdot, \cdot]\) denotes the commutator. We multiply (4.8a) and (4.8b) by \(\partial_x^k u\) and \(\partial_x^k q\), respectively, and add the resulting two equations, obtaining
\[
\{\frac{1}{2}(\partial_x^k u)^2\}_t + (\partial_x^k q)^2 + (\partial_x^{m+k} q)^2 + (F^k)_x = \frac{1}{2} f''(u)u_x(\partial_x^k u)^2 - \partial_x^k u [\partial_x^k, f'(u)]u_x,
\]
where
\[
F^k = \frac{1}{2} f'(u)(\partial_x^k u)^2 + \partial_x^k u \partial_x^k q + (-1)^m \sum_{j=0}^{m-1} (-1)^j \partial_x^{j+k} q \partial_x^{2m-j-1+k} q.
\]
We integrate this equality with respect to \(x\) and obtain
\[
\frac{d}{dt}\|\partial_x^k u\|_{L^2}^2 + c\|\partial_x^k q\|_{H^m}^2 \leq \left| \int_{\mathbb{R}} f''(u)u_x(\partial_x^k u)^2 \, dx \right| + 2 \left| \int_{\mathbb{R}} \partial_x^k u[\partial_x^k, f'(u)]u_x \, dx \right|.
\]
Here the first integral on the right-hand side of (4.9) is estimated as
\[
\left| \int_{\mathbb{R}} f''(u)u_x(\partial_x^k u)^2 \, dx \right| \leq K(\|u\|_{L^\infty})\|u_x\|_{L^\infty}\|\partial_x^k u\|_{L^2}^2,
\]
where \(K(\|u\|_{L^\infty})\) denotes a quantity depending on \(\|u\|_{L^\infty}\). On the other hand, using (2.2) and (2.5), we have
\[
\|\partial_x^k, f'(u)\|_{L^2} \leq C(\|\partial_x f'(u)\|_{L^\infty}\|\partial_x^k u\|_{L^2} + \|u_x\|_{L^\infty}\|\partial_x^k f'(u)\|_{L^2})
\]
\[
\leq K(\|u\|_{L^\infty})\|u_x\|_{L^\infty}\|\partial_x^k u\|_{L^2}
\]
for \(k \geq 1\). Therefore, we can estimate the second term on the right-hand side of (4.9) as
\[
2 \left| \int_{\mathbb{R}} \partial_x^k u[\partial_x^k, f'(u)]u_x \, dx \right| \leq 2\|\partial_x^k u\|_{L^2}\|\partial_x^k, f'(u)\|_{L^2} \leq K(\|u\|_{L^\infty})\|u_x\|_{L^\infty}\|\partial_x^k u\|_{L^2}^2.
\]
Substituting these estimates into (4.9) gives
\[
\frac{d}{dt}\|\partial_x^k u\|_{L^2}^2 + c\|\partial_x^k q\|_{H^m}^2 \leq K(\|u\|_{L^\infty})\|u_x\|_{L^\infty}\|\partial_x^k u\|_{L^2}^2,
\]
where \(1 \leq k \leq s\). This is the desired energy inequality for derivatives.

Next we show the time-weighted energy inequalities. To this end, we multiply (4.7) and (4.10) by \((1 + t)^\alpha\) and integrate with respect to \(t\). This yields
\[
(1 + t)^\alpha \|\partial_x^j u(t)\|_{L^2}^2 + c\int_0^t (1 + \tau)^\alpha \|\partial_x^j q(\tau)\|_{H^m}^2 \, d\tau \leq \|\partial_x^j u_0\|_{L^2}^2 + \alpha\int_0^t (1 + \tau)^{\alpha-1}\|\partial_x^j u(\tau)\|_{L^2}^2 \, d\tau
\]
\[
+ K(M(t)) \int_0^t (1 + \tau)^\alpha \|u_x(\tau)\|_{L^\infty}\|\partial_x^j u(\tau)\|_{L^2}^2 \, d\tau
\]
for \(j \geq 0\), where \(\alpha \in \mathbb{R}\) is not necessarily non-negative. Note that, when \(j = 0\), the third term on the right-hand side of this inequality is absent. Here we used \(j\) in place of \(k\). For the proof of our time-weighted energy inequality (4.6), it suffices to show the following two inequalities
for \( k \) with \( 0 \leq k \leq [s/m] 
\[
(1 + t)^{k-1/2} \| \partial_x^k u(t) \|_{H^{s-km}}^2 + \int_0^t (1 + \tau)^{k-1/2} \| \partial_x^k q(\tau) \|_{H^{s+m-km}}^2 \, d\tau \leq C \| u_0 \|_{H^s}^2 + K(M(t))M_1(t)D(t)^2, \tag{4.12}
\]
and we suppose that (4.12) holds true for \( k \). Then we show \( (4.12) \) for \( k-1 \). We put \( \alpha = -1/2 \) in (4.11) and add for \( j \) with \( 0 \leq j \leq s \). This yields
\[
(1 + t)^{-1/2} \| u(t) \|_{H^s}^2 + \frac{1}{2} \int_0^t (1 + \tau)^{-3/2} \| u(\tau) \|_{H^s}^2 \, d\tau + c \int_0^t (1 + \tau)^{-1/2} \| q(\tau) \|_{H^{s+m}}^2 \, d\tau \leq \| u_0 \|_{H^s}^2 + K(M(t))M_1(t)D(t)^2. \tag{4.14}
\]
Here the second term on the right-hand side is estimated as
\[
K(M(t)) \int_0^t (1 + \tau)^{-1/2} \| u(\tau) \|_{H^s} \| u(\tau) \|_{H^s}^2 \, d\tau \leq K(M(t))M_1(t) \int_0^t (1 + \tau)^{-3/2} \| u(\tau) \|_{H^s}^2 \, d\tau \leq K(M(t))M_1(t)D(t)^2.
\]
Therefore, (4.14) gives (4.12) and (4.13) for \( k = 0 \).

Next, let \( 1 \leq k \leq [s/m] \) and we suppose that (4.12) holds true for \( k-1 \). Then we show (4.13) for \( k \). From (1.1b) we see that
\[
\partial_t \partial_x^k u = -\partial_x^{k-1} (q + (-1)^m \partial_x^m q).
\]
Taking the \( H^{s-km} \) norm of this equality yields
\[
\| \partial_x^k u \|_{H^{s-km}} \leq C \| \partial_x^{k-1} q \|_{H^{s+2m-km}} = C \| \partial_x^{k-1} q \|_{H^{s+m-(k-1)m}}.
\]
Hence, by using (4.12) with \( k-1 \), we obtain
\[
\int_0^t (1 + \tau)^{k-3/2} \| \partial_x^k u(\tau) \|_{H^{s-km}}^2 \, d\tau \leq C \int_0^t (1 + \tau)^{(k-1)-1/2} \| \partial_x^{k-1} q(\tau) \|_{H^{s+m-(k-1)m}}^2 \, d\tau \leq C \| u_0 \|_{H^s}^2 + K(M(t))M_1(t)D(t)^2,
\]
which gives the desired estimate (4.13) for \( k \). Finally, we show (4.12) for \( k \). We put \( \alpha = k - 1/2 \) in (4.11) and add for \( j \) with \( k \leq j \leq s - km \). Then, using (4.13) with \( k \), we obtain
\[
(1 + t)^{k-1/2} \| \partial_x^k u(t) \|_{H^{s-km}}^2 + c \int_0^t (1 + \tau)^{k-1/2} \| \partial_x^k q(\tau) \|_{H^{s+m-km}}^2 \, d\tau \leq \| \partial_x^k u_0 \|_{H^s}^2 + (k - 1/2) \int_0^t (1 + \tau)^{k-3/2} \| \partial_x^k u(\tau) \|_{H^{s-km}}^2 \, d\tau + K(M(t)) \int_0^t (1 + \tau)^{k-1/2} \| u(\tau) \|_{L^\infty} \| \partial_x^k u(\tau) \|_{H^{s-km}}^2 \, d\tau \leq C \| u_0 \|_{H^s}^2 + K(M(t))M_1(t)D(t)^2,
\]
which proves (4.12) for \( k \). This completes the proof of Proposition 4.2.
4.2. Decay estimates for nonlinear problem

In this subsection, we show the optimal decay of solutions by applying the decay estimates for the semigroup $e^{tA}$. This optimal decay is measured by the time-weighted norm

$$N(t) := \sum_{k=0}^{[(s+1)/m]-2} \sup_{0 \leq \tau \leq t} (1 + \tau)^{1/4 + k/2} \| \partial_x^k u(\tau) \|_{L^2}. \quad (4.15)$$

To get the desired estimate for this $N(t)$, we use $M_0(t)$ in (4.4) and $E(t)$ in (4.3) in the form

$$\| \partial_x^k u(t) \|_{H^{s-km}} \leq E(t)(1 + t)^{-k/2 + 1/4}, \quad (4.16)$$

where $0 \leq k \leq [s/m]$.

**Proposition 4.3.** Let $s \geq 2m - 1$ and suppose that $u_0 \in H^s \cap L^1$. Let $(u, q)(x, t)$ be a solution to the initial value problem (1.1), (1.2). Then we have the following estimate:

$$N(t) \leq CE_1 + K(M(t))N(t)^2 + K(M(t))M_0(t)E(t), \quad (4.17)$$

where $E_1 = \|u_0\|_{H^s} + \|u_0\|_{L^1}$, and $K(M(t))$ is a function depending on $M(t)$.

**Proof.** We recall that the problem (1.1), (1.2) is transformed to the integral equation (3.6), where $g(u) = f(u) - f(0) - f'(0)u = O(|u|^2)$. Applying $\partial_x^k$ to (3.6) and taking the $L^2$ norm, we have

$$\| \partial_x^k u(t) \|_{L^2} \leq \| \partial_x^k e^{tA}u_0 \|_{L^2} + \int_0^{t/2} \| \partial_x^k e^{(t-\tau)A} g(u)(\tau) \|_{L^2} d\tau$$

$$+ \int_{t/2}^t \| \partial_x e^{(t-\tau)A} \partial_x^k g(u)(\tau) \|_{L^2} d\tau$$

$$=: I_1 + I_2 + I_3. \quad (4.18)$$

where $0 \leq k \leq [(s+1)/m] - 2$. For the first term $I_1$ on the right-hand side of (4.18), by applying (3.7) with $l = (k + 1)(m - 1)$, $\phi = u_0$, we have

$$I_1 \leq C(1 + t)^{-1/4 - k/2} (\| u_0 \|_{H^{s-m}} + \| u_0 \|_{L^1}),$$

where we have used the relation $k + (k + 1)(m - 1) \leq s - m$ for $k \leq [(s+1)/m] - 2$. Next, for the term $I_2$, we apply (3.7) with $k$ replaced by $k + 1$, and with $l = (k + 1)(m - 1)$, $\phi = g(u)$. This gives

$$I_2 \leq C \int_0^{t/2} (1 + t - \tau)^{-3/4 - k/2} \| g(u)(\tau) \|_{L^1} d\tau$$

$$+ C \int_0^{t/2} (1 + t - \tau)^{-1/2 - k/2} \| \partial_x^{(k+1)m} g(u)(\tau) \|_{L^2} d\tau$$

$$=: I_{21} + I_{22}. \quad (4.19)$$

For the term $I_{21}$, we see easily that

$$\| g(u)(\tau) \|_{L^1} \leq K (\| u(\tau) \|_{L^\infty}) \| u(\tau) \|_{L^2} \leq K (M(t))N(t)^2 (1 + \tau)^{-1/2}. $$
Therefore, we have
\[
I_{21} \leq K(M(t))N(t)^2 \int_0^{t/2} (1 + t - \tau)^{-3/4 - k/2} (1 + \tau)^{-1/2} \, d\tau \\
\leq K(M(t))N(t)^2 (1 + t)^{-1/4 - k/2}.
\]

On the other hand, for the term \(I_{22}\), by applying (2.6) and using (4.16) with \(k = 1\), we find that
\[
\|\partial_x^{(k+1)m} g(u)(\tau)\|_{L^2} \leq K(\|u(\tau)\|_{L^\infty}) \|u(\tau)\|_{L^2} \|\partial_x^{(k+1)m} u(\tau)\|_{L^2} \\
\leq K(M(t))\|u(\tau)\|_{L^\infty} \|\partial_x u(\tau)\|_{H^{s-m}} \\
\leq K(M(t))M_0(t)E(t)(1 + \tau)^{-3/4},
\]
where we have used the fact that \((k + 1)m \leq s + 1 - m\) for \(0 \leq k \leq [(s + 1)/m] - 2\).

Consequently, we obtain
\[
I_{22} \leq K(M(t))M_0(t)E(t) \int_0^{t/2} (1 + t - \tau)^{-3/4 - k/2} (1 + \tau)^{-1/2} \, d\tau \\
\leq K(M(t))M_0(t)E(t)(1 + t)^{-1/4 - k/2}.
\]

Next we estimate the term \(I_3\). We apply (3.7) with \(k = 1, l = m - 1, \phi = \partial_x^k g\), obtaining
\[
I_3 \leq C \int_{t/2}^t (1 + t - \tau)^{-3/4} \|\partial_x^k g(u)(\tau)\|_{L^1} \, d\tau \\
+ C \int_{t/2}^t (1 + t - \tau)^{-1/2} \|\partial_x^{k+m} g(u)(\tau)\|_{L^2} \, d\tau \\
=: I_{31} + I_{32}.
\] (4.20)

For the term \(I_{31}\), we see from (2.6) that
\[
\|\partial_x^k g(u)(\tau)\|_{L^1} \leq K(\|u(\tau)\|_{L^\infty}) \|u(\tau)\|_{L^2} \|\partial_x^k u(\tau)\|_{L^2} \\
\leq K(M(t))N(t)^2 (1 + \tau)^{-1/2 - k/2}.
\]

Thus we have
\[
I_{31} \leq K(M(t))N(t)^2 \int_{t/2}^t (1 + t - \tau)^{-3/4} (1 + \tau)^{-1/2 - k/2} \, d\tau \\
\leq K(M(t))N(t)^2 (1 + t)^{-1/4 - k/2}.
\]

For the term \(I_{32}\), we also apply (2.6) and use (4.16) with \(k\) replaced by \(k + 1\). This gives
\[
\|\partial_x^{k+m} g(u)(\tau)\|_{L^2} \leq K(\|u(\tau)\|_{L^\infty}) \|u(\tau)\|_{L^\infty} \|\partial_x^{k+m} u(\tau)\|_{L^2} \\
\leq K(M(t))\|u(\tau)\|_{L^\infty} \|\partial_x^{k+1} u(\tau)\|_{H^{s-(k+1)m}} \\
\leq K(M(t))M_0(t)E(t)(1 + \tau)^{-3/4 - k/2}.
\]

Here we have used the fact that
\[
k + m \leq s + (k + 1) - (k + 1)m \quad \text{for} \ 0 \leq k \leq [(s + 1)/m] - 2.
\]
Therefore, we have
\[ I_{32} \leq K(M(t))M_0(t)E(t) \int_{t/2}^{t} (1 + t - \tau)^{-1/2} (1 + \tau)^{-3/4-k/2} d\tau \]
\[ \leq K(M(t))M_0(t)E(t)(1 + t)^{-1/4-k/2}. \]
Substituting all these estimates into (4.18), we arrive at the estimate
\[ (1 + t)^{1/4+k/2} \| \partial^k u(t) \|_{L^2} \]
\[ \leq C(\| u_0 \|_{H^{s-m}} + \| u_0 \|_{L^1}) + K(M(t))N(t)^2 + K(M(t))M_0(t)E(t), \]
where \( 0 \leq k \leq \lfloor (s + 1)/m \rfloor - 2 \). This implies the desired estimate (4.17) and hence the proof of Proposition 4.3 is complete.

4.3. Proof of Theorem 4.1

We see that \( M(t) \leq CE(t) \) for \( s \geq m \), \( M_0(t) \leq CN(t) \) for \( s \geq 3m - 1 \), and \( M_1(t) \leq CN(t) \) for \( s \geq 4m - 1 \). Therefore, for \( s \geq 4m - 1 \), we have from Propositions 4.2 and 4.3 that
\[ E(t)^2 + D(t)^2 \leq CE_0^2 + K(E(t))N(t)D(t)^2, \]
\[ N(t) \leq CE_1 + K(E(t))N(t)^2 + K(E(t))N(t)E(t), \]
where \( E_0 = \| u_0 \|_{H^s}, E_1 = \| u_0 \|_{H^s} + \| u_0 \|_{L^1} \), and \( K(E(t)) \) is a function depending on \( E(t) \).
Now we put \( X(t) := E(t) + D(t) + M(t) \) and reduce (4.21) to
\[ X(t)^2 \leq CE_1^2 + K(X(t))X(t)^3. \]
This inequality can be solved as \( X(t) \leq CE_1 \) if \( E_1 \) is sufficiently small. Consequently, we have \( N(t) \leq CE_1 \) and hence
\[ \| \partial^k u(t) \|_{L^2} \leq CE_1(1 + t)^{-1/4-k/2} \]
for \( 0 \leq k \leq \lfloor (s + 1)/m \rfloor - 2 \). This is the desired decay estimate (4.1). On the other hand, the decay estimate (4.2) for \( q \) can be obtained by Lemma 2.3 and (4.1). In fact, we have
\[ \| \partial^k q(t) \|_{H^{2m}} \leq C\| \partial^k u(t) \|_{L^2} \leq CE_1(1 + t)^{-3/4-k/2}, \]
where \( 0 \leq k \leq \lfloor (s + 1)/m \rfloor - 3 \). Thus the proof of Theorem 4.1 is complete.

5. Large-time behavior

In the section, we consider large-time behavior of the solution \( (u, q)(x, t) \) to the initial value problem (1.1), (1.2). For this purpose, we define the nonlinear diffusion wave for (1.1). Let \( w = W(x, t; M) \) be the self-similar solution to the Burgers equation
\[ w_t + (w^2/2)_x = w_{xx}, \]
which satisfies the integral condition \( \int_{\mathbb{R}} W(x, t; M) \, dx = M \). It is well known that \( W(x, t; M) \) is given explicitly as
\[ W(x, t; M) = \frac{1}{\sqrt{t}} \frac{(e^{M/2} - 1)e^{-\xi^2}}{\sqrt{\pi} + (e^{M/2} - 1) \int_{\xi}^{\infty} e^{-\eta^2} \, d\eta}, \quad \xi = \frac{x}{\sqrt{4t}}. \]
We define a function \( v(x, t) \) by
\[
v(x, t) := \beta^{-1}W(x - \alpha(t + 1), t + 1; \beta M),
\]
(5.3)
where \( \alpha = f'(0) \) and \( \beta = f''(0) \). Here we have assumed that \( \beta = f''(0) > 0 \) and changed the time variable as \( t \to t + 1 \) to avoid the singularity at \( t = 0 \). We see that this \( v(x, t) \) has the conserved quantity \( \int_{\mathbb{R}} v(x, t) \, dx = M \) and becomes a solution to
\[
v_t + \left( \alpha v + \frac{\beta}{2} v^2 \right)_x = 0.
\]
(5.4)
When \( M = \int_{\mathbb{R}} u_0(x) \, dx \), we call \( v(x, t) \) defined by (5.3) the nonlinear diffusion wave for (1.1), (1.2).

Our result on the asymptotic behavior of the global solution constructed in Theorem 4.1 is then stated as follows.

**THEOREM 5.1. (Large time behavior)** Let \( f''(0) > 0 \). Let \( s \geq 4m - 1 \) and assume that \( u_0 \in H^s \cap L^1 \) and \( E_2 = \|u_0\|_{H^s} + \|u_0\|_{L^1} \). Let \( (u, q)(x, t) \) be the global solution to the problem (1.1), (1.2) which was constructed in Theorem 4.1, and let \( v(x, t) \) be the nonlinear diffusion wave defined by (5.3) with \( \int_{\mathbb{R}} u_0(x) \, dx = M \). Then, for any \( \epsilon > 0 \), there is a small positive constant \( \delta_1 \) such that if \( E_1 \leq \delta_1 \), then we have the following asymptotic relations:
\[
\| \partial^k_x (u - v)(t) \|_{L^2} \leq CE_2(1 + t)^{-3/4 - k/2 + \epsilon}
\]
(5.5)
for \( k \) with \( 0 \leq k \leq (s + 1)/m - 3 \) and
\[
\| \partial^k_x (q + v)(t) \|_{H^{2m}} \leq CE_2(1 + t)^{-5/4 - k/2 + \epsilon}
\]
(5.6)
for \( k \) with \( 0 \leq k \leq (s + 1)/m - 4 \).

We recall that our solution to (1.1), (1.2) satisfies the integral equation (3.6). Also, our nonlinear diffusion wave \( v \) satisfies the viscous conservation laws (5.4), which can be transformed into the integral equation
\[
v(t) = e^{tA_0} v_0 - \int_0^t e^{(t-\tau)A_0} \partial_x \left( \frac{\beta}{2} v^2(\tau) \right) d\tau,
\]
(5.7)
where \( e^{tA_0} \) is the semigroup defined in (3.12) and \( v_0(x) := \beta^{-1}W(x - \alpha, 1; \beta M) \). Since the integral \( M = \int_{\mathbb{R}} u_0(x) \, dx \) has the bound \( |M| \leq \|u_0\|_{L^1} \), we see easily that
\[
\| \partial^k_x v_0 \|_{L^2} + \| v_0 \|_{L^1} \leq C \| u_0 \|_{L^1},
\]
\[
\| \partial^k_x v(t) \|_{L^2} \leq C \| u_0 \|_{L^1}(1 + t)^{-1/4 - k/2}
\]
(5.8)
for any \( k \geq 0 \). Now, we subtract (5.7) from (3.6). The result is written as
\[
u(t) - v(t) = (e^{tA} - e^{tA_0})u_0 + e^{tA_0} (u_0 - v_0)
\]
\[- \int_0^t \left( e^{(t-\tau)A} - e^{(t-\tau)A_0} \right) \partial_x g(u)(\tau) \, d\tau
\]
\[- \int_0^t e^{(t-\tau)A_0} \partial_x \left( \frac{\beta}{2} (u^2 - v^2)(\tau) \right) \, d\tau - \int_0^t e^{(t-\tau)A_0} \partial_x h(u)(\tau) \, d\tau,
\]
(5.9)
where \( g(u) = f(u) - f(0) - f'(0)u = O(|u|^2) \) and \( h(u) := g(u) - (\beta/2)u^2 = O(|u|^3) \).

To derive a quantitative decay estimate for the difference \( u - v \), we introduce the quantity

\[
L(t) := \sum_{k=0}^{[(s+1)/m]-3} \sup_{0 \leq \tau \leq t} (1 + \tau)^{3/4 + k/2 - \epsilon} \| \partial_x^k (u - v)(\tau) \|_{L^2},
\]

(5.10)

where \( \epsilon \) is a fixed positive number.

**Proposition 5.2.** Let \( s \geq 3m - 1 \) and assume that \( u_0 \in H^s \cap L_1^1 \). Let \( (u, q)(t, x) \) be a solution to the problem (1.1), (1.2) satisfying \( \|u(t)\|_{L^\infty} \leq \delta \) for some constant \( \delta \), and let \( v(t, x) \) be the nonlinear diffusion wave given by (5.7). Then, for any fixed \( \epsilon > 0 \), we have the following estimate:

\[
L(t) \leq C(\|u_0\|_{H^s} + \|u_0\|_{L^1} + \|u_0\|_{L^1}^2) + CN(t)^2 + CN(t)E(t) + C\epsilon N(t)^3 + C\epsilon(N(t) + \|u_0\|_{L^1})L(t),
\]

(5.11)

where \( C\epsilon \) is a positive constant depending on \( \epsilon \).

**Proof.** Let \( 0 \leq k \leq (s + 1)/m \). We apply \( \partial_x^k \) to (5.9) and take the \( L^2 \) norm, obtaining

\[
\| \partial_x^k (u - v)(t) \|_{L^2} \leq \| \partial_x^k (e^{tA} - e^{tA_0})u_0 \|_{L^2} + \| \partial_x^k e^{tA_0} (u_0 - v_0) \|_{L^2}
\]

\[
+ \int_0^{t/2} \| \partial_x^k+1 (e^{(t-\tau)A} - e^{(t-\tau)A_0})g(u)(\tau) \|_{L^2} d\tau
\]

\[
+ \int_{t/2}^t \| (e^{(t-\tau)A} - e^{(t-\tau)A_0})\partial_x^k+1 g(u)(\tau) \|_{L^2} d\tau
\]

\[
+ \int_0^{t/2} \| \partial_x^k (e^{(t-\tau)A_0} - e^{(t-\tau)A})(\beta/2)(u_0^2 - v_0^2)(\tau) \|_{L^2} d\tau
\]

\[
+ \int_{t/2}^t \| \partial_x^k e^{(t-\tau)A_0} \partial_x^k (\beta/2)(u_0^2 - v_0^2)(\tau) \|_{L^2} d\tau
\]

\[
+ \int_0^{t/2} \| \partial_x^k (e^{(t-\tau)A_0} h(u)(\tau)) \|_{L^2} d\tau + \int_{t/2}^t \| \partial_x^k e^{(t-\tau)A_0} \partial_x^k h(u)(\tau) \|_{L^2} d\tau
\]

\[
=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\]

(5.12)

We estimate each term on the right-hand side of (5.12) as follows. For \( I_1 \), we apply (3.18) with \( l = (k + 2)(m - 1) \), \( \phi = u_0 \) to get

\[
I_1 \leq C(1 + t)^{-1/4 - m/2} \|u_0\|_{L^1} + C(1 + t)^{-1-k/2} \|\partial_x^{(k+2)m-2}u_0\|_{L^2}
\]

\[
\leq C(1 + t)^{-1-k/2}(\|u_0\|_{H^s} + \|u_0\|_{L^1}),
\]

where we have used the relation \((k + 2)m - 2 \leq s - m - 1 \leq s\). For the second term \( I_2 \), we recall the fact that \( \int_{\mathbb{R}} (u_0 - v_0)(x) \, dx = 0 \) and apply (3.17) with \( l = k \), \( \phi = u_0 - v_0 \). This gives

\[
I_2 \leq C(1 + t)^{-3/4 - k/2} \|u_0 - v_0\|_{L^1} + Ce^{-ct} \|\partial_x^k(u_0 - v_0)\|_{L^2}
\]

\[
\leq C(1 + t)^{-3/4 - k/2}(\|u_0\|_{H^s} + \|u_0\|_{L^1}),
\]

where we have used (5.8).
Next we estimate the term $I_3$. Applying the decay estimate (3.18) with $k$ replaced by $k + 1$ and with $l = (k + 2)(m - 1)$, $\phi = g(u)$, we obtain

\[ I_3 \leq C \int_0^{t/2} (1 + t - \tau)^{-3/4 - m - k/2} \| g(u)(\tau) \|_{L^1} \, d\tau \]
\[ + C \int_0^{t/2} (1 + t - \tau)^{-1 - k/2} \| \partial_x^{(k+2)m-1} g(u)(\tau) \|_{L^2} \, d\tau \]
\[ =: I_{31} + I_{32}. \]

For $I_{31}$, we see that $\| g(u)(\tau) \|_{L^1} \leq C \| u(\tau) \|_{L^2} \leq CN(t)^2 (1 + \tau)^{-1/2}$. Hence we have

\[ I_{31} \leq CN(t)^2 \int_0^{t/2} (1 + t - \tau)^{-3/4 - m - k/2} (1 + \tau)^{-1/2} \, d\tau \]
\[ \leq CN(t)^2 (1 + t)^{-1/4 - m - k/2}. \]

To estimate $I_{32}$, we apply (2.6) and obtain

\[ \| \partial_x^{(k+2)m-1} g(u)(\tau) \|_{L^2} \leq C \| u(\tau) \|_{L^\infty} \| \partial_x^{(k+2)m-1} u(\tau) \|_{L^2} \]
\[ \leq C \| u(\tau) \|_{L^\infty} \| \partial_x u(\tau) \|_{H^{s-m}} \]
\[ \leq CM_0(t) E(t) (1 + \tau)^{-3/4}, \]

where we have used (4.16) with $k = 1$ since $(k + 2)m - 1 = s + 1 - m$ for $4k \leq (s + 1)/m - 34$.

Therefore, we have

\[ I_{32} \leq CM_0(t) E(t) \int_0^{t/2} (1 + t - \tau)^{-1 - k/2} (1 + \tau)^{-3/4} \, d\tau \]
\[ \leq CM_0(t) E(t) (1 + t)^{-3/4 - k/2}. \]

For the term $I_4$, we apply (3.18) with $k = 0$, $l = m - 1$, $\phi = \partial_x^{k+1} g(u)$ to obtain

\[ I_4 \leq C \int_{t/2}^t (1 + t - \tau)^{-1/4 - m} \| \partial_x^{k+1} g(u)(\tau) \|_{L^1} \, d\tau \]
\[ + C \int_{t/2}^t (1 + t - \tau)^{-1/2} \| \partial_x^{k+m} g(u)(\tau) \|_{L^2} \, d\tau \]
\[ =: I_{41} + I_{42}. \]

Here, using (2.6) on the first term $I_{41}$ yields

\[ \| \partial_x^{k+1} g(u)(\tau) \|_{L^1} \leq C \| u(\tau) \|_{L^2} \| \partial_x^{k+1} u(\tau) \|_{L^2} \leq CN(t)^2 (1 + \tau)^{-1/2}. \]

Consequently, we have

\[ I_{41} \leq CN(t)^2 \int_{t/2}^t (1 + t - \tau)^{-1/4 - m} (1 + \tau)^{-1/2} \, d\tau \]
\[ \leq CN(t)^2 (1 + t)^{-1/2}. \]
By using (2.6) also for $I_{42}$, we have
\[
\| \partial_{x}^{k+m} g(u) \|_{L^2} \leq C \| u(\tau) \|_{L^\infty} \| \partial_{x}^{k+m} u(\tau) \|_{L^2} \\
\leq C \| u(\tau) \|_{L^\infty} \| \partial_{x}^{k+2m} u(\tau) \|_{H^{-(k+2)m}} \\
\leq CM_0(t) E(t) (1 + \tau)^{-5/4-k/2},
\]
where we have also used (4.16) with $k$ replaced by $k + 2$ since $k + m \leq s + (k + 2) - (k + 2)m$ for $k \leq [(s + 1)/m] - 3$. Hence, $I_{42}$ is estimated as
\[
I_{42} \leq CM_0(t) E(t) \int_{t/2}^{t} (1 + t - \tau)^{-1/2} (1 + \tau)^{-5/4-k/2} \; d\tau \\
\leq CM_0(t) E(t) (1 + t)^{-3/4-k/2}. 
\]

For the term $I_5$, we apply (3.15) with $k$ replaced by $k + 1$ and with $l = k$, $\phi = u^2 - v^2$. This yields
\[
I_5 \leq C \int_{t/2}^{t} (1 + t - \tau)^{-3/4-k/2} \| (u^2 - v^2)(\tau) \|_{L^1} \; d\tau \\
+ C \int_{0}^{t/2} e^{-c(t-\tau)(t-\tau)} (t-\tau)^{-1/2} \| \partial_{x}^{k}(u^2 - v^2)(\tau) \|_{L^2} \; d\tau \\
=: I_{51} + I_{52}.
\]
Here we use (5.8) to get
\[
\|(u^2 - v^2)(\tau)\|_{L^1} \leq \|(u + v)(\tau)\|_{L^2} \|(u - v)(\tau)\|_{L^2} \\
\leq C \left( N(t) + \|u_0\|_{L^1} \right) L(t) (1 + \tau)^{-1+\epsilon}.
\]
Similarly, by using (2.1) and (5.8), we have
\[
\| \partial_{x}^{k}(u^2 - v^2)(\tau) \|_{L^2} \leq \| \partial_{x}^{k}(u^2)(\tau) \|_{L^2} + \| \partial_{x}^{k}(v^2)(\tau) \|_{L^2} \\
\leq C \left( M_0(t) N(t) + \|u_0\|_{L^2} \right) (1 + \tau)^{-3/4-k/2}.
\]
Therefore, we obtain
\[
I_{51} \leq C \left( N(t) + \|u_0\|_{L^1} \right) L(t) \int_{0}^{t/2} (1 + t - \tau)^{-3/4-k/2}(1 + \tau)^{-1+k/2+\epsilon} \; d\tau \\
\leq C_\epsilon \left( N(t) + \|u_0\|_{L^1} \right) L(t) (1 + \tau)^{-3/4-k/2+\epsilon},
\]
where $C_\epsilon$ is a constant depending on $\epsilon$, and
\[
I_{52} \leq C \left( M_0(t) N(t) + \|u_0\|_{L^2} \right) \int_{0}^{t/2} e^{-c(t-\tau)(t-\tau)} (t-\tau)^{-1/2} (1 + \tau)^{-3/4-k/2} \; d\tau \\
\leq C \left( M_0(t) N(t) + \|u_0\|_{L^2} \right) e^{-ct}.
\]

For the term $I_6$, by applying (3.14) with $k = 1$, $\phi = \partial_{x}^{k}(u^2 - v^2)$, we have
\[
I_6 \leq C \int_{t/2}^{t} (t - \tau)^{-3/4} \| \partial_{x}^{k}(u^2 - v^2)(\tau) \|_{L^1} \; d\tau.
\]
Here, using (2.1) and (2.6), we have
\[ \| \partial_x^k (u^2 - v^2) (\tau) \|_{L^1} \leq C (N(t) + \| u_0 \|_{L^1}) L(t) (1 + \tau)^{-1 - k/2 + \varepsilon}. \]

Therefore, we obtain
\[ I_6 \leq C (N(t) + \| u_0 \|_{L^1}) L(t) \int_{t/2}^{t} (t - \tau)^{-3/4} (1 + \tau)^{-1 - k/2 + \varepsilon} \, d\tau \]
\[ \leq C (N(t) + \| u_0 \|_{L^1}) L(t) (1 + t)^{-3/4 - k/2 + \varepsilon}. \]

The term \( I_7 \) is estimated similarly as \( I_5 \). In fact, we have
\[ I_7 \leq C \int_0^{t/2} (1 + t - \tau)^{-3/4 - k/2} \| h(u)(\tau) \|_{L^1} \, d\tau \]
\[ + C \int_0^{t/2} e^{-c(t-\tau)} (t - \tau)^{-1/2} \| \partial_x^k h(u)(\tau) \|_{L^2} \, d\tau \]
\[ =: I_{71} + I_{72}. \]

Here, applying (2.6), we see that
\[ \| h(u)(\tau) \|_{L^1} \leq C M_0(t) N(t)^2 (1 + \tau)^{-1}. \]
Similarly, we have
\[ \| \partial_x^k h(u)(\tau) \|_{L^2} \leq C M_0(t) N(t)^2 (1 + \tau)^{-3/4 - k/2} \] by (2.6). Therefore, we obtain
\[ I_{71} \leq C M_0(t) N(t)^2 \int_0^{t/2} (1 + t - \tau)^{-3/4 - k/2} (1 + \tau)^{-1} \, d\tau \]
\[ \leq C M_0(t) N(t)^2 (1 + t)^{-3/4 - k/2 + \varepsilon} \log(2 + t) \]
\[ \leq C \varepsilon M_0(t) N(t)^2 (1 + t)^{-3/4 - k/2 + \varepsilon}. \]
\[ I_{72} \leq C M_0(t) N(t)^2 \int_0^{t/2} e^{-c(t-\tau)} (t - \tau)^{-1/2} (1 + \tau)^{-3/4 - k/2} \, d\tau \]
\[ \leq C M_0(t) N(t)^2 e^{-c t}. \]

Finally, we can estimate the term \( I_8 \) just in the same way as \( I_6 \). In fact, we have
\[ I_8 \leq C \int_{t/2}^{t} (t - \tau)^{-3/4} \| \partial_x^k h(u)(\tau) \|_{L^1} \, d\tau \]
\[ \leq C M_0(t) N(t)^2 \int_{t/2}^{t} (t - \tau)^{-3/4} (1 + \tau)^{-1 - k/2} \, d\tau \]
\[ \leq C M_0(t) N(t)^2 (1 + t)^{-3/4 - k/2}. \]

We substitute all these estimates into (5.12). Then, using the relation \( M_0(t) \leq C N(t) \) for \( s \geq 3m - 1 \), we can deduce the desired inequality (5.11). Thus the proof of Proposition 5.2 is complete.

**Proof of Theorem 5.1.** Let \( s \geq 4m - 1 \). Then we have already proved in Theorem 4.1 that
\[ E(t) + N(t) \leq C E_1, \]
where \( E_1 = \| u_0 \|_{H^s} + \| u_0 \|_{L^1} \). Thus, (5.11) becomes
\[ L(t) \leq C E_2 + C E_1^2 + C \varepsilon E_1^3 + C \varepsilon E_1 L(t), \]
where $E_2 = \|u_0\|_{H^s} + \|u_0\|_{L^1}$, and $C_\epsilon$ is a constant depending on $\epsilon$. This inequality implies that $L(t) \leq CE_2$ if $E_1$ is suitably small, say $E_1 \leq \delta_1$ with $\delta_1$ depending on $\epsilon$. This proves (5.5). To show (5.6) for $q$, we rewrite (1.1b) in the form

$$(-1)^m \partial_x^{2m} + 1)(q + v_x) = -(u - v)_x + (-1)^m \partial_x^{2m} v_x.$$ 

Then, applying the elliptic estimate (2.10), we have

$$\|\partial_x^k(q + v_x)(t)\|_{H^{2m}} \leq C \|\partial_x^{k+1}(u - v)(t)\|_{L^2} + C \|\partial_x^{k+2m+1} v(t)\|_{L^2}$$

$$\leq CE_2(1 + t)^{-5/4-k/2+\epsilon},$$

where $0 \leq k \leq [(s + 1)/m] - 4$. This completes the proof of Theorem 5.1. \qed

REFERENCES


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