ZETA POLYNOMIALS OF TYPE IV CODES OVER RINGS OF ORDER FOUR

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Abstract. We extend the definition of zeta function and zeta polynomial to codes defined over finite rings with respect to a specified weight function. Moreover, we also investigate the Riemann hypothesis analogue for Type IV codes over any of the rings \( \mathbb{Z}_4, \mathbb{F}_2 + u\mathbb{F}_2 \) and \( \mathbb{F}_2 + v\mathbb{F}_2 \). Although, for small lengths, there are only a few actual Type IV codes over \( \mathbb{Z}_4, \mathbb{F}_2 + u\mathbb{F}_2 \) or \( \mathbb{F}_2 + v\mathbb{F}_2 \) that satisfy the Hamming distance upper bound \( 2(1 + \lfloor n/6 \rfloor) \), we will show that zeta polynomials corresponding to these weight enumerators that meet this bound satisfy the Riemann hypothesis analogue property.

1. Introduction

The concept of zeta functions and zeta polynomials for linear codes over finite fields was first introduced by Duursma [5]. He conjectured that the Riemann hypothesis analogue (RHA) is true for some divisible codes with minimum distance satisfying the Mallows–Sloane bounds. In fact, he was able to show that extremal Type IV codes (over \( \mathbb{F}_4 \) with Hamming weights divisible by 2) with length divisible by 6 indeed satisfy RHA [7]. On the other hand, Dougherty et al [3] defined Type IV codes over a ring \( R \) of order four to be self-dual codes with even Hamming weights. They considered the finite ring \( R \) to be any of \( \mathbb{Z}_4, \mathbb{F}_2 + u\mathbb{F}_2 \) and \( \mathbb{F}_2 + v\mathbb{F}_2 \). A number of actual Type IV codes over these rings were given in [1, 2, 3, 8, 9, 10]. The basis of this paper are the papers mentioned above with the objective of studying the behavior of zeta polynomials of codes defined over rings of order four, i.e. \( \mathbb{Z}_4, \mathbb{F}_2 + u\mathbb{F}_2 \) and \( \mathbb{F}_2 + v\mathbb{F}_2 \). Computations of zeta polynomials of all self-dual codes of length at most 16 defined over these rings will be shown. More importantly, we will also prove that the RHA property holds for any extremal Type IV code over a ring of order four when its length is \( n = 6k \) or \( 6k - 2 \) for any integer \( k > 0 \).

2. General zeta polynomials

Let \( R \) be a commutative quasi-Frobenius (QF) ring of order \( m \) and let \( V = \{ (x_1, \ldots, x_n) \mid x_i \in R, \ 1 \leq i \leq n \} \). By a code \( C \) of length \( n \) over \( R \), we mean an \( R \)-submodule of \( V \). Each \( n \)-tuple \( c \in C \) is called a codeword of \( C \). The inner product of \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) in \( V \) is given by

\[ x \cdot y = \sum_{i=1}^{n} x_i y_i. \]

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y = (y_1, \ldots, y_n) in V is
\[ x \cdot y = x_1 y_1 + \cdots + x_n y_n. \]

The dual code of C is
\[ C^\perp = \{ y \in V \mid x \cdot y = 0 \text{ for all } x \in C \}. \]

A code C is self-orthogonal if \( C \subseteq C^\perp \); it is self-dual if \( C = C^\perp \).

Let \( x = (x_1, \ldots, x_n) \in V \) and \( r \in \mathbb{R} \). Define \( n_r(x) \) as
\[ n_r(x) := |\{ i \mid x_i = r \}|. \]

For a given weight function \( a \) from \( \mathbb{R} \) into \( \mathbb{R}_{\geq 0} \), \( a : r \mapsto a_r \) such that \( a_0 = 0 \), we define the weight of \( x = (x_1, \ldots, x_n) \in V \) as
\[ w(x) := \sum_{r \in \mathbb{R}} a_r n_r(x). \]

For example, if \( \mathcal{R} = \mathbb{Z}_4 \) and \( a_r = 1 \) for all non-zero \( r \in \mathcal{R} \), then \( w(x) \) gives the Hamming weight of \( x \); and if \( a_1 = a_3 = 1 \) while \( a_2 = 2 \), then \( w(x) \) yields the Lee weight of \( x \).

The distance between two \( n \)-tuples \( x \) and \( y \) with respect to the weight function \( a \) is given by
\[ d_a(x, y) = w(x - y). \]

The minimum distance of \( C \) associated with the weight function \( a \) is
\[ d_a := \min \{ w(x) \mid x \in C \text{ and } x \neq 0 \}. \]

For any \( x \in V \), we define the support of \( x \) to be the set
\[ \text{supp}(x) := \{ i \mid x_i \neq 0 \}. \]

In other words, \( \text{supp}(x) \) is the set of all positions where the components of \( x \) are non-zero. Note that if \( A = \max \{ a_r \mid r \in \mathcal{R} \} \), then
\[ w(x) \leq A |\text{supp}(x)|. \]

Moreover, we have \( \max_{x \in V} w(x) = nA \).

The raw weight enumerator of \( C \) with respect to the weight function \( a \) is given by
\[ a(Y) = \sum_{x \in C} Y^{w(x)}. \]

If the minimum distance of \( C \) is \( d_a = d \), then \( a(Y) \) has the homogenized form
\[ a(X, Y) = X^{nA} + \sum_{i = d}^{nA} b_i X^{nA-i} Y^i. \]

The coefficient \( b_i \) in this expression merely gives the number of codewords in \( C \) that are of weight \( i \). Hence, \( B_a = \{ 1 = b_0, b_1, \ldots, b_{nA} \} \) gives the weight distribution of \( C \) with respect to the weight function \( a \).

We are now ready to give an analogous definition of zeta function and zeta polynomial of a code over a finite commutative QF ring. For the following definition, we use the notation \( [T^k] \) to refer to the coefficient \( a_k \) of \( T^k \) in the series expansion \( \sum_{i=0}^{\infty} a_k T^k \).
Definition 2.1. Let $C$ be a code over a commutative QF ring of order $m$ with length $n$. For a given weight function $a$ and minimum distance $d_a = d$, let $a(X, Y)$ be the corresponding homogeneous weight enumerator. We define the zeta polynomial of $C$, 

$$ P_a(T) = \sum_{j=0}^{r} p_j T^j, $$

to be the unique polynomial of degree at most $nA - d$ such that

$$ \left[ T^{nA-d} \right] \frac{P_a(T)}{(1-T)(1-mT)} (xT + y(1-T))^n = \frac{a(X, Y) - X^{nA}}{m-1}. \tag{1} $$

The rational function

$$ Z(T) = \frac{P_a(T)}{(1-T)(1-mT)} $$

is called the zeta function of $C$.

The existence and uniqueness of $P_a(T)$ can be shown by using linear algebra, which involves solving a linear system in $nA - d + 1$ unknowns.

2.1. MacWilliams identity for codes over rings

It is also worth while mentioning the following known theorem, which will become helpful in the succeeding discussions of this paper. The readers are directed to [11] for an extensive discussion on this topic. The theorem is an application of results from [15]. The notation $hwe_C(x, y)$ refers to the Hamming weight enumerator associated with a code $C$ over the QF ring $\mathcal{R}$. Equation (2) is popularly known as the MacWilliams identity.

**Theorem 2.1.** (Wood) For a finite Frobenius ring $\mathcal{R}$ and a submodule $C$ of $\mathcal{R}^n$,

$$ hwe_C(x, y) = \frac{1}{|C^\perp|} hwe_{C^\perp} (x + (|\mathcal{R}| - 1)y, x - y). \tag{2} $$

It is clear from this theorem that codes over rings of the same order satisfy the same MacWilliams identity.

For the sake of establishing conditions that will avoid the degeneracy cases of codes that will be discussed in this paper, we assume from here on that $d \geq 2$ and $d^\perp \geq 2$.

3. MDS weight enumerators and zeta polynomials associated with the Hamming weight function

We give here the Singleton bound for a general weight function for codes defined over a finite commutative ring $\mathcal{R}$. For more detailed discussions in this section, readers are referred to [13].

**Theorem 3.1.** (Shiromoto) Let $C$ be a code over a finite commutative QF ring $\mathcal{R}$. Let $w$ be a general weight function on $C$ with maximum $a_r$-value $A$. Suppose the minimum distance of $C$ is $d$, then

$$ \left\lfloor \frac{d - 1}{A} \right\rfloor \leq n - \log_{|\mathcal{R}|} |C|. $$
If we apply this theorem to codes over a commutative ring of order $m$ and use the Hamming weight function, then $A = 1$ so that
\[ d \leq n - \log_m |C| + 1. \] (3)

An equivalent form of this inequality gives an upper bound to the number of codewords for a given length $n$ and minimum distance $d$.
\[ |C| \leq m^{n-d} + 1. \] (4)

Codes satisfying this upper bound are called maximum distance separable codes or MDS codes. By applying (3) to $C$ and its dual $C^\perp$ (with minimum distance $d^\perp$), we obtain the following relation, which is independent of the order of the underlying finite ring:
\[ d + d^\perp \leq n + 2. \] (5)

If a weight enumerator satisfies the condition $d + d^\perp = n + 2$, then it is called an MDS weight enumerator. The following discussion shows that, for a given length $n$ and minimum distance $d$, a unique MDS weight enumerator $M_{n,d}$ can be determined. Moreover, there is a basis of Hamming weight enumerators of degree $n$ and minimum distance $d$ that consists of MDS weight enumerators. In the following theorem, we give an explicit method of solving for $M_{n,d}$ without necessarily associating it with an existing code. Yet, for convenience, we add an additional parameter $m$ that might represent the order of the underlying finite ring.

**Theorem 3.2.** Let $n$ be a positive integer and $2 \leq d < n$ be given. Then the MDS weight enumerator $M_{n,d,m}$ is given by
\[ M_{n,d,m} = M_{n,d} = x^n + \sum_{w=0}^{n} A_w x^{n-w} y^w, \]
where
\[ A_w = \sum_{i=d}^{n} (-1)^{i-w} \binom{n-i}{n-w} \binom{n}{i} (m^{i-d+1} - 1). \] (6)

**Proof.** We begin with the use of the MacWilliams identity relating the weight enumerators of a given code $C$ and its dual $C^\perp$:
\[ W_C(x, y) = \frac{1}{|C^\perp|} W_C(x + (m - 1)y, x - y). \]

Here, we put
\[ W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i \]
with $A_0 = 1$, $A_i = 0$ for $1 \leq i \leq d - 1$ and $A_d \neq 0$ and
\[ W_C^\perp(x, y) = \sum_{i=0}^{n} A'_i x^{n-i} y^i \]
with $A'_0 = 1$, $A'_i = 0$ for $1 \leq i \leq d^\perp - 1$ and $A_{d^\perp} \neq 0$. By requiring $d^\perp$ to satisfy the condition $d^\perp = n - d + 2$ so that $|C^\perp| = m^{d-1}$ we now have
\[ \sum_{i=0}^{n} A_i x^{n-i} y^i = \frac{1}{m^{d-1}} \sum_{i=0}^{n} A'_i (x + (m - 1)y)^{n-i} (x - y)^i. \] (7)
Next, we substitute \( y = 1 \) into (7) to obtain
\[
\sum_{i=0}^{n} A_i x^{n-i} = \frac{1}{m^{d-1}} \sum_{i=0}^{n} A_i'(x + m - 1)^{n-i} (x - 1)^i.
\] (8)

By differentiating \( \nu \) times both sides of (8) with respect to \( x \) and evaluating at \( x = 1 \), we obtain
\[
\sum_{i=0}^{\nu n} \binom{\nu}{i} A_{n-i} = \binom{n}{\nu} m^{n-d+\nu+1} - \nu - 1.
\] (9)

Writing (9) as
\[
\sum_{i=0}^{n-\nu} \binom{n-i}{\nu} A_i = \binom{n}{\nu} m^{n-d+\nu+1}
\] (10)
and substituting \( \nu = n - d, n - d + 1, \ldots, n \) will lead us to the values of \( A_w \) for \( d \leq w \leq n \), so that we obtain the desired result:
\[
A_w = \sum_{i=d}^{w} (-1)^{i-w} \binom{n-i}{n-w} \binom{n}{i} (m^{i-d+1} - 1).
\] (11)

Thus, with \( A_0 = 1 \) and \( A_i = 0 \) for \( 1 \leq i \leq d - 1 \) we are able to determine the unique MDS weight enumerator \( M_{n,d} \) given the parameters \( n \) and \( d \).

A direct consequence of the above theorem is stated in Corollary 3.2.1. In this discussion, we describe the action of the linear transformation
\[
\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
on a polynomial \( f(x, y) \) in the following manner:
\[
(\phi \cdot f)(x, y) = f((x y)\phi) = f(ax + cy, bx + dy).
\]

**Corollary 3.2.1.** Let \( n \) be a positive integer and \( 2 \leq d \leq n \) be given. Consider \( \sigma \) to be the transformation associated with the MacWilliams identity so that
\[
\sigma = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 1 \\ m-1 & -1 \end{pmatrix}.
\]

Then \( \sigma \cdot M_{n,d} = m^{n/2-d+1} M_{n,n-d+2} \).

By writing
\[
M_{n,d} = x^n + \sum_{w=d}^{n} A_w x^{n-w} y^w,
\]
where \( A_w \) is given by (6), we see that \( \{M_{n,2}, M_{n,3}, \ldots, M_{n,n}\} \) is linearly independent. Moreover, if
\[
W(x, y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i
\]
is a weight enumerator of degree \( n \) with minimum distance \( d \), then
\[
W(x, y) = p_d M_{n,d} + p_{d+1} M_{n,d+1} + \cdots + p_n M_n,
\] (12)
for some constants \( p_i \), that is, the above set of MDS weight enumerators form a basis for any weight enumerators of given degree and minimum distance. It is now our interest to relate the coefficients \( p_i \) in the expansion (12) with the zeta polynomials of the underlying code.

**Lemma 3.3.** The zeta polynomial associated with \( M_{n,d} \) for \( 2 \leq d \leq n \) is \( P(T) = 1 \).

**Proof.** We expand \( \frac{(xT + (1 - T)y)^n}{(1 - T)(1 - mT)} \) using

\[
\frac{1}{(1 - T)(1 - mT)} = \left( \sum_{i=0}^{\infty} T^i \right) \left( \sum_{j=0}^{\infty} m^j T^j \right) = \frac{1}{m - 1} \sum_{l=0}^{\infty} (m^{l+1} - 1)T^l
\]

and

\[
(xT + y(1 - T))^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i (1 - T)^i
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^j \binom{n}{i} \binom{i}{j} x^{n-i} y^i T^{n-i+j}.
\]

Hence, by changing the index using \( j = a - n + i \) we obtain

\[
\frac{(xT + y(1 - T))^n}{(1 - T)(1 - mT)} = \frac{1}{m - 1} \left( \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^j \binom{n}{i} \binom{i}{j} x^{n-i} y^i T^{n-i+j} \right) \times \left( \sum_{l=0}^{\infty} (m^{l+1} - 1)T^l \right)
\]

\[
= \frac{1}{m - 1} \left( \sum_{a=0}^{n} \sum_{i=a-n}^{n} (-1)^{a+n+i} \binom{n}{i} \binom{i}{n-a} x^{n-i} y^i T^a \right) \times \left( \sum_{l=0}^{\infty} (m^{l+1} - 1)T^l \right)
\]

\[
= \frac{1}{m - 1} \left( \sum_{a=0}^{n} \left( \sum_{i=a-n}^{n} (-1)^{a+n+i} \binom{n}{i} \binom{i}{n-a} x^{n-i} y^i T^a \right) \right) \times \left( \sum_{l=0}^{\infty} (m^{l+1} - 1)T^l \right).
\]

The last expression in the above computation can be simplified further by writing

\[
\frac{1}{m - 1} \sum_{k=0}^{\infty} \left( \sum_{a+l=k} \left( \sum_{i=a-n}^{n} (-1)^{a+n+i} \binom{n}{i} \binom{i}{n-a} x^{n-i} y^i (m^{l+1} - 1) \right) T^k.\]
Now, the coefficient of $x^{n-i}y^i$ in the expansion

$$\left[ T^{n-d} \right] \frac{(xT + y(1-T))^n}{(1-T)(1-mT)} (m-1)$$

is

$$\sum_{j=d}^{i} (-1)^{j-d} \binom{n}{i} \binom{i}{j-d} (m^{i-j+1} - 1).$$

When $k = i + d - j$, this coefficient becomes

$$\sum_{k=d}^{i} (-1)^{k-i} \binom{n}{i} \binom{i}{i-k} (m^{k-d+1} - 1) = \sum_{k=d}^{i} (-1)^{k-i} \binom{n}{k} \binom{n-k}{i-k} (m^{k-d+1} - 1).$$

The last of this expansion tells us that

$$M_{n,d} - x^n = \left[ T^{n-d} \right] \frac{(xT + y(1-T))^n}{(1-T)(1-mT)} (m-1)$$

$$= \left[ T^{n-d} \right] \frac{1}{(1-T)(1-mT)} (xT + y(1-T))^n (m-1).$$

Hence, the zeta polynomial corresponding to $M_{n,d}$ is given by $P(T) = 1$.

It is now clear to us how the zeta polynomial of a code is related to its weight enumerator written as a linear combination of the $M_{n,i}$. Thus, if

$$W_C(x, y) = p_d M_{n,d} + p_{d+1} M_{n,d+1} + \cdots + p_n M_{n,n},$$

then the zeta polynomial of $C$ is

$$P(T) = p_d + p_{d+1} T + \cdots + p_n T^n.$$
LEMMA 4.1. Let $C$ be a self-dual code over a finite commutative QF ring of order $m$. Then the zeta polynomial $P(T)$ of degree $2g$ of $C$ satisfies

$$P(T) = m^g T^{2g} P(1/mT). \tag{13}$$

The value $g$ is referred to as the genus of the code.

Proof. Let $W$ be the weight enumerator of $C$ written as

$$W = \sum_{i=d}^{n} p_i M_{n,i}.$$ 

Put

$$\sigma = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 1 \\ m-1 & -1 \end{pmatrix}.$$ 

Since $C$ is self-dual, then

$$\sigma \cdot W = W.$$ 

Computing both sides of this equation will yield the following:

$$\sigma \cdot W = \sum_{i=d}^{n} p_i (\sigma \cdot M_{n,i})$$

$$= \sum_{i=d}^{n} p_i (m^{n/2-i+1} M_{n,n-i+2})$$

$$= \sum_{j=1}^{n-d+2} p_{n+2-j} (m^{-n/2+j-1} M_{n,j})$$

$$= \sum_{i=d}^{n} p_i M_{n,i}$$

$$= W.$$ 

Thus, by the linear independence of the MDS weight enumerators $\{M_{n,j}\}_{1 \leq j \leq n}$, we see that $p_i = 0$ for $i > n - d + 2$ and

$$p_i = m^{-n/2-i+1} p_{n-i+2} \text{ for } d \leq i \leq n - d + 2.$$ 

These conditions on $p_i$ are equivalent to the relation

$$P(T) = m^g T^{2g} P(1/mT).$$

It is also clear that the degree of $P(T)$ becomes $r = 2g = n - 2d + 2$. 

In the following theorem, we relate zeta polynomials of self-dual codes to self-inversive polynomials.

THEOREM 4.2. The zeta polynomial of a self-dual code written as a function of $t = \sqrt{mT}$ is self-inversive.
Proof. Apply the substitution \( t = \sqrt{mT} \) to the functional equation
\[
P(T) = m^8 T^{2g} P(1/mT)
\]
and obtain
\[
f(t) = P(t/\sqrt{m}) = t^{2g} P(1/\sqrt{mt}).
\]
We see that \( t^{2g} f(1/t) = P(t/\sqrt{m}) \). Hence,
\[
f(t) = t^{2g} f(1/t).
\]
Since all the coefficients involved in these polynomials are real numbers, then we see that the zeta polynomial \( P(T) \), written in terms of \( t = \sqrt{mT} \), is indeed a self-inversive polynomial. \( \Box \)

5. Rings of order four

In the succeeding discussions, we limit ourselves to certain rings of order four. In particular, we consider the three rings \( \mathbb{Z}_4 \), \( \mathbb{F}_2 + u\mathbb{F}_2 \), and \( \mathbb{F}_2 + v\mathbb{F}_2 \) and use a common notation \( R \) to denote any of them. This section will discuss known results on codes defined over \( R \). From here on, any code \( C \) is taken as a code over \( R \).

5.1. Weight functions and weight enumerators

Three popular weight functions assigned to the codewords of \( C \) are the Hamming weight, the Euclidean weight and the Lee weight. The Hamming weight of a codeword is equal to the number of its non-zero components. Table 1 gives the Lee weight of each of the elements of the three rings \( \mathbb{Z}_4 \), \( \mathbb{F}_2 + u\mathbb{F}_2 \), and \( \mathbb{F}_2 + v\mathbb{F}_2 \).

Another weight function of codes over \( \mathbb{Z}_4 \) and \( \mathbb{F}_2 + u\mathbb{F}_2 \) is the Euclidean weight function, which is simply defined to be the square of the Lee weight. For example, the Euclidean weight of \( u \in \mathbb{F}_2 + u\mathbb{F}_2 \) is \( 2^2 = 4 \). Now, the Lee weight (respectively Euclidean weight) of a codeword is the rational sum of the Lee weights (respectively Euclidean weights) of its components. The Hamming and Lee distances of two codewords \( x \) and \( y \) are the Hamming and Lee weights of \( x - y \), respectively.

Two codes are equivalent if one can be obtained from the other by permuting the coordinates and, if necessary, by interchanging the elements 1 and 3 of some coordinates for codes over \( \mathbb{Z}_4 \) and interchanging 1 and \( 1 + u \) of some coordinates for codes over \( \mathbb{F}_2 + u\mathbb{F}_2 \). In the case of \( \mathbb{F}_2 + v\mathbb{F}_2 \), two codes \( C \) and \( C' \) can be permutation equivalent or \( C \) can be permutation equivalent to the code obtained from \( C' \) by interchanging \( v \) and \( 1 + v \) in all coordinates. Two of the known weight enumerators of codes over \( R \) are the Hamming weight enumerator and the symmetrized weight enumerator. These are defined as follows.

<table>
<thead>
<tr>
<th>Ring</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_4 )</td>
<td>0 1 2 3</td>
</tr>
<tr>
<td>( \mathbb{F}_2 + u\mathbb{F}_2 )</td>
<td>0 1 ( u ) ( 1 + u )</td>
</tr>
<tr>
<td>( \mathbb{F}_2 + v\mathbb{F}_2 )</td>
<td>0 ( v ) 1 ( 1 + v )</td>
</tr>
<tr>
<td>Lee weight</td>
<td>0 1 2 1</td>
</tr>
</tbody>
</table>
Hamming weight enumerator of a code of length \(n\) over \(\mathbb{Z}_4\), \(\mathbb{F}_2 + u\mathbb{F}_2\), or \(\mathbb{F}_2 + v\mathbb{F}_2\):

\[
\text{hwe}_C(x, y) = \sum_{c \in C} x^{n - \text{wt}(c)} y^{|c|}.
\]

The symbol \(\text{wt}(c)\) refers to the Hamming weight of the codeword \(c\).

Symmetrized weight enumerator for a code over \(\mathbb{Z}_4\) or \(\mathbb{F}_2 + u\mathbb{F}_2\):

\[
\text{swe}_C(x, y, z) = \sum_{c \in C} x^{n_0(c)} y^{n_1(c)} z^{n_2(c)}.
\]

The symbol \(n_i(c)\) corresponds to the number of components of \(c \in C\) that are \(i\) in \(\mathbb{Z}_4\).

For \(\mathbb{F}_2 + u\mathbb{F}_2\), \(n_0(c), n_1(c), n_2(c),\) and \(n_3(c)\) correspond to the number of components in \(c\) that are 0, 1, \(u\), and \(1 + u\), respectively.

Symmetrized weight enumerator for a code over \(\mathbb{F}_2 + v\mathbb{F}_2\):

\[
\text{swe}_C(x, y, z) = \sum_{c \in C} x^{n_0(c)} y^{n_1(c)} z^{n_2(c)}.
\]

The symbol \(n_i(c)\) corresponds to the number of components of \(c \in C\) that have Lee weight \(i\) in \(\mathbb{F}_2 + v\mathbb{F}_2\).

MacWilliams identities relate the weight enumerators of a code and its dual. The following are known for all codes over commutative rings of order four:

1. \(\text{hwe}_{C^\perp}(x, y) = \frac{1}{|C|} \text{hwe}_C(x + 3y, x - y);\)
2. \(\text{swe}_{C^\perp}(x, y, z) = \frac{1}{|C|} \text{swe}_C(x + 2y + c, x - z, x - 2y + z).\)

Self-dual codes over \(\mathbb{Z}_4\) containing the all-one vector and with the property that all Euclidean weights are divisible by eight are called Type II codes. For codes over \(\mathbb{F}_2 + u\mathbb{F}_2\), we say that a self-dual code is Type II if all the Lee weights are multiples of four. Self-dual codes over \(R\) are said to be Type I if they are not Type II. A certain family of self-dual codes, called Type IV codes, is the main focus of this paper. These are self-dual codes satisfying the property that each codeword has even Hamming weight. If a Type IV code is Type II as well, we shall call it a Type IV–II code; if it is Type I as well, we call it Type IV–I.

5.2. Generator matrices

An efficient way of describing the codewords of a code is through the use of a generator matrix, which basically generates all the members of a code given its length \(n\). Generally speaking, if \(G\) is a \(k \times n\) matrix with entries from \(R\), then all codewords of the code associated with \(G\) can be computed as \(\alpha_k \cdot G\), where \(\alpha_k\) runs over all \(k\)-tuples from \(R^k\). The following are known facts about the generator matrices of \(\mathbb{Z}_4\) and \(\mathbb{F}_2 + u\mathbb{F}_2\).

Any code over \(\mathbb{Z}_4\) is permutation equivalent to a code \(C\) with generator matrix of the form

\[
\begin{bmatrix}
I_k & A & B_1 + 2B_2 \\
0 & 2I_k & 2D
\end{bmatrix},
\]

where \(A, B_1, B_2,\) and \(D\) are \((1, 0)\) matrices (that is, matrices whose entries are either 0 or 1). This code is said to be of type \(4^k/2^{k_2}\).
Two binary codes $C^{(1)}$ and $C^{(2)}$ called residue code and torsion code, respectively, are associated with the code $C$. Their generator matrices are given by

$$G_{C^{(1)}} = \begin{bmatrix} I_k & A & B_1 \end{bmatrix},$$

and

$$G_{C^{(2)}} = \begin{bmatrix} I_k & A & B_1 \\ 0 & I_k & D \end{bmatrix}.$$  \hspace{1cm} (14)

Any code over $\mathbb{F}_2 + u\mathbb{F}_2$ is permutation equivalent to a code $C$ with generator matrix of the form

$$\begin{bmatrix} I_k & A & B_1 + uB_2 \\ 0 & uI_k & uD \end{bmatrix},$$

where $A$, $B_1$, $B_2$, and $D$ are matrices over $\mathbb{F}_2$. This code is said to be of type $4^{k_1}2^{k_2}$.

Again, two binary codes – the residue code $C^{(1)}$ and torsion code $C^{(2)}$ – are associated with any code $C$ over $\mathbb{F}_2 + u\mathbb{F}_2$. Their generator matrices also take the same form as (14) and (15). For both rings $\mathbb{Z}_4$ and $\mathbb{F}_2 + u\mathbb{F}_2$, it is known that, if a code $C$ is self-dual, then $C^{(1)}$ is doubly even and $C^{(2)} = C^{(1)\perp}$.

5.3. Facts about Type IV codes over $R$

We list here some important facts about Type IV codes over $R$ as discussed in [3]. We skip discussing the proofs of these results since the details can be found in that paper.

(1) Let $C$ be a code over $\mathbb{Z}_4$.
   
   (a) The number of 2s in each codeword of $C$ is even.
   
   (b) The residue code $C^{(1)}$ contains the all-one vector 1.
   
   (c) A Type IV code $C$ is Type IV–II if and only if all the Hamming weights of $C^{(1)}$ are multiples of eight.
   
   (d) If $C$ is a Type IV code of length $n$ then all its Lee weights are divisible by four and its Gray image $\phi(C)$ is a self-dual Type II binary code.
   
   (e) A Type IV code over $\mathbb{Z}_4$ of length $n$ exists if and only if $n \equiv 0 \pmod{4}$.
   
   (f) There is no Type IV code of type $4^{n/2}$.

(2) Let $C$ be a code over $\mathbb{F}_2 + u\mathbb{F}_2$.

   (a) The residue code $C^{(1)}$ contains the all-one vector 1.
   
   (b) A Type IV code $C$ over $\mathbb{F}_2 + u\mathbb{F}_2$ is Type IV–II if and only if $C^{(1)}$ is doubly even.

5.4. Invariant rings of Type IV codes over $R$

We consider here the invariant rings of Type IV codes over $R$. Since Type IV codes have even Hamming weights and are self-dual, we investigate first the invariant rings of groups
generated by combinations of the following matrices:

\[
M_1 = \frac{1}{2} \begin{pmatrix}
1 & 2 & 1 \\
1 & 0 & -1 \\
1 & -2 & 1
\end{pmatrix},
\]

\[
M_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & i & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
M_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\]

\[
M_4 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
M_5 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

Denote by \( G_1 \) the group generated by \( M_1, M_2 \) and \( M_3 \) and by \( G_2 \) the group generated by \( M_1, M_4 \) and \( M_5 \). Then the following are the computed Molien series associated with \( G_1 \) and \( G_2 \), respectively:

\[
M_{G_1}(\lambda) = \frac{1}{(1 - \lambda^4)(1 - \lambda^8)(1 - \lambda^{12})},
\]

\[
M_{G_2}(\lambda) = \frac{1}{(1 - \lambda^2)(1 - \lambda^4)(1 - \lambda^6)}.
\]

Upon defining

\[
f_n = \frac{1}{2}[(x + z)^n + (x - z)^n] + 2^{n-1} y^n,
\]

the following lemma can be shown.

**Lemma 5.1.** (Invariant rings of \( G_1 \) and \( G_2 \)) One has:

\[
\mathbb{C}[x, y, z]^{G_1} = \mathbb{C}[f_4, f_8, f_{12}], \quad \mathbb{C}[x, y, z]^{G_2} = \mathbb{C}[f_2, f_4, f_6].
\]

Note: A Type IV code over \( \mathbb{Z}_4 \) is invariant under the MacWilliams identity and so it is invariant under matrix \( M_1 \). Moreover, the number of 2s in each codeword is even so that it is invariant under \( M_3 \). Lastly, the number of 1s and 3s in each codeword is divisible by four and hence it is also invariant under \( M_2 \). On the other hand, a Type IV code over \( \mathbb{F}_2 + u\mathbb{F}_2 \) is invariant under \( M_1, M_4 \) and \( M_5 \).

**Lemma 5.2.** The weight enumerators of Type IV codes over \( R \) belong to the invariant ring \( \mathbb{C}[g_2, g_6] \) where

\[
g_2 = \frac{1}{2}[(x + y)^2 + (x - y)^2] + 2y^2 = x^2 + 3y^2
\]

and

\[
g_6 = \frac{1}{2}[(x + y)^6 + (x - y)^6] + 2y^5 = x^6 + 15x^4y^2 + 15x^2y^4 + 33y^6.
\]

**Proof.** Since a weight enumerator of a Type IV code over \( R \) satisfies the properties

1. \( A_C(x, y) = A_{C^\perp}(x, y) = \frac{1}{|C|} A_C(x + 3y, x - y) \),

2. \( A_C(x, y) = A_C(x, -y) \),

\[\square\]
then $A_C(x, y)$ is invariant under each of the following matrices:

$$M_7 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}, \quad M_8 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

The invariant ring of the group generated by matrices $M_7$ and $M_8$ yields

$$\mathbb{C}[x, y]^{(M_7, M_8)} = \mathbb{C}[g_2, g_6]. \qed$$

### 6. Zeta function of a code over $R$

Duursma introduced zeta polynomials and zeta functions of linear codes over finite fields. We extend this definition to codes over $R$ using the following definition. Note that the underlying weight function in this definition is the Hamming weight function.

**Definition 6.1.** Let $C$ be a code of length $n$ over $R$ with minimum Hamming distance $d$. The polynomial $P(T)$ of degree at most $n - d$ satisfying

$$\frac{P(T)}{(1 - T)(1 - 4T)} (xT + y(1 - T))^n = \cdots + \left(\frac{\text{hwe}_C(x, y) - x^n}{3}\right) T^{n-d} + \cdots$$  \hspace{1cm} (16)

is called the zeta polynomial associated with $C$, while the rational function

$$Z(T) = \frac{P(T)}{(1 - T)(1 - 4T)}$$

is called the zeta function of $C$.

Let $\sum_{i=0}^{\infty} a_i T^i$ be a formal power series in $\mathbb{C}[[T]]$. Using the notation $[T^k]$ to refer to the coefficient $a_k$ of $T^k$, equation (16) can be written as follows:

$$[T^{n-d}] \frac{P(T)}{(1 - T)(1 - 4T)} (xT + y(1 - T))^n = \frac{\text{hwe}_C(x, y) - x^n}{3}. \hspace{1cm} (17)$$

In order to compute the zeta polynomial $P(T)$, let $P_i(T) = \sum_{i=0}^{n-d} p_i T^i \in \mathbb{C}[T]$ so that

$$[T^{n-d}] P_i(T) (xT + y(1 - T))^n = \frac{1}{3} (\text{hwe}_C(x, y) - x^n). \hspace{1cm} (18)$$

We put

$$[T^i] P_i(T) = [T^i] \frac{P(T)}{(1 - T)(1 - 4T)} \quad (0 \leq i \leq n - d), \hspace{1cm} (19)$$

so that

$$[T^i] P_i(T) (1 - T)(1 - 4T) = [T^i] P(T) \quad (0 \leq i \leq n - d). \hspace{1cm} (20)$$

Then (18) will yield

$$\sum_{i=0}^{n-d} \binom{n}{n-d-i} p_i (x - y)^{n-d-i} y^{d+i} = \frac{1}{3} \sum_{i=d}^{n} A_i x^{n-i} y^{i}, \hspace{1cm} (21)$$

where $A_i$ is the number of codewords in $C$ that have Hamming weight $i$. 

As an example, consider the type $4^{1}2^{2}$ code $D_{4}^{⊕}$ over $\mathbb{Z}_{4}$ with generator matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 2 & 0 & 2 \\
0 & 0 & 2 & 2
\end{bmatrix}.
$$

This code is self-dual with Hamming weight enumerator given by $hwe_{D_{4}^{⊕}}(x, y) = x^{4} + 6x^{2}y^{2} + 9y^{4}$. This tells us that the minimum distance of $D_{4}^{⊕}$ is $d = 2$. Using (21), we obtain

$$
\sum_{i=0}^{2} \binom{4}{2 - i} p_i(x - y)^{2-i}y^{2+i} = 2x^2 + 3y^4.
$$

By solving for the values of $p_0$, $p_1$, and $p_2$, we get the zeta polynomial $P(T) = 1 - 2T + 4T^2$ corresponding to the code $D_{4}^{⊕}$.

The concept of zeta functions is known for algebraic curves over a finite field. In fact, if $C$ is a non-singular projective curve of genus $g$ over $\mathbb{F}_q$, then its congruence zeta function $Z(T)$ has rational representation

$$
Z(T) = \frac{P(T)}{(1 - T)(1 - qT)}.
$$

The numerator $P(T)$ is also referred to as the zeta polynomial corresponding to the non-singular projective curve $C$. One well-known property of such zeta functions is stated in the Hasse–Weil theorem, which states that all zeros of the zeta function have absolute value $1/\sqrt{q}$. This theorem is also known as the Riemann hypothesis for curves. As Duursma pointed out, it is then natural to ask whether this holds for the zeta functions of codes over rings. Thus, we define the Riemann hypothesis analogue (RHA) for codes over $R$ in the following manner. Note that the weight function involved is the Hamming weight function.

**Definition 6.2.** A self-dual code over $R$ satisfies the **Riemann hypothesis analogue** (or has the **RHA property**) if all the zeros of the corresponding zeta polynomial have absolute value $1/2$.

For illustration, let us refer to the self-dual code $D_{4}^{⊕}$ with zeta polynomial $P(T) = 1 - 2T + 4T^2 = (T + \frac{1}{4}(1 - i \sqrt{3}))(T - \frac{1}{4}(1 - i \sqrt{3}))$. Now, since each of the roots $\pm \frac{1}{4}(1 - i \sqrt{3})$ has absolute value $1/2$, then $D_{4}^{⊕}$ has the RHA property.

### 7. On extremal Type IV codes over $R$

We discuss here some properties of weight enumerators corresponding to Type IV codes with minimum distance satisfying the Singleton bound for codes defined over rings of order four. As seen in Section 3 when Theorem 3.1 is applied to codes over rings of order $m$ with respect to the Hamming distance, the bound obtained coincides with the Singleton upper bound that pertains to codes defined over finite fields. Thus, codes defined from alphabets (elements of a ring or a field) having the same cardinality have the same bound.

As was discussed in Section 5.3, the weight enumerators of Type IV codes over rings of order four belong to the invariant ring $\mathbb{C}[g_2, g_6]$ where $g_2 = x^2 + 3y^2$ and $g_6 = x^6 + 15x^4y^2 + 15x^2y^4 + 33y^6$. We use Duursma’s technique [7] in studying their properties.

For a given homogeneous polynomial $p \in \mathbb{C}[x, y]$, we use the notation $p(D)$ to refer to the differential operator defined by replacing each $x$ by $\partial/\partial x$ and each $y$ by
The idea is to look for two polynomials $p$ and $s$ in $\mathbb{C}[x, y]$ satisfying $s(x, y)|p(x, y)(D)A(x, y)$ for a weight enumerator $A(x, y)$. The following relations involving linear transformations are then used:

(1) $(u, v) = (x, y) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$;

(2) $(\partial/\partial x \quad \partial/\partial y) = (\partial/\partial u \quad \partial/\partial v) \begin{pmatrix} a & c \\ b & d \end{pmatrix}$;

(3) for a linear transformation $(u, v) = (x, y)\phi$, we have $A(x, y)\phi = A(u, v)$, $p(x, y)(D) = p((u, v)\phi^T)(D)$ and, moreover,

$$p((u, v)\phi^T)(D)A(u, v) = p(x, y)(D)A((x, y)\phi).$$

Now for a weight enumerator that corresponds to a Type IV code over $R$, we obtain the following observations.

(i) Each Hamming weight enumerator $A(x, y)$ of degree $n$ and minimum distance $d$ can be written in the form

$$A(x, y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i.$$  

With this form, it is easy to see that

$$y^{d-1}|y(D)A(x, y). \quad (22)$$

(ii) Because of self-duality, the weight enumerator of a Type IV code $C$ over $R$ has the property that $A_C^\perp(x, y) = A_C(x, y)$. Now using (22), we have $v^{d-1}|v(D)A(u, v)$ and by defining the linear transformation

$(x, y)\sigma = (u, v)$ where $\sigma = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$,

we obtain the relation

$$(x - y)^{d-1}|(3x - y)(D)A(x, y). \quad (23)$$

(iii) Since the Hamming weight of each codeword in a Type IV code over $R$ is even, then its weight enumerator $A(x, y)$ satisfies the property that $A(x, y) = A(x, -y)$. With the use of (23), and the linear transformation

$(u, v) = (x, y)\tau$ where $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

we obtain the relation

$$(x + y)^{d-1}|(3x + y)(D)A(x, y). \quad (24)$$

From the relations (22), (23) and (24), we arrive at the following result.

**Lemma 7.1.** *For a weight enumerator $A(x, y)$ corresponding to a Type IV code $C$ of length $n$ and minimum distance $d$, we have*

$$y^{d-3}(x^2 - y^2)^{d-3}|y(y^2 - 9x^2)(D)A(x, y). \quad (25)$$
For the following discussions, we consider a subset \( A \subseteq \mathbb{C}[g_2, g_6] \) with elements of the form

\[
A(x, y) = x^n + \sum_{i=0}^{n} A_i x^{n-i} y^i, \quad A_d \neq 0
\]

(26)

\[
= \sum_{j=0}^{\lfloor n/6 \rfloor} a_j g_2^{n/2 - 3j} g_6^j.
\]

(27)

We shall refer to such an element of \( A \) as a formal weight enumerator of degree \( n \) and minimum distance \( d \). Note that polynomials in \( A \) may or may not correspond to actual Type IV codes over \( R \).

Of utmost importance for us now are those polynomials in \( A \) that satisfy the equation

\[
d = 2 \lfloor n/6 \rfloor + 2,
\]

which gives the Singleton bound for the minimum distance of codes over \( R \) with given length \( n \) [13]. It is not difficult to establish the uniqueness of the polynomial in \( A \) of given degree \( n \) and minimum distance \( d = 2 \lfloor n/6 \rfloor + 2 \). From here on, we shall call polynomials of this type extremal weight enumerators or simply extremal polynomials. Note that, in the extremal case, we may put

\[
n = 6k + 2\ell, \quad \ell = 0, 1, 2,
\]

in order to obtain the relation

\[
n = 3(d - 2) + 2\ell, \quad \ell = 0, 1, 2.
\]

By applying Lemma 7.1, we look for the cofactor \( m(x, y) \) of \( y^{d-3}(x^2 - y^2)^{d-3} \) such that

\[
y^{d-3}(x^2 - y^2)^{d-3} m(x, y) = y(y^2 - 9x^2)(D)A(x, y),
\]

(28)

for a polynomial \( A(x, y) \in A \). By examining the degrees of both sides of equation (28), we obtain

\[
\deg m(x, y) = (n - 3) - (3d - 9) = n - 3d + 6.
\]

In the extremal case, we see that \( \deg m(x, y) = 2\ell, \ell = 0, 1, 2. \)

**Lemma 7.2.** An extremal polynomial \( A(x, y) \in A \) satisfies

\[
(d - 2)3A_d y^{d-3}(x^2 - y^2)^{d-3}(x^2 + 3y^2)^{\ell} = y(y^2 - 9x^2)(D)A(x, y),
\]

(29)

where the symbol \( (a)_n \) means \( (a)_n = a(a + 1) \cdots (a + n - 1) \).

**Proof.** We note that by writing

\[
A(x, y) = x^n + \sum_{i=2k}^{n/2} A_{2i} x^{n-2i} y^{2i}
\]

so that its minimum distance is \( d = 2k \), we obtain

\[
y(y^2 - 9x^2)(D)A(x, y) = (d - 2)3A_d x^{n-d} y^{d-3} + \sum_{j=0}^{(n-d)/2-1} c_j x^{n-d-2j} y^{d-1+2j},
\]
where
\[ c_j = (d + 2j)^3A_{d+2j+2} - 9(d + 2j)(n - d - 1 - 2j)A_{d+2j}. \]

Thus, we can express \( y(y^2 - 9x^2)(D)A(x, y) \) as
\[ y(y^2 - 9x^2)(D)A(x, y) = (d - 2)^3A_d y^{d-3}(x^2 - y^2)^{d-3}m_1(x, y), \]
so that the polynomial \( m_1(x, y) \) that we are seeking becomes monic with degree equal to \( 2\ell, \ell = 0, 1, 2 \). It is easy to see that \( m_1(x, y) \) is homogeneous satisfying \( m_1(x, y) = m_1(x, -y) \), that is, every exponent of \( y \) in its expansion is even. Moreover, from the fact that the polynomial \( y(y^2 - 9x^2) \) is \( \sigma^T \)-invariant, we know that \( m_1(x, y) \) is \( \sigma \)-invariant. When \( \ell = 0 \), it follows that \( m_1(x, y) = 1 \). Now, in order to solve for \( m_1(x, y) \) for \( \ell = 1, 2 \), we put
\[ m_1(x, y) = \begin{cases} x^2 + Ay^2 & \text{if } \ell = 1, \\ x^4 + Ax^2y^2 + By^4 & \text{if } \ell = 2. \end{cases} \]

Upon solving for the unknowns \( A \) and \( B \), we obtain
\[ m_1(x, y) = (x^2 + 3y^2)^\ell, \quad \ell = 0, 1, 2, \]
which gives the desired result (29).

We now examine zeta polynomials that are associated with extremal polynomials from \( A \). We associate a zeta polynomial \( P(T) \) to a polynomial \( A(x, y) \) in \( A \) so that the following equation is satisfied:
\[ \left[ T^{n-d} \right] \frac{P(T)}{(1 - T)(1 - 4T)}(xT + y(1 - T))^n = \frac{A(x, y) - x^n}{3}. \]  

(30)

We define \( p(x, y) = y(y^2 - 9x^2) \) and apply the differential operator \( p(x, y)(D) \) on both sides of (30). This will then yield
\[ \left[ T^{n-d} \right](n - 2)^3P(T)(1 + 2T)(xT + y(1 - T))^n - \frac{1}{3}(d - 2)^3A_d y^{d-3}(x^2 - y^2)^{d-3}(x^2 + 3y^2)^\ell. \]  

(31)

Using the substitutions \( x = 1 + T \) and \( y = T \) and the equation
\[ Q(T) = \sum_{i=0}^{n-2d+3} q_i T^i = P(T)(1 + 2T), \]
we obtain
\[ \sum_{i=0}^{n-2d+3} q_i \left( \frac{n - 3}{d - 3 + i} \right) T^i T^{d-3} = \frac{1}{3}(d - 2)^3A_d (1 + 2T)^{d-3}(1 + 2T + 4T^2)^\ell. \]  

(33)

By dividing both sides of (33) by \( T^{d-3} \), we obtain
\[ \sum_{i=0}^{n-2d+3} q_i \left( \frac{n - 3}{d - 3 + i} \right) T^i = \frac{1}{3}(d - 2)^3A_d (1 + 2T)^{d-3}(1 + 2T + 4T^2)^\ell. \]  

(34)

Put \( m = d - 3 \) so that
\[ n - 3 = 3(d - 3) + 2\ell = 3m + 2\ell. \]
Then (34) can be written as
\[ \sum_{i=0}^{m+2\ell} q_i \binom{3m+2\ell}{m+i} T^i = \frac{1}{3} (d-2)^3 A_d (1 + 2T)^{m} (1 + 2T + 4T^2)^\ell. \] (35)

and in the special case \( \ell = 0 \) we obtain
\[ \sum_{i=0}^{m} q_i \binom{3m}{m+i} T^i = \frac{1}{3} (d-2)^3 A_d (1 + 2T)^{m}. \] (36)

At this point, we use a result from [7, Section 5.3], which will lead us to the proof of Theorem 7.3. By letting \( \tau = 2T \) we can write (36) as
\[ \sum_{i=0}^{m} q_i \binom{3m}{m+i} \left( \frac{\tau}{2} \right)^i = K (1 + \tau)^{m}. \]

With \( Q(T) = (1 + 2T) P(T) = R(\tau) \), it can be shown that all zeros of \( Q(T) \) have absolute value 1/2. Readers can study [4] to learn more about this topic on zeros of hypergeometric polynomials.

**Theorem 7.3.** The zeta polynomial of an extremal polynomial in \( A \) of degree \( n = 6k \), \( k > 0 \), satisfies the RHA property.

Assume that \( P(T) \) is a zeta polynomial that corresponds to an extremal polynomial \( A(x, y) \) in \( A \) with degree \( n = 6k \), \( k > 0 \). Then the minimum distance of this polynomial is \( d = 2k + 2 \) and it satisfies the equation
\[ [T^{n-d}] \frac{P(T)}{(1-T)(1-4T)} (xT + y(1-T))^n = A(x, y) - x^n. \] (37)

Define
\[ p_1(x, y) = \frac{1}{3(n-1)^2} (3x^2 + y^2) \]
and apply the differential operator \( p_1(x, y)(D) \) on both sides of (37). We obtain
\[ [T^{(n-2)-(d-2)}] \frac{P(T)(1 - 2T + 4T^2)/3}{(1-T)(1-4T)} (xT + y(1-T))^{n-2} = \frac{A_1(x, y) - x^{n-2}}{3}, \] (38)

where \( A_1(x, y) = p_1(x, y)(D)(A(x, y)) \). Observe that \( A_1(x, y) \) is the unique polynomial in \( A \) with degree \( n - 2 \) and minimum distance \( d - 2 \). This comes from the fact that \( p_1(x, y) \) is \( \sigma_T \)-invariant so that \( p_1(x, y)(D)A(x, y) \) yields a polynomial that is \( \sigma \)-invariant. Moreover, the resulting polynomial \( p_1(x, y)(D)A(x, y) \) has minimum distance \( d - 2 \) and it is clear that the exponent of \( y \) in its expansion is always even. Now based on (38), the zeta polynomial of \( A_1(x, y) \) is given by
\[ P_1(T) = \frac{1}{3} P(T)(1 - 2T + 4T^2). \]

This observation then implies the following as a corollary to Theorem 7.3.

**Corollary 7.3.1.** The zeta polynomial corresponding to an extremal polynomial in \( A \) of degree \( n = 6k - 2 \), \( k > 0 \), satisfies the RHA property.
7.1. Computational results for the case \( n = 6k - 4 \)

Although no direct proof is given to support the claim that the RHA works for all even degrees \( n \), we verify for large values of \( n \), where \( n \equiv 2 \pmod{6} \), that the zeta polynomials associated with weight enumerators of Type IV codes have roots with absolute value \( 1/2 \). Using Mathematica, the author is able to show that the RHA is satisfied for this remaining case where \( n \leq 500 \). The following computation can be used in order to prove the satisfaction of RHA in all cases.

Consider the extremal formal weight enumerator of degree \( n = 50 \equiv 2 \pmod{6} \) given by
\[
A(x, y) = x^{50} + 17665725x^{32}y^{18} + 263312280x^{30}y^{20} + 5024262600x^{28}y^{22}
+ 60932396700x^{26}y^{24} + 548391570300x^{24}y^{26} + 3608521110600x^{22}y^{28}
+ 17232025872600x^{20}y^{30} + 59433734900925x^{18}y^{32}
+ 145851800500350x^{16}y^{34} + 250060528842600x^{14}y^{36}
+ 291301768812600x^{12}y^{38} + 221848653702060x^{10}y^{40}
+ 104349325657500x^{8}y^{42} + 27798443379000x^{6}y^{44}
+ 3625599033000x^{4}y^{46} + 173602582650x^{2}y^{48} + 1273241133y^{50}.
\]
The corresponding zeta polynomial is
\[
P(T) = \frac{1}{8268616329}(2697 + 37758T + 28240272^2 + 1476216T^3 + 6001128T^4
+ 200445127^5 + 567903047^6 + 1390519047^7 + 2971979527^8
+ 5562076167^9 + 9086448647^10 + 12828487687^11 + 15362887687^12
+ 15116451847^13 + 11567185927^14 + 6186270727^15 + 1767505927^16).
\]

By letting \( t = 2T \), we can write the above equation as
\[
f(t) = \frac{t^8}{8268616329}\left(\frac{2321859}{2} + 1086343(t + t^{-1}) + \frac{1774697}{2}(t^2 + t^{-2})
+ 626391(t^3 + t^{-3}) + \frac{750141}{2}(t^4 + t^{-4}) + 184527(t^5 + t^{-5})
+ \frac{141201}{2}(t^6 + t^{-6}) + 18879(t^7 + t^{-7}) + 2697(t^8 + t^{-8})\right).
\]

Now, if we let \( t = e^{i\theta} \) so that \( 2\cos m\theta = e^{im\theta} + e^{-im\theta} \) (\( m \in \mathbb{Z} \)), we obtain the equation
\[
g(\theta) = \frac{e^{8i\theta}}{8268616329}\left(\frac{2321859}{2} + 2172686\cos \theta + 1774697\cos 2\theta + 1252782\cos 3\theta
+ 750141\cos 4\theta + 369054\cos 5\theta + 141201\cos 6\theta
+ 37758\cos 7\theta + 5394\cos 8\theta\right).
\]
By solving for the roots of the last equation, we get

\[ \theta = \pm 1.183\,04, \pm 1.5147, \pm 1.776\,68, \pm 2.018\,79, \pm 2.251, \pm 2.477\,48, \pm 2.7005, \]
\[ \pm 2.921\,51. \]

All these zeros yield the absolute value

\[ |2T| = |t| = |e^{i\theta}| = 1, \]

so that it can be seen that the original zeta polynomial \( P(T) \) has roots all of whose absolute values are equal to 1/2. Thus, this implies the satisfaction of the RHA for the case when \( n = 50 \).

8. Zeta polynomials of some classified Type IV codes

We show here the behavior of classified Type IV codes over the rings \( \mathbb{Z}_4 \) and \( \mathbb{F}_2 + u\mathbb{F}_2 \) as enumerated in [3]. In particular, we include a complete description of the family of Klemm codes whose zeta polynomials behave in a consistent manner. We also show some examples that belong to the family \( C(m, r) \), which carry the same feature as other Type IV codes. Note that the codes included in this section are not necessarily extremal.

8.1. Classified Type IV codes \( \mathbb{Z}_4 \) and \( \mathbb{F}_2 + u\mathbb{F}_2 \)

In Tables 2–5 that follow, the symbol \( \times \) is used to indicate that all roots of the computed zeta polynomial have absolute value 1/2 except for two other positive roots whose product is 1/4; the symbol \( \checkmark \) is used to indicate satisfaction of the RHA. It is also indicated whether the listed code is Type I or Type II as defined in Section 5. Moreover, we identify an extremal case by placing the symbol \( (\text{ext}) \) beside the name of the code. Observe that these codes that satisfy extremality have the RHA property. But there are also a few non-extremal cases that demonstrate RHA satisfaction.

### Table 2. Classified Type IV \( \mathbb{Z}_4 \)-codes up to length 16.

<table>
<thead>
<tr>
<th>Length</th>
<th>Mallows–Sloane bound</th>
<th>Code</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>( K_4 ) (ext)</td>
<td>2</td>
</tr>
<tr>
<td>8 (Type I)</td>
<td>4</td>
<td>( K_4^2 )</td>
<td>2</td>
</tr>
<tr>
<td>8 (Type II)</td>
<td>4</td>
<td>( K_8 )</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>( K_4^3, K_4 + K_8, K_{12}, [12, 3] - 3d4b )</td>
<td>2</td>
</tr>
<tr>
<td>16 (Type I)</td>
<td>6</td>
<td>( K_4^4, K_4^2 + K_8, K_4 + K_{12}, K_4 + [12, 3] - 3d4b, C_{16,5}, C_{16,9} )</td>
<td>2</td>
</tr>
<tr>
<td>16 (Type II)</td>
<td>6</td>
<td>( K_8^2, K_{16}, 3_-, f3, 4_-f4, 5_-f5^* )</td>
<td>2, 4*</td>
</tr>
</tbody>
</table>
Table 3. Zeta polynomials for codes in Table 2.

<table>
<thead>
<tr>
<th>Code</th>
<th>Zeta polynomial $P(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_4$ (ext.)</td>
<td>$\frac{1}{5}(1 - 2T + 4T^2)$</td>
</tr>
<tr>
<td>$K_4^2$</td>
<td>$\frac{1}{5}(5 - 10T - 16T^2 - 40T^3 - 64T^4 - 160T^5 + 320T^6)$</td>
</tr>
<tr>
<td>$K_8$</td>
<td>$\frac{1}{5}(1 - 2T - 4T^2 - 8T^3 - 16T^4 - 32T^5 + 64T^6)$</td>
</tr>
<tr>
<td>$K_4^2$</td>
<td>$\frac{1}{5}(7 - 14T - 28T^2 - 56T^3 - 104T^4 - 208T^5 + 416T^6)$</td>
</tr>
<tr>
<td>$K_4 + K_8$</td>
<td>$\frac{1}{1485}(255 - 510T - 1028T^2 - 2040T^3 - 3912T^4 - 7760T^5)$</td>
</tr>
<tr>
<td></td>
<td>$- 15 648T^6 - 32 640T^7 - 65 792T^8 - 130 560T^9 + 261 120T^{10})$</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>$\frac{1}{3}(1 - 2T - 4T^2 - 8T^3 - 16T^4 - 32T^5 + 64T^6 - 128T^7)$</td>
</tr>
<tr>
<td></td>
<td>$- 256T^8 - 512T^9 + 1024T^{10})$</td>
</tr>
<tr>
<td>$[12, 3] - 3d4b$</td>
<td>$\frac{1}{3655}(315 - 630T - 1316T^2 - 2520T^3 - 4224T^4$</td>
</tr>
<tr>
<td></td>
<td>$- 8000T^5 - 16 896T^6 - 40 320T^7 - 84 224T^8$</td>
</tr>
<tr>
<td></td>
<td>$+ 161 280T^9 + 322 560T^{10} )$</td>
</tr>
<tr>
<td>$K_4^2$</td>
<td>$\frac{1}{2295}(1 - 2T + 4T^2)(143 - 1188T^2 - 3520T^3 - 4320T^4)$</td>
</tr>
<tr>
<td>$K_4 + K_8$</td>
<td>$\frac{1}{2295}(1 - 2T + 4T^2)(143 - 1188T^2 - 3520T^3 - 4320T^4)$</td>
</tr>
<tr>
<td></td>
<td>$+ 1728T^5 + 13 760T^6 + 69 127T^7 - 69 127T^8 - 225 280T^9$</td>
</tr>
<tr>
<td></td>
<td>$- 304 128T^{10} + 585 728T^{12})$</td>
</tr>
<tr>
<td>$K_4 + [12, 3] - 3d4b$</td>
<td>$\frac{1}{2295}(1 - 2T + 4T^2)(1001 - 8250T^2 - 24 508T^3 - 30 648T^4)$</td>
</tr>
<tr>
<td></td>
<td>$+ 9408T^5 + 89 152T^6 + 37 632T^7 - 490 368T^8$</td>
</tr>
<tr>
<td></td>
<td>$- 1568 512T^9 - 2112 000T^{10} + 4100 096T^{12})$</td>
</tr>
<tr>
<td>$K_4 + K_8$</td>
<td>$\frac{1}{2295}(1 - 2T + 4T^2)(1001 - 8250T^2 - 24 508T^3 - 30 648T^4)$</td>
</tr>
<tr>
<td></td>
<td>$+ 9408T^5 + 89 152T^6 + 37 632T^7 - 490 368T^8$</td>
</tr>
<tr>
<td></td>
<td>$- 1568 512T^9 - 2112 000T^{10} + 4100 096T^{12})$</td>
</tr>
<tr>
<td>$C_{16, 5}$</td>
<td>$\frac{1}{3505}(1 - 2T + 4T^2)(5005 - 41 382T^2 - 12 280T^3)$</td>
</tr>
<tr>
<td></td>
<td>$- 152 508T^4 + 52 080T^5 + 459 200T^6 + 208 320T^7$</td>
</tr>
<tr>
<td></td>
<td>$- 2440 128T^8 - 7859 456T^9 - 105 93 792T^{10} + 20 500 480T^{12})$</td>
</tr>
<tr>
<td>$C_{16, 9}$</td>
<td>$\frac{1}{4075}(1 - 2T + 4T^2)(1001 - 8404T^2 - 24 816T^3 - 29 752T^4)$</td>
</tr>
<tr>
<td></td>
<td>$+ 15 455T^5 + 105 280T^6 + 61 824T^7 - 476 032T^8$</td>
</tr>
<tr>
<td></td>
<td>$- 1588 224T^9 - 2151 424T^{10} + 4100 096T^{12})$</td>
</tr>
<tr>
<td>$K_8^2$</td>
<td>$\frac{1}{6435}(1 - 2T + 4T^2)(1001 - 7920T^2 - 23 848T^3 - 32 496T^4)$</td>
</tr>
<tr>
<td></td>
<td>$- 3264T^5 + 55 360T^6 - 13 056T^7 - 519 396T^8$</td>
</tr>
<tr>
<td></td>
<td>$- 1526 272T^9 - 2027 520T^{10} + 4100 096T^{12})$</td>
</tr>
<tr>
<td>$K_{16}$</td>
<td>$\frac{1}{2145}(1 - 2T + 4T^2)(1 - 8T^2 - 24T^3 - 32T^4 + 64T^5 - 512T^6)$</td>
</tr>
<tr>
<td></td>
<td>$- 1536T^7 - 2048T^8 + 4096T^{12})$</td>
</tr>
<tr>
<td>$3_{f3}$</td>
<td>$\frac{1}{2145}(1 - 2T + 4T^2)(143 - 1100T^2 - 3344T^3 - 4808T^4)$</td>
</tr>
<tr>
<td></td>
<td>$- 1632T^5 + 4800T^6 + 6528T^7 - 76 928T^8 - 214 016T^9$</td>
</tr>
<tr>
<td></td>
<td>$- 281 600T^{10} + 585 728T^{12})$</td>
</tr>
<tr>
<td>$4_{f4}$</td>
<td>$\frac{1}{4075}(1 - 2T + 4T^2)(143 - 990T^2 - 3124T^3 - 5388T^4)$</td>
</tr>
<tr>
<td></td>
<td>$- 5712T^5 - 6080T^6 - 22 848T^7 - 86 208T^8 - 199 936T^9$</td>
</tr>
<tr>
<td></td>
<td>$- 253 440T^{10} + 585 728T^{12})$</td>
</tr>
<tr>
<td>$5_{f5}$</td>
<td>$\frac{1}{729}(1 - 2T + 4T^2)(11 + 22T - 58T^2 - 408T^3 - 1088T^4)$</td>
</tr>
<tr>
<td></td>
<td>$- 1632T^5 - 928T^6 + 1408T^7 + 2816T^8)$</td>
</tr>
</tbody>
</table>
TABLE 4. Classified Type IV $\mathbb{F}_2 + u\mathbb{F}_2$-codes up to length 16.

<table>
<thead>
<tr>
<th>Length</th>
<th>Mallows–Sloane bound</th>
<th>Code</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (Type I)</td>
<td>2</td>
<td>$K_2$ (ext)</td>
<td>2</td>
</tr>
<tr>
<td>4 (Type I)</td>
<td>2</td>
<td>$2K_2$ (ext)</td>
<td>2</td>
</tr>
<tr>
<td>4 (Type II)</td>
<td>2</td>
<td>$K_4$ (ext)</td>
<td>2</td>
</tr>
<tr>
<td>6 (Type I)</td>
<td>4</td>
<td>$K_6$, [6, 3]_3d2d, [6, 3]_3d2a, [6, 2]_d4d2a</td>
<td>2</td>
</tr>
<tr>
<td>8 (Type I)</td>
<td>4</td>
<td>$HMI_{8,1}$, $HMI_{8,2}$, $HMI_{8,3}$, $HMI_{8,4}$, $HMI_{8,5}$, $HMI_{8,6}$</td>
<td>2</td>
</tr>
<tr>
<td>8 (Type II)</td>
<td>4</td>
<td>$K_8$, [8, 2]_2d4, [8, 3]_d8a, [8, 4]_e8a (ext)</td>
<td>2, 4$^*$</td>
</tr>
<tr>
<td>10 (Type I)</td>
<td>4</td>
<td>$HMI_{10,1}$, $HMI_{10,2}$, ..., $HMI_{10,14}$</td>
<td>2</td>
</tr>
<tr>
<td>12 (Type I)</td>
<td>6</td>
<td>$HMI_{12}$</td>
<td>4</td>
</tr>
<tr>
<td>12 (Type II)</td>
<td>6</td>
<td>$A_{12,6}$, $A_{12,8}$, $B_{12,1}$, $C_{12,1}$, $C_{12,4}$, $C_{12,8}$, $C_{12,20}$, $E_{12,1}$, $E_{12,6}$, $F_{12,3}$, $EE_{12,1}$, $EE_{12,5}$, $DE_{12,1}$, $DE_{12,3}$</td>
<td>2</td>
</tr>
<tr>
<td>14 (Type I)</td>
<td>6</td>
<td>$HMI_{14}$</td>
<td>4</td>
</tr>
<tr>
<td>16 (Type II)</td>
<td>6</td>
<td>$C_{821}$, $C_{822}$, $C_{831}$, $C_{832}$, $C_{833}$, $C_{841}$, $C_{842}$, $C_{851}$</td>
<td>4</td>
</tr>
</tbody>
</table>

TABLE 5. Zeta polynomials for codes in Table 4.

<table>
<thead>
<tr>
<th>Code</th>
<th>Zeta polynomial $P(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_2$ (ext)</td>
<td>$\frac{1}{2}(1 - 2T + 4T^2)$</td>
</tr>
<tr>
<td>$2K_2$ (ext), $K_4$ (ext)</td>
<td>$\frac{1}{3}(1 - 2T - 4T^2 - 8T^3 + 16T^4)$</td>
</tr>
<tr>
<td>$K_6$</td>
<td>$\frac{1}{4}(1 + 4T^2)(1 - 2T + 4T^2)$</td>
</tr>
<tr>
<td>[6, 3]_3d2d</td>
<td>$\frac{1}{5}(1 - 2T - 3T^2 - 8T^3 + 16T^4)$</td>
</tr>
<tr>
<td>[6, 3]_3d2a</td>
<td>$\frac{1}{6}(1 - 2T - 2T^2 - 8T^3 + 16T^4)$</td>
</tr>
<tr>
<td>$HMI_{8,1}$</td>
<td>$\frac{5}{95}(5 - 10T - 2T^2 - 40T^3 - 8T^4 - 160T^5 + 320T^6)$</td>
</tr>
<tr>
<td>$HMI_{8,2}$</td>
<td>$\frac{1}{17}(2 - 4T - 5T^2 - 16T^3 - 20T^4 - 64T^5 + 128T^6)$</td>
</tr>
<tr>
<td>$HMI_{8,3}$, $HMI_{8,4}$, [8, 3]_d8a</td>
<td>$\frac{1}{35}(5 - 10T - 16T^2 - 40T^3 - 64T^4 - 160T^5 + 320T^6)$</td>
</tr>
<tr>
<td>$HMI_{8,5}$</td>
<td>$\frac{1}{27}(3T - 6T^2 - 11T^3 - 24T^4 - 44T^5 + 96T^6)$</td>
</tr>
<tr>
<td>$HMI_{8,6}$, [8, 2]_2d4</td>
<td>$\frac{1}{7}(5 - 10T - 9T^2 - 40T^3 - 36T^4 - 160T^5 + 320T^6)$</td>
</tr>
<tr>
<td>$K_8$</td>
<td>$\frac{1}{6}(1 - 2T - 4T^2 - 8T^3 - 16T^4 - 32T^5 + 64T^6)$</td>
</tr>
<tr>
<td>[8, 4]_e8a (ext)</td>
<td>$\frac{1}{7}(1 + 4T^2)$</td>
</tr>
<tr>
<td>$HMI_{10,1}$</td>
<td>$\frac{1}{2}(1 - 2T + 4T^2)(1 - 8T^2 - 24T^3 - 32T^4 + 64T^6)$</td>
</tr>
<tr>
<td>$HMI_{10,3}$</td>
<td>$\frac{1}{12}(1 - 2T + 4T^2)(49 - 384T^2 - 1160T^3 - 1536T^4 + 3136T^6)$</td>
</tr>
<tr>
<td>$HMI_{10,5}$</td>
<td>$\frac{1}{25}(1 - 2T + 4T^2)(17 - 132T^2 - 400T^3 - 528T^4 + 1088T^6)$</td>
</tr>
<tr>
<td>$HMI_{10,7}$, $HMI_{10,9}$, $HMI_{10,11}$, $HMI_{10,14}$</td>
<td>$\frac{1}{189}(1 - 2T + 4T^2)(7 - 48T^2 - 152T^3 - 192T^4 + 448T^6)$</td>
</tr>
<tr>
<td>$HMI_{10,6}$, $HMI_{10,12}$</td>
<td>$\frac{1}{105}(1 - 2T + 4T^2)(7 - 52T^2 - 160T^3 - 208T^4 + 448T^6)$</td>
</tr>
<tr>
<td>Code</td>
<td>Zeta polynomial $P(T)$</td>
</tr>
<tr>
<td>------------</td>
<td>---------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>HMI$1_{10,8}$</td>
<td>$\frac{1}{35}(1 - 2T + 4T^2)(1 - 6T^2 - 20T^3 - 24T^4 + 64T^6)$ ×</td>
</tr>
<tr>
<td>HMI$1_{10,9}$</td>
<td>$\frac{1}{35}(1 - 2T + 4T^2)(91 - 969T^2 - 2120T^3 - 2784T^4 + 5824T^6)$ ×</td>
</tr>
<tr>
<td>HMI$1_{10,13}$</td>
<td>$\frac{1}{35}(1 - 2T + 4T^2)(7 - 12T^2 - 80T^3 - 48T^4 + 448T^6)$ ×</td>
</tr>
<tr>
<td>HMI$1_{12}$</td>
<td>$\frac{1}{35}(1 - 2T + 4T^2)(1 + 2T + 6T^2 + 8T^3 + 16T^4)$ ✓</td>
</tr>
<tr>
<td>A$1_{12,6}$</td>
<td>$\frac{1}{35}(45 - 90T - 184T^2 - 360T^3 - 636T^4 - 1240T^5 - 2544T^6)$ ×</td>
</tr>
<tr>
<td>A$1_{12,8}$</td>
<td>$\frac{1}{35}(105 - 210T - 434T^2 - 840T^3 - 1116T^4 - 2120T^5)$ ×</td>
</tr>
<tr>
<td>B$1_{12,1}$</td>
<td>$\frac{1}{35}(75 - 150T - 314T^2 - 600T^3 - 906T^4 - 1700T^5 - 3624T^6)$ ×</td>
</tr>
<tr>
<td>C$1_{12,1}$</td>
<td>$\frac{1}{35}(315 - 630T - 1316T^2 - 2520T^3 - 4224T^4 - 8000T^5)$ ×</td>
</tr>
<tr>
<td>C$1_{12,4}$</td>
<td>$\frac{1}{35}(105 - 210T - 448T^2 - 840T^3 - 1200T^4 - 2176T^5)$ ×</td>
</tr>
<tr>
<td>C$1_{12,8}$, C$1_{12,20}$</td>
<td>$\frac{1}{35}(35 - 70T - 154T^2 - 280T^3 - 296T^4 - 408T^5 - 1184T^6)$ ×</td>
</tr>
<tr>
<td>E$1_{12,1}$</td>
<td>$\frac{1}{7}(1 - 2T - 4T^2 - 8T^3 - 16T^4 - 32T^5 - 64T^6 - 128T^7)$ ×</td>
</tr>
<tr>
<td>E$1_{12,6}$</td>
<td>$\frac{1}{7}(21 - 42T - 70T^2 - 168T^3 - 360T^4 - 832T^5 - 1440T^6)$ ×</td>
</tr>
<tr>
<td>F$1_{12,3}$</td>
<td>$\frac{1}{7}(21 - 42T - 98T^2 - 168T^3 - 132T^4 - 528T^6)$ ×</td>
</tr>
<tr>
<td>EE$1_{12,1}$</td>
<td>$\frac{1}{7}(7 - 14T - 28T^2 - 56T^3 - 104T^4 - 416T^6 - 896T^7)$ ×</td>
</tr>
<tr>
<td>EE$1_{12,5}$</td>
<td>$\frac{1}{7}(15 - 30T - 58T^2 - 120T^3 - 192T^4 - 400T^5 - 768T^6)$ ×</td>
</tr>
<tr>
<td>DE$1_{12,1}$</td>
<td>$\frac{1}{7}(255 - 510T - 1028T^2 - 2040T^3 - 3912T^4 - 7760T^5)$ ×</td>
</tr>
<tr>
<td>DE$1_{12,3}$</td>
<td>$\frac{1}{7}(525 - 1050T - 2072T^2 - 4200T^3 - 7368T^4 - 14960T^5)$ ×</td>
</tr>
<tr>
<td>HMI$1_{14}$</td>
<td>$\frac{1}{527}(6 + T^2 - 46T^3 - 42T^4 - 184T^5 + 16T^6 + 1536T^8)$ ×</td>
</tr>
<tr>
<td>C$8_{21}$</td>
<td>$\frac{1}{527}(1 - 2T + 4T^2)(11 + 22T - 58T^2 - 408T^3 - 1088T^4)$ ×</td>
</tr>
<tr>
<td>C$8_{22}$, C$8_{41}$</td>
<td>$\frac{1}{527}(1 - 2T + 4T^2)(121 + 242T - 566T^2 - 4200T^3)$ ×</td>
</tr>
<tr>
<td>C$8_{31}$</td>
<td>$\frac{1}{527}(1 - 2T + 4T^2)(209 + 418T - 1054T^2 - 7560T^3)$ ×</td>
</tr>
<tr>
<td>C$8_{32}$, C$8_{33}$, C$8_{42}$, C$8_{51}$</td>
<td>$\frac{1}{527}(1 - 2T + 4T^2)(11 + 22T - 46T^2 - 360T^3 - 960T^4)$ ×</td>
</tr>
</tbody>
</table>
8.2. Klemm codes over $\mathbb{Z}_4$ and $\mathbb{F}_2 + u\mathbb{F}_2$

We discuss here the behavior of zeta polynomials that correspond to a family of Type IV codes called Klemm codes $K_n$ defined over the rings $\mathbb{Z}_4$ and $\mathbb{F}_2 + u\mathbb{F}_2$. We note that the generator matrix of $K_n$ takes the following form:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
0 & z & 0 & 0 & \ldots & 0 & z \\
0 & 0 & z & 0 & \ldots & 0 & z \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & z & z
\end{bmatrix}.
$$

For the case of $\mathbb{Z}_4$, we set $z = 2$; and for $\mathbb{F}_2 + u\mathbb{F}_2$, we set $z = u$, where $u^2 = 0$. The weight enumerator corresponding to $K_n$ (for $\mathbb{Z}_4$, $n \equiv 0 \pmod{4}$, and for $\mathbb{F}_2 + u\mathbb{F}_2$, $n \equiv 0 \pmod{2}$) is given by

$$
w(x, y) = \frac{n}{2} \sum_{i=0}^{n/2} x^{n-2i} y^{2i} + 2^{n-1} y^n = \frac{1}{2} \left( (x + y)^n + (x - y)^n \right) + 2^{n-1} y^n. \quad (39)
$$

To determine the zeta polynomial corresponding to the weight enumerator of the Klemm code $K_n$ given in equation (39), we use the equality

$$
P_1(T) = \frac{P(T)}{(1 - T)(1 - 4T)}
$$

where $P_1(T) = \sum p_j T^j$. The idea is to look for the coefficients of $P_1(T)$ so that

$$
[T^i] P(T) = [T^i] P_1(T)(1 - T)(1 - 4T) \quad \text{for } 0 \leq i \leq n - 2. \quad (40)
$$

By Definition 6.1, we obtain the equation

$$
\sum_{j=0}^{n-2} p_j \binom{n}{j+2} (x - y)^{n-2-j} y^{j+2} = \frac{1}{3} \sum_{i=1}^{n/2} x^{n-2i} y^{2i} + 2^{n-1} y^n.
$$

This yields the system of equations

$$
\sum_{i=0}^{k-2} p_i (-1)^{k-2-i} \binom{n}{i+2} \binom{n-i-2}{j-i-2} = \begin{cases} 
0, & \text{if } k = 3, 5, \ldots, n - 1, \\
\frac{1}{3} \binom{n}{k}, & \text{if } k = 2, 4, \ldots, n - 2, \\
\frac{1}{3} (2^{n+1} + 1), & \text{if } k = n,
\end{cases}
$$

with solution

$$
p_0 = \frac{1}{3},
$$

$$
p_r = p_{r-1} + \frac{2^r}{3}, \quad r \geq 1.
$$
Hence, we obtain
\[ P_1(T) = \frac{1}{3} + T + \frac{7}{3}T^2 + 5T^3 + \frac{31}{3}T^4 + 21T^5 + \frac{127}{3}T^6 + \ldots \]
and by using (40) we get the desired zeta polynomial of \( K_n \), which is
\[ P(T) = \frac{1}{3} [(2T)^{n-2} - \sum_{i=3}^{n-1} (2T)^{n-i} + 1]. \]

The following lemma gives the location of the zeros of a certain type of polynomial of degree \( p = 2m \).

**Lemma 8.1.** Let \( p > 2 \) be even. The polynomial
\[ f(z) = z^p - z^{p-1} - z^{p-2} - \cdots - z + 1 \]
has \( p - 2 \) roots on the unit circle and two positive roots \( \alpha \) and \( \alpha' \) with \( \alpha \cdot \alpha' = 1 \).

**Proof.** Let \( p = 2m \) and write \( f \) as
\[ f(z) = F(z) - G(z), \]
where
\[ F(z) = z^p + z^{p-1} + \cdots + z + 1, \]
\[ G(z) = 2z(z^{p-2} + z^{p-1} + \cdots + z + 1). \]

Note that the non-zero roots of \( F \) and \( G \) can be enumerated, respectively, as
\[ F : e^{i\alpha_k}, \text{ with } \alpha_k = \frac{2k\pi}{p+1}, \quad k = 1, 2, \ldots, p; \quad (41) \]
\[ G : e^{i\beta_k}, \text{ with } \beta_k = \frac{2k\pi}{p-1}, \quad k = 1, 2, \ldots, p-2. \quad (42) \]

All these zeros are located on the unit circle.

Now, using the transformation \( z = e^{i\theta} \) and the function
\[ g(s, \theta) = 2 \cos s\theta + 2 \cos(s-1)\theta + \cdots + 2 \cos \theta + 1, \]
we obtain the equation
\[ f(e^{i\theta}) = e^{im\theta}[g(m, \theta) - 2g(m-1, \theta)]. \]

The zeros of \( f \) can then be located by taking note of the sign changes that occur in the expression
\[ Q(\theta) = g(m, \theta) - 2g(m-1, \theta) \]
for \( \theta \) values obtained from the zeros of \( F \) and \( G \) in (41) and (42). The idea is to use the continuity of \( Q \) and the intermediate value theorem to find intervals \((\theta_v, \theta_u) \) containing roots of \( f \). (See Figure 1 illustrating sign changes on values between two consecutive values of \( \theta \) for each of \( g(m, \theta) \) and \( g(m-1, \theta) \).) Listing the values of \( \theta \) in increasing order yields the sign changes in \( Q \) shown in Table 6. This shows the \( p - 2 \) sign changes occurring
in \(Q\) giving the locations of \(\theta\) values that correspond to the zeros of \(f\) located on the unit circle. Hence, a zero \(e^{i\theta}\) of \(f\) exists in \((\beta_k, \alpha_{k+1})\), \(k = 1, \ldots, p/2 - 1\) and \((\alpha_k, \beta_{k-1})\), \(k = p/2 + 1, \ldots, p - 1\).

Clearly, \(f\) has exactly \(p\) zeros in the complex plane and hence there are still two remaining zeros unaccounted for. Now, observe that \(f\) also has exactly two sign changes, which implies that it can have none or two positive roots by Descartes’ rule of signs [14]. Observe that \(f(0) > 0\) and \(f(1) < 0\) and therefore, by applying the intermediate value theorem, we can say that \(f\) has a positive root between 0 and 1. In fact, since \(f\) is self-inversive, then if \(\alpha > 0\) is a root then so is \(\alpha' = 1/\alpha\). This tells us that \(f\) has exactly two positive roots with the relation \(\alpha \cdot \alpha' = 1\).

As observed from the tables shown in this section, there are two prominent cases pertaining to the behavior of a zeta polynomial of a Type IV code over \(R\) with degree 2\(g\). We shall use the notation \((0, 2g, 0)\) to indicate that all 2\(g\) zeros of this zeta polynomial are on the circle \(|T| = 1/2\) and hence demonstrating satisfaction of the RHA property. On the other hand, we use the notation \((1, 2g - 2, 1)\) to refer to the second case wherein the zeta polynomial has all roots on the circle \(|T| = 1/2\) except for two which are located one inside and the other outside the circle.

Now, for the family of codes \(K_n\), we obtain the following result as a direct application of Lemma 8.1.

**Theorem 8.2.** The zeta polynomial of the Klemm code \(K_n\) (over \(\mathbb{Z}_4\) or \(\mathbb{F}_2 + u\mathbb{F}_2\)) is of type \((1, n - 4, 1)\) when \(n > 4\).

**Proof.** Set \(z = 2T\) so that we can write the zeta polynomial \(P(T)\) as

\[
f(z) = \frac{1}{4}(z^{n-2} - z^{n-3} - z^{n-4} - \cdots - z + 1).
\]

By using this transformation and applying Lemma 8.1 we are led to the conclusion that all zeros \(T_0\) of \(P(T)\) satisfy the property that \(|T_0| = 1/2\) except for two positive real roots.  \(\square\)
We note that, for $K_4$, we obtain a zeta polynomial all of whose roots have absolute value equal to $1/2$ and is thus of type $(0, 2, 0)$. Now, for illustration we compute the zeta polynomial of $K_{12}$.

**Example 8.1.** The zeta polynomial of the code $K_{14}$ is given by

$$P(T) = \frac{1}{3}(2T)^{14} - (2T)^{13} - \cdots - (2T)^2 - 2T + 1).$$

Figure 2 shows the locations of the zeros of $P(T)$ with respect to the circle $|T| = 1/2$. This clearly shows that $P(T)$ is of type $(1, 8, 1)$ with reference to the circle $|T| = 1/2$.

### 8.3. $C_{m,r}$ codes

Another family of Type IV codes over $\mathbb{Z}_4$ or $\mathbb{F}_2 + u\mathbb{F}_2$ that can be constructed is the $C_{m,r}$ [$3$]. The bilevel construction uses binary Reed–Muller codes and is given by

$$C_{m,r} := \text{RM}(r, m) + u\text{RM}(m - r - 1, m), \quad \text{for } 3r \leq m - 1.$$ 

When the underlying ring is $\mathbb{Z}_4$, then $z = 2$; for the ring $\mathbb{F}_2 + u\mathbb{F}_2$, $z = u$, with $u^2 = 0$.

We note here that, given appropriate values of $r$ and $m$, $C_{m,r}$ produces a code of length $2^m$ having $4^r2^t$, with $s = \sum_{i=0}^{r} \binom{m}{i}$ and $t = \sum_{i=0}^{m-r-1} \binom{m}{i}$, codewords. The code $C_{m,0}$ yields the Klemm code $K_{2^m}$. For small values of $m$, one can verify that the computed zeta polynomials are of type $(1, 2g, 1)$, that is, all of the roots have absolute value $1/2$ except for two positive roots that have product equal to $1/4$.

Figure 3 shows the zeta polynomial of $C_{4,1}$ defined over $\mathbb{Z}_4$ and the behavior of its roots:

$$P(T) = \frac{1}{429}(11264T^{10} - 3712T^8 - 3264T^7 - 2016T^6$$
$$- 1088T^5 - 504T^4 - 204T^3 - 58T^2 + 11).$$

### 9. Implication of results to coding theory

The results of this paper show an interesting behavior of the zeta polynomials of Type IV codes over rings of order four. Computational results show that their roots are either of
FIGURE 3. Location of zeroes of $P(T)$.

type $(0, 2g, 0)$ or $(1, 2g - 2, 1)$. Of greater interest is the behavior of zeta polynomials that correspond to Hamming weight enumerators in the extremal case. These polynomials are the ones that describe codes that have the RHA property. As an implication of this, we see that, in the extremal case, when the roots of the zeta polynomial (of degree $2g$) are $\alpha_1, \alpha_2, \ldots, \alpha_{2g}$, then

$$\sum_{i=1}^{2g} |\alpha_i| = 2g \cdot \frac{1}{2} = g.$$ 

Moreover, if we write $P(T) = p_0(1 + a_1 T + a_2 T^2 + \cdots)$ so that $-a_1$ represents the sum of all the roots of $P(T)$ and then use Definition 6.1, we find expressions that explicitly give the values of the coefficients of $x^{n-d} y^d$ and $x^{n-d-1} y^{d+1}$ of the weight enumerator. Since the minimum distance is $d$, we have

$$A_d = p_0 \binom{n}{d} > 0$$

and

$$A_{d+1} = p_0 \binom{n}{d+1} (a_1 + 4 - d)$$

becomes positive when

$$d \leq a_1 + 4.$$ 

Moreover, since $a_1 \leq |a_1| \leq g$, we obtain the relation

$$d \leq g + 4,$$

which gives a link between the minimum distance and the degree of the zeta polynomial. With this same view, we obtain an upper bound for the minimum distance in terms of the sum of the absolute value of the roots of the zeta polynomial.

As for the non-extremal cases, we observe a consistent behavior of the roots of the zeta polynomial. With the knowledge of $P(T/2)$ being self-inversive, we see that all roots are located on the circle $|T| = 1/2$ except possibly for two positive roots whose product is equal to $1/4$. 


It is worthwhile mentioning that the above findings are consistent with the results concerning codes over finite fields in the general case as discussed by Duursma in his paper [6].

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