APPARENT CONTOURS WITH MINIMAL NUMBER OF SINGULARITIES

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Abstract. For a stable map $\varphi: M \rightarrow S^2$ of a closed and connected surface into the sphere, let $c(\varphi)$, $i(\varphi)$ and $n(\varphi)$ denote the numbers of cusps, fold curve components and nodes respectively. In this paper, in a given homotopy class, we will determine the minimal triples (c, i, n) , (n, c, i) and (i, n, c) with respect to the lexicographic order.

1. Introduction

Let M be a closed and connected surface and N a connected surface. Let $\varphi: M \rightarrow N$ be a C^∞ map. Define the set of singular points of φ as

$$S(\varphi) = \{p \in M \mid \text{rank } d\varphi_p < 2\}.$$

We call $\varphi(S(\varphi))$ the *apparent contour* (or *contour* for short) of φ and denote it by $\gamma(\varphi)$. When $S(\varphi) \subset M$ consists of one component, $\gamma(\varphi)$ is said to be *irreducible*.

A C^∞ map $\varphi: M \rightarrow N$ is said to be *stable* if it satisfies the following two properties.

- (1) The map germ at each $p \in M$ is C^∞ right–left equivalent to one of the map germs at $0 \in \mathbb{R}^2$ below:
 - $(a, x) \mapsto (a, x)$: p is a regular point;
 - $(a, x) \mapsto (a, x^2)$: p is a fold point;
 - $(a, x) \mapsto (a, x^3 + ax)$: p is a cusp point.

Hence, $S(\varphi)$ is a finite disjoint union of circles.

- (2) For each $q \in \gamma(\varphi)$, the map germ $(\varphi|_{S(\varphi)}, \varphi^{-1}(q) \cap S(\varphi))$ is right–left equivalent to one of the three multi-germs as depicted in Figure 1.

According to a classical result of Whitney [10], stable maps form an open everywhere dense set in the space of all C^∞ maps $M \rightarrow N$. Thus, for each homotopy class of a C^∞ map $M \rightarrow N$, there is a stable map.

In this paper, we consider stable maps with singular points.

When φ is stable, $S(\varphi)$ is called the *fold curve* of φ , and the numbers of cusps, fold curve components and nodes on $\gamma(\varphi)$ are denoted by $c(\varphi)$, $i(\varphi)$ and $n(\varphi)$ respectively. A stable map that has no cusp is called a *fold map*.

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FIGURE 1. The multi-germs of $\varphi|_{S(\varphi)}$.

An oriented closed surface (or a non-orientable closed surface) of genus g , that is a connected sum of g copies of the two-dimensional torus T^2 (respectively g copies of the real projective plane $\mathbb{R}P^2$), is denoted by Σ_g (respectively F_g). The two-dimensional sphere and the plane are denoted by S^2 and $\mathbb{R}^2$ respectively.

The apparent contours of stable maps $M \rightarrow \mathbb{R}^2$ or S^2 have been studied by some authors [1, 2, 4, 5]. Recall the following proposition in [2].

PROPOSITION 1.1. [2] *Let $f: \Sigma_g \rightarrow S^2$ be a degree d stable map whose contour is irreducible. Then:*

- (1) *the contour $\gamma(f)$ has at least two cusps if $|d| \leq g$ and $d \not\equiv g \pmod{2}$;*
- (2) *the contour $\gamma(f)$ has at least $2(|d| - g)$ cusps if $|d| \geq g$.*

Proposition 1.1 shows that there is a relation among the number of fold curve components and the number of cusps for a stable map between surfaces.

We recall that, for each pair (d, g) satisfying $|d| \leq g$ and $d \equiv g \pmod{2}$, a degree d fold map $\Sigma_g \rightarrow S^2$ whose contour is irreducible was obtained in [2].

We remark that, for a stable map $\varphi: M \rightarrow S^2$, the number of cusps of φ and the Euler characteristic of M have the same parity by a classical result of Thom [7].

In this paper, for a given C^∞ map $\varphi_0: M \rightarrow N$, we study three kinds of apparent contours of stable maps that are homotopic to φ_0 . One is a (c, i, n) -minimal contour of φ_0 . The others are an (n, c, i) -minimal contour of φ_0 and an (i, n, c) -minimal contour of φ_0 .

Let $\varphi_0: M \rightarrow N$ be a C^∞ map and $\varphi: M \rightarrow N$ be a stable map. The contour $\gamma(\varphi)$ is called a (c, i, n) -minimal contour, an (n, c, i) -minimal contour and an (i, n, c) -minimal contour of φ_0 if the triples $(c(\varphi), i(\varphi), n(\varphi))$, $(n(\varphi), c(\varphi), i(\varphi))$ and $(i(\varphi), n(\varphi), c(\varphi))$ are the smallest with respect to the lexicographic order among the stable maps that are homotopic to φ_0 respectively.

We recall that by virtue of Hopf's theorem (for example, see [3]), two C^∞ maps $\Sigma_g \rightarrow S^2$ (or $F_g \rightarrow S^2$) are homotopic if and only if their degrees (respectively modulo two degrees) coincide. Thus, the homotopy classes of a degree d stable map $\Sigma_g \rightarrow S^2$ and a modulo two degree d_2 stable map $F_g \rightarrow S^2$ are determined by the pairs (d, g) and (d_2, g) respectively.

The following theorem concerning a (c, i, n) -minimal contour is the first result of this paper.

FIGURE 2. The (c, i, n) -minimal contours of C^∞ map $\Sigma_g \rightarrow S^2$.

THEOREM 1.2.

- (1) Let $f: \Sigma_g \rightarrow S^2$ be a stable map of degree d . Then, $\gamma(f)$ is (c, i, n) -minimal if and only if the triple (c, i, n) for $\gamma(f)$ is one of the items below:

$$(c, i, n) = \begin{cases} \text{(i)} (0, |d| + 1, 0) \\ \quad \text{if } g = 0; \\ \text{(ii)} (0, 2, 0) \\ \quad \text{if } (|d|, g) = (0, 1); \\ \text{(iii)} (0, 1, |d| + g + 2) \\ \quad \text{if } g \neq 0, |d| \leq g \text{ and } d \equiv g \pmod{2}; \\ \text{(iv)} (0, 2, |d| + g + 1) \\ \quad \text{if } g \neq 0, |d| < g \text{ and } d \not\equiv g \pmod{2} \text{ except } (|d|, g) = (0, 1); \\ \text{(v)} (0, |d| - g + 1, 2g + 2) \\ \quad \text{if } g \neq 0, |d| \geq g. \end{cases}$$

Figure 2(a) with $\ell = |d| + 1$ and $\ell = 2$ shows (c, i, n) -minimal contours of the cases (i) and (ii) respectively. Figure 2(b) with $(j, n) = (0, |d| + g + 2)$, $(j, n) = (1, |d| + g + 1)$ and $(j, n) = (|d| - g, 2g + 2)$ shows (c, i, n) -minimal contours of the cases (iii), (iv) and (v) respectively.

- (2) Let $h: F_g \rightarrow S^2$ be a stable map of modulo two degree d_2 . Then, $\gamma(h)$ is (c, i, n) -minimal if and only if the triple (c, i, n) for $\gamma(h)$ is one of the items below:

$$(c, i, n) = \begin{cases} (1, 1, (g + 5)/2) & \text{if } d_2 = 1 \text{ and } g \text{ is odd;} \\ (0, 1, (g + 6)/2) & \text{if } d_2 = 1 \text{ and } g \text{ is even;} \\ (1, 1, (g + 7)/2) & \text{if } d_2 = 0 \text{ and } g \text{ is odd;} \\ (0, 1, (g + 8)/2) & \text{if } d_2 = 0 \text{ and } g \text{ is even, and } g/2 \text{ is odd;} \\ (0, 1, (g + 4)/2) & \text{if } d_2 = 0 \text{ and } g \text{ is even, and } g/2 \text{ is even.} \end{cases}$$

Theorem 1.2 shows the following corollary.

COROLLARY 1.3. Let $f: \Sigma_g \rightarrow S^2$ be a degree d stable map whose contour is (c, i, n) -minimal. Then:

FIGURE 3. The (n, c, i) -minimal contours of C^∞ maps $M \rightarrow S^2$.

- (1) *the number $n(f)$ is an even number;*
- (2) *the number $i(f)$ and $(\chi(\Sigma_g)/2) + |d|$ have the same parity, where $\chi(\Sigma_g)$ denotes the Euler characteristic of Σ_g .*

Note that there is a stable map $\Sigma_g \rightarrow S^2$ that has an odd number of nodes.

Yamamoto [8] determined the minimal number of fold curve components of fold maps $\Sigma_g \rightarrow \Sigma_h$. Theorem 1.2 determines the minimal number of nodes among fold maps $\Sigma_g \rightarrow S^2$ such that the number of fold curve components is minimal.

The following theorem concerning an (n, c, i) -minimal contour is the second result of this paper.

THEOREM 1.4.

- (1) *Let $f: \Sigma_g \rightarrow S^2$ be a stable map of degree d . Then, $\gamma(f)$ is an (n, c, i) -minimal contour if and only if the triple (n, c, i) for $\gamma(f)$ is*

$$(n, c, i) = (0, 0, |d| + g + 1).$$

Figure 3(a) with $\ell_1 = |d| + g + 1$ shows an (n, c, i) -minimal contour of f .

- (2) *Let $h: F_g \rightarrow S^2$ be a stable map of modulo two degree one. Then, $\gamma(h)$ is an (n, c, i) -minimal contour if and only if the triple (n, c, i) for $\gamma(h)$ is*

$$(n, c, i) = \begin{cases} \text{(i)} (0, 0, (g+4)/2) & \text{if } g \text{ is even;} \\ \text{(ii)} (0, 1, (g+3)/2) & \text{if } g \text{ is odd.} \end{cases}$$

Figure 3(a) with $\ell_1 = (g+4)/2$ and Figure 3(b) with $\ell_2 = (g+1)/2$ show (n, c, i) -minimal contours of the cases (i) and (ii) respectively.

The following theorem concerning an (i, n, c) -minimal contour is the third result of this paper.

THEOREM 1.5.

- (1) *Let $f: \Sigma_g \rightarrow S^2$ be a stable map of degree d . Then, $\gamma(f)$ is an (i, n, c) -minimal contour if and only if the triple (i, n, c) for $\gamma(f)$ is one of the items below:*

$$(i, n, c) = \begin{cases} \text{(i)} (1, 0, 0) & \text{if } d = g = 0; \\ \text{(ii)} (1, 0, 2(g+2)) & \text{if } d = 0; \\ \text{(iii)} (1, 0, 2(|d| + g)) & \text{otherwise.} \end{cases}$$

FIGURE 4. The (i, n, c) -minimal contour of C^∞ maps $M \rightarrow S^2$.

Figure 4 with $c = 0$, $c = 2(g + 2)$ and $c = 2(|d| + g)$ shows (i, n, c) -minimal contours of the cases (i), (ii) and (iii) respectively.

- (2) Let $h: F_g \rightarrow S^2$ be a stable map of modulo two degree d_2 . Then, $\gamma(h)$ is an (i, n, c) -minimal contour if and only the triple (i, n, c) for $\gamma(h)$ is

$$(i, n, c) = (1, 0, -2d_2 + g + 4).$$

Figure 4 with $c = (-2d_2 + g + 4)$ shows an (i, n, c) -minimal contour of h .

This paper is organized as follows. In Section 2, we prepare some notions concerning the apparent contour of a stable map between surfaces. In Section 3, Theorem 1.2 is proved. In Section 4, Theorem 1.4 is proved. In Section 5, Theorem 1.5 is proved.

Throughout this paper, all surfaces are connected and of class C^∞ , and all maps are of class C^∞ . The symbols $\deg$ and $\deg_2$ denote the mapping degree and the modulo two mapping degree respectively. For a topological space X , the symbol $\chi(X)$ denotes the Euler characteristic of X .

2. Preparation

In the following, we prepare some notions concerning the apparent contour of a stable map of a closed surface into the sphere.

Let M be a closed surface and $\varphi: M \rightarrow S^2$ a stable map with singular points. Let $S(\varphi) = S_1 \cup \dots \cup S_\ell$ be the decomposition of $S(\varphi)$ into the connected components and set $\gamma_i = \varphi(S_i)$ ($i = 1, \dots, \ell$). Note that $\gamma(\varphi) = \gamma_1 \cup \dots \cup \gamma_\ell$. Denote by $n_1(\varphi)$ the sum of the number of self-intersection points of γ_i ($i = 1, \dots, \ell$) and $n_2(\varphi)$ the sum of the number of points $\gamma_i \cap \gamma_j$ for all i and j with $i \neq j$. Note that $n_2(\varphi)$ is an even number and that $n(\varphi) = n_1(\varphi) + n_2(\varphi)$. Let $m(\varphi)$ be the smallest number of elements in the set $\varphi^{-1}(y)$, where $y \in S^2$ runs over all regular values of φ . Fix a regular value ∞ such that $\varphi^{-1}(\infty)$ consists of $m(\varphi)$ points. For each γ_i , denote by U_i the component of $S^2 \setminus \gamma_i$ that contains ∞ . Note that $\partial U_i \subset \gamma_i$.

Orient γ_i so that, at each fold point image, the surface is ‘folded to the left’. More precisely, for a point $y \in \gamma_i$ that is not a cusp or a node of γ_i , choose a normal vector v of γ_i at y such that $\varphi^{-1}(y')$ contains more elements than $\varphi^{-1}(y)$, where y' is a regular value of φ close to y in the direction of v . Let τ be a tangent vector of γ_i at y with respect to the above orientation of γ_i . Then, orient S^2 by the ordered pair (τ, v) . It is easy to see that this gives a well-defined orientation of S^2 .

FIGURE 5. A positive node and a negative node.

Definition 2.1. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is said to be *positive* if the normal orientation v at y points toward U_i ; otherwise, it is said to be *negative*.

A component γ_i is said to be *positive* if all points of $\partial U_i \setminus \{\text{cusps, nodes}\}$ are positive; otherwise, γ_i is said to be *negative*. The numbers of positive and negative components are denoted by i^+ and i^- respectively. Note that there is at least one negative component unless $S(\varphi) = \emptyset$.

Definition 2.2. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is called an *admissible starting point* if

- (1) y is a positive point of a positive component γ_i or
- (2) y is a negative point of a negative component γ_i .

Note that, for each i , there always exists an admissible starting point in γ_i .

Definition 2.3. Let $y \in \gamma_i$ be an admissible starting point. Suppose that $Q \in \gamma_i$ is a node, and let $\alpha: [0, 1] \rightarrow \gamma_i$ be a parameterization that is consistent with the orientation and is singular only when the image is a cusp such that $\alpha^{-1}(y) = \{0, 1\}$. Then, there are two numbers $0 < t_1 < t_2 < 1$ satisfying $\alpha(t_1) = \alpha(t_2) = Q$.

We say that Q is *positive* if the orientation of S^2 at Q defined by the ordered pair $(\alpha'(t_1), \alpha'(t_2))$ coincides with that of S^2 at Q ; we say that Q is *negative*, otherwise. See Figure 5 for details.

The numbers of positive and negative nodes on γ_i are denoted by N_i^+ and N_i^- respectively. The definition of a positive (or negative) node of γ_i depends on the choice of an admissible starting point y . However, it is known that the algebraic number $N_i^+ - N_i^-$ does not depend on the choice of y ; see [9] for details. Thus, the algebraic number

$$N^+ - N^- = \sum_{i=1}^k (N_i^+ - N_i^-)$$

is well defined. Note that nodes arising from $\gamma_i \cap \gamma_j$ ($i \neq j$) play no role in the computation: $n_1(\varphi) = N^+ + N^-$.

FIGURE 6. Proof of Lemma 2.6.

Then, we have the following formula in [2].

PROPOSITION 2.4. [2] *For a stable map $\varphi: M \rightarrow S^2$ of a closed surface of genus g , we have*

$$g = \varepsilon(M) \left[(N^+ - N^-) + \frac{c(\varphi)}{2} + (1 + i^+ - i^-) - m(\varphi) \right], \quad (2.1)$$

where $\varepsilon(M)$ is equal to 1 if M is orientable and 2 if M is not orientable.

In the following, we assume that $\gamma_i \cap \gamma_j = \emptyset$ for all $i \neq j$. Denote by $U_\infty \subset S^2 \setminus \gamma(\varphi)$ the component that contains ∞ . Denote by γ_1 the component of $\gamma(\varphi)$ that contains ∂U_∞ . Note that γ_1 is a negative component of φ .

LEMMA 2.5. *If γ_1 has a node, then it has a negative node.*

Proof. Let us go along γ_1 starting from an admissible starting point y , following the canonical orientation of γ_1 . When we pass through a positive node of γ_1 for the first time, the number of points in the inverse image decreases by two; see Figure 5 for details. This is a contradiction. $\square$

LEMMA 2.6. *If a positive component γ_i has a node, then it has a positive node.*

Proof. Let us go along γ_i starting from an admissible starting point y , following the canonical orientation of γ_i . If the first node that we encounter is a negative node (see Figure 5), then the point $y' \in \gamma_i$ close to the node as in Figure 6 is in ∂U_i and y' is a negative point. This is a contradiction. $\square$

COROLLARY 2.7. *If the number of negative components of $\gamma(\varphi)$ is equal to one and $\gamma(\varphi)$ has a node, then it has a negative node.*

Proof. We may assume that γ_1 has no node, and that γ_i is a positive component that has a node and is adjacent to the same component of $S^2 \setminus \gamma(\varphi)$ as γ_1 . Then, by the proof of Lemma 2.6, there is a point $q \in S^2 \setminus \gamma(\varphi)$ that is close to the node and whose inverse image consists of $m(\varphi) - 2$ points (see Figure 7). This is a contradiction. $\square$

3. Proof of Theorem 1.2

For a C^∞ map $\Sigma_g \rightarrow S^2$ of degree d , by chaining the orientation of Σ_g , we obtain a C^∞ map $\Sigma_g \rightarrow S^2$ of degree $-d$. In the following, we assume that $d \geq 0$.

FIGURE 7. Proof of Corollary 2.7.

We remark that the notion of an *essential* contour of a C^∞ map $M \rightarrow N$ was introduced in [5]. Let $\varphi_0: M \rightarrow N$ be a C^∞ map between surfaces and $\varphi: M \rightarrow N$ a stable map. The contour $\gamma(\varphi)$ is called an *essential contour* of φ_0 if the triple $(i(\varphi), c(\varphi), n(\varphi))$ is the smallest with respect to the lexicographic order among the stable maps homotopic to φ_0 . The following theorem was obtained in [2].

THEOREM 3.1.

- (1) For each $d \geq 0$, the triple (i, c, n) for an essential contour of a C^∞ map $\Sigma_g \rightarrow S^2$ of degree d is one of the items below:

$$(i, c, n) = \begin{cases} (1, 2d, 0) & \text{if } g = 0; \\ (1, 0, d + g + 2) & \text{if } g \neq 0, d \leq g \text{ and } d \equiv g \pmod{2}; \\ (1, 2, d + g + 1) & \text{if } g \neq 0, d < g \text{ and } d \not\equiv g \pmod{2}; \\ (1, 2(d - g), 2g + 2) & \text{if } g \neq 0, d \geq g. \end{cases}$$

- (2) The triple (i, c, n) for an essential contour of a C^∞ map $F_g \rightarrow S^2$ of modulo two degree d_2 is one of the items below:

$$(i, c, n) = \begin{cases} (1, 1, (g + 5)/2) & \text{if } d_2 = 1 \text{ and } g \text{ is odd}; \\ (1, 0, (g + 6)/2) & \text{if } d_2 = 1 \text{ and } g \text{ is even}; \\ (1, 1, (g + 7)/2) & \text{if } d_2 = 0 \text{ and } g \text{ is odd}; \\ (1, 0, (g + 8)/2) & \text{if } d_2 = 0 \text{ and } g \text{ is even, and } g/2 \text{ is odd}; \\ (1, 0, (g + 4)/2) & \text{if } d_2 = 0 \text{ and } g \text{ is even, and } g/2 \text{ is even}. \end{cases}$$

Note that, if an essential contour of a C^∞ map $M \rightarrow S^2$ has no cusps or exactly one cusp, then the essential contour is also a (c, i, n) -minimal contour. Thus, Theorem 1.2(1) with the case (iii) and 1.2(2) have already been proved.

Let us prove Theorem 1.2(1) except the case (iii).

We prepare some stable maps $\Sigma_g \rightarrow S^2$ which we employ in the proof of Theorem 1.2. Figure 8 shows the following proposition for a stable map $f: \Sigma_g \rightarrow S^2$ with singular points.

PROPOSITION 3.2.

- (1) By attaching a handle vertically to f (Figure 8(a)), we obtain a stable map $f_1: \Sigma_{g+1} \rightarrow S^2$ whose quadruple $(i, c, n, \deg)$ is equal to $(i(f) + 1, c(f), n(f), \deg f)$.

FIGURE 8. Proof of Proposition 8.

FIGURE 9. A stable map $p_g: \Sigma_g \rightarrow \mathbb{R}^2$.

- (2) By attaching a cover vertically to f (Figure 8(b)), we obtain a stable map $f_2: \Sigma_g \rightarrow S^2$ whose quadruple $(i, c, n, \deg)$ is equal to $(i(f) + 1, c(f), n(f), \deg f + 1)$.

Define a degree zero stable map $f'_{0,1}: T^2 \rightarrow S^2$ by $f'_{0,1} = \iota \circ p_1$ where p_1 is defined by Figure 9 with $g = 1$ and ι is the inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \cup \{\infty\} = S^2$.

For each pair $(d, g) \neq (0, 1)$ satisfying $d < g$ and $d \not\equiv g \pmod{2}$, there is a degree d stable map $f_{d,g-1}: \Sigma_{g-1} \rightarrow S^2$ whose contour is in Figure 2(b) with $(j, n) = (0, d + g + 1)$. Then, by attaching a handle vertically (Proposition 3.2(1)) to $f_{d,g-1}$, we obtain a degree d stable map $f'_{d,g}: \Sigma_g \rightarrow S^2$ whose contour is in Figure 2(b) with $(j, n) = (1, d + g + 1)$.

For each pair (d, g) satisfying $d \geq g \neq 0$, by attaching $d - g$ covers vertically (Proposition 3.2(2)) to a degree g stable map $f_{g,g}: \Sigma_g \rightarrow S^2$ whose contour is in Figure 2(b) with $(j, n) = (0, 2g + 2)$, we obtain a degree d stable map $f'_{d,g}: \Sigma_g \rightarrow S^2$ whose contour is in Figure 2(b) with $(j, n) = (d - g, 2g + 2)$. On the other hand, by attaching covers vertically (Proposition 3.2(2)) to the degree one stable map $\psi_{1,0}: S^2 \rightarrow S^2$ defined by Figure 10 with $g = 0$, we obtain a degree $d \geq 1$ stable map $f'_{d,0}: S^2 \rightarrow S^2$ whose contour is in Figure 2(a) with $\ell = d + 1$.

Define $f'_{0,0}: S^2 \rightarrow S^2$ by $f'_{0,0} = \iota \circ p_0$ where p_0 is defined by Figure 9 with $g = 0$, and $\iota: \mathbb{R}^2 \hookrightarrow S^2$ is the inclusion. The triple (c, i, n) for $\gamma(f'_{0,0})$ is equal to $(0, 1, 0)$.

FIGURE 10. A stable map $\psi_{1,g}: \Sigma_g \rightarrow S^2$.

Thus, we obtain a degree d stable map $f'_{d,g}: \Sigma_g \rightarrow S^2$ whose triple (c, i, n) is equal to the items below:

$$(c, i, n) = \begin{cases} (0, d+1, 0) & \text{if } g = 0; \\ (0, 2, 0) & \text{if } (d, g) = (0, 1); \\ (0, 2, d+g+1) & \text{if } g \neq 0, d < g \text{ and } d \not\equiv g \pmod{2}; \\ (0, d-g+1, 2g+2) & \text{if } g \neq 0, d \geq g. \end{cases}$$

Proof of Theorem 1.2(1) except the case (iii). By Proposition 1.1(1), a (c, i, n) -minimal contour of a C^∞ map $\Sigma_g \rightarrow S^2$ of degree d consists of at least two components if $g \neq 0$, $d < g$ and $d \not\equiv g \pmod{2}$. Thus, the contour $\gamma(f'_{0,1})$ is (c, i, n) -minimal and the case (ii) is proved.

Let us prove Theorem 1.2(1) with (iv). Let $f: \Sigma_g \rightarrow S^2$ be a degree d fold map whose contour consists of two components. We divide this case into the following two parts (1) and (2).

(1) *The case of $n_2(f) = 0$.*

(1-1) When (i^+, i^-) for $\gamma(f)$ is equal to $(1, 1)$, then by (2.1), we have

$$d + g - 1 \leq N^+ - N^-.$$

Therefore we have

$$n_1(f) \geq d + g - 1 + 2N^-. \quad (3.1)$$

By (3.1), we have $n_1(f) > 0$. By Corollary 2.7, we have

$$n_1(f) \geq d + g - 1 + 2N^- \geq d + g + 1.$$

(1-2) When (i^+, i^-) for $\gamma(f)$ is equal to $(0, 2)$, then, by (2.1), we have

$$d + g + 1 \leq N^+ - N^-.$$

Therefore we have

$$n_1(f) \geq d + g + 1 + 2N^- \geq d + g + 1.$$

(2) *The case of $n_2(f) \neq 0$.*

When (i^+, i^-) for $\gamma(f)$ is equal to $(2 - \lambda, \lambda)$ ($\lambda = 1, 2$), then by (2.1), we have $d + g + 2\lambda - 3 \leq N^+ - N^-$. Therefore we have

$$n_1(f) \geq d + g + 2\lambda - 3 + 2N^- \geq d + g - 1.$$

Thus,

$$n(f) = n_1(f) + n_2(f) \geq d + g - 1 + 2 = d + g + 1.$$

These show that $\gamma(f'_{d,g})$ is (c, i, n) -minimal if $g \neq 0$, $d < g$ and $d \not\equiv g \pmod{2}$ except $(d, g) = (0, 1)$. Thus, the case (iv) is proved.

On the other hand, there is the following lemma.

LEMMA 3.3. *Let $f: \Sigma_g \rightarrow S^2$ be a degree d stable map whose contour consists of μ components ($\mu \neq 0$). If $|d| \geq g$ and $\mu < |d| - g + 1$, then $\gamma(f)$ has at least*

$$2(|d| - g - \mu + 1)$$

cusps.

Proof. Apply the following result of Quine [6]. For a stable map $\varphi: M \rightarrow N$ between oriented surfaces, we have

$$\chi(M) - 2\chi(M_-) + \sum_{q_k: \text{cusp of } f} \text{sign}(q_k) = (\deg \varphi)\chi(N),$$

where M_- denotes the closure of the set of regular points x such that the differential df_x reverses the orientation, and $\text{sign}(q_k)$ ($= \pm 1$) is the sign of a cusp q_k of f . See [6] for details.

In our situation, we have:

$$\sum_{q_k: \text{cusp of } f} \text{sign}(q_k) = 2(d + g - 1 + \chi((\Sigma_g)_-)).$$

Since $(\Sigma_g)_-$ has μ boundary components, we have

$$2 - 2g - \mu \leq \chi((\Sigma_g)_-) \leq \mu.$$

Thus, we have

$$2(d - g - \mu + 1) \leq \sum_{q_k: \text{cusp of } f} \text{sign}(q_k) \leq 2(d + g + \mu - 1).$$

Then, the result follows immediately. $\square$

By Lemma 3.3, the fold curve of a degree d fold map $\Sigma_g \rightarrow S^2$ ($d \geq g$) consists of at least $d - g + 1$ components. Thus, the contour $\gamma(f'_{d,0})$ is (c, i, n) -minimal and the case (i) is proved.

By combining similar arguments as the above and the proof of the case (iv), we can show that the contour $\gamma(f'_{d,g})$ is (c, i, n) -minimal for each pair (d, g) satisfying $g \neq 0$, $d \geq g$. We omit the proof here. Thus, the case (v) is proved.

Therefore, we have completed the proof of Theorem 1.2. $\square$

4. Proof of Theorem 1.4

As well as the proof of Theorem 1.2(1), assume $d \geq 0$.

We prepare some stable maps $M \rightarrow S^2$ which we shall employ in the proof of Theorem 1.4.

Figure 10 defines a degree one stable map $\psi_{1,g}: \Sigma_g \rightarrow S^2$ whose contour is in Figure 3(a) with $\ell_1 = g + 2$ for each $g \geq 0$. By attaching covers vertically (Proposition 3.2(2)) to $\psi_{1,g}$ inductively, we obtain a degree d stable map $\psi_{d,g}: \Sigma_g \rightarrow S^2$ whose contour is in Figure 3(a) with $\ell_1 = d + g + 1$ for each $d \geq 1$ and $g \geq 0$. On the other hand, for each $g \geq 0$, a map $\psi_{0,g}: \Sigma_g \rightarrow S^2$ that is defined by $\psi_{0,g} = \iota \circ p_g$ is a degree zero stable map, where p_g is in Figure 9 and ι is the inclusion $\mathbb{R}^2 \hookrightarrow S^2$. The contour $\gamma(\psi_{0,g})$ is in Figure 3(a) with $\ell_1 = g + 1$. Thus, for each $d \geq 0$ and $g \geq 0$, we obtain a degree d stable map $\psi_{d,g}: \Sigma_g \rightarrow S^2$ whose triple (n, c, i) is

$$(n, c, i) = (0, 0, d + g + 1).$$

There is a modulo two degree one stable map $\phi_{1,2}: F_2 \rightarrow S^2$ whose contour is in Figure 3(a) with $\ell_1 = 3$. By attaching handles vertically to $\phi_{1,2}$ inductively, we obtain a modulo two degree one stable map $\phi_{1,g}: F_g \rightarrow S^2$ whose contour is in Figure 3(a) with $\ell_1 = (g + 4)/2$ for each even number g .

For the case that g is an odd number, we prepare the stable map below.

PROPOSITION 4.1. *There is a modulo two degree one stable map $\phi_{1,1}: \mathbb{R}P^2 \rightarrow S^2$ whose contour is in Figure 3(b) with $\ell_2 = 1$.*

Proof. To construct the stable map, divide $\mathbb{R}P^2$ into three parts; a Möbius band $\tilde{M}$, an annulus A , and a disk D . Define $A \rightarrow \mathbb{R}^3$ and $\tilde{M} \rightarrow \mathbb{R}^3$ by Figure 11(a) and (b) respectively. Note that each map has a Whitney umbrella point. By sticking the boundary of $\tilde{M}$ to the boundary of A along the solid line in Figure 11(a) and (b), we obtain a stable map $\phi: M \rightarrow \mathbb{R}^3$ that has two Whitney umbrella points, where M is homeomorphic to a Möbius band. Then, we define $\phi_{1,1}|_M: M \rightarrow S^2$ by $\iota \circ \pi \circ \phi$, where $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the projection and $\iota: \mathbb{R}^2 \hookrightarrow S^2$ is the inclusion. Finally, by sticking the boundary of M to the boundary of D along the dotted line (Figure 12), we obtain the desired map $\phi_{1,1}: \mathbb{R}P^2 \rightarrow S^2$. $\square$

By attaching handles vertically (Proposition 3.2) to $\phi_{1,1}$, we obtain a modulo two degree one stable map $\phi_{1,g}: F_g \rightarrow S^2$ whose contour is in Figure 3(b) with $\ell_2 = (g + 1)/2$ for each odd number g .

Thus, for each $g \geq 1$, we have a modulo two degree one stable map $\phi_{1,g}: F_g \rightarrow S^2$ whose triple (n, c, i) is

$$(n, c, i) = \begin{cases} (0, 0, (g + 4)/2) & \text{if } g \text{ is an even number,} \\ (0, 1, (g + 3)/2) & \text{if } g \text{ is an odd number.} \end{cases}$$

Proof of Theorem 1.4(1). By (2.1), for a stable map $f: \Sigma_g \rightarrow S^2$ whose contour has no node and no cusp, we have

$$d + g + 1 \leq i(f).$$

Therefore, the contour $\gamma(\psi_{d,g})$ is (n, c, i) -minimal for each d and g . Thus, Theorem 1.4(1) has been proved. $\square$

FIGURE 11.

FIGURE 12.

For each g , we can prove that the contour $\gamma(\phi_{1,g})$ is (n, c, i) -minimal in a similar way as Theorem 1.4(1). Thus, Theorem 1.4(2) is proved. We omit the proof here.

Finally in this section, we pose the following problem.

Problem 4.2. Let us study an (n, c, i) -minimal contour of a C^∞ map $F_g \rightarrow S^2$ of modulo two degree zero.

5. Proof of Theorem 1.5

We prepare some stable maps $M \rightarrow S^2$ which we shall employ in the proof of Theorem 1.5. Figure 13 shows the following proposition for a stable map $f: \Sigma_g \rightarrow S^2$ with singular points.

PROPOSITION 5.1.

- (1) By attaching a handle horizontally to f (Figure 13(a)), we obtain a stable map $f_3: \Sigma_{g+1} \rightarrow S^2$ whose quadruple $(i, c, n, \deg)$ is equal to $(i(f), c(f) + 2, n(f), \deg f)$.

FIGURE 13.

- (2) By attaching a cover horizontally (Figure 13(b)) to f , we obtain a stable map $f_4: \Sigma_g \rightarrow S^2$ whose quadruple $(i, c, n, \deg)$ is equal to $(i(f), c(f) + 2, n(f), \deg f + 1)$.

Define a degree zero stable map $u_{0,0}: S^2 \rightarrow S^2$ by $\iota \circ p_0$ where p_0 is in Figure 9 with $g = 0$ and $\iota: \mathbb{R}^2 \hookrightarrow S^2$ is the inclusion.

By making a pleat (Figure 14) to id_{S^2} , we obtain a degree one stable map $u_{1,0}: S^2 \rightarrow S^2$ whose contour is in Figure 4(a) with $c = 2$, where id_{S^2} is the identity map on S^2 . By attaching handles or covers horizontally (Proposition 5.1(1) and (2)) to $u_{1,0}$ inductively, we obtain a degree d stable map $u_{d,g}: \Sigma_g \rightarrow S^2$ whose contour is in Figure 4(a) with $c = 2(d + g)$ for each $d \geq 1$ and $g \geq 0$. By attaching a cover that is turned over to $u_{1,1}$, we obtain a degree zero stable map $u_{0,1}: T^2 \rightarrow S^2$ whose contour is in Figure 4(a) with $c = 6$. By attaching handles horizontally (Proposition 3.2(1)) inductively to $u_{0,1}$, we obtain a degree zero stable map $u_{0,g}: \Sigma_g \rightarrow S^2$ whose contour is in Figure 4(a) with $c = 2(g + 2)$. Thus, for each g , we have a degree d stable map $u_{d,g}: \Sigma_g \rightarrow S^2$ whose triple (i, n, c) is

$$(i, n, c) = \begin{cases} (1, 0, 0) & \text{if } d = g = 0; \\ (1, 0, 2(g + 2)) & \text{if } d = 0; \\ (1, 0, 2(d + g)) & \text{otherwise.} \end{cases}$$

We constructed modulo two degree one stable maps $h_{1,1}: \mathbb{R}P^2 \rightarrow S^2$ and $h_{1,2}: F_2 \rightarrow S^2$ whose contours are in Figure 4 with $c = 3$ and $c = 4$ in [2] respectively. By applying a similar argument as $u_{d,g}$ to $h_{1,1}$ and $h_{1,2}$, we obtain a modulo two degree d_2 stable map $\eta_{d_2,g}: F_g \rightarrow S^2$ whose contour is in Figure 4 with $c = -2d_2 + g + 4$ for each d_2 and g .

Proof of Theorem 1.5(1). The contour $\gamma(u_{0,0})$ is trivially (i, n, c) -minimal. Thus, the case (i) is proved.

Note that there is no stable map $f: \Sigma_g \rightarrow S^2$ with $m(f) = 0$ whose contour is irreducible, and whose contours have no node and have cusps by the geometrical reason of a cusp. By combining this reason and (2.1), for the contour of a degree zero stable map $f: \Sigma_g \rightarrow S^2$ whose contour is irreducible and has no nodes, we have

$$2(g + 2) \leq 2(g + m(f)) = c.$$

FIGURE 14. Pleat.

 FIGURE 15. The (i, n, c) -minimal contours of stable maps $M \rightarrow \mathbb{R}^2$.

Thus, the contour $\gamma(u_{0,g})$ is (i, n, c) -minimal for each $g > 0$, and the case (ii) is proved.

By (2.1), for the contour of a degree d stable map $f: \Sigma_g \rightarrow S^2$ whose contour is irreducible and has no nodes, we have

$$2(d + g) \leq 2(g + m(f)) = c.$$

Thus, the contour $\gamma(u_{d,g})$ is (i, n, c) -minimal for each $d \geq 1$ and $g \geq 0$, and the case (iii) is proved.

Therefore, the proof of Theorem 1.5(1) has been completed. $\square$

We can prove that $\gamma(\eta_{d_2,g})$ is (i, n, c) -minimal by a similar argument as the case (1), for each d_2 and g . Thus, Theorem 1.5(2) is proved. We omit the proof here.

The following theorem is also obtained by similar arguments.

THEOREM 5.2.

- (1) *Let $f: \Sigma_g \rightarrow \mathbb{R}^2$ be a stable map. Then, the contour $\gamma(f)$ is (i, n, c) -minimal if and only if the triple (i, n, c) for $\gamma(f)$ is*

$$(i, n, c) = \begin{cases} \text{(i) } (1, 1, 2g + 2) & \text{if } g \geq 1; \\ \text{(ii) } (1, 0, 0) & \text{if } g = 0. \end{cases}$$

Figure 15(a) with $c = 2g + 2$ and 15(b) show (i, n, c) -minimal contours of the cases (i) and (ii) respectively.

- (2) Let $h: F_g \rightarrow \mathbb{R}^2$ be a stable map. The contour $\gamma(h)$ is (i, n, c) -minimal if and only if the triple (i, n, c) for $\gamma(h)$ is

$$(i, n, c) = (1, 1, g + 2).$$

Figure 15(a) with $\ell = g + 2$ shows an (i, n, c) -minimal contour of h .

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