TRIVIALITY OF A 2-KNOT WITH ONE OR TWO SHEETS

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Dedicated to Professor Takao Matumoto on his 60th birthday

Abstract. The sheet number of a 2-knot is an analogous quantity to the crossing number of a 1-knot. We prove that (i) a 2-knot is trivial if and only if the sheet number is equal to one, and (ii) there is no 2-knot with the sheet number two.

1. Introduction

The sheet number of a 2-knot K , denoted by $\text{sh}(K)$, is an analogous quantity to the crossing number of a 1-knot [4]. A diagram of K consists of a finite number of sheets, which are used as the generators of a Wirtinger representation of the 2-knot group $G(K) = \pi_1(\mathbb{R}^4 \setminus K)$. The sheet number is the minimal number of sheets for all diagrams of K . It is known that if $G(K) \not\cong \mathbb{Z}$, then it holds that $\text{sh}(K) \geq 4$ (cf. [6, 7]). In particular, the equality holds for the spun trefoil. However, the 2-knots with $\text{sh}(K) \leq 3$ have not been studied. The aim of this paper is to prove the following.

THEOREM 1.1. *We have the following results.*

- (i) *A 2-knot K is trivial if and only if $\text{sh}(K) = 1$.*
- (ii) *There is no 2-knot K with $\text{sh}(K) = 2$.*

If K is trivial, then it is represented by an embedded 2-sphere in $\mathbb{R}^3$ and we have $\text{sh}(K) = 1$. Hence, we may prove that if $\text{sh}(K) \leq 2$, then K is trivial. We remark that it is still an open problem whether there is a 2-knot K with $\text{sh}(K) = 3$.

This paper is organized as follows. In Section 2, we review several definitions concerning a diagram of a 2-knot. In Section 3, we study a diagram with no triple point. Sections 4, 5 and 6 are devoted to studying a diagram with several triple points. In particular, we prove that a 2-knot represented by a diagram with one sheet (Proposition 5.3) or two sheets (Proposition 6.3) is trivial. Carter–Saito’s book [1] will help the reader to understand 2-knots and their diagrams.

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FIGURE 1. An edge and six-valent and one-valent vertices.

2. Definitions

2.1. Generic projection

Throughout this paper, we work in the piecewise linear category. A 2-knot is a two-dimensional sphere embedded in $\mathbb{R}^4$ locally flatly. Two 2-knots are *equivalent* if there is an ambient isotopy of $\mathbb{R}^4$ which maps one to the other.

Let $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the projection defined by $\pi(x, y, z, h) = (x, y, z)$ which drops the height h . By a slight perturbation of K if necessary, we may assume that the projection image $\pi(K)$ is generic. Namely, any point of $\pi(K)$ is a regular point, a double point, a triple point or a branch point. We denote by $\Gamma(K)$ the set of double points, triple points and branch points of $\pi(K)$, and call it the *multiple point set*.

2.2. Edge and curve of $\Gamma(K)$

The multiple point set $\Gamma(K)$ is regarded as a disjoint union of connected graphs and embedded circles in $\mathbb{R}^3$. The valency of each vertex of $\Gamma(K)$ is one or six corresponding to a branch point or a triple point, respectively. See Figure 1.

By connecting the edges which are diagonal at triple points, we obtain *double point curves* (arcs and circles) such that the endpoints of a double point arc are a pair of branch points, and the curves pass through each triple point three times. Figure 2 shows neighborhoods of a double point arc and a double point circle. We remark that each curve may have self-intersections. We say that a double point curve is *simple* if it does not pass through any triple point.

2.3. Diagram of 2-knot

Let $\pi|_K : K \rightarrow \mathbb{R}^3$ be the restriction of π to K , and p a double point of $\pi(K)$. The preimage $(\pi|_K)^{-1}(p)$ consists of a pair of points p_- and p_+ with $h(p_-) < h(p_+)$. The point p_- is called the *lower point* of p , and p_+ the *upper point* of p . There are two intersecting disks near p containing p_- and p_+ , which are called the *upper* and *lower disks*, respectively.

Let $\Lambda_-(K)$ and $\Lambda_+(K)$ denote the closures of the sets of lower and upper points, respectively. Put $\Lambda(K) = \Lambda_-(K) \cup \Lambda_+(K)$. Then it holds that $\Lambda(K) = (\pi|_K)^{-1}(\Gamma(K))$ by definition.

FIGURE 2. Neighborhoods of (a) a double point arc and (b) a double point circle.

FIGURE 3. Crossing information near a double point, a triple point t and a branch point b of $D(K)$.

We take a neighborhood N of $\Lambda_-(K)$ in K as shown in Figure 3. By removing $\pi(\text{int } N)$ from $\pi(K)$, we indicate crossing information near $\Gamma(K)$ so that we obtain a *diagram* of K , which is denoted by $D(K) = \pi(K) \setminus \pi(\text{int } N)$. Here, $\text{int } N$ means the interior of N . In the diagram $D(K)$, the lower disk near a double point is divided into two pieces. See the top row of Figure 3, where the curves of $\Lambda_-(K)$ and $\Lambda_+(K)$ are drawn by solid and dotted lines, respectively.

Let t be a triple point of $\pi(K)$. The preimage $(\pi|_K)^{-1}(t)$ consists of three points t_1 , t_2 and t_3 with $h(t_1) < h(t_2) < h(t_3)$. The points t_1 , t_2 and t_3 are called the *bottom*, *middle* and *top points* of t , respectively. There are three intersecting disks near t containing t_1 , t_2 and t_3 , which are called *bottom*, *middle* and *top disks*. In $D(K)$, the bottom and middle disks are divided into four and two pieces, respectively. See the second row of Figure 3.

Let b be a branch point of $\pi(K)$. The preimage $(\pi|_K)^{-1}(b)$ consists of a single point b_* . The sign $\varepsilon(b)$ of a branch point b is defined by crossing information of the edge of $\Gamma(K)$ near b as shown in the bottom row of Figure 3. We remark that if b and b' are the endpoints of a double point arc of $\Gamma(K)$, then it always holds that $\varepsilon(b) = -\varepsilon(b')$.

Each connected component of $D(K)$ is called a *sheet*. There is a one-to-one correspondence between the set of sheets of $D(K)$ and that of connected regions of the

FIGURE 4. Neighborhoods of (a) a double point arc A and (b) its preimage $A_- \cup A_+$.

complement $K \setminus \Lambda_-(K)$. The *sheet number* of K is the minimal number of sheets for all diagrams of 2-knots equivalent to K . We denote it by $\text{sh}(K)$.

For a diagram $D = D(K)$, we denote by $\text{sh}(D)$ and $t(D)$ the numbers of sheets and triple points of D , respectively.

Let K and K' be 2-knots with the diagrams $D = D(K)$ and $D' = D(K')$. We say that D is *equivalent* to D' if K is equivalent to K' . Similarly to the Reidemeister moves for 1-knot diagrams, Roseman [8] introduced seven kinds of moves for 2-knot diagrams, and proved that D and D' are equivalent if and only if they are related by a finite sequence of the Roseman moves.

2.4. Edge and curve of $\Lambda_\varepsilon(K)$

The lower/upper graph $\Lambda_\varepsilon(K)$ ($\varepsilon = \pm$) is regarded as a disjoint union of connected graphs and embedded circles in K . More precisely, for an edge e of $\Gamma(K)$, we denote by e_- and e_+ the closures of the lower and upper points of any double points on e , which are called the *lower* and *upper edges* of e , respectively. The edges of the graph $\Lambda_\varepsilon(K)$ consists of such e_ε .

The preimage b_* of a branch point b is a one-valent vertex of $\Lambda_-(K)$ and $\Lambda_+(K)$ both. The bottom point t_1 of a triple point t is a four-valent vertex of $\Lambda_-(K)$, and the top point t_3 is a four-valent vertex of $\Lambda_+(K)$. Moreover, the middle point t_2 is a two-valent vertex of $\Lambda_-(K)$ and $\Lambda_+(K)$ both. See Figure 3 again.

Consider a double point curve $L = \bigcup_{i=1}^n e(i)$ of $\Gamma(K)$ which is a union of the edges $e(1), e(2), \dots, e(n)$. We denote by L_- and L_+ the unions $\bigcup_{i=1}^n e_-(i)$ and $\bigcup_{i=1}^n e_+(i)$ of the lower and upper edges, and call them the *lower* and *upper point curves* of L , respectively. By definition, it holds that $\pi(L_-) = \pi(L_+) = L$. If L is a double point arc, then L_- and L_+ are immersed arcs in K with $\partial L_- = \partial L_+$. If L is a double point circle, then L_- and L_+ are immersed circles in K .

We give an example. Assume that $\Gamma(K)$ has a double point arc A connecting between branch points b and b' as shown in Figure 4(a). Here, A passes through three triple points t , t' and t'' such that the disks normal to A are bottom, middle and top, respectively. Then the lower point arc A_- of A passes through the middle point t_2 and the bottom points t'_1 and t''_1 as in Figure 4(b). Similarly, the upper point arc A_+ passes through the top points t_3 and t'_3 and the middle point t''_2 .

FIGURE 5. A branch point, which is TM-, TB- or MB-adjacent to a triple point.

2.5. Adjacency of branch point and triple point

In a graph, if the endpoints of an edge e are vertices v and w , that is, $e = \overline{vw}$, then we say that e is *incident* to v and w , or v is *adjacent* to w by e .

Consider the multiple point set $\Gamma(K)$ of a diagram $D(K)$. There are six edges of $\Gamma(K)$ incident to a triple point t . We say that a branch point b is called *TM-*, *TB-* or *MB-adjacent* to t if b is adjacent to t by an edge e which is the intersection of the top and middle, top and bottom or middle and bottom disks near t , respectively. See Figure 5.

2.6. Innermost disk and circle

Let $\ell_1, \dots, \ell_m$ be disjoint circles embedded in a 2-sphere S^2 . We say that $\Delta \subset S^2$ is an *innermost disk* if there is a circle ℓ_k with $\partial\Delta = \ell_k$ and $\text{int } \Delta \cap (\bigcup_{i=1}^m \ell_i) = \emptyset$, where $\text{int } \Delta$ means the interior of Δ . Also, ℓ_k is called the *innermost circle* for Δ . It is easy to see that there are at least two innermost disks Δ and Δ' . In particular, if $m = 1$, then it holds that $\partial\Delta = \partial\Delta' = \ell_1$ and $\Delta \cup \Delta' = S^2$.

3. Diagrams with no triple point

Throughout this section, we assume that a 2-knot diagram $D = D(K)$ has no triple point. Let $A(1), \dots, A(m)$ be the double point arcs of $\Gamma(K)$ and $C(1), \dots, C(n)$ the double point circles. We remark that all double point curves are simple by assumption. Let $A_-(i)$ and $A_+(i)$ be the lower and upper point arcs of $A(i)$, and $C_-(j)$ and $C_+(j)$ the lower and upper point circles of $C(j)$. Put $A_*(i) = A_-(i) \cup A_+(i)$. Then $\Lambda(K)$ is a disjoint union of $m + 2n$ embedded circles in K ;

$$A_*(1), \dots, A_*(m), C_-(1), \dots, C_-(n) \text{ and } C_+(1), \dots, C_+(n).$$

Then it is easy to see that $sh(D) = n + 1$.

Assume that $A_*(i)$ is an innermost circle of $\Lambda(K)$. Let Δ be an innermost disk for $A_*(i)$. Since $\pi(A_-(i)) = \pi(A_+(i)) = A(i)$, the 2-sphere $\pi(\Delta)$ bounds a 3-ball in $\mathbb{R}^3 \cup \{\infty\}$ whose interior misses D . We define a move (a) which eliminates $\pi(\Delta)$ from D as shown in Figure 6. This is one of the Roseman moves which does not change the equivalence class.

Assume that both of $C_-(j)$ and $C_+(j)$ are innermost circles of $\Lambda(K)$. Let Δ_- and Δ_+ be innermost disks for $C_-(j)$ and $C_+(j)$, respectively. Since $\pi(C_-(i)) = \pi(C_+(i)) = C(i)$, the 2-sphere $\pi(\Delta_-) \cup \pi(\Delta_+)$ bounds a 3-ball in $\mathbb{R}^3 \cup \{\infty\}$ whose interior misses D . We define a move (b) which separates the disk $\pi(\Delta_-)$ from $\pi(\Delta_+)$ with decreasing the number of sheets

FIGURE 6. Move (a), which eliminates $\pi(\Delta)$ from D . FIGURE 7. Move (b), which separates the disk $\pi(\Delta_-)$ from $\pi(\Delta_+)$ with decreasing the number of sheets by one.

by one as shown in Figure 7. This is also one of the Roseman moves which does not change the equivalence class of the 2-knot K .

LEMMA 3.1. *Let $D = D(K)$ be a 2-knot diagram. If $sh(D) \leq 2$ and $t(D) = 0$, then K is trivial.*

Proof. We use the notations as above. If $sh(D) = 1$, then we have $n = 0$. Assume that $m > 0$. Since $\Lambda(K) = \bigcup_{i=1}^m A_*(i)$ is a disjoint union of m circles embedded in K , we can find an innermost one among them to which we can apply a move (a). By repeating this procedure, we obtain an embedded 2-sphere in $\mathbb{R}^3$ which induces that K is trivial.

If $sh(D) = 2$, then we have $n = 1$. If either of $C_-(1)$ or $C_+(1)$ is not innermost among the circles of $\Lambda(K) = (\bigcup_{i=1}^m A_*(i)) \cup C_-(1) \cup C_+(1)$, then we have an innermost circle $A_*(i)$ for some i . Hence, by repeating moves (a) for such circles, we may assume that both of $C_-(1)$ and $C_+(1)$ are innermost. Then we apply a move (b) to obtain a diagram D' with $sh(D') = 1$ and $t(D') = 0$, which induces that K is trivial by the first case. $\square$

4. Triple points with three branch points

Assume that a 2-knot diagram $D = D(K)$ has a triple point t which is TM-adjacent to a branch point b . We define the move (c) which pulls b down through the bottom disk near t as shown in Figure 8, where we illustrate the case of $\varepsilon(b) = +1$. For the case of $\varepsilon(b) = -1$, we take the mirror image of the figure. We remark that this is represented by a combination of two Roseman moves.

LEMMA 4.1. *If D' is obtained from D by a move (a), then we have $sh(D') \leq sh(D)$ and $t(D') < t(D)$.*

FIGURE 8. Move (c), which pulls b down through the bottom disk near t . FIGURE 9. A pair of branch points b and b' , which are TB-adjacent to t .

Proof. We have $t(D') = t(D) - 1$ by definition. Let X and Y be the sheets which belong to the bottom disk near t as in Figure 8. Since they join together after the move, $sh(D') = sh(D)$ if $X = Y$ and $sh(D') = sh(D) - 1$ if $X \neq Y$. $\square$

LEMMA 4.2. (Cf. [2, 3, 5]) *If D is a 2-knot diagram, then there is no triple point t such that two branch points are TB-adjacent to t .*

Proof. Assume that there is a triple point t such that two branch points b and b' are TB-adjacent to t . Let A be the double point arc of $\Gamma(K)$ whose endpoints are b and b' . Put $A_* = A_- \cup A_+$, where A_- and A_+ are the lower and upper point arcs of A . Let L and L' be the double point curves of $\Gamma(K)$ which contain the intersections of middle and bottom and top and middle disks near t , respectively. See Figure 9. By definition, we have $A_* \cap L_- = \{t_1\}$ and $A_* \cap L'_+ = \{t_3\}$, where t_1 and t_3 are the bottom and top points of t , respectively.

If L is a double point circle of $\Gamma(K)$, then L_- is a lower point circle of $\Lambda_-(K)$ with $A_* \cap L_- = \{t_1\}$. We remark that $L_- \neq L'_+$. Since K is homeomorphic to a 2-sphere, it is impossible. Similarly, if L' is a double point circle, then we have a contradiction.

Assume that L and L' are double point arcs which may be the same. If $L \neq L'$, then $L_- \cup L'_+$ is a circle of $\Lambda(K)$ with $A_* \cap (L_- \cup L'_+) = \{t_1\}$, which is a contradiction. Consider the case of $L = L'$. Since K is orientable, we may assume that the double point curve L passes through t twice as shown in Figure 10(a). We splice the circle $L_- \cup L'_+$ at t_2 to obtain a pair of immersed circles, say C and C' , which contains t_1 and t_3 , respectively. Then we have $A_* \cap C = \{t_1\}$ and $A_* \cap C' = \{t_3\}$, which is impossible on the 2-sphere K . See Figure 10(b). $\square$

Assume that D has a triple point t which is MB-adjacent to branch points b and b' and TB-adjacent to a branch point b'' . As shown in Figure 11, we define a move (d), which pulls b

FIGURE 10. Neighborhoods of (a) L passing through t twice and (b) its preimage $L_- \cup L_+$.

FIGURE 11. Move (d), which pulls b down through the top disk near t , slides a 1-handle to eliminate a double point arc and makes a pair of branch points again.

down through the top disk near t , slides a 1-handle to eliminate a double point arc, and makes a pair of branch points again. In the figure, we illustrate the case of $\varepsilon(b) = +1$, $\varepsilon(b') = -1$ and $\varepsilon(b'') = -1$.

LEMMA 4.3. *If D' is obtained from D by a move (d), then we have $sh(D') \leq sh(D)$ and $t(D') < t(D)$.*

Proof. It is easy to see that the neighborhood of t in D consists of a single sheet, and so does the neighborhood after the move (d). $\square$

5. TM-free diagrams

We say that a 2-knot diagram D is *TM-free* if D has no triple point which is TM-adjacent to a branch point. If D is not a TM-free diagram, then we can apply a move (a) to decrease $t(D)$ without increasing $sh(D)$ by Lemma 4.1. Hence, it is natural to consider a TM-free diagram to study the sheet number.

LEMMA 5.1. *Let b be a branch point of D , and $b_* = (\pi|_K)^{-1}(b)$ be the preimage of b . If D is a TM-free diagram, then the following are equivalent:*

- (i) *b is adjacent to a triple point in $\Gamma(K)$;*
- (ii) *b_* is adjacent to a four-valent vertex in $\Lambda_-(K)$.*

Proof. (i) $\Rightarrow$ (ii). Assume that b is TB- or MB-adjacent to a triple point t by an edge e in $\Gamma(K)$. Then the one-valent vertex b_* is adjacent to the four-valent vertex t_1 by the lower edge e_- in $\Lambda_-(K)$, where t_1 is the bottom point of t . See Figure 4 again.

(ii) $\Rightarrow$ (i). Assume that b is adjacent to a branch point b' by an edge e in $\Gamma(K)$. Then b_* is adjacent to the one-valent vertex b'_* by the lower edge e_- in $\Lambda_-(K)$. $\square$

We remark that a branch point b is TM-adjacent to a triple point t by an edge e in $\Gamma(K)$, then b_* is adjacent to the two-valent vertex t_2 in $\Lambda_-(K)$, where t_2 is the middle point of t .

Let $b_0(D)$ denote the number of branch points each of which is adjacent to some triple point in $\Gamma(K)$.

LEMMA 5.2. *If D is a TM-free diagram with $t(D) > 0$, then it holds that*

$$b_0(D) \geq 2(t(D) + 2 - sh(D)).$$

Proof. Consider the lower graph $\Lambda_-(K)$ in K . Recall that $\Lambda_-(K)$ is a disjoint union of connected graphs and embedded circles in K . We use the following notation for the components of $\Lambda_-(K)$:

- μ , the number of connected graphs each of which contains some four-valent vertices;
- ν , the number of connected graphs which contain no four-valent vertex;
- λ , the number of embedded circles.

Since D is TM-free, $b_0(D)$ is equal to the number of one-valent vertices of the graphs in the definition of μ by Lemma 5.1. On the other hand, each graph in the definition of ν consists of a single edge with two one-valent vertices. We remark that the number of connected regions of the complement $K \setminus \Lambda_-(K)$ is equal to $sh(D)$. Hence, it holds that

$$(b_0(D) + 2\nu + t(D)) - \frac{b_0(D) + 2\nu + 4t(D)}{2} + sh(D) = (\mu + \nu + \lambda) + 1,$$

or, equivalently,

$$b_0(D) = 2(t(D) + \mu + \lambda + 1 - sh(D)).$$

Since $t(D) > 0$, we have $\mu \geq 1$ by definition and, hence, the inequality in the lemma holds. $\square$

We remark that the inequality in Lemma 5.2 holds for any diagram which may not be TM-free, as it always satisfies (ii) $\Rightarrow$ (i) in Lemma 5.1.

PROPOSITION 5.3. *Let $D = D(K)$ be a 2-knot diagram. If $sh(D) = 1$, then K is trivial.*

Proof. We first assume that $t(D) > 0$. If D is not TM-free, then there is an equivalent diagram D' with $sh(D') = 1$ and $t(D') < t(D)$ by Lemma 4.1. If D is TM-free, then we have $b_0(D) \geq 2t(D) + 2$ by Lemma 5.2. Hence, there is a triple point t of D which is adjacent to at least three branch points. By Lemma 4.2, t is MB-adjacent to two branch points and TB-adjacent to a branch point. By Lemma 4.3, there is an equivalent diagram D' with $sh(D') = 1$ and $t(D') < t(D)$. By repeating the argument as above, we may assume that $sh(D) = 1$ and $t(D) = 0$. Hence, K is trivial by Lemma 3.1. $\square$

FIGURE 12. (a) Move (e_1) for $\varepsilon(b) = -\varepsilon(b')$ and (b) move (e_2) for $\varepsilon(b) = \varepsilon(b')$, each of which pulls b down through the top disk near t .

6. Triple points with two branch points

Assume that D has a triple point t which is MB-adjacent to a branch point b and TB-adjacent to a branch point b' . We define a move (e_1) for $\varepsilon(b) = -\varepsilon(b')$ and a move (e_2) for $\varepsilon(b) = \varepsilon(b')$ each of which pulls b down through the top disk near t . See Figure 12, where the deformations for $\Lambda(K)$ are also illustrated. In particular, the deformations restricted to $\Lambda_-(K)$ are the same under the moves (e_1) and (e_2) .

LEMMA 6.1. *If D' is obtained from D by a move (e_1) or (e_2) , then we have $sh(D') \leq sh(D) + 1$ and $t(D') < t(D)$. In particular, $sh(D') = sh(D) + 1$ if and only if $\Lambda_-(K)$ has a form as shown in Figure 13.*

FIGURE 13. Deformation of $\Lambda_-(K)$, for which a move (e_1) or (e_2) increases the number of sheets by one.

Proof. Let $\sigma(D)$ be the number of connected components of $\Lambda_-(K)$, and $b(D)$ the number of branch points of D . Similarly to the proof of Lemma 5.2, it holds that

$$\begin{cases} (b(D) + t(D)) - \frac{b(D) + 4t(D)}{2} + sh(D) = \sigma(D) + 1, \\ (b(D') + t(D')) - \frac{b(D') + 4t(D')}{2} + sh(D') = \sigma(D') + 1. \end{cases}$$

On the other hand, it is easy to see that $b(D') = b(D)$ and $t(D') = t(D) - 1$. Hence, we obtain

$$sh(D') - sh(D) = \sigma(D') - \sigma(D) - 1.$$

Since $\sigma(D') - \sigma(D) \leq 3 - 1 = 2$, we have $sh(D') - sh(D) \leq 1$. In particular, the equality holds if and only if $\sigma(D') - \sigma(D) = 2$; that is, the bottom point t_1 and the middle point t_2 belong to the same component in $\Lambda_-(K)$ and the three edges after the move belong to mutually different components. Hence, we obtain the case as in Figure 13. $\square$

Let t be a triple point of D , and t_1 , t_2 and t_3 the bottom, middle and top points of t , respectively. If t_1 and t_2 are adjacent by a lower edge e_- of $\Lambda_-(K)$, then the upper edge e_+ has the same endpoint t_3 . See Figure 14. Let Δ be the disk in K bounded by e_+ . If $\text{int } \Delta \cap \Lambda(K) = \emptyset$, then we can define a move (f) which shrinks the edge $\pi(e_-) = \pi(e_+)$ to eliminate the triple point t and make a pair of branch points. This is represented by a combination of Roseman moves.

LEMMA 6.2. *If D' is obtained from D by a move (f), then we have $sh(D') \leq sh(D)$ and $t(D') < t(D)$.*

Proof. It is easy to see that the neighborhood of t in D consists of five sheets, and the pair on each side of the top disk joins together after the move (f). $\square$

PROPOSITION 6.3. *Let $D = D(K)$ be a 2-knot diagram. If $sh(D) = 2$, then K is trivial.*

Proof. We first assume that $t(D) > 0$. If D is not TM-free, then there is an equivalent diagram D' with $sh(D') \leq 2$ and $t(D') < t(D)$ by Lemma 4.3. In particular, if $sh(D') = 1$, then K is trivial by Proposition 5.3.

Consider the case that D is TM-free with $t(D) > 0$. By Lemma 5.2, it holds that $b_0(D) \geq 2t(D)$. Similarly to the proof of Proposition 5.3, if D has a triple point which is adjacent to at least three branch points, then we can decrease $t(D)$ without increasing $sh(D)$. Hence, it is sufficient to consider the case that $b_0(D) = 2t(D)$ and every triple point t is adjacent to just two branch points.

FIGURE 14. (a) A lower edge between t_1 and t_2 and an upper edge whose end points are t_3 , and (b) move (f), which eliminates t and makes a pair of branch points.

FIGURE 15. (a) Types of triple points, and (b) a lower graph $\Lambda_-(K)$ with $sh(D) = 2$.

Let $t(i)$ ($i = 1, 2, \dots, n = t(D)$) be the triple points of D , and $b(i)$ and $b'(i)$ the branch points adjacent to $t(i)$ by edges $e(i)$ and $e'(i)$ in $\Gamma(K)$, respectively. By Lemma 4.2, each triple point satisfies one of the following two types:

- (I) $t(i)$ is MB-adjacent to $b(i)$ and TB-adjacent to $b'(i)$;
- (II) $t(i)$ is MB-adjacent to $b(i)$ and $b'(i)$.

In the lower graph $\Lambda_-(K)$, each four-valent vertex $t_1(i)$ is adjacent to one-valent vertices $b_*(i)$ and $b'_*(i)$ by the lower edges $e_-(i)$ and $e'_-(i)$, respectively. In particular, if $t(i)$ is of type (I) (or, type (II)), then $e_-(i)$ and $e'_-(i)$ are consecutive (or diagonal) near $t_1(i)$. See Figure 15(a).

By assumption, we have $\mu = 1$ (the number of graphs with four-valent vertices) and $\lambda = 0$ (the number of embedded circles) in the proof of Lemma 5.2. Hence, the lower graph $\Lambda_-(K)$ is a disjoint union of connected graphs

$$G \text{ and } E_-(1), E_-(2), \dots, E_-(\nu)$$

such that G contains all four-valent vertices and each $E_-(j)$ consists of a single lower edge with two one-valent vertices. We remark that the subgraph of G ,

$$C_- = G \setminus \left[\left(\bigcup_{i=1}^n (e_-(i) \cup e'_-(i)) \right) \cup \left(\bigcup_{i=1}^n (b_*(i) \cup b'_*(i)) \right) \right],$$

is homeomorphic to a circle. See Figure 15(b).

FIGURE 16. The union $\Lambda(K)$ of $\Lambda_-(K)$ and $\Lambda_+(K)$.

If D has a triple point $t(i)$ of type (I), then we can apply a move (e_1) or (e_2) near $t(i)$. Since both of $t_1(i)$ and $t_2(i)$ belong to C_- , we obtain an equivalent diagram D' with $sh(D') \leq 2$ and $t(D') < t(D)$ by Lemma 6.1. (More precisely, it is not difficult to see that $sh(D') = 2$ in this case.)

We consider the case that every triple point of D is of II-type. Put $A_-(i) = e_-(i) \cup e'_-(i)$ ($i = 1, 2, \dots, n$). Then the lower graph $\Lambda_-(K)$ is a union of:

- the lower point circle C_- embedded in K ;
- the lower point arcs $A_-(i)$ ($1 \leq i \leq n$) with $C_- \cap A_-(i) = \{t_1(i)\}$; and
- the lower point arcs $E_-(j)$ ($1 \leq j \leq \nu$) disjoint from C_- and $A_-(i)$.

Hence, the upper graph $\Lambda_+(K)$ is a union of:

- the upper point circle C_+ with $\pi(C_+) = \pi(C_-)$;
- the upper point arcs $A_+(i)$ ($1 \leq i \leq n$) with $\pi(A_+(i)) = \pi(A_-(i))$; and
- the upper point arcs $E_+(j)$ ($1 \leq j \leq \nu$) with $\pi(E_+(j)) = \pi(E_-(j))$.

Since the double point arc $A(i) = \pi(A_-(i))$ passes through a triple point $t(i)$ only, the union $A_*(i) = A_-(i) \cup A_+(i)$ is an embedded circle in K with $C_- \cap A_+(i) = \{t_2(i)\}$. We remark that the union $E_*(j) = E_-(j) \cup E_+(j)$ is also an embedded circle disjoint from other components of $\Lambda(K)$. See Figure 16.

If there is a disk $\Delta \subset K$ with $\partial\Delta = E_*(j)$ and $\text{int } \Delta \cap \Lambda(K) = \emptyset$ for some j , then we can apply a move (a) to eliminate $E_*(j)$ from $\Lambda(K)$. Hence, we may assume that $\Lambda(K)$ does not have such an innermost circle $E_*(j)$.

Consider the embedded circles $A_*(1), \dots, A_*(n)$ in K . If $n > 1$, there is a pair of innermost disks $\delta(a)$ and $\delta(b)$ with $\partial\delta(a) = A_*(a)$ and $\partial\delta(b) = A_*(b)$, respectively. Then for each $i \in \{a, b\}$, $t_1(i)$ and $t_2(i)$ are connected by the lower edge $f_-(i) = \delta(i) \cap C_-$ and, hence, the corresponding upper edge $f_+(i)$ has the same endpoint $t_3(i)$. Each loop $f_+(i)$ divides K into two disks. Let $\Delta(i)$ be the one of the disks such that $\text{int } \Delta(i) \cap C_+ = \emptyset$. See Figure 17.

Since the edges $f_+(a)$ and $f_+(b)$ belong to C_+ , it holds that $\text{int } \Delta(a) \cap \text{int } \Delta(b) = \emptyset$. We may assume that $\text{int } \Delta(a) \cap (C_- \cup (\bigcup_{i=1}^n A_*(i))) = \emptyset$. Since there is no innermost circle $E_*(j)$ by assumption, it holds that $\text{int } \Delta(a) \cap (\bigcup_{j=1}^{\nu} E_*(j)) = \emptyset$ and, hence, $\text{int } \Delta(a) \cap \Lambda(K) = \emptyset$. Then we can apply a move (f) near the triple point $t(a)$ to decrease $t(D)$ without increasing $sh(D)$ by Lemma 6.2.

FIGURE 17. The upper point circle C_+ , which bounds a pair of disks $\Delta(a)$ and $\Delta(b)$.

If $n = 1$, we take a pair of disks $\delta(1)$ and $\delta'(1)$ with $\partial\delta(1) = \partial\delta'(1) = A_*(1)$ and $\delta(1) \cup \delta'(1) = K$. Then we can apply a similar argument for $\delta(1)$ and $\delta'(1)$ instead of $\delta(a)$ and $\delta(b)$ easily.

By repeating the above argument to decrease $t(D)$, we may assume that $sh(D) = 2$ and $t(D) = 0$. Then K is trivial by Lemma 3.1. $\square$

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