A WIENER INTEGRAL APPROACH TO NON-COMMUTATIVE HARMONIC OSCILLATORS

Setsuo TANIGUCHI

(Received 23 April 2009)

Abstract. A Wiener integral approach to the non-commutative harmonic oscillator, i.e. an approach via Brownian motions and matrix-valued stochastic differential equations, will be given. Such an approach is a continuation and an extension of the one made by the author in an earlier paper (Kyushu J. Math. 62 (2008), 63–68).

1. Introduction

Let $C_1^\infty(\mathbb{R}; \mathbb{C}^2)$ be the space of all $C^\infty$ functions

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \mathbb{R} \to \mathbb{C}^2,$$

which are of at most polynomial growth as are their derivatives of all orders. The non-commutative harmonic oscillator for $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha, \beta > 0$ is by definition the differential operator $Q_{(\alpha, \beta)}$ of $C_1^\infty(\mathbb{R}; \mathbb{C}^2)$ to itself defined by

$$Q_{(\alpha, \beta)}f(x) = -\frac{1}{2} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left( f''(x) - x^2 f(x) \right) + J\left( x f'(x) + \frac{1}{2} f(x) \right)$$

for $x \in \mathbb{R}$, $f \in C_1^\infty(\mathbb{R}; \mathbb{C}^2)$,

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $f'$ and $f''$ denote the first and second derivatives of $f$. For preceding studies on $Q_{(\alpha, \beta)}$, see [2]. In [2], the author gave a Wiener integral representation of the semigroup associated with $-Q_{(\alpha, \alpha)}$, i.e. in the case $\alpha = \beta$. In this paper, we shall continue a Wiener integral approach to $-Q_{(\alpha, \beta)}$.

To state our result more precisely, we introduce some notation. Let $(\mathcal{W}, \mu)$ be the classical one-dimensional Wiener space; $\mathcal{W}$ is the space of all continuous functions $w : [0, \infty) \to \mathbb{R}$ with $w(0) = 0$, and $\mu$ is the Wiener measure on $\mathcal{W}$. For $t \geq 0$, we denote by $B_t$ the coordinate function on $\mathcal{W}$ at time $t$; $B_t(w) = w(t)$, $w \in \mathcal{W}$. For $n \in \mathbb{N}$, we use $\mathbb{R}^{n \times n}$ to indicate the space of all real $n \times n$ matrices, and the Hilbert–Schmidt norm of $A \in \mathbb{R}^{n \times n}$

2000 Mathematics Subject Classification: Primary 60H30, 60H20.

Keywords: Wiener integral; non-commutative harmonic oscillator; matrix-valued stochastic differential equation.

© 2009 Faculty of Mathematics, Kyushu University
is denoted by $\|A\|$. Fix $x \in \mathbb{R}$ arbitrarily. Define an $\mathbb{R}^{2 \times 2}$-valued stochastic process $\{M_t^x\}_{t \geq 0}$ to be the unique solution to the following stochastic differential equation (SDE):

$$
\begin{align*}
    dM_t^x &= M_t^x (-J) \left( \frac{(x + \sqrt{\alpha} B_t)}{\sqrt{\alpha}} \begin{pmatrix} 0 \\ (x + \sqrt{\beta} B_t) / \sqrt{\beta} \end{pmatrix} \right) dB_t \\
    &+ \frac{1}{2} M_t^x \left\{ - \left( \frac{\alpha(x + \sqrt{\alpha} B_t)^2}{\sqrt{\alpha}} \begin{pmatrix} 0 \\ \beta(x + \sqrt{\beta} B_t)^2 \end{pmatrix} - J \right) \right\} dt,
\end{align*}
$$

$M_0^x = I$,

where $I$ is the $2 \times 2$ unit matrix. The existence and uniqueness of the solution are well known [1]. The goal of this paper is to show the following result.

**Theorem 1.**

(i) For any $p > 1$, there exists $t_p > 0$ such that

$$
\sup_{t \leq t_p} \mathbb{E}[\|M_t^x\|^p] < \infty,
$$

where $\mathbb{E}$ stands for the expectation with respect to $\mu$.

(ii) For every $f \in C^\infty_1(\mathbb{R}; \mathbb{C}^2)$, it holds that

$$
\mathbb{E} \left[ M_t^x \left( \begin{array}{c} f_1(x + \sqrt{\alpha} B_t) \\ f_2(x + \sqrt{\beta} B_t) \end{array} \right) \right] - f(x) \rightarrow -Q_{(\alpha,\beta)} f(x) \quad \text{as } t \to 0.
$$

The proof of the theorem will be given in Section 2, and some remarks in Section 3.

2. Wiener integral – proof of Theorem 1

In this section, we shall give the proof of Theorem 1. We start with the observation that implies the first assertion of Theorem 1.

**Proposition 1.** Let $n \in \mathbb{N}$, and $\{A_t\}_{t \geq 0}$ and $\{D_t\}_{t \geq 0}$ be $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^{n \times n}$-valued continuous processes. Suppose that there exist $T > 0$ and $K > 0$ such that

$$
\|A_t\|^2 + \|D_t\| \leq K \left( 1 + \sup_{u \leq t} |B_u|^2 \right) \quad \text{for every } t \in [0, T].
$$

Let $N_t$ be the unique solution to the $\mathbb{R}^{n \times n}$-valued SDE

$$
\begin{align*}
    dN_t &= N_t A_t dB_t + N_t D_t dt, \\
    N_0 &= I,
\end{align*}
$$

where $I$ is the $n \times n$ unit matrix. Then, for each $p > 0$, there exists $t_p > 0$ such that

$$
\sup_{t \leq t_p} \mathbb{E}[\|N_t\|^p] < \infty.
$$

**Proof.** We first deal with the case that $n = 1$. Then we have that

$$
N_t = \exp \left( \int_0^t A_s dB_s + \int_0^t \left\{ D_s - \frac{A_s^2}{2} \right\} ds \right).
$$
By the Schwartz inequality, we obtain that
\[
\mathbb{E}[N_t^p] \leq \left( \mathbb{E} \left[ \exp \left( 2p \int_0^t A_s \, dB_s - 2p^2 \int_0^t A^2_s \, ds \right) \right] \right)^{1/2} \\
\times \left( \mathbb{E} \left[ \exp \left( \int_0^t \{2pD_s + p(2p - 1)A^2_s \} \, ds \right) \right] \right)^{1/2}.
\]
Due to the supermartingale property of exponential local martingales and the assumption, the right-hand side is dominated by
\[
\left( \mathbb{E} \left[ \exp \left( p \max \{2, |2p - 1|\} KT \left( 1 + \sup_{u \leq t} |B_u|^2 \right) \right) \right] \right)^{1/2}.
\]
This implies the desired assertion when \(n = 1\).

Let \(n \geq 2\). Define \(h \in C^\infty(\mathbb{R}^{n\times n}; \mathbb{R})\) by
\[
h(x) = (1 + \|x\|^2)^{1/2}/\sqrt{1 + n}, \quad x = (x^{ij})_{1 \leq i, j \leq n} \in \mathbb{R}^{n\times n}.
\]
Then we have that
\[
\frac{\partial h}{\partial x^{ij}} = (1 + \|x\|^2)^{-1/2}x^{ij}/\sqrt{1 + n},
\]
\[
\frac{\partial^2 h}{\partial x^{ij} \partial x^{k\ell}} = ((1 + \|x\|^2)^{-1/2}\delta_{ik}\delta_{j\ell} - (1 + \|x\|^2)^{-3/2}x^{ij}x^{k\ell})/\sqrt{1 + n},
\]
where \(\delta_{ij}\) denotes the Kronecker delta. Set
\[
\nu_t = h(N_t) = (1 + \|N_t\|^2)^{1/2}/\sqrt{1 + n}.
\]
Note that the SDE (4) reads as
\[
N_{ij}^t = \delta_{ij} + \sum_{\alpha = 1}^n \int_0^t N_i^{\alpha t} A_j^{\alpha t} \, dB_s + \sum_{\alpha = 1}^n \int_0^t N_i^{\alpha t} D_j^{\alpha t} \, ds, \quad 1 \leq i, j \leq n,
\]
where \(N_{ij}^t, A_{ij}^t,\) and \(D_{ij}^t\) are the \((i, j)\)th components of \(N_t, A_t,\) and \(D_t,\) respectively. By virtue of the Itô formula, we obtain that
\[
d\nu_t = \nu_t a_t \, dB_t + \nu_t b_t \, dt, \quad \nu_0 = 1,
\]
where
\[
a_t = \sum_{\alpha, i, j = 1}^n \frac{N_{ij}^{i\alpha t} N_{ij}^{j\alpha t}}{1 + \|N_t\|^2} A_{ij}^{\alpha t},
\]
\[
b_t = \sum_{\alpha, i, j = 1}^n \frac{N_{ij}^{i\alpha t} N_{ij}^{j\alpha t}}{1 + \|N_t\|^2} D_{ij}^{\alpha t} + \frac{1}{2} \sum_{\alpha, \beta, i, j, k, \ell = 1}^n \left\{ \delta_{ik}\delta_{j\ell} - \frac{N_{ij}^{i\alpha t} N_{ij}^{j\alpha t}}{1 + \|N_t\|^2} \right\} \frac{N_{ij}^{i\beta t} N_{ij}^{j\beta t}}{1 + \|N_t\|^2} A_{ij}^{\alpha t} A_{ij}^{\beta t}.
\]
Due to the assumption (3), there exists \(K' > 0\) such that
\[
a_t^2 + |b_t| \leq K' \left( 1 + \sup_{u \leq t} |B_u|^2 \right), \quad t \in [0, T].
\]
By the previous observation in the case when \(n = 1,\) for each \(p > 0\) there exists \(t_p > 0\) so that
\[
\sup_{t \leq t_p} \mathbb{E}[\nu_t^p] < \infty.
\]
Since \(\|N_t\| \leq \sqrt{1 + n} \nu_t,\) we arrive at the desired estimation (5).
Remark 1. In repetition of the proof, we see that, if there are $K > 0$ and $\gamma < 2$ such that

$$\|A_t\|^2 + \|D_t\| \leq K \left(1 + \sup_{s \leq t} |B_s|^\gamma\right), \quad t \in [0, T],$$

then $\sup_{t \leq T} \mathbb{E}[\|N_t\|^p] < \infty$ for any $p > 0$.

We now proceed to the proof of the second assertion in Theorem 1.

Proof of Theorem 1(ii). Due to the Itô formula, we have that

$$d \left[ M_t^x \left( f_1(x + \sqrt{\alpha} B_t) \right) \right] = M_t^x \Phi_t^x \, dt + M_t^x \Psi_t^x \, dB_t,$$

where

$$\Phi_t^x = \frac{1}{2} \left\{ - \left( \begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array} \right) \left( (x + \sqrt{\alpha} B_t)^2 f_1(x + \sqrt{\alpha} B_t) \right) - J \left( f_1(x + \sqrt{\alpha} B_t) \right) \right\}$$

$$\Phi_t^x = \frac{1}{2} \left\{ \left( \begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array} \right) \left( f_1'(x + \sqrt{\alpha} B_t) \right) \right\} + (-J) \left( (x + \sqrt{\alpha} B_t) f_1'(x + \sqrt{\alpha} B_t) \right)$$

$$\Psi_t^x = \left( \begin{array}{cc} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{array} \right) \left( f_1'(x + \sqrt{\alpha} B_t) \right)$$

$$\Psi_t^x = \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right) \left( f_2'(x + \sqrt{\beta} B_t) \right)$$

By the first assertion of Theorem 1, there exists $t_4 > 0$ such that

$$\sup_{t \leq t_4} \mathbb{E}[\|M_t^x\|^4] < \infty.$$ 

This implies that

$$\mathbb{E} \left[ \int_0^{t_4} \|M_t^x\|^2 \left(1 + \sup_{s \leq t} |B_s|^\gamma\right)^q \, dt \right] < \infty \quad \text{for every } q \in [0, \infty).$$

(7)

Hence the stochastic process $\{\int_0^t M_s^x \Psi_s^x \, dB_s\}_{t \in [0, t_4]}$ is a martingale. It then follows from this and (6) that

$$\mathbb{E} \left[ \int_0^t M_s^x \left( f_1(x + \sqrt{\alpha} B_t) \right) \right] - f(x) = \int_0^t \mathbb{E}[M_s^x \Phi_s^x] \, ds, \quad t \in [0, t_4],$$

(8)

where the integrability of $M_s^x \Phi_s^x$ is due to (7). By (7) again, it holds that

$$\sup_{s \leq t_4} \mathbb{E}[\|M_s^x \Phi_s^x\|^2] < \infty,$$

which implies that $\{M_s^x \Phi_s^x\}_{s \in [0, t_4]}$ is uniformly integrable. Thus, the mapping $s \mapsto \mathbb{E}[M_s^x \Phi_s^x]$ is continuous. Since $\Phi_0^x = -Q_{(\alpha, \beta)} f(x)$, the identity (8) then implies the second assertion. \[\square\]

3. Remarks

In this section, we shall give several remarks.
First suppose that \( \alpha = \beta \). We then have that
\[
M^x_t = \text{Exp} \left( -\left\{ \frac{1}{2} B^2_t + \frac{x}{\sqrt{\alpha}} B_t \right\} J - \left\{ \frac{\alpha^2 - 1}{2} \int_0^t \left( \frac{x}{\sqrt{\alpha}} + B_s \right)^2 ds \right\} I \right),
\]
where \( \text{Exp}(A) = \sum_{n=0}^{\infty} A^n/n! \), \( A \in \mathbb{R}^{2 \times 2} \). In this case, for sufficiently small \( \varepsilon > 0 \),
\[
T_t f(x) = \mathbb{E} \left[ M^x_t \left( f_1(x + \sqrt{\alpha} B_t) \right) \right], \quad t \in [0, \varepsilon],
\]
determines the semigroup \( \{T_t\}_{t \in [0, \varepsilon]} \) associated with \(-Q_{(\alpha,\alpha)}\). The second assertion in Theorem 1 follows from the semigroup property of \( \{T_t\}_{t \in [0, \varepsilon]} \). For details, see [2].

Next let \( \alpha \neq \beta \). In this case, the semigroup behind Theorem 1 is slightly different from the above one. Namely, for \( z = (x, y) \in \mathbb{R}^2 \), define \( \{Z^z_t\}_{t \geq 0} \) by
\[
Z^z_t = (Z_t^{z,1}, Z_t^{z,2}) = (x + \sqrt{\alpha} B_t, y + \sqrt{\beta} B_t).
\]
Let \( N^z_t \) be the unique solution to the \( \mathbb{R}^{2 \times 2} \)-valued SDE
\[
dN^z_t = N^z_t (-J) \begin{pmatrix} Z_t^{z,1}/\sqrt{\alpha} & 0 \\ 0 & Z_t^{z,2}/\sqrt{\beta} \end{pmatrix} dB_t + \frac{1}{2} N^z_t \left\{ - \left( \begin{array}{cc} \alpha (Z_t^{z,1})^2 & 0 \\ 0 & \beta (Z_t^{z,2})^2 \end{array} \right) - J \right\} dt,
\]
\( N^z_0 = I \).

It holds that
\[
N^z_t = M^x_t.
\]
Let \( C^\infty_\uparrow (\mathbb{R}^2; \mathbb{C}^2) \) be the space of all \( C^\infty \) functions from \( \mathbb{R}^2 \) to \( \mathbb{C}^2 \), which are of at most polynomial growth as are their derivatives of all orders. For
\[
g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in C^\infty_\uparrow (\mathbb{R}^2; \mathbb{C}^2),
\]
define \( S_t g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by
\[
S_t g(z) = \mathbb{E}[N^z_t g(Z^z_t)], \quad z \in \mathbb{R}^2.
\]
By Proposition 1, for sufficiently small \( \varepsilon > 0 \), \( S_t, t \in [0, \varepsilon] \), are well defined. For \( s \geq 0 \), define \( \theta_s : \mathcal{W} \rightarrow \mathcal{W} \) by \( (\theta_s w)(t) = w(t + s) - w(s), w \in \mathcal{W}, t \geq 0 \). Due to the uniqueness of the \( \mathbb{R}^{2 \times 2} \)-valued SDE defining \( N^z_t \), we have that
\[
N^z_{t+s} = N^z_t (N^z_s \circ \theta_s)|_{\xi = Z^z_t}.
\]
As an application of the Itô formula, it follows from this that \( \{S_t\}_{t \in [0, \varepsilon]} \) is the semigroup associated with the differential operator \( \mathcal{L}_{(\alpha, \beta)} \) on \( C^\infty_\uparrow (\mathbb{R}^2; \mathbb{C}^2) \) given by
\[
\mathcal{L}_{(\alpha, \beta)} = \frac{1}{2} \left\{ \alpha \frac{\partial^2}{\partial x^2} + 2\sqrt{\alpha\beta} \frac{\partial}{\partial x} \frac{\partial}{\partial y} + \beta \frac{\partial^2}{\partial y^2} \right\}
+ (-J) \begin{pmatrix} x/\sqrt{\alpha} & 0 \\ 0 & y/\sqrt{\beta} \end{pmatrix} \left\{ \sqrt{\alpha} \frac{\partial}{\partial x} + \sqrt{\beta} \frac{\partial}{\partial y} \right\}
+ \frac{1}{2} \left\{ - \left( \begin{array}{cc} \alpha x^2 & 0 \\ 0 & \beta y^2 \end{array} \right) - J \right\}.
For $f \in C^\infty_\uparrow(\mathbb{R}; \mathbb{C}^2)$, define $g^f \in C^\infty_\uparrow(\mathbb{R}^2; \mathbb{C}^2)$ by
\[
g^f(z) = \begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix}, \quad z = (x, y) \in \mathbb{R}^2.
\]
It is easily seen that, for every $x \in \mathbb{R}$,
\[
\mathcal{L}_{(\alpha, \beta)} g^f((x, x)) = -Q_{(\alpha, \beta)} f(x), \quad S_t g^f((x, x)) = \mathbb{E} \left[ M_t^x \begin{pmatrix} f_1(x + \sqrt{\alpha} B_t) \\ f_2(x + \sqrt{\beta} B_t) \end{pmatrix} \right].
\]
Thus, the second assertion in Theorem 1 follows from the semigroup property of $\{S_t\}_{t \in [0, \varepsilon]}$:
\[
\frac{1}{t} \{S_t g^f((x, x)) - f(x)\} \to \mathcal{L}_{(\alpha, \beta)} g^f((x, x)) = -Q_{(\alpha, \beta)} f(x).
\]
If $\alpha = \beta$, then
\[
S_t g^f((x, x)) = T_t f(x), \quad x \in \mathbb{R}.
\]
In this sense, the observation made in the previous section is a continuation and an extension of that made in [2].

REFERENCES