ASYMPTOTIC EXPANSION OF SOLUTIONS TO THE NONLINEAR SCHRÖDINGER EQUATION WITH POWER NONLINEARITY

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Abstract. We study an asymptotic expansion near $t = \infty$ of the solution to the Cauchy problem for the nonlinear Schrödinger equation with repulsive short-range nonlinearity of power type. We construct two kinds of approximate solution with asymptotic expansions. The first is an accurate approximate solution and of abstract form. The second is the approximation of the first and of explicit form. The sharpness of these approximations strongly depend on the fractional part of the power of the nonlinearity. In particular, if the power is an integer, we obtain a complete expansion of the solution.

1. Introduction

This paper is devoted to the study of the asymptotic behavior of the solution to the Cauchy problem of the nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u = f(u), \quad u(0, x) = \phi, \quad (1.1)$$

where u is a complex valued unknown function of $(t, x) \in \mathbb{R}^{1+N}$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbb{R}^N$, and f denotes a nonlinear interaction given by a complex-valued function f on $\mathbb{C}$. In this paper, we assume that f is a single-power nonlinearity satisfying the Gauge invariance and repulsive conditions. Namely we assume that f is of the form

$$f(u) = |u|^\alpha u \quad (1.2)$$

with a positive number α .

There is a large amount of literature on the Cauchy problem and the scattering theory for (1.1) (see, for instance, [2–13, 15–17], and references therein). The usual scattering theory for (1.1) compares the full dynamics given by the solutions to (1.1) and the free dynamics described by the free propagator $U(t) = \exp it\Delta$. In the case of repulsive interactions, the existence and asymptotic completeness of wave operators for (1.1) with (1.2) have been proved in the space $X_{1,1} = H^1 \cap \mathcal{F}(H^1)$ for any α with $\gamma(N) \leq \alpha < \alpha_0(N)$, where H^1 is the Sobolev space of order one on $\mathbb{R}^N$ defined by $H^1 = (1 - \Delta)^{-1/2} L^2(\mathbb{R}^N)$, $\mathcal{F}$ denotes the Fourier transform, $\gamma(N) = (2 - N + \sqrt{N^2 + 12N + 4})/(2N)$, and $\alpha_0(N) = 4/(N - 2)$ if $N \geq 3$ and $\alpha_0(N) = \infty$ if $N \leq 2$. From the view point of the asymptotic behavior of the

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solution u , the scattering theory means that the nonlinear effect becomes small near $t = \pm\infty$, and so that the solution asymptotically behaves as a free solution. In other words, if we rewrite (1.1) as

$$u(t) = U(t)\phi_+ + i \int_t^\infty U(t-s)f(u(s)) ds \quad (1.3)$$

with the scattered state for positive time $\phi_+ = \lim_{t \rightarrow \infty} U(-t)u(t) \in X_{1,1}$, then the scattering theory implies that the latter term converges to zero as $t \rightarrow \infty$. It can be said that $U(t)\phi_+$ is the first approximate solution of u in large time. The second approximate solution to (1.1) with (1.2) for scattering is given in [12] for the case $1 \leq N \leq 5$ and $\gamma(N) < \alpha < \alpha_0(N)$ under a suitable regularity condition on the data. The second approximation term was first constructed in [13] under several restrictive assumptions except for the case of integrable systems (see also [11]). For (1.1) with the Hartree-type nonlinearity, the second and higher approximate solutions are understood in the long-range as well as in the short-range case (see [18]).

In this paper, we give two sequences of functions under the assumption that $\alpha \in (\gamma(N), \alpha_0(N))$. We restrict our attention to positive time. We note beforehand that the results in this paper also hold for negative time. The first is the sequence $\{w_n(t)\}_{n=0}$. The function w_0 is the free solution $e^{it\Delta}\phi_+$. The $(n+1)$ th approximate solution w_n , which is defined by a recursive formula from w_0 , approximates $u - \sum_{k=0}^{n-1} w_k$ for all $n \geq 1$, and therefore the sums $\sum_{k=0}^n w_k$ yield the approximation of the solution for (1.1) with (1.2) near $t = \infty$. Although the sums of w_n are good approximate solutions, these functions w_n themselves take very complicated abstract form. Then, under the suitable regularity of the initial data, we construct the functions of the phase-amplitude form, denoted by V_n , which approximate the previous w_n near $t = \infty$. This approximation of w_n not only clarifies the asymptotic form of w_n as $t \rightarrow \infty$, but also gives us the lower estimates of w_n and $u - \sum_{k=0}^n w_k$.

The accuracy of approximation of u by $\sum_{k=0}^n w_k$ strongly depends on the fractional part of the power α . The upper bound of the accuracy goes to infinity as α tends to an integer from below. In particular, if the power is an integer, then the approximation can be made accurate arbitrarily. Furthermore, it gives a complete expansion of the solution u , that is, the infinite sum $\sum_{k=0}^\infty w_k$ converges to the true solution in strong topology (see Theorem 1). In the special case that α is an even number, we have slightly different constructions of w_n which are based on the property that $|u|^\alpha u$ can be written as $u^a \bar{u}^b$ with $a, b \in \mathbb{N}$. The sum $\sum_{k=0}^n w_k$ converges to the solution in a topology stronger than that in the previous case. In particular, it does so in $L^r(\mathbb{R}^N)$ for all $r \in [2, \infty]$.

The approximation of w_n by V_n is possible only for $n = 0, 1$ in general. However, in the special case when α is an even number, we can construct V_n for all $n \geq 0$ and so the lower bound estimates of w_n and $u - \sum_{k=0}^n w_k$ hold for all n (see Theorem 4).

In [12], the authors have used a second approximate solution of u , that is, an approximation of $u - w_0$. For our notation, the function V_1 (the approximation of w_1) corresponds to it. Our result is better in the following two respects. First, we also show the same asymptotics for $N \leq 8$ (see Theorem 3 and Remark 1.7). Second, we obtain several ways to construct a more accurate approximate solution. If n is large then the sum $\sum_{k=1}^n w_k$ gives us better approximation of $u - w_0$ in some cases with $X_{1,1}$ data (see Remark 1.3). Moreover, under the suitable regularity, w_1 alone provides a better approximation (see Theorem 2).

To state our main result precisely, we introduce the following notation. For any r with $1 \leq r \leq \infty$, $L^r = L^r(\mathbb{R}^N)$ denotes the Lebesgue space on $\mathbb{R}^N$. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty$, $H_r^s = (1 - \Delta)^{-s/2} L^r$ and $\dot{H}_r^s = (-\Delta)^{-s/2} L^r$ denote the Sobolev space defined in terms of Bessel potentials and the homogeneous Sobolev space defined in terms of Riesz potentials, respectively. We write $H^s = H_2^s$ and $\dot{H}^s = \dot{H}_2^s$, for simplicity. For any $s \in \mathbb{R}$ and any r with $1 \leq r \leq \infty$, $\dot{B}_r^s = \dot{B}_{r,2}^s$ denotes the homogeneous Besov space. For $a, b \in \mathbb{R}$ we denote by $a \vee b$ and $a \wedge b$ the maximum and minimum, respectively. For the free propagator $U(t) = \exp(it\Delta)$ we use the factorization for $t \neq 0$: $U(t) = M(t)D(t)\mathcal{F}M(t)$, where $M(t) = \exp(i|x|^2/(4t))$ is the modulation operator defined by the multiplication by $\exp(i|x|^2/(4t))$, $D(t)$ is the dilation operator defined by $(D(t)\psi)(x) = (2it)^{-N/2}\psi(x/(2t))$, and $\mathcal{F}$ is the Fourier transform defined by $(\mathcal{F}\psi)(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-ix \cdot \xi} \psi(x) dx$. For any $\rho, \rho' \in \mathbb{R}$, $X_{\rho, \rho'} = H^{\rho'} \cap \mathcal{F}(H^\rho)$ denotes the weighted Sobolev space and $\mathcal{X}_{\rho, \rho'}$ denotes the associated function space for solution to (1.1) defined by

$$\mathcal{X}_{\rho, \rho'} = \{u \in C(\mathbb{R}; X_{\rho, \rho'}) : u \in L_{\text{loc}}^q(\mathbb{R}; L^r \cap \dot{B}_r^{\rho'}), M^{-1}u \in L_{\text{loc}}^q(\mathbb{R}; \dot{B}_r^{\rho'})\}$$

for any q, r with $0 \leq 2/q = \delta(r) < 1$,

where $\delta(r) := N(1/2 - 1/r)$. A pair of numbers (q, r) is called admissible if $2 \leq r \leq 2N/(N-2)$ ($2 \leq r \leq \infty$ if $N = 1$ and $2 \leq r < \infty$ if $N = 2$), and $2/q = \delta(r)$. The pair $(\infty, 2)$ is always admissible. For any r with $2 \leq r \leq \infty$ and $T > 0$, $Y^r(T)$ denotes the function space defined by

$$Y^r(T) := \{u \in \mathcal{S}' : \|u\|_{Y^r(T)} = \|D^{-1}M^{-1}u\|_{L^\infty([T, \infty); L^r(\mathbb{R}^N))} < \infty\}. \quad (1.4)$$

For any $\rho \in \mathbb{R}$ and any $T > 0$, $\Sigma^\rho(T)$ denotes the function space defined as follows:

$$\Sigma^\rho(T) := \{u \in \mathcal{S}' : \|u\|_{\Sigma^\rho(T)} = \|D^{-1}M^{-1}u\|_{L^\infty([T, \infty); H^{\rho r}(\mathbb{R}^N))} < \infty\}. \quad (1.5)$$

The embedding $\Sigma^\rho(T) \hookrightarrow Y^r(T)$ holds for all $r \in [2, \infty]$ if $\rho > N/2$, by the Sobolev embedding. The transform $v(t) := D^{-1}(t)M^{-1}(t)u(t)$ is equivalent to the pseudoconformal transform except for the inversion of time variable and was introduced in [8]. Moreover, the norm $\|u\|_{Y^r(T)}$ is equivalent to $\sup_{t \geq T} (t^{\delta(r)} \|u(t)\|_{L^r(\mathbb{R}^N)})$, and the norm $\|u\|_{\Sigma^\rho(T)}$ is equivalent to the norm $\|u\|_{L^\infty([T, \infty); L^2)} + \| |2t|^\rho M(-t)u \|_{L^\infty([T, \infty); \dot{H}^\rho)}$, which is used in [4]. The $\|\cdot\|_{\Sigma^\rho(T)}$ norm is almost the same as the $\|\cdot\|_{\mathcal{X}_{\rho, \rho'}}$ norm corresponding to the case $\rho' = 0$, $q = \infty$, and $r = 2$.

Our results rely on the following existence result from [12] (see also [4]).

PROPOSITION 1.1. (Proposition 1.5 in [12]) *Let α and N satisfy $\gamma(N) < \alpha < \alpha_0(N)$ and $N \geq 1$. Let $\phi \in X_{1,1}$ and let $u \in \mathcal{X}_{1,1}$ be the unique solution of (1.1) and (1.2) with $u(0) = \phi$. Let $\phi_\pm \in X_{1,1}$ be the corresponding asymptotic states at $t = \pm\infty$, respectively. Let ρ satisfy $1 \leq \rho < (\alpha + 1) \wedge 2$. Assume further that $\phi \in X_{\rho,1}$. Then, $u \in \mathcal{X}_{\rho,1}$, $\phi_\pm \in X_{\rho,1}$, and $U(-t)u(t) \rightarrow \phi_\pm$ in $X_{\rho,1}$ as $t \rightarrow \pm\infty$. In particular, $u \in \Sigma^\rho(T)$ for $T > 0$.*

1.1. Main result 1: the general case

1.1.1. Asymptotics with w_n for $X_{1,1}$, data. We begin with the construction of functions w_n whose sums yield the asymptotics of the solution to (1.1) with (1.2). For an initial datum ϕ belonging to $X_{1,1}$, global existence, existence of the asymptotic states $\phi_\pm$, and the asymptotic

completeness in $X_{1,1}$ are known. For the construction of w_n , we first define the $(\lceil \alpha \rceil + 1)$ -linear operator G by

$$G(u_1, \dots, u_{\lceil \alpha \rceil + 1})(t) := i \int_t^\infty U(t-s) \prod_{k=1}^{\lceil \alpha \rceil + 1} u_k(s) ds, \quad (1.6)$$

where $\lceil \alpha \rceil$ denotes the minimum integer larger than or equal to α , that is,

$$\lceil \alpha \rceil := \min\{m \in \mathbb{Z} : m \geq \alpha\}.$$

This is not the usual Gauss symbol $[\alpha]$ which denotes the maximum number smaller than or equal to α . We denote $\alpha/\lceil \alpha \rceil$ by β . By the definition of $\lceil \alpha \rceil$, it always holds that $(\lceil \alpha \rceil - 1)/\lceil \alpha \rceil < \beta \leq 1$. We note that (1.1) with (1.2) can be written as

$$u = U(t)\phi_+ + G(|u|^\beta, \dots, |u|^\beta, u). \quad (1.7)$$

Now, the sequence of functions $\{w_n(t)\}_{n=0}$ is given in terms of ϕ_+ as follows:

$$w_0 = U(t)\phi_+, \quad w_n = \sum_{i_1 + \dots + i_{\lceil \alpha \rceil + 1} = n-1} G(\tilde{w}_{i_1}, \dots, \tilde{w}_{i_{\lceil \alpha \rceil}}, w_{i_{\lceil \alpha \rceil + 1}}), \quad (1.8)$$

where $\tilde{w}_0 = |w_0|^\beta$ and $\tilde{w}_n = |\sum_{k=0}^n w_k|^\beta - |\sum_{k=0}^{n-1} w_k|^\beta$ for $n \geq 1$. Since all the indices in the definition of w_n are smaller than n , w_n is well defined. Each function w_n takes a complicated abstract form. For example, $w_1(t)$ is equal to

$$G(\tilde{w}_0, \dots, \tilde{w}_0, w_0) = i \int_t^\infty U(t-s) |U(s)\phi_+|^\alpha U(s)\phi_+ ds$$

and $w_2(t)$ is equal to

$$\begin{aligned} & \sum_{i_1 + \dots + i_{\lceil \alpha \rceil + 1} = 1} G(\tilde{w}_{i_1}, \dots, \tilde{w}_{i_{\lceil \alpha \rceil}}, w_{i_{\lceil \alpha \rceil + 1}}) \\ &= i \lceil \alpha \rceil \int_t^\infty U(t-s_1) \left[|U(s_1)\phi_+|^{\alpha-\beta} U(s_1)\phi_+ \right. \\ & \quad \times \left(\left| \int_{s_1}^\infty U(s_1-s_2) |U(s_2)\phi_+|^\alpha U(s_2)\phi_+ ds_2 + U(s_1)\phi_+ \right|^\beta - |U(s_1)\phi_+|^\beta \right) \Big] ds_1 \\ & \quad - \int_t^\infty U(t-s_1) \left(\int_{s_1}^\infty U(s_1-s_2) |U(s_2)\phi_+|^\alpha U(s_2)\phi_+ ds_2 \right) |U(s_1)\phi_+|^\alpha ds_1. \end{aligned}$$

Now, let us state our first main result. We use the notation $e(n) := \sum_{k=1}^n \beta^{k-1}$.

THEOREM 1. *Let $N \geq 1$. Let $\gamma(N) < \alpha < \alpha_0(N)$. Put $\theta = N\alpha/2$. Let $\phi \in X_{1,1}$ and let $u \in \mathcal{X}_{1,1}$ be the unique solution to (1.1) and (1.2) with $u(0) = \phi$. Let $\phi_+ \in X_{1,1}$ be the corresponding asymptotic states at $t = \infty$ and define w_n as (1.8). Let $I = [T, \infty)$ with $T \gg 1$. Then, it holds for all r with $\delta(r) \in [0, (\theta - 1) \wedge 1)$, admissible pairs (q_0, r_0) , and $n \geq 0$ that*

$$\|w_n\|_{Y^r(T)} \leq CT^{(1-\theta+(\delta(\alpha+2)\vee\delta(r)))e(n)}, \quad (1.9)$$

$$\|w_n\|_{L^{q_0}(I; L^{r_0})} \leq CT^{(1-\theta+\delta(\alpha+2))e(n)-\delta(\alpha+2)/2}, \quad (1.10)$$

and

$$\left\| u - \sum_{k=0}^n w_k \right\|_{Y^r(T)} \leq CT^{(1-\theta+(\delta(\alpha+2)\vee\delta(r))e(n+1))}, \quad (1.11)$$

$$\left\| u - \sum_{k=0}^n w_k \right\|_{L^{q_0}(I; L^{r_0})} \leq CT^{(1-\theta+\delta(\alpha+2))e(n+1)-\delta(\alpha+2)/2}. \quad (1.12)$$

Moreover, if α is an integer then

$$u = \sum_{n=0}^{\infty} w_n \quad \text{in } Y^r(T) \cap L^{q_0}(I; L^{r_0}).$$

Remark 1.1. We have $\theta > 1$ if and only if $\alpha > 2/N$. We have $\theta > 1 + \delta(\alpha + 2)$ if and only if $\theta > \gamma(N)$. Note that $2/N < \gamma(N) < 4/N < \alpha_0(N)$.

Remark 1.2. The powers of T appearing in (1.9)–(1.12) depend on $e(n) = \sum_{k=1}^n \beta^{k-1}$ which satisfies $\lim_{n \rightarrow \infty} e(n) = (1 - \beta)^{-1}$ if $\beta < 1$ and $\lim_{n \rightarrow \infty} e(n) = \infty$ if $\beta = 1$. Since $\beta = \alpha/\lceil \alpha \rceil$, it means that the sharpness of approximation strongly depends on the fractional part of the power α . The upper bound of the accuracy goes to infinity as α tends to an integer from below. In particular, if the power is an integer then the approximation can be made accurate arbitrarily.

Remark 1.3. In [12], they construct a second approximate function V_+ such that $\|u - w_0 - V_+\|_{L^{q_0}([T, \infty); L^{r_0})} = o(T^{1-\theta})$ for all admissible pairs (q_0, r_0) under the condition that $\phi \in X_{\rho,1}$ with $\rho = 1$ in the case $N = 1$ or the case where $N = 2$ and $\alpha < 2$, and with ρ satisfying $\delta(2(\alpha + 1)) \vee \delta((\alpha + 1)(\alpha + 2)) \vee (2\delta(\alpha + 2)) \vee 1 < \rho < (\alpha + 1) \wedge 2$ in the case $N = 2$ and $\alpha \geq 2$ or the case $3 \leq N \leq 5$. Theorem 1 is better than this result in the following respects.

- (1) There is no restriction on the space dimensions N , and we only need $\phi \in X_{1,1}$ for all $\alpha \in (\gamma(N), \alpha_0(N))$.
- (2) The best decay rate given by (1.12) is $(1 - \theta + \delta(\alpha + 2))/(1 - \beta) - \delta(\alpha + 2)/2$ if $\beta < 1$, and it is $-\infty$ if $\beta = 1$. This power becomes smaller than $(1 - \theta)$ in the following cases: $\alpha > (1 + \sqrt{89})/4 \in (\gamma(1), 3)$ if $N = 1$, $\alpha > (-1 + \sqrt{69})/4 \in (\gamma(2), 2)$ if $N = 2$, $\alpha > 7/6 \in (\gamma(3), 2)$ if $N = 3$, and $\alpha > (3 - 4N + \sqrt{17N^2 + 40N + 16})/4N \in (\gamma(N), 1)$ if $4 \leq N \leq 9$.

1.1.2. Sharp asymptotics for $N \leq 5$. The estimates in Theorem 1 are not sharp, while they hold for large n . Next we state more accurate estimates for $n = 0, 1, 2$ under a suitable regularity condition on ϕ .

THEOREM 2. *Let $1 \leq N \leq 5$ and $\alpha \in (\gamma(N), \alpha_0(N))$. Put $\theta = \alpha N/2$. Let ρ satisfy $1 \leq \rho < (\alpha + 1) \wedge 2$ and*

$$\frac{N\alpha}{2(\alpha - \beta + 1)} \vee \left(\frac{N}{2} - \frac{2(1 - \beta)(\theta - 1)}{\alpha} \right) \leq \rho. \quad (1.13)$$

Let $\phi \in X_{\rho,1}$ and let $u \in \mathcal{X}_{\rho,1}$ be the unique solution to (1.1) and (1.2) with $u(0) = \phi$. Let $\phi_+ \in X_{\rho,1}$ be the corresponding asymptotic states at $t = \infty$ and define w_n as (1.8). Let r satisfy $0 \leq \delta(r) < (\theta - 1) \wedge 1$ and (q_0, r_0) be an admissible pair. Put $T \gg 1$ and

$I = [T, \infty)$. Then, it holds that

$$\|w_n\|_{Y^r(T)} \leq CT^{p+\delta(r)}, \quad \|w_n\|_{L^{q_0}(I; L^{r_0})} \leq CT^p$$

for $n = 0, 1, 2$, where $p = (1 - \theta)e(n)$ and

$$\left\| u - \sum_{k=0}^n w_k \right\|_{Y^r(T)} \leq CT^{q+\delta(r)}, \quad \left\| u - \sum_{k=0}^n w_k \right\|_{L^{q_0}(I; L^{r_0})} \leq CT^q$$

for $n = 0, 1$, where $q = (1 - \theta)e(n + 1)$. Moreover, if $\alpha > 1$ then

$$\|u - w_0 - w_1 - w_2\|_{L^{q_0}(I; L^{r_0})} = o(T^{(1-\theta)(1+\beta)}).$$

Remark 1.4. There exists $\rho \in [1, (\alpha + 1) \wedge 2)$ so that (1.13) is satisfied in the following cases: $\alpha \in (\gamma(N), \alpha_0(N))$ if $N \leq 3$; $\alpha \in (\gamma(4), \alpha_0(4))$ and $\alpha \neq 1$ if $N = 4$; and $\alpha \in (1, \alpha_0(5))$ if $N = 5$.

Remark 1.5. Some estimates in Theorem 2 are sharp in such a sense that we have lower L^2 estimates. Indeed, Theorem 2 and Theorem 3 given below imply

$$\begin{aligned} \|w_1\|_{Y^2(T)} &\geq \|V_1\|_{Y^2(T)} - \|w_1 - V_1\|_{Y^2(T)} = cT^{1-\theta}, \\ \|u - w_0\|_{Y^2(T)} &\geq \|w_1\|_{Y^2(T)} - \|u - w_0 - w_1\|_{Y^2(T)} = cT^{1-\theta}. \end{aligned}$$

A similar argument also gives a lower L^2 bound for w_0 . If α is even number then we have similar lower estimates on w_2 and $u - w_0 - w_1$ (see Proposition 6.2).

1.1.3. The approximation of w_1 . Next we consider the approximation of w_1 by V_1 .

THEOREM 3. Let $N \leq 8$ and $\alpha \in (\gamma(N), \alpha_0(N))$. Put $\theta = \alpha N/2$. Let ρ satisfy $1 \leq \rho < (\alpha + 1) \wedge 2$ and

$$\frac{\theta(1 - \beta) + \beta}{\alpha - \beta + 1} \vee \frac{\theta - \beta}{\alpha - \beta + 1} < \rho. \quad (1.14)$$

Let $\phi \in X_{\rho,1}$ and let $u \in \mathcal{X}_{\rho,1}$ be the unique solution to (1.1) and (1.2) with $u(0) = \phi$. Let $\phi_+ \in X_{\rho,1}$ be the corresponding asymptotic states at $t = \infty$ and define w_n as (1.8). Let $r \geq 2$ with $\delta(r) < (\theta - 1) \wedge 1$ and (q_0, r_0) be an admissible pair. For r and r_0 satisfying $\delta((\alpha + 1)r) \leq \rho$ and $\delta((\alpha + 1)r_0) \leq \rho$, it holds that

$$\|w_1 - V_1\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}), \quad (1.15)$$

$$\|u - w_0 - V_1\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}), \quad (1.16)$$

$$\|w_1 - V_1\|_{L^{q_0}(I; L^{r_0})} = o(T^{1-\theta}), \quad (1.17)$$

$$\|u - w_0 - V_1\|_{L^{q_0}(I; L^{r_0})} = o(T^{1-\theta}), \quad (1.18)$$

where

$$V_1(t) = \frac{it^{1-\theta}}{2^\theta(\theta - 1)} M(t) D(t) |\mathcal{F}\phi_+|^\alpha \mathcal{F}\phi_+.$$

Remark 1.6. There exists $\rho \in [1, (\alpha + 1) \wedge 2)$ so that (1.14) is satisfied in the following cases: $\alpha \in (\gamma(N), \alpha_0(N))$ if $N \leq 6$ and $\alpha \in (\gamma(N), 2/(N - 4))$ if $N = 7, 8$. We also note that (1.15)–(1.18) hold at least for $r = r_0 = 2$ since the condition $\delta((\alpha + 1)2) = N\alpha/2(\alpha + 1) < \rho$ is weaker than (1.14).

FIGURE 1. Admissible value of ρ for Theorem 3 in the case $N = 6$.

Remark 1.7. The condition (1.14) is needed for the estimate of $u - w_0 - w_1$. If (1.14) is satisfied then the $Y^r(T)$ norm and $L^{q_0}(I; L^{r_0})$ norm of $u - w_0 - w_1$ become the same order as (1.16) and (1.18), respectively (see the second part of Proposition 4.1). On the other hand, we need another condition when we estimate $w_1 - V_1$. It is $\delta((\alpha + 1)r) \leq \rho$. In [12], they require $f(\mathcal{F}\phi_+) \in L^r \cap \dot{H}_{r'}^{2\delta(r)+\varepsilon}$ for the same estimate, while here we require only $f(\mathcal{F}\phi_+) \in \dot{H}^{\delta(r)}$. This is an improvement. It is due to Proposition 2.5 (see Remark 2.1). Figure 1 shows the conditions of Theorem 3 in the case $N = 6$. Note that $\gamma(6) = (\sqrt{7} - 1)/3 = 0.548 \dots$ and $\alpha_0(6) = 1$. The solid curve shows the left-hand side of (1.14). Theorem 3 holds for α between this line and $\rho = \alpha + 1$ (dotted curve). What is proved in [12] is the asymptotics (1.18) with $(q_0, r_0) = (\infty, 2)$, $(2/\delta(\alpha + 2), \alpha + 2)$ (and therefore with any admissible pair satisfying $2 \leq r_0 \leq \alpha + 2$). The dash-dotted curve is the graph of $\rho = \delta((\alpha + 1)(\alpha + 2))$. We see that we can choose $\rho \geq \delta((\alpha + 1)(\alpha + 2))$ only in the case $\alpha < \sqrt{3} - 1$. Namely, (1.18) with $(q_0, r_0) = (2/\delta(\alpha + 2), \alpha + 2)$ does not hold if $\alpha \geq \sqrt{3} - 1$. Our Theorem 3 relaxes the regularity condition on data so that the asymptotics (1.18) holds for some $r_0 \geq 2$ beyond $\alpha = \sqrt{3} - 1$.

1.2. Main result 2: the case where α is an even number

By the fact that $|u|^\alpha u = u^a \bar{u}^b$ with $a, b \in \mathbb{N}$, we modify the definition of the $(\alpha + 1)$ -linear operator G as follows:

$$H(u_1, \dots, u_{\alpha+1})(t) = i \int_t^\infty U(t-s) u_1 \cdots u_{\alpha/2+1} \overline{u_{\alpha/2+2}} \cdots \overline{u_{\alpha+1}}(s) ds. \quad (1.19)$$

With this modified $\mathbb{R}$ -multilinear operator H , (1.1) with (1.2) can be written as

$$u(t) = U(t)\phi_+ + H(u, \dots, u). \quad (1.20)$$

We define the function $w_n(t)$ as follows:

$$w_0(t) = U(t)\phi_+, \quad w_n(t) = \sum_{i_1 + \dots + i_{\alpha+1} = n-1} H(w_{i_1}, \dots, w_{i_{\alpha+1}}), \quad (1.21)$$

and also define the function $V_n(t)$ as follows:

$$V_n(t) = \frac{(i)^n}{n!} \cdot \frac{t^{n(1-\theta)}}{2^{n\theta}(\theta-1)^n} M(t)D(t)|\mathcal{F}\phi_+|^{\alpha n} \mathcal{F}\phi_+. \quad (1.22)$$

We use the following notation: $a \asymp b$ holds if there exist positive constants c_1 and c_2 such that $c_1 b \leq a \leq c_2 b$.

THEOREM 4. *Let $N = 1, 2, 3$. Let α be a positive even number satisfying $\gamma(N) < \alpha < \alpha_0(N)$. Put $\theta = N\alpha/2$. Let ρ satisfy $\rho = 1$ if $N = 1$ and $N/2 < \rho < 2$ if $N = 2, 3$. Let $\phi \in X_{\rho,1}$ and let $u \in \mathcal{X}_{\rho,1}$ be the unique solution to (1.1) and (1.2) with $u(0) = \phi$. Let $\phi_+ \in X_{\rho,1}$ be the corresponding scattered state at $t = \infty$ and define w_n and V_n as (1.21) and (1.22), respectively. Then,*

$$\|w_n\|_{\Sigma^\rho(T)} \asymp T^{n(1-\theta)}, \quad (1.23)$$

$$\left\| u - \sum_{k=0}^n w_k \right\|_{\Sigma^\rho(T)} \asymp T^{(n+1)(1-\theta)}, \quad (1.24)$$

and

$$\|V_n\|_{\Sigma^\rho(T)} = CT^{n(1-\theta)}, \quad (1.25)$$

$$\|w_n - V_n\|_{\Sigma^\rho(T)} = o(T^{n(1-\theta)}) \quad (1.26)$$

as $T \rightarrow \infty$. Moreover, if T is sufficiently large then

$$u = \sum_{k=0}^{\infty} w_k \quad \text{in } \Sigma^\rho(T). \quad (1.27)$$

Furthermore, the above asymptotics hold with $\|\cdot\|_{Y^r(T)}$ for any $r \in [2, \infty]$.

Remark 1.8. In Theorem 4, the admissible value of α is $\alpha = 4, 6, 8, \dots$ if $N = 1$, $\alpha = 2, 4, 6, \dots$ if $N = 2$, and $\alpha = 2$ if $N = 3$. This is because $\gamma(1) = (1 + \sqrt{17})/2 \in (2, 3)$, $\gamma(2) = \sqrt{2}$, $\gamma(3) = 1$, $\alpha_0(1) = \alpha_0(2) = \infty$, and $\alpha_0(3) = 4$. We also note that $\alpha_0(N) \leq 2$ for $N \geq 4$. Therefore, there is no even number in $(\gamma(N), \alpha_0(N))$ if $N \geq 4$.

Remark 1.9. The asymptotics in Theorem 4 can be summarized as follows:

$$\begin{array}{ccccccc} u(t) & = & w_0(t) & + & w_1(t) & + & \dots + w_n(t) + \dots \\ & & \wr & & \wr & & \wr \\ & & V_0(t) & & V_1(t) & & V_n(t) \end{array}$$

The true solution u is expanded as the infinite sum of w_n , and each w_n is approximated by V_n . We also note that the sums of V_n do not necessarily yield the approximate solution.

Remark 1.10. Our regularity condition is required for the control of the L^∞ norm of $\mathcal{F}\phi_+$ with the Sobolev embedding. It seems to be natural for higher-order approximations. For example, to obtain the upper bound of $\|V_n; L^r\|$, we need the boundedness of $\|\mathcal{F}\phi_+; L^{r(an+1)}\|$. Hence, it is almost inevitable that $\mathcal{F}\phi_+$ belongs to L^∞ if we are concerned with large n . Then, we assume $\phi_+ \in X_{\rho,1}$ with $\rho > N/2$ which ensures the desired L^∞ boundedness of $\mathcal{F}\phi_+$ and, thanks to this assumption, we can estimate the L^r norm of the solution u , approximate solutions w_n and V_n , and their distance for all $2 \leq r \leq \infty$.

Remark 1.11. The construction of approximate solution w_n is based on the multilinearity of the nonlinearity. It is applicable to the equation of the form $u = w_0 + F(u)$, where F is a multilinear function (see Theorem 3.2). A typical example of the equation of this form is a Cauchy problem for a nonlinear partial differential equation $\partial_t u + Lu + f(u) = 0$ with $u(0, x) = u_0(x)$, where $(t, x) \in \mathbb{R}^{1+N}$, L is a linear (differential) operator, and f is a nonlinear function. With the virtue of the Duhamel principle, this equation is, at least formally, written as

$$u(t) = e^{-tL}u_0 - \int_0^t e^{-(t-s)L} f(u(s)) ds.$$

Examples of multilinear nonlinearity other than $|u|^\alpha u$ with positive even α are the Hartree type nonlinearity $(|x|^{-\gamma} * |u|^2)u$ and $\partial_x(u^2)$ which appears in Korteweg–de Vries (KdV) equations. In [14], the author applies this construction to the semi-classical nonlinear Schrödinger equation with Hartree-type nonlinearity.

The rest of the paper is constructed as follows. In Section 2, we first collect preliminary estimates. Sections 3 and 4 are devoted to the study of approximation by w_n . We first state abstract results of asymptotic expansion in Section 3, and then, applying them, we prove Theorem 1 and some results in Theorem 4. We also prove Theorem 2 in Section 4 without the abstract results. We study the asymptotic behavior of w_n in Sections 5 and 6. In Section 5, we see the derivation of the function V_n by a formal observation, and we finally prove Theorems 3 and the rest of the results in Theorem 4.

2. Preliminary estimates

2.1. Estimates for the proof of Theorems 1 and 2

LEMMA 2.1. *Let $N \geq 1$ and $\alpha > 2/N$. Let $G = G(u_1, \dots, u_{[\alpha]+1})$ be the $([\alpha] + 1)$ -linear operator defined by (1.6). Put $\theta = N\alpha/2$ and let $r \in [2, \infty]$ satisfy $0 \leq \delta(r) < 1$ and $\delta(r) < \theta - 1$. It holds that*

$$\|G\|_{Y^r(T)} \leq CT^{1-\theta+\delta(r)} \left(\prod_{k=1}^{[\alpha]} \| |u_k|^{1/\beta} \|_{Y^{s_k}(T)}^\beta \right) \|u_{[\alpha]+1}\|_{Y^{s_{[\alpha]+1}}(T)}, \quad (2.1)$$

where $s_k \in [2, \infty]$ ($k = 1, \dots, [\alpha] + 1$) satisfy

$$\beta \sum_{k=1}^{[\alpha]} \frac{1}{s_k} + \frac{1}{s_{[\alpha]+1}} + \frac{1}{r} = 1. \quad (2.2)$$

Proof. From the L^p – L^q estimate and the Hölder inequality, we obtain

$$\begin{aligned} \|G\|_{L^r} &\leq C \int_t^\infty |t-s|^{-\delta(r)} \left\| \prod_{k=1}^{\lceil \alpha \rceil + 1} u_k(s) \right\|_{L^{r'}} ds \\ &\leq C \int_t^\infty |t-s|^{-\delta(r)} \left(\prod_{k=1}^{\lceil \alpha \rceil} \| |u_k(s)|^{1/\beta} \|_{L^{s_k}}^\beta \right) \|u_{\lceil \alpha \rceil + 1}(s)\|_{L^{s_{\lceil \alpha \rceil + 1}}} ds, \end{aligned}$$

where $r \in [2, \infty]$ and s_k satisfy (2.2). By the definition of $Y^r(T)$, the right-hand side is bounded by

$$C \left(\prod_{k=1}^{\lceil \alpha \rceil} \| |u_k|^{1/\beta} \|_{Y^{s_k}(T)}^\beta \right) \|u_{\lceil \alpha \rceil + 1}\|_{Y^{s_{\lceil \alpha \rceil + 1}}(T)} \int_t^\infty |t-s|^{-\delta(r)} s^{-d}$$

for any $t \geq T$, where

$$\begin{aligned} d &= \beta \sum_{k=1}^{\lceil \alpha \rceil} \delta(s_k) + \delta(s_{\lceil \alpha \rceil + 1}) = \frac{N(\alpha + 2)}{2} - N \left(\beta \sum_{k=1}^{\lceil \alpha \rceil} \frac{1}{s_k} - \frac{1}{s_{\lceil \alpha \rceil + 1}} + \frac{1}{r} \right) - \delta(r) \\ &= \theta - \delta(r) \end{aligned}$$

by the definition of $\delta(s_k)$ and $\delta(r)$. The time integral is equal to $Ct^{1-\theta}$ with some positive constant C if $\delta(r) \in [0, 1)$ and $\delta(r) < \theta - 1$. Again by the definition of the space $Y^r(T)$ we obtain the result. $\square$

LEMMA 2.2. *Let $N \geq 1$ and $\alpha > 2/N$. Let $G = G(u_1, \dots, u_{\lceil \alpha \rceil + 1})$ be the $(\lceil \alpha \rceil + 1)$ -linear operator defined by (1.6). Put $\theta = N\alpha/2$. Let (q_0, r_0) be an admissible pair. Put $\theta = N\alpha/2$ and $I = [T, \infty)$. Suppose that $r \in [2, 2N/(N-2)]$ ($r \in [2, \infty]$ if $N = 1$, $r \in [2, \infty]$ if $N = 2$) and $s_k \in [2, \infty]$ satisfy (2.2). Then, it holds that*

$$\|G\|_{L^{q_0}(I; L^{r_0})} \leq CT^{1-\theta+\delta(r)/2} \left(\prod_{k=1}^{\lceil \alpha \rceil} \| |u_k|^{1/\beta} \|_{Y^{s_k}(T)}^\beta \right) \|u_{\lceil \alpha \rceil + 1}\|_{Y^{s_{\lceil \alpha \rceil + 1}}(T)} \quad (2.3)$$

if $\delta(r) < 2(\theta - 1)$. Moreover, for a subset $K \subset \{1, 2, \dots, \lceil \alpha \rceil\}$ and $d_K = \beta \sum_{k \in K} \delta(s_k)/2$, it holds that

$$\begin{aligned} \|G\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{1-\theta+\delta(r)/2+d_K} \left(\prod_{k \in K} \| |u_k|^{1/\beta} \|_{L^{2/\delta(s_k)}(I; L^{s_k})}^\beta \right) \\ &\quad \times \left(\prod_{k \notin K} \| |u_k|^{1/\beta} \|_{Y^{s_k}(T)}^\beta \right) \|u_{\lceil \alpha \rceil + 1}\|_{Y^{s_{\lceil \alpha \rceil + 1}}(T)} \end{aligned} \quad (2.4)$$

if $\delta(r)/2 + d_K < \theta - 1$, and that

$$\begin{aligned} \|G\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{1-\theta+\delta(r)/2+d_K+\delta(\lceil \alpha \rceil + 1)/2} \left(\prod_{k \in K} \| |u_k|^{1/\beta} \|_{L^{2/\delta(s_k)}(I; L^{s_k})}^\beta \right) \\ &\quad \times \left(\prod_{k \notin K} \| |u_k|^{1/\beta} \|_{Y^{s_k}(T)}^\beta \right) \|u_{\lceil \alpha \rceil + 1}\|_{L^{2/\delta(s_{\lceil \alpha \rceil + 1})}(I; L^{s_{\lceil \alpha \rceil + 1}})} \end{aligned} \quad (2.5)$$

if $\delta(r)/2 + d_K + \delta(s_{\lceil \alpha \rceil + 1})/2 < \theta - 1$.

This lemma is shown in a similar way to the previous lemma. We use the Strichartz estimate instead of the L^p – L^q estimate.

2.2. Estimates for the proof of Theorem 4

In this section, we assume that α is an even integer.

LEMMA 2.3. *Let $N = 1, 2, 3$. Let $\alpha \in (\gamma(N), \alpha_0(N))$ be an even integer. Let $H = H(u_1, \dots, u_{\alpha+1})$ be the $(\alpha + 1)$ -linear operator defined by (1.19). Let $\rho = 1$ if $N = 1$ and $\rho \in (N/2, 2)$ if $N = 2, 3$. Then, it holds for all $T > 0$ that*

$$\|H\|_{\Sigma^\rho(T)} \leq C_1 T^{1-\theta} \prod_{k=1}^{\alpha+1} \|u_k\|_{\Sigma^\rho(T)}, \quad (2.6)$$

where $\theta = N\alpha/2$.

Proof. By the definition of H , we calculate

$$\begin{aligned} D^{-1}(t)M^{-1}(t)H(t) \\ = iD^{-1}(t)M^{-1}(t) \int_t^\infty U(t-s)u_1 \cdots u_{\alpha/2+1} \overline{u_{\alpha/2+2}} \cdots \overline{u_{\alpha+1}}(s) ds. \end{aligned}$$

Let us note that $D^{-1}(t)M^{-1}(t)U(t-s) = U(1/4s - 1/4t)D^{-1}(s)M^{-1}(s)$. Then, it holds that

$$\|D^{-1}(t)M^{-1}(t)H(t)\|_{H^\rho} \leq C \int_t^\infty |2s|^{-N\alpha/2} \prod_{k=1}^{\alpha+1} \|D^{-1}(s)M^{-1}(s)u_k\|_{H^\rho} ds,$$

where we use the fact that the nonlinearity is Gauge-invariant and that the H^ρ is an algebra for $\rho > N/2$. Taking the L^∞ norm in time, we obtain the result. $\square$

2.3. Estimates for the approximation of w_n

Next we show the error estimates which are used when we estimate $w_n - V_n$. We use these results for the proof of Theorems 3 and 4.

LEMMA 2.4. *Let $U(t) = e^{it\Delta}$ and $r \in [2, \infty)$. There exists a real-valued continuous function h with $h(0) = 0$ such that*

$$\|(U(t) - 1)\phi\|_{L^r} \leq Ch(t)$$

if $\phi \in \dot{H}^{\delta(r)}$. Moreover, it holds for any $\phi \in H^s$ with $s > N/2$ that

$$\|(U(t) - 1)\phi\|_{L^\infty} \leq Ch(t).$$

Proof. The first estimate with $r = 2$ is clear because $\|(U(t) - 1)f\|_{L^2}$ is bounded by $2\|f\|_{L^2}$ and tends to zero as $t \rightarrow 0$. Since $U(t) - 1$ commutes with both $(-\Delta)^{s/2}$ and $(1 - \Delta)^{s/2}$, the same argument shows, for any $s \in \mathbb{R}$, that the norms $\|(U(t) - 1)\phi\|_{\dot{H}^s}$ and $\|(U(t) - 1)\phi\|_{H^s}$ tend to zero if ϕ belongs to $\dot{H}^s$ and H^s , respectively. Then, the results follow from the embedding $\dot{H}^{\delta(r)} \hookrightarrow L^r$ ($r \in [2, \infty)$) and $H^s \hookrightarrow L^\infty$ ($s > N/2$). $\square$

Remark 2.1. In [12], they use the estimate

$$\|(U(t) - 1)\psi\|_{L^r} \leq \begin{cases} Ct^{\delta(r)} \|\psi\|_{\dot{H}_{r'}^{4\delta(r)}}, & \text{if } N = 1, \\ Ct^{\varepsilon\delta(r)/(1-\varepsilon)} \|\psi\|_{\dot{H}_{r'}^{2\delta(r)/(1-\varepsilon)}}, & \text{if } N \geq 2, \end{cases}$$

where $\varepsilon > 0$ and r satisfies $0 \leq \delta(r) \leq 1 - \varepsilon$. The estimates in Lemma 2.4 are improvements of them and relax the regularity conditions on data.

For a real number $\sigma \geq 1$ and a spacetime function ϕ_+ , we define the *error term* E_σ as

$$E_\sigma(t) := M(t)D(t)i \int_t^\infty (U(1/4s - 1/4t) - 1)s^{\sigma(1-\theta)-1} |\mathcal{F}\phi_+|^{\alpha\sigma} \mathcal{F}\phi_+ ds. \quad (2.7)$$

PROPOSITION 2.5. *Let $N \geq 1$, $\alpha > 2/N$, and $\sigma \geq 1$. Let E_σ be defined by (2.7). Let $\mathcal{F}\phi_+ \in H^\rho$ for some ρ with $1 \leq \rho < (\alpha + 1) \wedge 2$. Let $r \in [2, \infty]$. Suppose that ρ satisfy*

$$\rho \geq \delta(r(\alpha\sigma + 1)) \quad \text{if } r < \infty, \quad \rho > \frac{N}{2} \quad \text{if } r = \infty.$$

Then, there exists a continuous function $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $h(0) = 0$ such that

$$\|E_\sigma\|_{Y^r(T)} \leq h(1/T)T^{\sigma(1-\theta)} \|\mathcal{F}\phi_+\|_{H^\rho}^{1+\alpha\sigma}, \quad (2.8)$$

$$\|E_\sigma\|_{L^q(I; L^r)} \leq h(1/T)T^{\sigma(1-\theta)-\delta(r)/2} \|\mathcal{F}\phi_+\|_{H^\rho}^{1+\alpha\sigma}, \quad (2.9)$$

where $q = 2/\delta(r)$ and $I = [T, \infty)$.

Proof. By the definition of E_σ , we have

$$\|E_\sigma(t)\|_{L^r} \leq Ct^{-\delta(r)} \int_t^\infty s^{\sigma(1-\theta)-1} \|\mathcal{U}(t, s)|\mathcal{F}\phi_+|^{\alpha\sigma} \mathcal{F}\phi_+\|_{L^r} ds,$$

where $\mathcal{U}(t, s) := U(1/4s - 1/4t) - 1$. If the L^r norm in the right-hand side is bounded by $h(1/s)\|\mathcal{F}\phi_+\|_{H^\rho}^{1+\alpha\sigma}$ for some continuous function h with $h(0) = 0$, we obtain the result. Hence, we prove this. Applying Lemma 2.4, we obtain

$$\|\mathcal{U}(t, s)|\mathcal{F}\phi_+|^{\alpha\sigma} \mathcal{F}\phi_+\|_{L^r} \leq h((1/4s - 1/4t)) \|\mathcal{F}\phi_+|^{\alpha\sigma} \mathcal{F}\phi_+\|_{\dot{H}^{\delta(r)}}$$

for $N \geq 1$ and $r \in [2, \infty)$. Moreover, we use the estimate

$$\begin{aligned} \|\mathcal{F}\phi_+|^{\alpha\sigma} \mathcal{F}\phi_+\|_{\dot{H}^{\delta(r)}} &\leq C \|\mathcal{F}\phi_+|^{\alpha\sigma} \mathcal{F}\phi_+\|_{B_2^{\delta(r)}} \leq C \|\mathcal{F}\phi_+\|_{L^p}^{\alpha\sigma} \|\mathcal{F}\phi_+\|_{B_q^{\delta(r)}} \\ &\leq C \|\mathcal{F}\phi_+\|_{\dot{H}^{\delta(p)}}^{\alpha\sigma} \|\mathcal{F}\phi_+\|_{H^{\delta(r)+\delta(q)}} \end{aligned}$$

for any $p, q \geq 2$ satisfying $\alpha\sigma/p + 1/q = 1/2$, that is, $\alpha\sigma\delta(p) + \delta(q) = N\alpha\sigma/2$, where we have used the usual embeddings between Besov and Triebel–Lizorkin spaces and nonlinear estimates in homogeneous Besov spaces [4]. We now choose p so that $\delta(p) = \delta(r) + \delta(q)$. Then,

$$\delta(p) = \frac{N\alpha\sigma}{2(\alpha\sigma + 1)} + \frac{\delta(r)}{\alpha\sigma + 1} = \delta(r(\alpha\sigma + 1))$$

since $\alpha\sigma\delta(p) + \delta(q) = N\alpha\sigma/2$. Thus, if $\delta(p) \leq \rho$ then we conclude that

$$\|\mathcal{F}\phi_+\|_{\dot{H}^{\delta(p)}}^{\alpha\sigma} \|\mathcal{F}\phi_+\|_{H^{\delta(r)+\delta(q)}} \leq \|\mathcal{F}\phi_+\|_{H^\rho}^{\alpha\sigma+1}.$$

Similarly, if $\rho > N/2$ then we have

$$\begin{aligned} \|\mathcal{U}(t, s)|\mathcal{F}\phi_+|^{\alpha\sigma} \mathcal{F}\phi_+\|_{L^\infty} &\leq Ch(1/4s - 1/4t) \|\mathcal{F}\phi_+|^{\alpha\sigma} \mathcal{F}\phi_+\|_{H^\rho} \\ &\leq Ch(1/4s - 1/4t) \|\mathcal{F}\phi_+\|_{H^\rho}^{\alpha\sigma+1}, \end{aligned}$$

which completes the proof. $\square$

PROPOSITION 2.6. *Let $N = 1, 2, 3$, $\alpha > 2/N$ be an even integer, and n be a non-negative integer. Let E_n be defined by (2.7) with $\sigma = n$. Let $\mathcal{F}\phi_+ \in H^\rho$ for some ρ with $N/2 < \rho < 2$. Then, there exists a continuous function $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $h(0) = 0$ such that*

$$\|E_n\|_{\Sigma^\rho(T)} \leq Ch(1/T)T^{n(1-\theta)}\|\mathcal{F}\phi_+\|_{H^\rho}^{\alpha n+1}.$$

Proof. By the definition of E_n , we deduce that

$$\begin{aligned} \|D^{-1}(t)M(-t)E_n(t)\|_{H^\rho} &\leq \int_t^\infty \|\mathcal{U}(t, s)s^{n(1-\theta)-1}|\mathcal{F}\phi_+|^{\alpha n}\mathcal{F}\phi_+\|_{H^\rho} ds \\ &\leq Ch(1/4t)t^{n(1-\theta)}\| |\mathcal{F}\phi_+|^{\alpha n}\mathcal{F}\phi_+ \|_{H^\rho} \\ &\leq Ch(1/4t)t^{n(1-\theta)}\|\mathcal{F}\phi_+\|_{H^\rho}^{\alpha n+1}. \end{aligned}$$

□

3. Abstract results

In this section, we consider an abstract semilinear equation with a k -linear nonlinearity for some $k \geq 2$. Specifically, we consider the abstract equation

$$u = L(f) + N_k(u, \dots, u), \quad (3.1)$$

where the initial data f take values in some data space D , the solution u takes values in some solution space S , the linear operator $L : D \rightarrow S$ is densely defined, and the k -linear operator $N_k : S \times \dots \times S \rightarrow S$ is also densely defined.

ASSUMPTION 3.1. *Let $(D, \|\cdot\|_D)$ be a Banach space of initial data, and $(S, \|\cdot\|_S)$ be a Banach space of spacetime functions. The operators L and N satisfy*

$$\|L(f)\|_S \leq C_0\|f\|_D$$

and

$$\|N_k(u_1, \dots, u_k)\|_S \leq \varepsilon \prod_{l=1}^k \|u_l\|_S$$

with positive constants C_0 and ε .

This assumption is the same as the ‘quantitative well-posedness’ introduced by Bejenaru and Tao. For the well-posedness result of (3.1), see [1]. We adopt the well-posedness result into the approximation of the solution to (3.1).

THEOREM 3.2. *Assume that (3.1) satisfies Assumption 3.1. Let u be a solution of (3.1) with $\|u\|_S \leq C_1\|f\|_D$ with some constant C_1 . If we define the approximate solution w_n by the recursive formula*

$$w_0 := L(f), \quad w_n := \sum_{\substack{i_1, \dots, i_k \geq 0 \\ i_1 + \dots + i_k = n-1}} N_k(w_{i_1}, \dots, w_{i_k}),$$

then we have the following boundedness and asymptotics:

$$\|w_n\|_S \leq I(n, k)\varepsilon^n(C_0\|f\|_D)^{n(k-1)} \quad (3.2)$$

$$\left\| u - \sum_{l=0}^n w_l \right\|_S \leq I(n+1, k)\varepsilon^{n+1}((C_0 \vee C_1)\|f\|_D)^{(n+1)(k-1)}, \quad (3.3)$$

where $I(n, k)$ is the generalized Catalan number defined by

$$I(n, k) = \frac{(kn)!}{(n(k-1)+1)!n!}. \quad (3.4)$$

Moreover, if ε is sufficiently small then w_n gives the absolutely convergent expansion

$$u = \sum_{n=0}^{\infty} w_n \quad \text{in } S.$$

We illustrate the proof below, because we quote some part of it later.

Proof. Let $I(n, k)$ be a sequence which satisfies $I(0, k) = 1$ and

$$I(n, k) = \sum_{i_1+\dots+i_k=n-1} \prod_{m=1}^k I(i_m, k). \quad (3.5)$$

We first prove the boundedness of w_n by induction. If (3.2) holds for $n = 0, 1, \dots, n_0 - 1$, then

$$\begin{aligned} \|w_{n_0}\|_S &= \left\| \sum_{i_1+\dots+i_k=n_0-1} N_k(w_{i_1}, \dots, w_{i_k}) \right\|_S \\ &\leq \varepsilon^{n_0} (C_0 \|f\|_D)^{n(k-1)} \sum_{i_1+\dots+i_k=n_0-1} \prod_{m=1}^k I(i_m, k) \\ &= I(n_0, k) \varepsilon^{n_0} (C_0 \|f\|_D)^{n_0(k-1)}. \end{aligned} \quad (3.6)$$

Therefore, it holds for $n = n_0$, and for all $n \geq 0$. We next prove the asymptotics (3.3). An elementary computation shows that

$$\begin{aligned} u - \sum_{l=0}^n w_l &= N_k(u, \dots, u) - \sum_{l=1}^n w_l \\ &= \sum_{m=1}^k \sum_{i_m+\dots+i_k=n-1} N_k\left(u, \dots, u, u - \sum_{l=0}^{i_m} w_l, w_{i_{m+1}}, \dots, w_{i_k}\right). \end{aligned} \quad (3.7)$$

Now, we consider the replacement $u \leftrightarrow w_0$ and $u - \sum_{j=0}^n w_j \leftrightarrow w_{n+1}$. Then the right-hand side is replaced by

$$\begin{aligned} &\sum_{m=1}^k \sum_{\substack{i_1=\dots=i_{m-1}=0, \\ i_m+\dots+i_k=n-1}} N_k(w_{i_1}, \dots, w_{i_{m-1}}, w_{i_m+1}, w_{i_{m+1}}, \dots, w_{i_k}) \\ &= \sum_{i_1+\dots+i_k=n} N_k(w_{i_1}, \dots, w_{i_k}) = w_{n+1}. \end{aligned}$$

Since u has the upper bound estimate $\|u\|_S \leq C_1 \|f\|_D$ which is the same as for w_0 up to a constant, we infer inductively that the upper bound estimates for $u - \sum_{l=0}^n w_l$ and for w_{n+1} are also the same up to a constant.

The convergence of $\sum_{n=0}^{\infty} w_n$ is nothing but the well-posedness result. In fact, conversely, the solution u is defined by $\sum_{n=0}^{\infty} w_n$. We now give a slightly different proof. It is known that $I(n, k)$, defined by (3.5), is the generalized Catalan number, and that $I(n, k) = (nk)!/(n(k-1)+1)!n!$ (see [9]). Therefore, we conclude that $I(n, k) \sim (k^k/(k-1)^{k-1})^n$ as $n \rightarrow \infty$. $\square$

When we try to apply this theorem to the Cauchy problem of the nonlinear Schrödinger equation, we need the assumption that α is an even number. Indeed, (1.1) with (1.2) can be written as

$$u(t) = U(t)\phi - i \int_0^t U(t-s)(|u|^\alpha u)(s) ds.$$

Moreover, if the scattered state $\phi_+ := \lim_{t \rightarrow \infty} U(-t)u(t)$ exists then this equation is also written as

$$u(t) = U(t)\phi_+ + i \int_t^\infty U(t-s)(|u|^\alpha u)(s) ds.$$

If α is an even integer then $|u|^\alpha u = u^{\alpha/2+1} \bar{u}^{\alpha/2}$ is $(\alpha+1)$ -linear, and so this equation can be written as

$$u = L(\phi) + N_{\alpha+1}(u, \dots, u).$$

Then, setting suitable data space D and solution space S , we can obtain the asymptotics of the solution from Theorem 3.2.

We want to extend this argument to the general case where α is not an even integer. However, if α is not an even integer, then the nonlinearity $|u|^\alpha u$ is no longer multilinear with respect to u . Hence, we make some modifications. Let $\lceil \alpha \rceil$ be the minimum integer larger than or equal to α , and let $\beta := \alpha / \lceil \alpha \rceil$. We split $|u|^\alpha u$ into $(|u|^\beta)^{\lceil \alpha \rceil} u$ and regard it as the product of one term of u and $\lceil \alpha \rceil$ terms of $|u|^\beta$. Thus, the equation which we consider becomes the following type:

$$u = L(\phi) + N_{\lceil \alpha \rceil+1}(|u|^\beta, \dots, |u|^\beta, u). \quad (3.8)$$

It is important that $\beta \leq 1$, because we use the following estimate: if $0 < b \leq 1$, then $|z_1|^b - |z_2|^b \leq |z_1 - z_2|^b$, for all $z_1, z_2 \in \mathbb{C}$. This inequality is the key for this modification.

Before treating (3.8), we consider the abstract equation

$$u = L(f) + N_k(|u|^b, \dots, |u|^b), \quad (3.9)$$

where $k \geq 2$, $0 < b \leq 1$, and the solution space S is a space of complex-valued spacetime functions. Here, we take k and b independently. We note that (3.8) can be regarded as a mixture type of (3.1) and (3.9).

ASSUMPTION 3.3. *Let $(D, \|\cdot\|_D)$ be a Banach space of initial data, and $(S, \|\cdot\|_S)$ be a Banach space of complex-valued spacetime functions. Moreover, the norm $\|\cdot\|_S$ is monotone, that is, if $|f| \leq |g|$ then $\|f\|_S \leq \|g\|_S$. The operator L satisfies*

$$\|L(f)\|_S \leq C_0 \|f\|_D$$

for some positive constant C_0 . There exists a constant ε such that

$$\|N_k(u_1, \dots, u_k)\|_S \leq \varepsilon \prod_{l=1}^k \| |u_l|^{1/b} \|_S^b.$$

THEOREM 3.4. *Suppose that (3.9) satisfies Assumption 3.3 with the same b . Assume further that the constant ε given by Assumption 3.3 is sufficiently small. Let u be a solution of (3.9) with $\|u\|_S \leq C_1 \|f\|_D$ with some constant C_1 . If we define the approximate solution w_n by*

the recursive formula

$$w_0 := L(f), \quad w_n := \sum_{\substack{i_1, \dots, i_k \geq 0 \\ i_1 + \dots + i_k = n-1}} N_k(\tilde{w}_{i_1}, \dots, \tilde{w}_{i_k}),$$

where $\tilde{w}_0 = |w_0|^b$ and $\tilde{w}_n = |\sum_{l=0}^n w_l|^b - |\sum_{l=0}^{n-1} w_l|^b$, then we have the following boundedness and asymptotics:

$$\|w_n\|_S \leq I_b(n, k) \varepsilon^{e(n)} (C_0 \|f\|_D)^{(k-1)e(n)} \quad (3.10)$$

$$\left\| u - \sum_{l=0}^n w_l \right\|_S \leq I_b(n+1, k) \varepsilon^{e(n+1)} ((C_0 \vee C_1) \|f\|_D)^{(k-1)e(n+1)}, \quad (3.11)$$

where $e(n) = 1 + b + \dots + b^{n-1}$ and $I_b(n, k)$ is defined by

$$I_b(0, k) = 1, \quad I_b(n, k) = \sum_{i_1 + \dots + i_k = n-1} \prod_{l=1}^k (I_b(i_l, k))^b. \quad (3.12)$$

Furthermore, if $b = 1$ then w_n gives the absolutely convergent expansion

$$u = \sum_{n=0}^{\infty} w_n \quad \text{in } S.$$

Proof. By the definition of $\tilde{w}_n$, we deduce that

$$\begin{aligned} |\tilde{w}_n| &\leq \left| \sum_{l=0}^n w_l \right|^b - \left| \sum_{l=0}^{n-1} w_l \right|^b \leq |w_n|^b, \\ \left| |u|^b - \sum_{l=0}^n \tilde{w}_l \right| &= \left| |u|^b - \left| \sum_{l=0}^n w_l \right|^b \right| \leq \left| u - \sum_{l=0}^n w_l \right|^b, \end{aligned}$$

and so that

$$\|\tilde{w}_n\|_S \leq \|w_n\|_S^b, \quad (3.13)$$

$$\left\| u - \sum_{l=0}^n \tilde{w}_l \right\|_S \leq \left\| u - \sum_{l=0}^n w_l \right\|_S^b \quad (3.14)$$

by the monotonicity of $\|\cdot\|_S$. Therefore, if (3.10) holds for $n = 0, 1, \dots, n_0 - 1$ then the calculation similar to (3.6) shows that

$$\begin{aligned} \|w_{n_0}\|_S &= \left\| \sum_{i_1 + \dots + i_k = n_0-1} N_k(\tilde{w}_{i_1}, \dots, \tilde{w}_{i_k}) \right\|_S \\ &\leq \varepsilon \sum_{i_1 + \dots + i_k = n_0-1} \prod_{m=1}^k (I_b(i_m, k) \varepsilon^{e(i_m)} (C_0 \|f\|_D)^{(k-1)e(i_m)})^b \\ &= I_b(n_0, k) \varepsilon^{e(n_0)} (C_0 \|f\|_D)^{(k-1)e(n_0)}, \end{aligned} \quad (3.15)$$

since $\sum_{m=1}^k be(i_m) \geq be(n_0 - 1) + (k - 1)be(0) = e(n_0)$.

Since N_k is k -linear, the same computation as (3.7) is possible, and so

$$\begin{aligned} u - \sum_{l=0}^n w_l &= N_k(|u|^b, \dots, |u|^b) - \sum_{l=1}^n w_l \\ &= \sum_{m=1}^k \sum_{i_m + \dots + i_k = n-1} N_k \left(|u|^b, \dots, |u|^b, |u|^b - \sum_{l=0}^{i_m} \tilde{w}_l, \tilde{w}_{i_m+1}, \dots, \tilde{w}_{i_k} \right). \end{aligned} \quad (3.16)$$

Hence, by replacing $|u|^b$ and $|u|^b - \sum_{l=0}^n \tilde{w}_l$ by $\tilde{w}_0$ and $\tilde{w}_{n+1}$, respectively, and using the estimate (3.13) instead of (3.14), we see that the upper bound estimates for $u - \sum_{l=0}^n w_l$ and w_{n+1} is the same up to a constant. The convergence result is clear since $I_b(n, k) = I(n, k)$ and $e(n) = n$ if $b = 1$. $\square$

Finally we treat the mixture-type equation (3.8). Let us consider the equation

$$u = L(f) + N_{k+1}(|u|^b, \dots, |u|^b, u). \quad (3.17)$$

We suppose that k and b are independent, which generalizes (3.8) slightly.

ASSUMPTION 3.5. *We assume that D , S , and L satisfy the same assumption as in Assumption 3.3. Moreover, there exists a constant ε such that*

$$\|N_{k+1}(u_1, \dots, u_{k+1})\|_S \leq \varepsilon \left(\prod_{l=1}^k \|u_l\|_S^{1/b} \right) \|u_{k+1}\|_S.$$

THEOREM 3.6. *Suppose that (3.17) satisfies Assumption 3.5 with the same b . Assume further that the constant ε given by Assumption 3.5 is sufficiently small. Let u be a solution of (3.17) with $\|u\|_S \leq C_1 \|f\|_D$ with some constant C_1 . If we define the approximate solution w_n by the recursive formula*

$$w_0 := L(f), \quad w_n := \sum_{\substack{i_1, \dots, i_{k+1} \geq 0 \\ i_1 + \dots + i_{k+1} = n-1}} N_{k+1}(\tilde{w}_{i_1}, \dots, \tilde{w}_{i_k}, w_{i_{k+1}}),$$

then we have the following boundedness and asymptotics:

$$\|w_n\|_S \leq I'_b(n, k+1) \varepsilon^{e(n)} (C_0 \|f\|_D)^{ke(n)} \quad (3.18)$$

$$\left\| u - \sum_{l=0}^n w_l \right\|_S \leq I'_b(n+1, k+1) \varepsilon^{e(n+1)} ((C_0 \vee C_1) \|f\|_D)^{ke(n+1)}, \quad (3.19)$$

where $e(n) = 1 + b + \dots + b^{n-1}$ and $I'_b(n, k+1)$ is defined by

$$\begin{aligned} I'_b(0, k+1) &= 1, \\ I'_b(n, k+1) &= \sum_{i_1 + \dots + i_{k+1} = n-1} \left(\prod_{l=1}^k (I'_b(i_l, k+1))^b \right) I'_b(i_{k+1}, k+1). \end{aligned}$$

Furthermore, if $b = 1$, then w_n gives the absolutely convergent expansion

$$u = \sum_{n=0}^{\infty} w_n \quad \text{in } S.$$

Proof. It suffices to mention that

$$\varepsilon^{e(n)}(C_0\|f\|_D)^{ke(n)} \leq 1 \quad \text{and} \quad \varepsilon^{e(n+1)}((C_0 \vee C_1)\|f\|_D)^{ke(n+1)} \leq 1$$

if ε is small. Then, using the inequality $x \leq x^b$ for $x \in [0, 1]$ and $b \in (0, 1]$, we can repeat the same computation as in Theorem 3.4. $\square$

Remark 3.1. There are several ways to split the nonlinearity $|u|^\alpha u$ into a product $u^{k_1} \bar{u}^{k_2} (\prod_i |u|^{b_i})$ with $b_i \leq 1$ and $k_1, k_2 \in \mathbb{N} \cup \{0\}$. However, the best way is $|u|^\alpha u = (|u|^\beta)^{[\alpha]} u$ since the same argument as in the proof of Theorem 3.6 shows that the accuracy of the asymptotics depends on the infimum of b_i .

4. Proofs of Theorems 1, 2, and 4

In this section we apply the abstract result in the previous section and prove Theorem 1 and the some results in Theorem 4. Speaking in detail, Theorem 1 follows from Theorem 3.6, and the upper bound estimates and the convergence result in Theorem 4 follows from Theorem 3.2. Theorem 2 is shown directly (without the abstract result).

4.1. The proof of Theorem 1

Proof of Theorem 1. We first note that (1.7) is of the form (3.17). We set $D = X_{\rho,1}$. All we have to do is to choose a suitable solution space S so that Assumption 3.5 holds. We first let $S = Y^2(T) \cap Y^{\alpha+2}(T)$. It follows from (2.1) and (2.3) that

$$\|G\|_{Y^{\alpha+2}(T)} \leq CT^{1-\theta+\delta(\alpha+2)} \left(\prod_{k=1}^{[\alpha]} \| |u_k|^{1/\beta} \|_{Y^{\alpha+2}(T)}^\beta \right) \|u_{[\alpha]+1}\|_{Y^{\alpha+2}(T)}$$

and

$$\|G\|_{L^\infty(I; L^2)} \leq CT^{1-\theta+\delta(\alpha+2)/2} \left(\prod_{k=1}^{[\alpha]} \| |u_k|^{1/\beta} \|_{Y^{\alpha+2}(T)}^\beta \right) \|u_{[\alpha]+1}\|_{Y^{\alpha+2}(T)},$$

respectively. Therefore, we see from $Y^2(T) = L^\infty(I; L^2)$ that

$$\|G\|_S \leq CT^{1-\theta+\delta(\alpha+2)} \left(\prod_{k=1}^{[\alpha]} \| |u_k|^{1/\beta} \|_S^\beta \right) \|u_{[\alpha]+1}\|_S.$$

Then, applying Theorem 3.6 and using $Y^2(T) \cap Y^{\alpha+2}(T) \hookrightarrow Y^r(T)$ for $r \in [2, \alpha+2]$, we obtain the result for $r \in [2, \alpha+2]$.

Let us proceed to the case $r > \alpha+2$, we let S be

$$Z^r(T) = \bigcap_{(\alpha+1)r/(r-1) \leq p \leq r} Y^p(T).$$

We apply (2.1) with $s_1 = \cdots = s_{[\alpha]+1} = (\alpha+1)r/(r-1)$ and $r = r$, and also with $s_1 = \cdots = s_{[\alpha]+1} = (\alpha+1)^2 r/(\alpha r + 1)$ and $r = (\alpha+1)r/(r-1)$. Then, we conclude that

$$\|G\|_S \leq CT^{1-\theta+\delta(r)} \left(\prod_{k=1}^{[\alpha]} \| |u_k|^{1/\beta} \|_S^\beta \right) \|u_{[\alpha]+1}\|_S.$$

Therefore, Theorem 3.6 provides the result. The results for $L^{q_0}(I; L^{r_0})$ norm are immediate consequences of the results for $Y^r(T)$ norm by virtue of (2.3). $\square$

4.2. The upper bound estimates in Theorem 4

Proof of Theorem 4. Equation (1.20) is of the form (3.1). We let $D = X_{\rho,1}$ and $S = \Sigma^\rho(T)$. Lemma 2.3 tells us that Assumption 3.1 holds. Then, Theorem 3.2 shows that

$$\|w_n\|_{\Sigma^\rho(T)} \leq CT^{n(1-\theta)}, \quad \left\| u - \sum_{k=0}^n w_k \right\|_{\Sigma^\rho(T)} \leq CT^{(n+1)(1-\theta)},$$

and

$$u = \sum_{n=0}^{\infty} w_n \quad \text{in } \Sigma^\rho(T).$$

By the embedding $\Sigma^\rho(T) \hookrightarrow Y^r(T)$, we see that these results also hold with $\|\cdot\|_{Y^r(T)}$ for all $r \in [2, \infty]$. $\square$

4.3. The proof of Theorem 2

We refine the estimate of w_n and of $u - \sum_{k=0}^n w_k$ for $n = 0, 1, 2$ by using the property $\mathcal{F}\phi_+ \in H^\rho$. We prove the following proposition which contains Theorem 2.

PROPOSITION 4.1. *Let $N \geq 1$ and $\alpha \in (\gamma(N), \alpha_0(N))$. Put $\theta = \alpha N/2$. Let $T \gg 1$ and $I = [T, \infty)$. Let ρ satisfy $1 \leq \rho < (\alpha + 1) \wedge 2$. Then, it holds for all $r \in [2, \infty)$ with $\delta(r) \leq \rho$ ($r \in [2, \infty]$ if $\rho > N/2$) and admissible pair (q_0, r_0) that*

$$\|w_0\|_{Y^r(T)} \leq C, \quad \|w_0\|_{L^{q_0}(I; L^{r_0})} \leq C, \quad \|u\|_{Y^r(T)} \leq C, \quad \|u\|_{L^{q_0}(I; L^{r_0})} \leq C.$$

In the following, let $r \in [2, \infty]$ satisfy $\delta(r) < (\theta - 1) \wedge 1$.

(1) Assume further that $N \leq 10$ and $N\alpha/2(\alpha + 1) \leq \rho$ then

$$\begin{aligned} \|w_1\|_{Y^r(T)} &\leq CT^{1-\theta+\delta(r)}, \quad \|w_1\|_{L^{q_0}(I; L^{r_0})} \leq CT^{1-\theta}, \\ \|u - w_0\|_{Y^r(T)} &\leq CT^{1-\theta+\delta(r)}, \quad \|u - w_0\|_{L^{q_0}(I; L^{r_0})} \leq CT^{1-\theta}. \end{aligned}$$

(2) Assume further that $N \leq 8$ and

$$\frac{\theta(1-\beta) + \beta}{\alpha - \beta + 1} \vee \frac{\theta - \beta}{\alpha - \beta + 1} < \rho. \quad (4.1)$$

Then,

$$\begin{aligned} \|w_2\|_{Y^r(T)} &= o(T^{1-\theta+\delta(r)}), \quad \|u - w_0 - w_1\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}), \\ \|w_2\|_{L^{q_0}(I; L^{r_0})} &= o(T^{1-\theta}), \quad \|u - w_0 - w_1\|_{L^{q_0}(I; L^{r_0})} = o(T^{1-\theta}). \end{aligned}$$

(3) Assume further that $N \leq 5$ and

$$\frac{N\alpha}{2(\alpha - \beta + 1)} \vee \left(\frac{N}{2} - \frac{2(1-\beta)(\theta - 1)}{\alpha} \right) \leq \rho \quad (4.2)$$

($\rho > N/2$ if $\beta = 1$). Then,

$$\begin{aligned} \|w_2\|_{Y^r(T)} &\leq CT^{(1-\theta)(1+\beta)+\delta(r)}, \\ \|u - w_0 - w_1\|_{Y^r(T)} &\leq CT^{(1-\theta)(1+\beta)+\delta(r)}, \\ \|w_2\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{(1-\theta)(1+\beta)}, \\ \|u - w_0 - w_1\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{(1-\theta)(1+\beta)}. \end{aligned}$$

Moreover, if $\alpha > 1$ then

$$\|u - w_0 - w_1 - w_2\|_{L^{q_0}(I; L^{r_0})} = o(T^{(1-\theta)(1+\beta)}).$$

Remark 4.1. The restriction on the dimension of N comes from the existence of ρ . By the existence result, ρ must be smaller than $(\alpha + 1) \wedge 2$. Then, the condition $N\alpha/(\alpha + 1) < (\alpha + 1) \wedge 2$ holds in the following cases: $\alpha \in (\gamma(N), \alpha_0(N))$ if $N \leq 8$ and $\alpha \in (\gamma(N), (N - 4 - \sqrt{N(N-8)})/4)$ if $N = 9, 10$. On the other hand, the left-hand side of (4.1) is bounded by $(\alpha + 1) \wedge 2$ in the cases $\alpha \in (\gamma(N), \alpha_0(N))$ if $N \leq 6$ and $\alpha \in (\gamma(N), 2/(N-4))$ if $N = 7, 8$, and the left-hand side of (4.2) is bounded by $(\alpha + 1) \wedge 2$ in the cases $\alpha \in (\gamma(N), \alpha_0(N))$ if $N \leq 3$, $\alpha \in (\gamma(4), \alpha_0(4))$ and $\alpha \neq 1$ if $N = 4$, and $\alpha \in (1, \alpha_0(5))$ if $N = 5$.

Remark 4.2. If $\alpha \in (\gamma(N), \alpha_0(N))$ satisfies $\alpha > 1$, then it holds that

$$\frac{\theta(1-\beta) + \beta}{\alpha - \beta + 1} \leq \frac{\theta - \beta}{\alpha - \beta + 1} \vee 1, \quad \frac{N}{2} - \frac{2(1-\beta)(\theta-1)}{\alpha} \leq \frac{N\alpha}{2(\alpha - \beta + 1)}.$$

Proof of Proposition 4.1. By the Sobolev embedding, for $r \in [2, \infty)$ with $\delta(r) \leq \rho$ ($r \in [2, \infty]$ if $\rho > N/2$), there exists a constant C_0 such that

$$\|u\|_{Y^r(T)} \leq \|u\|_{\Sigma^\rho(T)} \leq C_0, \quad (4.3)$$

$$\|w_0\|_{Y^r(T)} \leq \|w_0\|_{\Sigma^\rho(T)} \leq C_0. \quad (4.4)$$

Step 1. We first prove the bounds of $u - w_0$ and w_1 . Let us assume that $N\alpha/2(\alpha + 1) < \rho$. We deduce from (2.1), (4.3), and (4.4) that

$$\begin{aligned} \|u - w_0\|_{Y^r(T)} &= \|G(|u|^\beta, \dots, |u|^\beta, u)\|_{Y^r(T)} \leq CT^{1-\theta+\delta(r)} \|u\|_{Y^{(\alpha+1)r'}(T)}^{\alpha+1} \\ &\leq CT^{1-\theta+\delta(r)} \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} \|w_1\|_{Y^r(T)} &= \|G(\tilde{w}_0, \dots, \tilde{w}_0, w_0)\|_{Y^r(T)} \leq CT^{1-\theta+\delta(r_1)} \|w_0\|_{Y^{(\alpha+1)r'}(T)}^{\alpha+1} \\ &\leq CT^{1-\theta+\delta(r)} \end{aligned} \quad (4.6)$$

if $\delta((\alpha + 1)r') = (\theta - \delta(r))/(\alpha + 1) \leq \rho$. Therefore, if $N\alpha/2(\alpha + 1) \leq \rho$ then (4.5) and (4.6) hold for all r with $0 \leq \delta(r) < 1 \wedge (\theta - 1)$. Similarly, we see from (2.3), (4.3), and (4.4) that

$$\|u - w_0\|_{L^{q_0}(I; L^{r_0})} \leq CT^{1-\theta}, \quad (4.7)$$

$$\|w_1\|_{L^{q_0}(I; L^{r_0})} \leq CT^{1-\theta} \quad (4.8)$$

for all admissible pairs (q_0, r_0) .

Step 2. We proceed to the proof of the bounds of $u - w_0 - w_1$ and w_2 . Let us prove the second part of the proposition. Assume (4.1). An elementary calculation shows that

$$\begin{aligned} &\|u - w_0 - w_1\|_{Y^r(T)} \\ &\leq \|G(|u|^\beta, \dots, |u|^\beta, u - w_0)\|_{Y^r(T)} \\ &\quad + \left\| \sum_{\substack{1 \leq i \leq [\alpha] \\ i_1 = \dots = i_{[\alpha]+1} = 0}} G(|u|^\beta, \dots, |u|^\beta, |u|^\beta - \tilde{w}_{i_i}, \tilde{w}_{i_{i+1}}, \dots, \tilde{w}_{i_{[\alpha]}}, w_0) \right\|_{Y^r(T)} \\ &=: \|\mathcal{I}_1\|_{Y^r(T)} + \|\mathcal{I}_2\|_{Y^r(T)}. \end{aligned} \quad (4.9)$$

We first estimate $\mathcal{I}_2$. We deduce from (2.1), (4.5), (4.3), and (4.4) that

$$\begin{aligned} \|\mathcal{I}_2\|_{Y^r(T)} &\leq CT^{1-\theta+\delta(r)} \sum_{l=1}^{[\alpha]} \|u - w_0\|_{Y^{s_1}(T)}^\beta \|u\|_{Y^{s_2}(T)}^{\beta(l-1)} \|w_0\|_{Y^{s_2}(T)}^{\alpha-\beta l+1} \\ &\leq CT^{1-\theta+\beta(1-\theta+\delta(s_1))+\delta(r)} \end{aligned}$$

if $\delta(s_1) < (\theta - 1) \wedge 1$ and $\delta(s_2) = (\delta(r) - \beta\delta(s_1))/(\alpha - \beta + 1) \leq \rho$. Now, we let $\delta(s_1) = (\theta - 1) \wedge 1 - \varepsilon$ with $\varepsilon > 0$. Then, we have

$$\|\mathcal{I}_2\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}) \quad (4.10)$$

and

$$\delta(s_2) = \left(\frac{\theta(1-\beta) + \beta}{\alpha - \beta + 1} \vee \frac{\theta - \beta}{\alpha - \beta + 1} \right) + \frac{-\delta(r) + \beta\varepsilon}{\alpha - \beta + 1}.$$

If we take ε to be small then the condition $\delta(s_2) \leq \rho$ holds for $r = 2$ by assumption, and therefore it holds for all $r \geq 2$ with a suitable ε . Similarly, (2.3) provides that

$$\|\mathcal{I}_2\|_{L^{q_0}(I; L^{r_0})} = o(T^{1-\theta}). \quad (4.11)$$

We next estimate $\mathcal{I}_1$. Take a small constant $\varepsilon > 0$ and let r_1 satisfy $\delta(r_1) = 1 \wedge (\theta - 1) - \varepsilon$. Then, it follows that

$$\begin{aligned} \|\mathcal{I}_1\|_{Y^{r_1}(T)} &\leq CT^{1-\theta+\delta(r_1)} \|u\|_{Y^{s_3}(T)}^\alpha \|u - w_0\|_{Y^{s_4}(T)} \\ &\leq CT^{2(1-\theta)+\delta(r_1)+\delta(s_4)} \|u\|_{Y^{s_3}(T)}^\alpha. \end{aligned} \quad (4.12)$$

We take $s_4 = r_1$. Then,

$$\delta(s_3) = \frac{\theta - 2((\theta - 1) \vee 1)}{\alpha} + 2\varepsilon = \left\lfloor \frac{N}{2} - \frac{2}{\alpha} \right\rfloor + 2\varepsilon.$$

We deduce that the right-hand side is bounded by one if ε is small, so that $\|u\|_{Y^{s_3}(T)}$ is bounded. Then, (4.12) becomes

$$\|\mathcal{I}_1\|_{Y^{r_1}(T)} = o(T^{1-\theta+\delta(r_1)}). \quad (4.13)$$

Moreover, it follows from (2.4) that

$$\begin{aligned} \|\mathcal{I}_1\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{1-\theta+(\delta(r)+\delta(s_6))/2} \|u\|_{Y^{s_5}(T)}^\alpha \|u - w_0\|_{L^{2/\delta(s_6)}(I; L^{s_6})} \\ &\leq CT^{2(1-\theta)+(\delta(r)+\delta(s_6))/2} \|u\|_{Y^{s_5}(T)}^\alpha. \end{aligned} \quad (4.14)$$

We put $\delta(r) = \delta(s_6) = 1 \wedge (\theta - 1) - \varepsilon$. Then, $\delta(s_5) = \delta(s_3)$ and

$$\|\mathcal{I}_1\|_{L^{q_0}(I; L^{r_0})} = o(T^{1-\theta}). \quad (4.15)$$

Interpolating (4.13) and this with $(q_0, r_0) = (\infty, 2)$, we obtain

$$\|\mathcal{I}_1\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}) \quad (4.16)$$

for all $2 \leq r \leq r_1$. Hence, from (4.16), (4.10), (4.15), and (4.11), we finally obtain

$$\|u - w_0 - w_1\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}), \quad (4.17)$$

$$\|u - w_0 - w_1\|_{L^{q_0}(I; L^{r_0})} = o(T^{1-\theta}), \quad (4.18)$$

for all r with $0 \leq \delta(r) < 1 \wedge (\theta - 1)$ and all admissible pairs (q_0, r_0) . The estimate of w_2 is performed in the same way. Only replacing u and $u - w_0$ by w_0 and w_1 in the above estimation of $u - w_0 - w_1$, respectively, we obtain

$$\|w_2\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}), \quad (4.19)$$

$$\|w_2\|_{L^{q_0}(I; L^{r_0})} = o(T^{1-\theta}) \quad (4.20)$$

for all r with $0 \leq \delta(r) < 1 \wedge (\theta - 1)$ and all admissible pairs (q_0, r_0) .

Step 3. In this and the next step, we prove the third part. Assume further that (4.2) holds. We refine the estimates in the previous step. We first consider $\mathcal{I}_2$. We take $s_1 = 2$. Then, (4.10) becomes

$$\|\mathcal{I}_2\|_{Y^r(T)} \leq CT^{(1-\theta)(1+\beta)+\delta(r)} \quad (4.21)$$

and $\delta(s_2) = (\theta - \delta(r))/(\alpha - \beta + 1)$. We note that $\delta(s_2) \leq \rho$ holds for $r = 2$ by assumption, and therefore for all $r \geq 2$. In a similar way, we obtain

$$\|\mathcal{I}_2\|_{L^{q_0}(I; L^{r_0})} \leq CT^{(1-\theta)(1+\beta)}. \quad (4.22)$$

We next estimate $\mathcal{I}_1$. In (4.12), we take s_4 so that $\delta(s_4) = (\theta - 1)(1 - \beta)$. It is possible if $0 \leq (\theta - 1)(1 - \beta) < 1 \wedge (\theta - 1)$. We note that $\theta \geq 1$ if and only if $\alpha \geq 2/N$, and that $0 < \beta \leq 1$ always holds. Therefore, $0 \leq (\theta - 1)(1 - \beta) < \theta - 1$. Moreover, $(\theta - 1)(1 - \beta) < 1$ is equivalent to $N\alpha^2 - (N\lceil\alpha\rceil + 2)\alpha + 4\lceil\alpha\rceil > 0$, which is easy to verify. Moreover, we take ε so small that $\delta(r_1) > \delta(s_4)$. Then, it holds that

$$\delta(s_3) = \frac{\theta - \delta(r_1) - \delta(s_4)}{\alpha} < \frac{\theta - 2\delta(s_4)}{\alpha} = \frac{N}{2} - \frac{2(\theta - 1)(1 - \beta)}{\alpha}.$$

By assumption, we deduce that the right-hand side is bounded by ρ , so that $\|u\|_{Y^{s_3}(T)}$ is bounded. Thus, (4.12) becomes

$$\|\mathcal{I}_1\|_{Y^{r_1}(T)} \leq CT^{(1-\theta)(1+\beta)+\delta(r_1)}. \quad (4.23)$$

We next consider (4.14). Set $\delta(r) = \delta(s_6) = (\theta - 1)(1 - \beta) < 1 \wedge (\theta - 1)$. Then, $\delta(s_5) = N/2 - 2(\theta - 1)(1 - \beta)/\alpha \leq \rho$ by assumption and (4.14) becomes

$$\|\mathcal{I}_1\|_{L^{q_0}(I; L^{r_0})} \leq CT^{(1-\theta)(1+\beta)}. \quad (4.24)$$

Interpolating (4.23) and this with $(q_0, r_0) = (\infty, 2)$, we obtain

$$\|\mathcal{I}_1\|_{Y^r(T)} \leq CT^{(1-\theta)(1+\beta)+\delta(r)} \quad (4.25)$$

for all $2 \leq r \leq r_1$. Hence, from (4.25), (4.21), (4.24), and (4.22), we finally obtain

$$\|u - w_0 - w_1\|_{Y^r(T)} \leq CT^{(1-\theta)(1+\beta)}, \quad (4.26)$$

$$\|u - w_0 - w_1\|_{L^{q_0}(I; L^{r_0})} \leq CT^{(1-\theta)(1+\beta)} \quad (4.27)$$

for all r with $0 \leq \delta(r) < 1 \wedge (\theta - 1)$ and all admissible pairs (q_0, r_0) . The estimate of w_2 is performed in the same way. We obtain

$$\|w_2\|_{Y^r(T)} \leq CT^{(1-\theta)(1+\beta)}, \quad (4.28)$$

$$\|w_2\|_{L^{q_0}(I; L^{r_0})} \leq CT^{(1-\theta)(1+\beta)} \quad (4.29)$$

for all r with $0 \leq \delta(r) < 1 \wedge (\theta - 1)$ and all admissible pairs (q_0, r_0) .

Step 4. The estimate of $u - w_0 - w_1 - w_2$. From the existence result (Proposition 1.1), ρ must be smaller than $(\alpha + 1) \wedge 2$. If $\lceil \alpha \rceil = 1$, then the condition $N/2 - 2(1 - \beta)(\theta - 1)/\alpha < \alpha + 1$ is equivalent to $2(N - 1)\alpha^2 - (N + 6)\alpha + 4 < 0$, which does not hold for all $\alpha \in (\gamma(N), \alpha_0(N))$ with $\alpha \leq 1$ if $N \geq 5$. It means that we cannot take ρ so that (4.2) is satisfied if $\alpha \leq 1$ and $N \geq 5$. Hence, in the following, we only consider $N = 4$ when we are concerned with the case $\alpha \leq 1$.

We estimate w_3 instead of $u - w_0 - w_1 - w_2$ because these two estimates are shown in the same way according to the replacements $|u|^b \leftrightarrow \tilde{w}_0$, $|u|^b - \sum_{k=0}^{n-1} \tilde{w}_k \leftrightarrow \tilde{w}_n$ ($n \geq 1$), $u \leftrightarrow w_0$, and $u - \sum_{k=0}^{n-1} w_k \leftrightarrow w_n$ ($n \geq 1$). Now, w_3 is equal to

$$\sum_{i_1 + \dots + i_{\lceil \alpha \rceil + 1} = 2} G(\tilde{w}_{i_1}, \dots, \tilde{w}_{\lceil \alpha \rceil}, w_{\lceil \alpha \rceil + 1}).$$

There exist four cases:

- (1) one of $i_1, \dots, i_{\lceil \alpha \rceil}$ is two and the others are zero;
- (2) $i_{\lceil \alpha \rceil + 1}$ is two and the others are zero;
- (3) two of $i_1, \dots, i_{\lceil \alpha \rceil}$ are one and the others are zero;
- (4) $i_{\lceil \alpha \rceil + 1}$ is one and one of $i_1, \dots, i_{\lceil \alpha \rceil}$ is one, the others are zero.

We write $w_3 = \text{I} + \text{II} + \text{III} + \text{IV}$, where I, II, III, and IV are the sums of the terms which belong to the first, second, third, and fourth case, respectively. The third case occurs only if $\lceil \alpha \rceil \geq 2$. Therefore, we let III be identically zero if $\alpha \leq 1$. We deduce from (2.3), (4.4), and (4.28) that

$$\begin{aligned} \|\text{I}\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{1-\theta+(\delta(r)+\beta\delta(s_7))/2} \|w_2\|_{L^{2/\delta(s_7)}(I; L^{s_7})}^\beta \|w_0\|_{Y^{s_8}(T)}^{\alpha-\beta+1} \\ &\leq CT^{(1-\theta)(1+\beta+\beta^2)+(\delta(r)+\beta\delta(s_7))/2}. \end{aligned} \quad (4.30)$$

We set $\delta(r) = \delta(s_7) = ((2(\theta - 1)\beta^2/(1 + \beta)) \wedge (\theta/(1 + \beta)) \wedge 1) - \varepsilon$ with a small positive constant ε , which gives us

$$\|\text{I}\|_{L^{q_0}(I; L^{r_0})} = o(T^{(1-\theta)(1+\beta)}). \quad (4.31)$$

This choice is possible if $(\delta(r) + \beta\delta(s_7))/2 < \theta - 1$ and $\delta(s_8) = (\theta - \delta(r) - \beta\delta(s_7))/(\alpha - \beta + 1) \leq \rho$. The first condition is always satisfied because $(\delta(r) + \beta\delta(s_7))/2 < \beta^2(\theta - 1) \leq (\theta - 1)$ by definition. Taking ε small, one sees that the second condition holds if

$$f_1(\alpha) \vee f_2(\alpha) := \frac{\theta - \beta - 1}{\alpha - \beta + 1} \vee \frac{\theta - 2\beta^2(\theta - 1)}{\alpha - \beta + 1} < \rho. \quad (4.32)$$

Similarly,

$$\begin{aligned} \|\text{II}\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{1-\theta+(\delta(r)+\delta(s_9))/2} \|w_2\|_{L^{2/\delta(s_9)}(I; L^{s_9})} \|u\|_{Y^{s_{10}}(T)}^\alpha \\ &\leq CT^{(1-\theta)(2+\beta)+(\delta(r)+\delta(s_9))/2}. \end{aligned} \quad (4.33)$$

If we choose $\delta(r) = \delta(s_9) = 1 \wedge (\theta - 1) - \varepsilon$, then we obtain

$$\|\text{II}\|_{L^{q_0}(I; L^{r_0})} = o(T^{(1-\theta)(1+\beta)}). \quad (4.34)$$

The above choice is possible if $\delta(s_{10}) \leq \rho$, that is, if $|N/2 - 2/\alpha| < \rho$. We note that $|N/2 - 2/\alpha| < 1$ holds for all $\alpha \in (\gamma(N), \alpha_0(N))$. We next estimate III. Without any loss

of generality, we can assume that $i_1 = i_2 = 1$ and $i_3 = \dots = i_{[\alpha]+1} = 0$. It follows from (2.4) with $K = \{1, 2\}$, (4.4), (4.8) that

$$\begin{aligned} \|\text{III}\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{1-\theta+(\delta(r)+2\beta\delta(s_{11}))/2} \|w_1\|_{L^{2/\delta(s_{11})}(I; L^{s_{11}})}^{2\beta} \|w_0\|_{Y^{s_{12}}(T)}^{\alpha-2\beta+1} \\ &\leq CT^{(1-\theta)(1+2\beta)+(\delta(r)+2\beta\delta(s_{11}))/2}. \end{aligned} \quad (4.35)$$

Choosing $\delta(r) = \delta(s_{11}) = ((\theta - 1)2\beta/(2\beta + 1)) \wedge 1 - \varepsilon$, we obtain

$$\|\text{III}\|_{L^{q_0}(I; L^{r_0})} = o(T^{(1-\theta)(1+\beta)}). \quad (4.36)$$

This choice is possible if $\delta(s_{12}) \leq \rho$ and $(\delta(r) + 2\beta\delta(s_{11}))/2 < \theta - 1$, which follows from the condition

$$f_3(\alpha) \vee f_4(\alpha) := \frac{2\beta - (2\beta - 1)\theta}{\alpha - 2\beta + 1} \vee \frac{\theta - 2\beta - 1}{\alpha - 2\beta + 1} < \rho \quad (4.37)$$

and the definition of $\delta(r)$ and $\delta(s_{11})$, respectively. We turn to the estimate of IV. We consider the case where $i_1 = i_{[\alpha]+1} = 1$ and the others are zero. Then, applying (2.5), we obtain

$$\begin{aligned} \|\text{IV}\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{1-\theta+(\delta(r)+(\beta+1)\delta(s_{13}))/2} \|w_1\|_{L^{2/\delta(s_{13})}(I; L^{s_{13}})}^{\beta+1} \|w_0\|_{Y^{s_{14}}(T)}^{\alpha-\beta} \\ &\leq CT^{(1-\theta)(2+\beta)+(\delta(r)+(\beta+1)\delta(s_{13}))/2}. \end{aligned} \quad (4.38)$$

We first consider the case $[\alpha] \geq 2$. Then, we take the above powers so that $\delta(r) = \delta(s_{13}) = 2(\theta - 1)/(\beta + 2) \wedge 1 - \varepsilon$ and obtain

$$\|\text{IV}\|_{L^{q_0}(I; L^{r_0})} \leq o(T^{(1-\theta)(1+\beta)}). \quad (4.39)$$

This choice is admissible if

$$f_5(\alpha) \vee f_6(\alpha) := \frac{2 - \theta}{\alpha - \beta} \vee \frac{\theta - \beta - 2}{\alpha - \beta} < \rho. \quad (4.40)$$

We now assume $[\alpha] = 1$. Then,

$$\begin{aligned} \|\text{IV}\|_{L^{q_0}(I; L^{r_0})} &\leq CT^{(1-\theta)(2+\alpha)+(\delta(r)+(\alpha+1)\delta(s_{13}))/2} \\ &= CT^{(1-\theta)(1+\alpha)+1-\theta/2} \end{aligned}$$

since the powers r and s_{13} satisfy $\delta(r) + (\alpha + 1)\delta(s_{13}) = \theta$. Therefore, we obtain (4.39) if $\theta > 2$, which is equivalent to $\alpha > 4/N$. Since $N = 4$ and $\alpha \leq 1$, it is impossible. Summarizing (4.31), (4.34), (4.36), and (4.39), we finally obtain

$$\|w_3\|_{L^{q_0}(I; L^{r_0})} = o(T^{(1-\theta)(1+\beta)}). \quad (4.41)$$

The condition is $1 \leq N \leq 5$, $\gamma(N) \vee 1 < \alpha < \alpha_0(N)$, and $\max_{1 \leq i \leq 6} f_i(\alpha) < \rho$. We claim that these conditions can be summarized as $\rho \geq 1$ if $1 \leq N \leq 4$, and $\alpha > 1$, $\rho \geq 1$, and $\rho > f_2(\alpha)$ if $N = 5$. Let us first note that $\max_{1 \leq i \leq 6} f_i(\alpha) < N/2$ always holds. This is because $\delta(r) < N/2$ holds if $r < \infty$ in general. Here, by the definitions of s_8 , s_{12} , and s_{14} , they are finite. Hence, $\max_{1 \leq i \leq 6} f_i(\alpha) < N/2$. This implies that the condition $\rho \geq 1$ is sufficient if $N = 1, 2$. We next consider $N = 3$. Here $\gamma(3) = 1$ and $\alpha_0(3) = 4$. Regarding $[\alpha]$ as a constant, we consider that $f_i(\alpha)$ is a function only of α in the interval $([\alpha] - 1, [\alpha])$ for $i = 1, \dots, 6$. Then, by calculating directly, we verify for $[\alpha] = 2, 3, 4$ that the functions f_1, f_4, f_6 are

strictly increasing and the functions f_2, f_3, f_5 are strictly decreasing. Moreover, it holds that $\lim_{\alpha \uparrow [\alpha]} f_i(\alpha) \leq 1$ ($i = 1, 4, 6$) and $\lim_{\alpha \downarrow [\alpha]-1} f_i(\alpha) \leq 1$ ($i = 2, 3, 5$). The equal signs in the first limit and the second limit can hold only if $[\alpha] = 4$ and $[\alpha] = 2$, respectively. Therefore, we deduce that $\max_{1 \leq i \leq 6} f_i(\alpha) < 1$. The case $N = 4$ is handled in the same way. If $N = 5$, then the functions f_1, f_4, f_6 are strictly increasing and the functions f_2, f_3, f_5 are strictly decreasing. Moreover, it holds that $\lim_{\alpha \uparrow 2} f_i(\alpha) \leq 1$ ($i = 1, 4, 6$), $\lim_{\alpha \downarrow 1} f_i(\alpha) \leq 1$ ($i = 3, 5$), and $\lim_{\alpha \downarrow 1} f_2(\alpha) = 7/6 > 1$. Therefore, we need the condition $f_2(\alpha) < \rho$ and $\rho \geq 1$ in this case. However, the condition $f_2(\alpha) < \rho$ is weaker than (4.2) under $\rho \geq 1$. This can be seen by using a similar argument. In the end, we conclude that (4.41) holds if (4.2) holds and $\alpha > 1$. As stated above, the same proof gives us that

$$\|u - w_0 - w_1 - w_2\|_{L^{q_0}(I; L^{r_0})} = o(T^{(1-\theta)(1+\beta)}), \quad (4.42)$$

which completes the proof of Proposition 4.1, and so of Theorem 2. $\square$

5. The derivation of V_n

In the rest of the paper, we consider the approximation of w_n . In this section, we first discuss formally the approximation of w_n , and we see where V_n comes from. The rigorous proofs are in Section 6.

We first consider the case α as an even number. Let w_n be defined by (1.21). We begin with the approximation of $w_0 = U(t)\phi_+$. By the factorization of $U(t)$, it is written as

$$M(t)D(t)\mathcal{F}M(t)\phi_+ = M(t)D(t)(\mathcal{F}M(t)\mathcal{F}^{-1})\mathcal{F}\phi_+.$$

The operator $\mathcal{F}M(t)\mathcal{F}^{-1}$ is equal to $e^{-i(\Delta/4t)} = U(-1/4t)$. Now, let us regard this operator as the identity Id since we are dealing with very large t . Then, we obtain $V_0(t) = M(t)D(t)\mathcal{F}\phi_+$. Let us proceed to the approximation of w_1 . Since

$$\begin{aligned} w_1(t) &= H(w_0, \dots, w_0) = i \int_t^\infty U(t-s)|w_0(s)|^\alpha w_0(s) ds \\ &= iM(t)D(t) \int_t^\infty (\mathcal{F}M(t)M(s)^{-1}\mathcal{F}^{-1})D(s)^{-1}M(s)^{-1}|w_0(s)|^\alpha w_0(s) ds, \end{aligned}$$

replacing w_0 and the operator $\mathcal{F}M(t)M(s)^{-1}\mathcal{F}^{-1} = U(1/4s - 1/4t)$ with V_0 and Id , respectively, we obtain

$$\begin{aligned} V_1 &= iM(t)D(t) \int_t^\infty D(s)^{-1}M(s)^{-1}|V_0(s)|^\alpha V_0(s) ds \\ &= iM(t)D(t) \int_t^\infty |2s|^{-\theta}|\mathcal{F}\phi_+|^\alpha \mathcal{F}\phi_+ \\ &= i \frac{t^{1-\theta}}{2^\theta(\theta-1)} M(t)D(t)|\mathcal{F}\phi_+|^\alpha \mathcal{F}\phi_+. \end{aligned}$$

By a similar replacement, the function w_2 , which is written as

$$\begin{aligned} w_2(t) &= \sum_{i_1+\dots+i_{\alpha+1}=1} H(w_{i_1}, \dots, w_{i_{\alpha+1}}) \\ &= i \frac{\alpha+2}{2} \int_t^\infty U(t-s) |w_0(s)|^\alpha w_1(s) ds \\ &\quad + i \frac{\alpha}{2} \int_t^\infty U(t-s) |w_0(s)|^{\alpha-2} (w_0(s))^2 \overline{w_1(s)} ds, \end{aligned}$$

will be approximated by

$$\begin{aligned} &i \frac{\alpha+2}{2} M(t) D(t) \int_t^\infty D(s)^{-1} M(s)^{-1} |V_0(s)|^\alpha V_1(s) ds \\ &\quad + i \frac{\alpha}{2} M(t) D(t) \int_t^\infty D(s)^{-1} M(s)^{-1} |V_0(s)|^{\alpha-2} (V_0(s))^2 \overline{V_1(s)} ds \\ &= -\frac{1}{2} \left(\frac{t^{1-\theta}}{2^\theta(\theta-1)} \right)^2 |\mathcal{F}\phi_+|^{2\alpha} \mathcal{F}\phi_+. \end{aligned}$$

Similarly, since w_n is defined by

$$w_n(t) = \sum_{i_1+\dots+i_{\alpha+1}=n-1} H(w_{i_1}, \dots, w_{i_{\alpha+1}}),$$

the above argument implies that we should define

$$V_n(t) = \sum_{i_1+\dots+i_{\alpha+1}=n-1} \tilde{H}(V_{i_1}, \dots, V_{i_{\alpha+1}}),$$

where

$$\tilde{H}(u_1, \dots, u_{\alpha+1}) = i M(t) D(t) \int_t^\infty D(s)^{-1} M(s)^{-1} \prod_{l=1}^{\alpha/2+1} u_l(s) \prod_{l=\alpha/2+2}^{\alpha+1} \overline{u_l(s)} ds. \quad (5.1)$$

The function V_n defined by (1.22) is obtained by this inductive definition.

PROPOSITION 5.1. *Let V_n be defined by (1.22). Let $\tilde{H}$ be defined by (5.1). Then, V_n has the following alternative definition:*

$$V_0(t) = M(t) D(t) \mathcal{F}\phi_+, \quad V_n(t) = \sum_{i_1+\dots+i_{\alpha+1}=n-1} \tilde{H}(V_{i_1}, \dots, V_{i_{\alpha+1}}).$$

Proof. We show that V_n defined by this inductive definition is equal to (1.22). Putting $v_n = D^{-1} M^{-1} V_n$, we have the definition of v_n :

$$v_0(t) = \mathcal{F}\phi_+, \quad v_n(t) = \sum_{i_1+\dots+i_{\alpha+1}=n-1} i \int_t^\infty |2s|^{-\theta} \prod_{k=1}^{\alpha/2+1} v_{i_k} \prod_{k=\alpha/2+2}^{\alpha+1} \overline{v_{i_k}} ds.$$

We verify by induction that v_n takes the following form:

$$v_n(t) = J(n) \frac{t^{n(1-\theta)}}{2^{n\theta}(\theta-1)^n} |\mathcal{F}\phi_+|^{\alpha n} \mathcal{F}\phi_+,$$

where $J(n)$ is a sequence of complex numbers defined by $J(0) = 1$ and

$$J(n) = \frac{i}{n} \sum_{i_1 + \dots + i_{\alpha+1} = n-1} \prod_{k=1}^{\alpha/2+1} J(i_k) \prod_{k=\alpha/2+2}^{\alpha+1} \overline{J(i_k)}. \quad (5.2)$$

If $J(n) = (i)^n/n!$, then $V_n = MDv_n$ is identical to (1.22). Hence, we prove this.

We set the formal Taylor series $g(x) = \sum_{n=0}^{\infty} J(n)x^n$. If $x \in \mathbb{R}$, then this function g satisfies

$$\begin{aligned} |g|^\alpha g &= \sum_{n=0}^{\infty} \left(\sum_{i_1 + \dots + i_{\alpha+1} = n} \prod_{k=1}^{\alpha/2+1} J(i_k) \prod_{k=\alpha/2+2}^{\alpha+1} \overline{J(i_k)} \right) x^n \\ &= \sum_{n=0}^{\infty} (-i)(n+1)J(n+1)x^n = -ig', \end{aligned}$$

that is, $g' = i|g|^\alpha g$. We consider this equation for $x \in \mathbb{R}$. Multiplying $\bar{g}$ and taking the real part, we first deduce that $|g|$ is constant. Because $g(0) = 1$, the equation for g becomes $g' = ig$. Therefore, $g(x) = e^{ix}$. Then, we obtain $J(n) = (i)^n/n!$. $\square$

On the other hand, if α is not an even integer, then the definition of V_n for $n \geq 2$ becomes difficult. Let w_n be defined by (1.8). Since w_0 does not depend on α , the above argument gives $V_0(t) = M(t)D(t)\mathcal{F}\phi_+$. Let us proceed to the approximation of w_1 . Since

$$w_1(t) = G(|w_0|^\beta, \dots, |w_0|^\beta, w_0) = i \int_t^\infty U(t-s)|w_0(s)|^\alpha w_0(s) ds,$$

the previous formal replace argument suggests that

$$\begin{aligned} V_1 &= iM(t)D(t) \int_t^\infty D(s)^{-1} M(s)^{-1} |V_0(s)|^\alpha V_0(s) ds \\ &= i \frac{t^{1-\theta}}{2^\theta(\theta-1)} M(t)D(t) |\mathcal{F}\phi_+|^\alpha \mathcal{F}\phi_+. \end{aligned}$$

This is a natural extension of the previous even-number case. However, it is difficult to define V_2 . The function w_2 is written as

$$\begin{aligned} w_2(t) &= \sum_{i_1 + \dots + i_{\lceil \alpha \rceil + 1} = 1} G(\tilde{w}_{i_1}, \dots, \tilde{w}_{i_{\lceil \alpha \rceil}}, w_{i_{\lceil \alpha \rceil + 1}}) \\ &= \lceil \alpha \rceil i \int_t^\infty U(t-s)(|w_1(s) + w_0(s)|^\beta - |w_0(s)|^\beta) |w_0(s)|^{\alpha-\beta} w_0(s) ds \\ &\quad + i \int_t^\infty U(t-s) |w_0|^\alpha w_1(s) ds. \end{aligned}$$

Therefore, it will be approximated by

$$\begin{aligned} i \lceil \alpha \rceil M(t)D(t) \int_t^\infty D(s)^{-1} M(s)^{-1} (\sigma_1(s) |V_1(s)|^\beta) |V_0(s)|^{\alpha-\beta} V_0(s) ds \\ + iM(t)D(t) \int_t^\infty D(s)^{-1} M(s)^{-1} |V_0|^\alpha V_1(s) ds, \end{aligned} \quad (5.3)$$

where $\sigma_1(t, x) := \text{sign}(|w_1(t, x) + w_0(t, x)|^\beta - |w_0(t, x)|^\beta)$. This is because

$$||w_1(s) + w_0(s)|^\beta - |w_0(s)|^\beta - \sigma_1(s)|V_1(s)|^\beta| \leq |w_1(s) - V_1(s)|^\beta.$$

Note that

$$||w_1(s) + w_0(s)|^\beta - |w_0(s)|^\beta - |V_1(s)|^\beta| \leq |w_1(s) - V_1(s)|^\beta$$

does not hold. Then, the treatment of $\sigma_1(t)$ prevents our defining V_2 . However, using a completely different method, we can obtain V_2 only in the case where α is an even number. In fact, we see that if α is an even number, then V_2 defined by (1.22) approximates not only the function w_2 defined by (1.21) but also w_2 defined by (1.8) (see Proposition 6.2). This means that w_2 defined by (1.8) is approximated only by the second term of (5.3) and so that the first term of (5.3) is small.

6. The proofs of Theorems 3 and 4

Finally, we prove Theorem 3 and complete the proof of Theorem 4. We approximate w_n by V_n .

6.1. The proof of Theorem 3

PROPOSITION 6.1. *Let $N \geq 1$ and $\alpha \in (\gamma(N), \alpha_0(N))$. Put $\theta = \alpha N/2$, $T \gg 1$, and $I = [T, \infty)$. Let $V_0 = M(t)D(t)\mathcal{F}\phi_+$ and V_1 be given in Theorem 2. Let ρ satisfy $1 \leq \rho < (\alpha + 1) \wedge 2$. Then, it holds for all $r \in [2, \infty)$ with $\delta(r) \leq \rho$ ($r \in [2, \infty]$ if $\rho > N/2$) that*

$$\|V_0\|_{Y^r(T)} = C, \quad \|V_0\|_{L^{(2/\delta(r))}(I; L^r)} = CT^{-\delta(r)/2},$$

$$\|u - V_0\|_{Y^r(T)} = o(1), \quad \|u - V_0\|_{L^{(2/\delta(r))}(I; L^r)} = o(T^{-\delta(r)/2}).$$

Let $r \in [2, \infty]$ satisfy $\delta(r) < (\theta - 1) \wedge 1$ and (q_0, r_0) be an admissible pair. Then, it holds that

$$\|V_1\|_{Y^r(T)} = CT^{1-\theta}, \quad \|V_1\|_{L^{q_0}(I; L^{r_0})} = CT^{1-\theta-\delta(r)/2},$$

$$\|w_1 - V_1\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}), \quad \|w_1 - V_1\|_{L^{q_0}(I; L^{r_0})} = o(T^{1-\theta})$$

as long as r and r_0 satisfy $\delta((\alpha + 1)r) \leq \rho$ and $\delta((\alpha + 1)r_0) \leq \rho$, respectively.

Proof. It follows by the definition of V_0 that

$$\|V_0\|_{Y^r(T)} = 2^{-\delta(r)} \|\mathcal{F}\phi_+\|_{L^r} < \infty, \tag{6.1}$$

$$\|w_0 - V_0\|_{Y^r(T)} = \sup_{t \geq T} \|(\mathcal{F}(M - 1)\mathcal{F}^{-1})\mathcal{F}\phi_+\|_{L^r} = o(1) \tag{6.2}$$

for all $r \in [2, \infty)$ with $\delta(r) \leq \rho$ ($r \in [2, \infty]$ if $\rho > N/2$). Now, we turn to the estimate of V_1 . Assume that $\alpha N/2(\alpha + 1) \leq \rho$. An elementary calculation provides

$$\|V_1(t)\|_{L^r} = \frac{t^{1-\theta}}{2^\theta(\theta - 1)} (2t)^{-\delta(r)} \| |\mathcal{F}\phi_+|^\alpha \mathcal{F}\phi_+ \|_{L^r}.$$

The L^r norm in the right-hand side is bounded for all $r \in [2, \infty]$ with $\delta((\alpha + 1)r) \leq \rho$; there exists a non-zero constant depending on r such that

$$\|V_1\|_{Y^r(T)} = C_0 T^{1-\theta} \tag{6.3}$$

if $\mathcal{F}\phi_+$ is not identically zero. Similarly, one sees that

$$\|V_1\|_{L^{q_0}(I;L^{r_0})} = CT^{1-\theta-\delta(r_0)/2}. \quad (6.4)$$

Therefore, it follows from (2.1), (4.4), (6.1), and (6.2) that

$$\begin{aligned} & \|w_1 - V_1\|_{Y^r(T)} \\ & \leq \|G(\tilde{w}_0, \dots, \tilde{w}_0, w_0 - V_0)\|_{Y^r(T)} \\ & \quad + \sum_{\substack{1 \leq l \leq [\alpha] \\ i_1 = \dots = i_{[\alpha]+1} = 0}} \|G(\tilde{w}_{i_1}, \dots, \tilde{w}_{i_{l-1}}, \tilde{w}_{i_l} - |V_0|^\beta, |V_0|^\beta, \dots, |V_0|^\beta, V_0)\|_{Y^r(T)} \\ & \quad + \|E_1\|_{Y^r(T)} = o(T^{1-\theta+\delta(r)}) \end{aligned} \quad (6.5)$$

for r with $\delta(r) < 1 \wedge (\theta - 1)$, where we apply Lemma 2.5 and the same argument as in the proofs of (4.5) and (4.6). This inequality holds if $N\alpha/2(\alpha + 1) \leq \rho$ and $\delta((\alpha + 1)r) \leq \rho$. We note that $N\alpha/2(\alpha + 1) = \delta((\alpha + 1)2) \leq \delta((\alpha + 1)r)$ if $r \geq 2$, and so that the second condition is stronger. Similarly, (2.3) provides

$$\|w_1 - V_1\|_{L^{q_0}(I;L^{r_0})} = o(T^{1-\theta}), \quad (6.7)$$

where (q_0, r_0) is an admissible pair such that $\delta((\alpha + 1)r_0) \leq \rho$. $\square$

Proof of Theorem 3. This theorem immediately follows from Proposition 6.1 and the second part of Proposition 4.1 by the use of triangle inequality

$$\|u - w_0 - V_1\| \leq \|u - w_0 - w_1\| + \|w_1 - V_1\|. \quad \square$$

If α is an even number the we can obtain the asymptotic form of w_2 .

PROPOSITION 6.2. *Let $N = 1, 2, 3$. Let $\alpha \in (\gamma(N), \alpha_0(N))$ be an even number. Put $\theta = \alpha N/2$. Let $\rho \in [1, (\alpha + 1) \wedge 1]$ satisfy $\rho > N/2$. Let $T \gg 1$ and $I = [T, \infty)$. Let V_2 be a function defined by (1.22) with $n = 2$. Then,*

$$\begin{aligned} \|w_2 - V_2\|_{L^{q_0}(I;L^{r_0})} &= o(T^{2(1-\theta)}), \quad \|w_2(t)\|_{L^2} \asymp t^{2(1-\theta)}, \\ \|u(t) - w_0(t) - w_1(t)\|_{L^2} &\asymp t^{2(1-\theta)}. \end{aligned}$$

Proof. Let w_n and $\tilde{w}_n$ be functions defined by (1.8) and (1.21), respectively. One sees that $w_0 = \tilde{w}_0$ and $w_1 = \tilde{w}_1$. Then, from Theorem 4 (we prove this in Section 6.2), we deduce for all $r \in [2, \infty]$ that

$$\begin{aligned} \|u - w_0 - w_1 - \tilde{w}_2\|_{Y^r(T)} &\leq \|u - w_0 - w_1 - \tilde{w}_2\|_{\Sigma^\rho(T)} \leq CT^{3(1-\theta)}, \\ \|\tilde{w}_2 - V_2\|_{Y^r(T)} &\leq \|\tilde{w}_2 - V_2\|_{\Sigma^\rho(T)} = o(T^{2(1-\theta)}). \end{aligned}$$

By the definition of $Y^r(T)$, it means that

$$\begin{aligned} \|u - w_0 - w_1 - \tilde{w}_2\|_{L^{q_0}(I;L^{r_0})} &\leq CT^{3(1-\theta)-\delta(r_0)/2}, \\ \|\tilde{w}_2 - V_2\|_{L^{q_0}(I;L^{r_0})} &= o(T^{2(1-\theta)-\delta(r_0)/2}). \end{aligned}$$

Moreover, from Theorem 2, we also deduce that $\|u - w_0 - w_1 - w_2\|_{L^{q_0}(I; L^{r_0})} = o(T^{2(1-\theta)})$. Therefore, we conclude that

$$\begin{aligned} \|w_2 - V_2\|_{L^{q_0}(I; L^{r_0})} &\leq \|\tilde{w}_2 - V_2\|_{L^{q_0}(I; L^{r_0})} + \|u - w_0 - w_1 - \tilde{w}_2\|_{L^{q_0}(I; L^{r_0})} \\ &\quad + \|u - w_0 - w_1 - w_2\|_{L^{q_0}(I; L^{r_0})} = o(T^{2(1-\theta)}). \end{aligned}$$

The lower L^2 estimates hold from this and $\|V_2(t)\|_{L^2} = Ct^{2(1-\theta)}$. $\square$

6.2. The completion of the proof of Theorem 4

Completion of the proof of Theorem 4. By the definition of V_n (1.22), we have

$$\|V_n\|_{\Sigma^\rho(T)} = \frac{T^{n(1-\theta)}}{n!2^{n\theta}(\theta-1)^n} \| |\mathcal{F}\phi_+|^{\alpha n} \mathcal{F}\phi_+ \|_{H^\rho}.$$

Since the H^ρ norm of $|\mathcal{F}\phi_+|^{\alpha n+1} \mathcal{F}\phi_+$ is bounded by $C\|\mathcal{F}\phi_+\|_{H^\rho}^{\alpha n+1}$, they are positive finite. Therefore, there exists a constant C_0 such that $\|V_n; \Sigma^\rho(T)\| = C_0 T^{n(1-\theta)}$. This constant is positive unless ϕ_+ is identically zero. Similarly, we have $\|V_n; Y^r(T)\| = C_0 T^{n(1-\theta)}$.

Let us proceed to the proof of (1.26). We prove it by induction. Because $w_0 - V_0 = MDF(M-1)\phi_+$, one sees that if $\mathcal{F}\phi_+ \in H^\rho$, then

$$\|w_0 - V_0\|_{H^\rho} = \|(M(t) - 1)(1 + |x|^2)^{\rho/2} \phi_+\|_{L^2} \rightarrow 0$$

as $t \rightarrow 0$. Therefore, we have $\|w_0 - V_0\|_{\Sigma^\rho(T)} \rightarrow 0$ as $T \rightarrow \infty$. By the multilinearity of H , we have

$$\begin{aligned} w_n - V_n &= \sum_{i_1 + \dots + i_{\alpha+1} = n-1} \sum_{l=1}^{\alpha+1} H(w_{i_1}, \dots, w_{i_{l-1}}, w_{i_l} - V_{i_l}, V_{i_{l+1}}, \dots, V_{i_{\alpha+1}}) \\ &\quad + \left(\sum_{i_1 + \dots + i_{\alpha+1} = n-1} H(V_{i_1}, \dots, V_{i_{\alpha+1}}) - V_n \right). \end{aligned}$$

We denote the second term $\sum H(V_{i_1}, \dots, V_{i_{\alpha+1}}) - V_n$ by R_n . Since $\|w_n; \Sigma^\rho(T)\|, \|V_n; \Sigma^\rho(T)\| = O(T^{n(1-\theta)})$, if we assume that (1.26) holds for $n \leq n_0 - 1$, then we deduce from Lemma 2.3 that

$$\begin{aligned} \|w_n - V_n\|_{\Sigma^\rho(T)} &\leq \sum_{i_1 + \dots + i_{\alpha+1} = n-1} \sum_{l=1}^{\alpha+1} C_1 T^{1-\theta} \left(\prod_{k=1}^{l-1} \|w_{i_k}\|_{\Sigma^\rho(T)} \right) \|w_{i_l} - V_{i_l}\|_{\Sigma^\rho(T)} \\ &\quad \times \left(\prod_{k=l+1}^{\alpha+1} \|V_{i_k}\|_{\Sigma^\rho(T)} \right) + \|R_n\|_{\Sigma^\rho(T)} \\ &= \|R_n\|_{\Sigma^\rho(T)} + o(T^{n(1-\theta)}). \end{aligned}$$

Hence it suffices to show that $\|R_n\|_{\Sigma^\rho(T)} = o(T^{n(1-\theta)})$. We deduce from Proposition 5.1 that

$$\begin{aligned} R_n &= \sum_{i_1+\dots+i_{\alpha+1}=n-1} H(V_{i_1}, \dots, V_{i_{\alpha+1}}) - V_n \\ &= \sum_{i_1+\dots+i_{\alpha+1}=n-1} (H - \tilde{H})(V_{i_1}, \dots, V_{i_{\alpha+1}}) \\ &= \sum_{i_1+\dots+i_{\alpha+1}=n-1} iM(t)D(t) \int_t^\infty \mathcal{U}(t, s)D(s)^{-1}M(s)^{-1} \prod_{k=1}^{\alpha/2+1} V_{i_k} \prod_{k=\alpha/2+2}^{\alpha+1} \overline{V_{i_k}} ds, \end{aligned}$$

where $\mathcal{U}(t, s) = \mathcal{F}M(t)M(s)^{-1}\mathcal{F}^{-1} - 1$. Using the explicit definition of V_n and the definition of $J(n)$ ((1.22) and (5.2), respectively), we obtain

$$\begin{aligned} R_n &= iM(t)D(t) \int_t^\infty \mathcal{U}(t, s)|2s|^{-\theta} \left(\frac{s^{1-\theta}}{2^\theta(\theta-1)} \right)^{n-1} |\mathcal{F}\phi_+|^{\alpha n} \mathcal{F}\phi_+ ds \\ &\quad \times \sum_{i_1+\dots+i_{\alpha+1}=n-1} \prod_{k=1}^{\alpha/2+1} J(i_k) \prod_{k=\alpha/2+2}^{\alpha+1} \overline{J(i_k)} \\ &= \frac{1}{2^{n\theta}(\theta-1)^{n-1}} \left(iM(t)D(t) \int_t^\infty \mathcal{U}(t, s)s^{n(1-\theta)-1} |\mathcal{F}\phi_+|^{\alpha n} \mathcal{F}\phi_+ ds \right) \frac{n}{i} J(n) \\ &= \frac{(i)^{n-1}}{(n-1)!} \times \frac{1}{2^{n\theta}(\theta-1)^{n-1}} E_n(t). \end{aligned}$$

Then, Proposition 2.6 provides the desired estimate and completes the proof of (1.26). We finally obtain the lower bound estimates:

$$\|w_n\|_{\Sigma^\rho(T)} \geq \|V_n\|_{\Sigma^\rho(T)} - \|w_n - V_n\|_{\Sigma^\rho(T)} \geq cT^{n(1-\theta)}$$

and so

$$\left\| u - \sum_{k=0}^n w_k \right\|_{\Sigma^\rho(T)} \geq \|w_{n+1}\|_{\Sigma^\rho(T)} - \left\| u - \sum_{k=0}^{n+1} w_k \right\|_{\Sigma^\rho(T)} \geq cT^{(n+1)(1-\theta)}.$$

The lower bound estimates of $Y^r(T)$ also follow from $\|V_n; Y^r(T)\| = C_0 T^{n(1-\theta)}$ and the embedding $\Sigma^\rho(T) \hookrightarrow Y^r(T)$. $\square$

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