AN EXPLICIT FORMULA FOR LOCAL SINGULAR SERIES POLYNOMIALS

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Abstract. We give a new finite and explicit formula for the local singular series polynomial of a positive definite half-integral matrix of even size.

1. Introduction

The so-called local singular series polynomials $F_p(T; X)$ ($p$ a prime, $T$ a symmetric positive definite half-integral matrix of size $m$) play an important role both in the theory of quadratic forms as well as in the theory of modular forms.

For example, they occur in formulas for local densities and also enter in the Fourier coefficients of Siegel–Eisenstein series on $Sp_m(\mathbb{Z})$ and more generally (for $m$ even) in the Fourier coefficients of the Ikeda lift of an elliptic cuspidal Hecke eigenform.

There are various more or less explicit formulas for $F_p(T; X)$ in the literature. For example, Kitaoka [5] and Böcherer [1] expressed $F_p(T; X)$ as a finite sum of rather simple polynomials, with the sum running over certain superlattices of $\mathbb{Z}_m$, respectively $\mathbb{Z}$, attached to $T$. On the other hand, in a remarkable paper, Katsurada [4] proved a functional equation for $F_p(T; X)$ and as a consequence derived an explicit, though rather complicated, expression for it. However, from this explicit description the functional equation can no longer be recovered. The same applies to the formulas obtained in [1, 5].

In this paper we consider the case $m = 2n$ even. Let

$$ D_T := (-1)^n \det(2T) $$

be the discriminant of $T$. Then $D_T \equiv 0, 1 \pmod{4}$ and we shall write

$$ D_T = D_{T,0} f_T^2, \quad (1) $$

with $D_{T,0}$ a fundamental discriminant and $f_T \in \mathbb{N}$. We define

$$ \ell_p := \text{ord}_p f_T \quad (2) $$

and

$$ \tilde{F}_p(T; X) := X^{-\ell_p} F_p(T; p^{-n-1/2} X). \quad (3) $$

Then the functional equation of $F_p(T; X)$ can be nicely stated in saying that the Laurent polynomial $\tilde{F}_p(T; X)$ is symmetric of degree $\ell_p$.

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Our main result gives \( \tilde{F}_p(T; X) \) as an explicit linear combination of the Laurent polynomials

\[
\psi_j(X) := \frac{X^j - X^{-j}}{X - X^{-1}} \quad (j = 1, \ldots, \ell_p + 1).
\]

Note that the latter form a basis of the complex vector space of symmetric Laurent polynomials of degree at most \( \ell_p \). The coefficients in this linear combination involve numbers of certain superlattices of \( \mathbb{Z}^{2n} \) attached to \( T \) with given \( p \)-invariants, factorials and elementary divisor sums, and from a combinatorial point of view are rather simple.

The proof of our identity is based on earlier results given in [2] expressing \( \tilde{F}_p(T; X) \) for \( m \) even in terms of the \( \psi_j(X) \) \((j = 1, \ldots, \ell_p + 1)\) and on a combinatorial identity that expresses the elementary symmetric polynomials explicitly in terms of the power sum polynomials.

2. Local singular series

Throughout this paper, \( T \) denotes a symmetric positive definite half-integral matrix of size \( 2n \). If \( p \) is a prime, then the singular series attached to \( T \) and \( p \) is defined as

\[
b_p(T; s) := \sum_R \nu_p(R)^{-s} e_p(\text{tr}(TR)) \quad (s \in \mathbb{C}),
\]

where \( R \) runs over all symmetric \((2n, 2n)\) matrices with entries in \( \mathbb{Q}_p/\mathbb{Z}_p \) and \( \nu_p(R) \) is a power of \( p \) equal to the product of denominators of elementary divisors of \( R \). Furthermore, for \( x \in \mathbb{Q}_p \) we have put \( e_p(x) := e^{2\pi i x'} \), where \( x' \) denotes the fractional part of \( x \).

As is well known, \( b_p(T; s) \) is a product of two polynomials in \( p^{-s} \) with coefficients in \( \mathbb{Z} \). More precisely, put

\[
\gamma_p(T; X) := (1 - X) \left( 1 - p^n \left( \frac{D_{T,0}}{p} \right)^n X \right)^{-1} \prod_{j=1}^{n} (1 - p^{2j} X^2),
\]

where \( D_{T,0} \) is defined by (1) and \( \left( \frac{D_{T,0}}{p} \right) \) is the Jacobi symbol. Then

\[
b_p(T; s) = \gamma_p(T; p^{-s}) F_p(T; p^{-s}),
\]

where \( F_p(T; X) \) is a certain integral polynomial with constant term equal to 1. If we define \( \ell_p \) by (2) and \( \tilde{F}_p(T; X) \) by (3), then \( \tilde{F}_p(T; X) \) is a symmetric Laurent polynomial, i.e.

\[
\tilde{F}_p(T; X^{-1}) = \tilde{F}_p(T; X),
\]

and is of degree \( \ell_p \) [4].

3. Main result

For a prime \( p \) we denote by \( V = (\mathbb{F}_p^{2n}, q) \) the quadratic space over \( \mathbb{F}_p \) where \( q \) is the quadratic form obtained from the quadratic form \( x \mapsto T[x] := x'Tx \) \((x \in \mathbb{Z}^{2n}, \ x' = \text{transpose of } x)\) by reducing modulo \( p \). Write \( V = V_1 \oplus V_2 \) where \( V_2 \subset V \) is a maximal isotropic subspace and \( V_1 \subset V \) is a complementary subspace.
We let
\[ sp(T) := \dim V_2 \]
be the Witt index of \( V \) and put
\[ \lambda_p(T) := \begin{cases} 
0, & \text{if } sp(T) \text{ is odd or } sp(T) = 0, \\
1, & \text{if } sp(T) \text{ is even, } sp(T) > 0 \text{ and } V_1 \text{ is hyperbolic,} \\
-1, & \text{if } sp(T) \text{ is even, } sp(T) > 0 \text{ and } V_1 \text{ is not hyperbolic.} 
\end{cases} \]
(Note that our notation here and in the following is slightly different from that in [6].)

We set
\[ D(T) := GL_{2n} \mathbb{Z} \setminus \{ G \in M_{2n} \mathbb{Z} \cap GL_{2n} \mathbb{Q} \mid T[G^{-1}] \text{ half-integral} \}, \]
where \( GL_{2n} \mathbb{Z} \) operates by left-multiplication. Note that the map
\[ GL_{2n} \mathbb{Z} G \mapsto G^{-1} \mathbb{Z}^{2n} \]
is a bijection of \( D(T) \) onto the set of superlattices of \( \mathbb{Z}^{2n} \) on which the rational quadratic form
\[ x \mapsto 2T[x] \]
is even integral. Observe that \( D(T) \) is finite.

For non-negative integers \( j, v \) and \( \mu \) such that \( 0 \leq v \leq j, \ v \equiv j \pmod{2} \) and \( 1 \leq \mu \leq 2n \), we put
\[ N_{T,p}(j, v; \mu) := \begin{cases} 
# \{ G \in D(T) \mid |\det G| = p^{(j-v)/2}, \ sp(T[G^{-1}]) = \mu \} & \text{if } v \text{ is even,} \\
p^{(\mu-1)/2} \sum_{G \in D(T) \mid |\det G| = p^{(j-v)/2}, sp(T[G^{-1}]) = \mu} \lambda_p(T[G^{-1}]) & \text{if } v \text{ is odd.} 
\end{cases} \]

For \( \nu \in \mathbb{N} \) we set
\[ \tilde{\sigma}_\nu(n) := \sum_{d \mid n, d \equiv 1 \pmod{2}} d^\nu \quad (n \in \mathbb{N}). \]
We extend the definition to all \( x \in \mathbb{R} \) by
\[ \tilde{\sigma}_\nu(x) := 0 \quad (x \notin \mathbb{N}). \]

Finally, for a non-negative integer \( \ell \) we denote by \( L_\ell \) the \((\ell + 1)\)-dimensional complex vector space of symmetric Laurent polynomials
\[ \sum_{\nu=0}^{\ell} c_\nu(X^\nu + X^{-\nu}) \quad (c_\nu \in \mathbb{C}) \]
of degree at most \( \ell \). Then the polynomials \( \psi_j(X) \) (\( j = 1, \ldots, \ell + 1 \)) defined by (4) form a basis of \( L_\ell \), and so do the polynomials
\[ \psi_j(X) - \left( \frac{D_{T,0}}{p} \right) p^{-1/2} \psi_{j-1}(X) \quad (1 \leq j \leq \ell + 1) \]
(with the convention \( \psi_0(X) := 0 \)).

**Theorem.** Let \( \ell := \ell_p \). Then with the above definitions one has
\[ \tilde{F}_p(T; X) = \sum_{j=0}^{\ell} \alpha_j \left( \psi_{\ell+1-j}(X) - \left( \frac{D_{T,0}}{p} \right) p^{-1/2} \psi_{\ell-j}(X) \right), \]
where

\[ \alpha_j = \sum_{1 \leq \mu \leq 2n} N_{T, p}(j, \mu; \delta_{v,0} + \delta_{v,1} + (-1)^{v'} \varepsilon_{\mu}(v')) \]

and

\[ \varepsilon_{\mu}(v) = \sum_{m_1, m_2, \ldots, m_\nu \geq 0} \prod_{i=1}^\nu (-1)^{(i+1)m_i} (1 + \tilde{\sigma}_i(\nu^{[\mu-1]/2} - 1))^{m_i} / m_i! \]

Here \( v' := [v/2] \) and \( \delta_{v,0} \) and \( \delta_{v,1} \) denote the Kronecker delta.

**Proof.** According to [2, Theorem] one has

\[ \tilde{F}_p(T; X) = \sum_{j=0}^\ell \beta_j \left( \psi_{\ell+1-j}(X) - \left( \frac{DT,0}{p} \right)^{p^{-1/2}} \psi_{\ell-j}(X) \right), \]

where

\[ \beta_j = \sum_{0 \leq v \leq j/2} \sum_{G \in D(T)} \rho_{T[G^{-1]}(p^{j-2v})}, \]

or equivalently

\[ \beta_j = \sum_{0 \leq v \leq j} \sum_{G \in D(T) \mod \nu \equiv j \mod 2 \mid \det G \mid = p^{(j-v)/2}} \rho_{T[G^{-1}]}(p^v). \quad (5) \]

Here for any symmetric positive definite matrix \( S \) we define \( \rho_S(p^v) \) \((v \geq 0)\) by

\[ (1 - X^2)H_{n,p}(T; X) := \sum_{v=0}^{\infty} \rho_S(p^v)X^v \]

where

\[ H_{n,p}(T; X) := (1 + \lambda_p(S) p^{(s_p-1)/2} X) \prod_{i=1}^{[s_p-1]/2} (1 - p^{2i-1} X^2) \]

and we have written \( s_p = s_p(S) \). We have to show that \( \beta_j = \alpha_j \) for all \( j \).

Suppose that \( s_p(T[G^{-1}]) = 0 \) in (5). Then the quadratic space \( V/F_p \) attached to \( T[G^{-1}] \) reduced modulo \( p \) is anisotropic, hence non-singular, and it follows that

\[ \operatorname{ord}_p | \det G | = \ell. \]

However, in (5) we also have \( \operatorname{ord}_p | \det G | = (j - v)/2 \), and \( j \leq \ell \). Therefore this case cannot occur.

For \( s_p \geq 1 \) let us set

\[ g(Y) := (1 - Y) \prod_{i=1}^{[s_p-1]/2} (1 - p^{2i-1} Y). \]
Let \( N := [(s_p - 1)/2] + 1 \) be the degree of \( g(Y) \) and \( a_\mu \) \((0 \leq \mu \leq N)\) be its \( \mu \)th coefficient. Then for \( 0 \leq \nu \leq 2N + 1 \) we have

\[
\rho_S(p^\nu) = \begin{cases} 
  a_\mu & \text{if } \nu = 2\mu \text{ is even} \\
  \lambda_p(S) p^{(s_p-1)/2} a_\mu & \text{if } \nu = 2\mu + 1 \text{ is odd}
\end{cases}
\]  

(6)

(otherwise by definition \( \rho_S(p^\nu) = 0 \)).

On the other hand, the roots of \( Y^N g(Y^{-1}) \) are \( 1, p, p^3, \ldots, p^{2([(s_p-1)/2] - 1)} \), hence

\[
a_\mu = (-1)^\mu S_\mu(1, p, p^3, \ldots, p^{2([(s_p-1)/2] - 1)}) \quad (1 \leq \mu \leq N),
\]

(7)

where

\[
S_\mu(t_1, \ldots, t_N) = \sum_{1 \leq i_1 < i_2 \cdots < i_\mu \leq N} t_{i_1} \cdots t_{i_\mu} \quad (1 \leq \mu \leq N)
\]

is the \( \mu \)th elementary symmetric polynomial in the variables \( t_1, \ldots, t_N \). (Clearly \( a_0 = 1 \).)

Let us denote by

\[
P_\mu(t_1, \ldots, t_N) := t_1^\mu + \cdots + t_N^\mu \quad (1 \leq \mu \leq N)
\]

the power sums.

**Lemma.** For each \( \mu = 1, \ldots, N \) one has

\[
S_\mu(t_1, \ldots, t_N) = \sum_{m_1, m_2, \ldots, m_\mu \geq 0 \atop m_1 + 2m_2 + \cdots + \mu m_\mu = \mu} (-1)^{m_2 + m_4 + \cdots} \frac{1}{m_1! \cdots m_\mu!} P_1(t_1, \ldots, t_N)^{m_1} \cdots P_\mu(t_1, \ldots, t_N)^{m_\mu}.
\]

(8)

**Remarks.** (i) In a more invariant way, (8) can be stated as

\[
S_\mu = \sum_{X \in K(S_\mu)} (-1)^{\deg X} \frac{\#X}{\#S_\mu} P_1^{m_1} \cdots P_\mu^{m_\mu},
\]

where \( S_\mu \) is the symmetric group in \( \mu \) letters, \( K(S_\mu) \) is the set of conjugacy classes of \( S_\mu \) and a conjugacy class \( X \) is given by the cycle type \((m_1, m_2, \ldots, m_\mu)\) with \( m_i \) the number of \( i \)-cycles. Finally, \( \deg X = \sum_{j \equiv 0 \pmod{2}} m_j \) is the degree of \( X \).

(ii) Identity (8) can be found in [3] and there is attributed to MacMahon.

Clearly one has

\[
P_\mu(1, p, p^3, \ldots, p^{2([(s_p-1)/2] - 1)}) = 1 + \tilde{\sigma}_\mu(p^{2([(s_p-1)/2] - 1)}),
\]

and hence from (6), (7) and (8) we conclude, for \( 2 \leq \nu \leq 2N + 1 \), that

\[
\rho_S(p^\nu) = \epsilon_{S, \nu} (-1)^{\nu} \sum_{m_1, m_2, \ldots, m_\nu \geq 0 \atop m_1 + 2m_2 + \cdots + \nu m_\nu = \nu} (-1)^{m_2 + m_4 + \cdots} \frac{1}{m_1! \cdots m_\nu!} (1 + \tilde{\sigma}_1(p^{2([(s_p-1)/2] - 1)}))^{m_1} \cdots \times (1 + \tilde{\sigma}_\nu(p^{2([(s_p-1)/2] - 1)}))^{m_\nu}.
\]

(9)
where
\[
\epsilon_{S,\nu} := \begin{cases} 
1, & \text{if } \nu \text{ is even}, \\
\frac{\lambda_p(S)p^{(s_p-1)/2}}{2}, & \text{if } \nu \text{ is odd}.
\end{cases}
\]

Clearly \( \rho_S(p^\nu) = \epsilon_{S,\nu}(-1)^\nu \) if \( \nu = 0, 1 \).

We apply (9) with \( S = T[G^{-1}], G \in D(T) \). Checking the definition when \( \lambda_p(S) \) is zero and observing (5) we then obtain that \( \beta_j = \alpha_j \) for all \( j \). This proves the theorem.

REFERENCES


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