ON THE DUALITY FOR MULTIPLE ZETA-STAR VALUES OF HEIGHT ONE

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Abstract. An alternative proof of the duality property of multiple zeta-star values of height one is given by using the generating function studied by Aoki, Kombu and Ohno. A relation between the special values of the generalized hypergeometric function and the gamma function is also given.

1. Introduction and the theorems

For any multi-index $k = (k_1, \ldots, k_n)$ ($k_i \in \mathbb{Z}_{>0}, k_1 \geq 2$), the weight $\text{wt}(k)$, depth $\text{dep}(k)$, and height $\text{ht}(k)$ are defined by the integers $k = k_1 + \cdots + k_n$, $n$, and $s = \# \{ i | k_i \geq 2 \}$, respectively.

For any multi-index $k = (k_1, \ldots, k_n)$, the multiple zeta value $\zeta(k)$ is defined by

$$\zeta(k) = \zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$$

and the multiple zeta-star value $\zeta^*(k)$ is defined by

$$\zeta^*(k) = \zeta^*(k_1, \ldots, k_n) = \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

In general, $k$, $n$, and $s$ satisfy the conditions $k \geq n + s$ and $n \geq s \geq 1$.

Multiple zeta values and multiple zeta-star values were introduced and studied by Euler [5] in the old days. In the past two decades, many authors have studied multiple zeta values and multiple zeta-star values, and a number of relations among them have been found [2–4, 6, 10, 12, 14]. There are important properties for multiple zeta values, so-called sum, cyclic sum, and duality formulas. For multiple zeta-star values, sum and cyclic sum formulas are also known to be important for understanding the structure of $\mathbb{Q}$-algebras spanned by multiple zeta-star values. On the other hand, the corresponding property of the duality formula for multiple zeta-star values was not known until recently. We introduce the duality formula of multiple zeta values, which was proved by using the iterated integral representation of multiple zeta values. Such an integral representation is closely related to the invariants of knot theory. To state the duality formula of multiple zeta values, we define dual

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indices. Any multi-index $\mathbf{k}$ is written in the form

$$\mathbf{k} = (a_1 + 1, 1, \ldots, 1, a_2 + 1, 1, \ldots, 1, \ldots, a_s + 1, 1, \ldots, 1)$$

with some positive integers $a_1, b_1, \ldots, a_s, b_s$. Then the dual index $\mathbf{k}'$ of $\mathbf{k}$ is defined by

$$\mathbf{k}' = (b_s + 1, 1, \ldots, 1, b_{s-1} + 1, 1, \ldots, 1, \ldots, b_1 + 1, 1, \ldots, 1).$$

For this pair of indices, the following theorem is well known (cf. [14]).

**THEOREM 1.** (Duality formula) For any index $\mathbf{k}$ and its dual index $\mathbf{k}'$, the following equality holds:

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}').$$

In particular

$$\zeta(n + 1, 1, \ldots, 1) = \zeta(m + 1, 1, \ldots, 1)$$

is the duality formula for multiple zeta values of height one. Our purpose is to find an analogous relation to Theorem 1 among multiple zeta-star values. Until recently many authors thought that such a duality property does not exist for multiple zeta-star values. But, in 2007, Kaneko conjectured and Ohno proved the following property, which can be understood as a kind of duality of height one (cf. [9]).

**THEOREM 2.** (Kaneko–Ohno) [9] For any integers $m, n \geq 2$, we have

$$\zeta^*(n, 1, \ldots, 1) - (-1)^{n+m} \zeta^*(m, 1, \ldots, 1) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \ldots].$$

Ohno’s proof of Theorem 2 is a combination of some known specific theorems. In this paper, we give an alternative proof of Theorem 2 that uses a formula of Aoki, Kombu and Ohno [1]. In fact, we use a generating function of multiple zeta-star values defined by

$$\Phi^*_0(x, y, z) = \sum_{k,n,s} X_0(k, n, s) x^{k-n-s} y^{n-s} z^{2s-2},$$

where $X_0(k, n, s)$ denotes the sum of all multiple zeta-star values $\zeta^*(\mathbf{k})$ with $\text{wt}(\mathbf{k}) = k$, $\text{dep}(\mathbf{k}) = n$, and $\text{ht}(\mathbf{k}) = s$, and we show directly the following generating function expression of Theorem 2 pointed out by Kaneko [8].

**THEOREM 3.** The generating function $\Phi^*_0(x, y, z)$ of multiple zeta-star values satisfies the following equality:

$$\frac{1}{y} \Phi^*_0(-x, y, 0) - \frac{1}{y} \Phi^*_0(-x, 0, 0) - \frac{1}{x} \Phi^*_0(-y, x, 0) + \frac{1}{x} \Phi^*_0(-y, 0, 0)$$

$$= \frac{1}{xy} \left( -\psi(x) + \psi(y) - \pi \left( \cot(\pi x) - \cot(\pi y) \right) \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \right),$$

where $\psi(x)$ is the di-gamma function defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$
In particular,

$$\zeta^*(n, 1, \ldots, 1) - (-1)^{n+m} \zeta^*(m, 1, \ldots, 1),$$

for $m, n \geq 2$, can be expressed as a polynomial in $\zeta(2)$, $\zeta(3)$, \ldots with rational coefficients.

Remark. The functions in the right-hand side of the equality in Theorem 3 are known to have the following expansions:

$$\pi \cot(\pi x) = \frac{1}{x} - \sum_{n=1}^{\infty} \zeta(2n)x^{2n-1},$$

$$\psi(x) = -\left(\frac{1}{2x} + \gamma + \frac{\pi}{2} \cot(\pi x) + \sum_{n=1}^{\infty} \zeta(2n+1)x^{2n}\right),$$

$$\Gamma(1 - x) = \exp\left(\gamma x + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n}x^n\right).$$

2. The alternative proof of Theorem 3

To prove Theorem 3, we use the following fact given in [1], which studies the generating function of multiple zeta-star values of fixed weight, depth, and height with a similar method to [13].

THEOREM 4. (Aoki–Kombu–Ohno) [1] The generating function $\Phi^*_0(x, y, z)$ has the following integral representation:

$$\Phi^*_0(x, y, z) = \int_0^1 \left\{ s^{-\beta}(1-s)^{y-1} \frac{\Gamma(\beta - \alpha)\Gamma(x - \alpha - \beta + 1)}{\Gamma(1 - \alpha)\Gamma(x - \alpha + 1)} F(\alpha, x - \alpha, \beta + 1; s) 
+ s^{-\alpha}(1-s)^{\gamma-1} \frac{\Gamma(\alpha - \beta)\Gamma(x - \alpha - \beta + 1)}{\Gamma(1 - \beta)\Gamma(x - \beta + 1)} F(\beta, \alpha - \beta, \gamma + 1; s) \right\} ds,$$

where $\alpha$ and $\beta$ are defined by the relations $x + y = \alpha + \beta$ and $xy - z^2 = \alpha\beta$, and where

$$F(\alpha, \beta, \gamma; s) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_nn!} s^n$$

denotes the Gauss hypergeometric function, and $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ denotes the Pochhammer symbol.

Now we start the proof of Theorem 3. First, we rewrite the left-hand side of Theorem 3 term by term by using Theorem 4. We consider only $z = 0$, so we obtain $\{\alpha, \beta\} = \{x, y\}$. 

In this paper, we set \( \alpha = x \) and \( \beta = y \). At first we consider \( \Phi_0^{*}(x, y, 0) \) as follows:

\[
\Phi_0^{*}(x, y, 0) = \frac{\Gamma(y-x)\Gamma(1-y)}{\Gamma(1-x)\Gamma(1)} \int_0^1 s^{-y}(1-s)^{y-1} F(x, 0, x - y + 1; s) \, ds \\
+ \frac{\Gamma(x-y)\Gamma(1-y)}{\Gamma(1-y)\Gamma(x-y+1)} \int_0^1 s^{-x}(1-s)^{y-1} F(y, y-x, y-x+1; s) \, ds \\
= \frac{\Gamma(y-x)\Gamma(1-y)}{\Gamma(1-x)} \int_0^1 s^{-y}(1-s)^{y-1} \, ds \\
+ \frac{1}{x-y} \int_0^1 s^{-x}(1-s)^{y-1} F(y, y-x, y-x+1; s) \, ds.
\]

So we get

\[
\frac{1}{y} \Phi_0^{*}(-x, y, 0) = \frac{\Gamma(x+y)\Gamma(1-y)}{y\Gamma(1+x)} \int_0^1 s^{-y}(1-s)^{y-1} \, ds \\
- \frac{1}{y} \frac{\Gamma(x+y)\Gamma(1-y)}{\Gamma(1+y)} \int_0^1 s^{-x}(1-s)^{y-1} F(y, x+y, x+y+1; s) \, ds,
\]

\[
\frac{1}{x} \Phi_0^{*}(-y, x, 0) = -\frac{\Gamma(x+y)\Gamma(1-x)}{x\Gamma(1+y)} \int_0^1 s^{-x}(1-s)^{x-1} \, ds \\
+ \frac{1}{x} \frac{\Gamma(x+y)\Gamma(1-y)}{\Gamma(1+y)} \int_0^1 s^{-y}(1-s)^{x-1} F(x, x+y, x+y+1; s) \, ds,
\]

and

\[
\frac{1}{x} \Phi_0^{*}(-y, 0, 0) = -\frac{1}{xy} \left( \int_0^1 s^{-y}(1-s)^{x-1} - (1-s)^{-1} \, ds \right).
\]

We have the following two lemmas for these functions.

**Lemma 1.** The generating function \( \Phi_0^{*} \) has the following expressions in terms of the gamma and the di-gamma functions:

\[
\text{(1)} \quad -\frac{1}{y} \Phi_0^{*}(-x, 0, 0) + \frac{1}{x} \Phi_0^{*}(-y, 0, 0) = \frac{1}{xy} \left( \psi(y) + \frac{1}{y} - \psi(x) - \frac{1}{x} \right);
\]

\[
\text{(2)} \quad \frac{1}{y} \Phi_0^{*}(-x, y, 0) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(1+x+y)} \sum_{n=0}^{\infty} \frac{(y)_n(x)_n}{(1+x+y)_n(n+x+y)n!} \frac{x+n}{y} \\
+ \frac{\Gamma(x+y)\Gamma(1-y)^2\Gamma(y)}{xy\Gamma(x)};
\]

\[
\text{(3)} \quad \frac{1}{x} \Phi_0^{*}(-y, x, 0) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(1+x+y)} \sum_{n=0}^{\infty} \frac{(y)_n(x)_n}{(1+x+y)_n(n+x+y)n!} \frac{y+n}{x} \\
- \frac{\Gamma(x+y)\Gamma(1-x)^2\Gamma(x)}{xy\Gamma(y)}.
\]
Proof. The relation of part (1) is shown immediately:

\[-\frac{1}{y} \Phi_0^*(-x, 0, 0) + \frac{1}{x} \Phi_0^*(-y, 0, 0) = \frac{1}{xy} \left( \int_0^1 \frac{s^x}{1-s} - \frac{s^y}{1-s} \, ds \right)\]

\[= \frac{1}{xy}(\psi(y+1) - \psi(x+1))\]

\[= \frac{1}{xy}\left( \psi(y) + \frac{1}{y} - \psi(x) - \frac{1}{x} \right).\]

The left-hand side of part (2) is as follows:

\[-\frac{1}{y} \Phi_0^*(-x, y, 0) = -\frac{1}{y(x+y)} \int_0^1 s^x(1-s)^{y-1} \sum_{n=0}^{\infty} \frac{(y)_n(x+y)_n}{(1+x+y)n!} s^n \, ds\]

\[+ \frac{\Gamma(x+y)\Gamma(1-y)}{xy\Gamma(x)} \frac{\Gamma(y)\Gamma(1-y)}{\Gamma(1)}\]

\[= -\frac{1}{y(x+y)} \sum_{n=0}^{\infty} \frac{(y)_n(x+y)_n}{(1+x+y)n!} \int_0^1 s^{x+n}(1-s)^{y-1} \, ds\]

\[+ \frac{\Gamma(x+y)\Gamma(1-y)^2\Gamma(y)}{xy\Gamma(x)}.
\]

Using the gamma function expression of the beta function, the first term of the right-hand side of the above equation becomes

\[-\frac{1}{y} \sum_{n=0}^{\infty} \frac{(y)_n}{(n+x+y)n!} \frac{\Gamma(y)\Gamma(x+n+1)}{\Gamma(x+n+1+y)}\]

\[= -\frac{\Gamma(1+x+y)\Gamma(y)}{y(1+x+y)} \sum_{n=0}^{\infty} \frac{(y)_n(1+x)_n}{(1+x+y)n!(n+x+y)n!}\]

\[= -\frac{\Gamma(x+y)\Gamma(y)}{\Gamma(1+x+y)} \sum_{n=0}^{\infty} \frac{(y)_n(x)_n}{(1+x+y)n!(n+x+y)n!} \frac{x+n}{y}.
\]

So we get part (2). We obtain part (3) by exchanging the variables \(x\) and \(y\) in part (2). \(\square\)

It is not easy to evaluate the first terms of the right-hand sides of the second and third relations of Lemma 1 by using known quantities. But if we add up the right-hand sides term by term, we obtain equalities as follows.

LEMMA 2. The following equalities are valid:

(1) \[-\frac{\Gamma(x)\Gamma(y)}{\Gamma(1+x+y)} \sum_{n=0}^{\infty} \frac{(y)_n(x)_n}{(1+x+y)n!(n+x+y)n!} \frac{x+n}{y}\]

\[+ \frac{\Gamma(x+y)\Gamma(y)}{\Gamma(1+x+y)} \sum_{n=0}^{\infty} \frac{(y)_n(x)_n}{(1+x+y)n!(n+x+y)n!} \frac{y+n}{x} = \frac{1}{xy}\left( \frac{1}{x} - \frac{1}{y} \right);\]

(2) \[\frac{\Gamma(x+y)\Gamma(1+y)\Gamma(y)}{xy\Gamma(x)} - \frac{\Gamma(x+y)\Gamma(1-x)\Gamma(x)}{xy\Gamma(y)}\]

\[= \frac{1}{xy}\left(-\pi(\cot(\pi x) - \cot(\pi y))\right) \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}.\]
Proof. (1) The left-hand side of the first relation means

\[-\frac{\Gamma(x)\Gamma(y)}{\Gamma(1+x+y)} \sum_{n=0}^{\infty} \frac{(y)_n(x)_n (x-y)(x+y+n)}{(1+x+y)_n(n+x+y)_n n!} \frac{1}{xy}\]

\[= -\frac{(x-y)\Gamma(x)\Gamma(y)}{xy\Gamma(1+x+y)} F(x, y, 1+x+y; 1) \]

\[= -\frac{(x-y)\Gamma(x)\Gamma(y)}{xy\Gamma(1+x+y)} \frac{\Gamma(1+x+y)\Gamma(1)}{\Gamma(1+x)\Gamma(1+y)} \]

\[= -\frac{x-y}{x^2y^2}.\]

So we obtain the first relation.

(2) By a direct computation, we have

\[\frac{1}{xy} \left\{\frac{\Gamma(x+y)(\Gamma(1-y)^2\Gamma(y)^2 - \Gamma(1-x)^2\Gamma(x)^2)}{\Gamma(x)\Gamma(y)}\right\} \]

\[= \frac{1}{xy} \left\{\frac{\Gamma(x+y)(\Gamma(1-y)^2\Gamma(y)^2 - \Gamma(1-x)^2\Gamma(x)^2)\Gamma(1-x-y)}{\Gamma(x)\Gamma(1-x)\Gamma(y)\Gamma(1-y)} \times \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}\right\}.\]

Applying the formula

\[\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},\]

the right-hand side of the above equality is

\[\frac{1}{xy} \left\{\frac{\sin \pi x \sin \pi y \pi}{\pi \sin \pi(x+y)} \left\{\frac{\pi^2}{\sin^2 \pi y} - \frac{\pi^2}{\sin^2 \pi x}\right\} \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \right\} \]

\[= \frac{\pi}{xy} \left\{\frac{\sin \pi x}{\sin \pi y \sin \pi(x+y)} - \frac{\sin \pi y}{\sin \pi x \sin \pi(x+y)}\right\} \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \right\}.\]

The part in the braces of the right-hand side of the above equality can be rewritten as

\[\frac{\sin^2 \pi x - \sin^2 \pi y}{\sin \pi x \sin \pi y \sin \pi(x+y)} = -(\cot \pi x - \cot \pi y).\]

Applying this equation to (\*), we readily obtain part (2) of Lemma 2. \hfill \Box

Now we conclude our proof of Theorem 3.

Proof of Theorem 3. After rewriting the left-hand side of the equality of Theorem 3 by using Lemmas 1 and 2, we obtain

\[\frac{1}{y} \Phi_0^*(-x, y, 0) - \frac{1}{y} \Phi_0^*(-x, 0, 0) - \frac{1}{x} \Phi_0^*(-y, x, 0) + \frac{1}{x} \Phi_0^*(-y, 0, 0) \]

\[= \frac{1}{xy} \left(-\psi(x) + \psi(y) - \pi(\cot(\pi x) - \cot(\pi y)) \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}\right).\] \hfill \Box

Moreover, if we use another expression of \(\Phi_0^*(x, y, z)\) given by [1, Remark 3.2], then Theorem 3 can be rewritten as follows. It gives a relation between the generalized
hypergeometric function

\[ _3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; s) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{(\beta_1)_n(\beta_2)_n n!} s^n \]

and the gamma functions.

**COROLLARY OF THEOREM 3.** The generalized hypergeometric function \(_3F_2\) satisfies the following equality:

\[
\begin{align*}
\frac{1}{(1 + x)(1 - y)y} & \quad _3F_2(1, 1, 1 + x; 2 + x, 2 - y; 1) \\
- \frac{1}{(1 + x)y} & \quad _3F_2(1, 1, 1 + x; 2 + x, 2; 1) \\
- \frac{1}{(1 + y)(1 - x)x} & \quad _3F_2(1, 1, 1 + y; 2 + y, 2 - x; 1) \\
+ \frac{1}{(1 + y)x} & \quad _3F_2(1, 1, 1 + y; 2 + y, 2; 1) \\
= \frac{1}{xy} \left( -\psi(x) + \psi(y) - \pi(\cot(\pi x) - \cot(\pi y)) \frac{\Gamma(1 - x)\Gamma(1 - y)}{\Gamma(1 - x - y)} \right).
\end{align*}
\]

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