A NOTE ON NON-NEGATIVITY OF THE i th Δ -GENUS OF QUASI-POLARIZED VARIETIES

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Abstract. Let (X, L) be a quasi-polarized variety of dimension n . In this paper, we will study non-negativity of the i th Δ -genus $\Delta_i(X, L)$. We will prove the following. Assume that X is a normal Gorenstein variety such that the irrational locus of X consists of at most finite points and the dimension of the singular locus of X is less than or equal to the dimension of the base locus of $|L|$. If i is greater than the dimension of the base locus of $|L|$, then $\Delta_i(X, L)$ is non-negative. We will also give a lower bound for $\Delta_i(X, L)$ when (X, L) is a polarized abelian variety.

1. Introduction

Let X be a projective variety of dimension n defined over the complex number field and let L be a line bundle on X . If L is nef and big (respectively ample), then (X, L) is called a *quasi-polarized (respectively polarized) variety*. Furthermore, if X is smooth and L is nef and big (respectively ample), we say that (X, L) is a quasi-polarized (respectively polarized) *manifold*. For this (X, L) , there are some invariants, for example, the sectional genus $g(L)$ and the Δ -genus $\Delta(L)$ (see [7]). Fujita studied polarized varieties by using these invariants, and he gave many interesting results (see [7] in detail). On the other hand, in order to study polarized varieties more deeply, the author extended these invariants.

In [8], we defined the i th sectional geometric genus $g_i(X, L)$ of (X, L) for every integer i with $0 \leq i \leq n$, which is a generalization of the degree L^n and the sectional genus $g(L)$ of (X, L) . (We note that $g_0(X, L) = L^n$, $g_1(X, L) = g(L)$ and $g_n(X, L) = h^n(\mathcal{O}_X)$.) Some properties of the i th sectional geometric genus which have been obtained in [8] also show that the i th sectional geometric genus is a natural generalization of the sectional genus.

On the other hand, in [12], we defined the i th Δ -genus $\Delta_i(X, L)$ for every integer i with $0 \leq i \leq n$. This gives a generalization of the Δ -genus. Namely, if $i = 1$, then $\Delta_1(X, L)$ is the Δ -genus $\Delta(L)$ of (X, L) . Furthermore, in [12] we studied some properties of $\Delta_i(X, L)$. For example, if X is smooth and $\text{Bs } |L| = \emptyset$, then some properties of $\Delta_i(X, L)$ are similar to those of the Δ -genus $\Delta(L)$ of (X, L) (see [12, Section 3] or Section 3 in this paper). So $\Delta_i(X, L)$ has some good properties under the assumption that X is smooth and $\text{Bs } |L| = \emptyset$.

As the next step, we want to know whether or not the i th Δ -genus for general quasi-polarized varieties has good properties. For example, does the inequality $\Delta_i(X, L) \geq 0$ hold

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for $2 \leq i \leq n$? Unfortunately, the answer is negative. There exists an example of (X, L) with $\Delta_i(X, L) < 0$ (see [12, Section 4]). Hence it is interesting and important to consider when the i th Δ -genus is non-negative. For example, in [13], if (X, L) is a polarized toric variety, then we can show that $\Delta_i(X, L) \geq 0$ (see [13, Theorem 3.3]).

So, in this paper, we study non-negativity of the i th Δ -genus for quasi-polarized varieties. First, if X is a normal Gorenstein variety, L is nef and big, $\dim \text{Irr}(X) \leq 0$, $\dim \text{Sing}(X) \leq \dim \text{Bs } |L| \leq n - 1$ and $i \geq \dim \text{Bs } |L| + 1$, then we prove that $\Delta_i(X, L) \geq 0$ (see Theorem 4.1), where $\text{Sing}(X)$ (respectively $\text{Irr}(X)$) denotes the singular locus of X (respectively the irrational locus of X (see Definition 2.2)). Furthermore, we consider the case where $K_X \equiv 0$ (see Proposition 4.1). In particular, we investigate the case where (X, L) is a polarized Abelian variety, and we get a lower bound for $\Delta_i(X, L)$ (see Theorem 4.2). In particular, we see that $\Delta_i(X, L) > 0$ in this case. We will review in Section 3 the i th sectional geometric genus and the i th Δ -genus for the reader's convenience.

2. Preliminaries

Definition 2.1. Let (X, L) be a quasi-polarized variety of dimension n . Then L has a k -ladder if there exists a sequence of irreducible and reduced subvarieties $X \supset X_1 \supset \cdots \supset X_k$ such that $X_i \in |L_{i-1}|$ for $1 \leq i \leq k$, where $X_0 := X$, $L_0 := L$ and $L_i := L|_{X_i}$. Here we note that $\dim X_j = n - j$. Then let $r_{p,q} : H^p(X_q, L_q) \rightarrow H^p(X_{q+1}, L_{q+1})$ be the natural map.

Remark 2.1. Let X be a projective variety of dimension $n \geq 3$ and let L be a nef and big line bundle on X . Assume that X is normal and Cohen–Macaulay, $\dim \text{Sing}(X) \leq m$ and $\dim \text{Bs } |L| \leq m$, where m is an integer with $m \leq n - 3$. Then there exists a member $X_1 \in |L|$ such that X_1 satisfies the following properties:

- (2.1.1) X_1 is irreducible by Bertini's theorem (see [16, Theorem 17.16]);
- (2.1.2) X_1 is Cohen–Macaulay by [7, Section 2 in Chapter 0]. In particular, X_1 is reduced;
- (2.1.3) $\dim \text{Sing}(X_1) \leq m$ and $\dim \text{Bs } |L_1| \leq m$ by Bertini's theorem (see [16, Theorem 17.16]). (Here $L_1 := L|_{X_1}$.)

Hence we see that X_1 is normal by [3, Section 5.8 in Chapter IV] or [7, (2.8) Fact in Chapter 0]. Therefore, by carrying out this process, we see that L has an $(n - m - 2)$ -ladder $X \supset X_1 \supset \cdots \supset X_{n-m-2}$ such that X_j is normal and Cohen–Macaulay for every integer j with $1 \leq j \leq n - m - 2$. Moreover, if X is Gorenstein, then so is each X_j .

Definition 2.2. Let X be a normal projective variety of dimension n , and let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities of X . Then we set

$$\text{Irr}(X) := \bigcup_{i>0} \text{Supp}(R^i \pi_*(\mathcal{O}_{\tilde{X}}))$$

and we call this set the *irrational locus* of X . Here we note that $\text{Irr}(X)$ does not depend on the desingularization of X .

THEOREM 2.1. *Let X be a normal projective variety of dimension n and let L be a nef and big line bundle on X . Then $h^i(\omega_X \otimes L) = 0$ for $i > \max\{0, \dim \text{Irr}(X)\}$, where ω_X is the dualizing sheaf of X .*

Proof. See [22, (0.2.1) Theorem]. □

3. Review on the i th sectional geometric genus and the i th Δ -genus of quasi-polarized varieties

Here we are going to review the i th sectional geometric genus and the i th Δ -genus of quasi-polarized varieties (X, L) for every integer i with $0 \leq i \leq \dim X$. Up until now, many investigations of (X, L) via the sectional genus and the Δ -genus have been given. In order to analyze (X, L) more deeply, we have extended these notions. First, in [8, Definition 2.1] we gave an invariant called the i th sectional geometric genus which is thought to be a generalization of the sectional genus. Here we recall the definition of this invariant.

Notation 3.1. Let (X, L) be a quasi-polarized variety of dimension n , and let $\chi(tL)$ be the Euler–Poincaré characteristic of tL . Then $\chi(tL)$ is a polynomial in t of degree n , and we set

$$\chi(tL) = \sum_{j=0}^n \chi_j(X, L) \binom{t+j-1}{j}.$$

Definition 3.1. [8, Definition 2.1] Let (X, L) be a quasi-polarized variety of dimension n . Then, for any integer i with $0 \leq i \leq n$, the i th sectional geometric genus $g_i(X, L)$ of (X, L) is defined by the following:

$$g_i(X, L) = (-1)^i (\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

Remark 3.1.

- (1) If $i = 0$ (respectively $i = 1$), then $g_i(X, L)$ is equal to the degree (respectively the sectional genus) of (X, L) .
- (2) If $i = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$ and $g_n(X, L)$ is independent of L .

Then the i th sectional geometric genus satisfies the following properties.

THEOREM 3.1. [9, Propositions 2.1 and 2.3, and Theorem 2.4] *Let X be a projective variety of dimension $n \geq 2$ and let L be a nef and big line bundle on X . Assume that $h^t(-sL) = 0$ for every integers t and s with $0 \leq t \leq n-1$ and $1 \leq s$, and $|L|$ has an $(n-i)$ -ladder for an integer i with $1 \leq i \leq n$. Then the i th sectional geometric genus has the following properties:*

- (1) $g_i(X_j, L_j) = g_i(X_{j+1}, L_{j+1})$ for every integer j with $0 \leq j \leq n-i-1$ (here we use the notation in Definition 2.1);
- (2) $g_i(X, L) \geq h^i(\mathcal{O}_X)$.

In particular, from Theorem 3.1(1) and Remark 3.1(2) we see that if (X, L) satisfies the assumption in Theorem 3.1, then the i th sectional geometric genus is the geometric genus of the i -dimensional projective variety X_{n-i} . This is the reason why we call this invariant the i th sectional geometric genus. From Theorem 3.1 we see that the i th sectional geometric genus is expected to have properties similar to those of the geometric genus of i -dimensional projective varieties. For other results concerning the i th sectional geometric genus see, for example, [8–11]. The following result will be used later.

THEOREM 3.2. *Let X be a projective variety with $\dim X = n$ and let L be a nef and big line bundle on X .*

(1) For any integer i with $0 \leq i \leq n-1$, we have

$$g_i(X, L) = \sum_{j=0}^{n-i-1} (-1)^{n-j} \binom{n-i}{j} \chi(-(n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

(2) Assume that X is smooth. Then for any integer i with $0 \leq i \leq n-1$, we have

$$g_i(X, L) = \sum_{j=0}^{n-i-1} (-1)^j \binom{n-i}{j} h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

Proof. (1) By the same argument as in the proof of [8, Theorem 2.2], we obtain

$$\begin{aligned} \chi_{n-i}(X, L) &= \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n-i}{j} \chi(-(n-i-j)L) \\ &= \sum_{j=0}^{n-i-1} (-1)^{n-i-j} \binom{n-i}{j} \chi(-(n-i-j)L) + \chi(\mathcal{O}_X). \end{aligned}$$

Hence by Definition 3.1, we get the assertion.

(2) By using the Serre duality and the Kawamata–Viehweg vanishing theorem, we get the assertion from (1). $\square$

As the next step, we want to generalize the notion of the Δ -genus. Several generalizations can be considered from various points of view. Here we will give a generalization of the Δ -genus from the following point of view. For the case of $\Delta(X, L)$, the following result has been obtained.

THEOREM 3.3. (See e.g. [7, §3 in Chapter I]) *Let X be a projective variety of dimension $n \geq 2$ and let L be a nef and big line bundle on X . We use the notation in Definition 2.1. If $|L|$ has an $(n-1)$ -ladder and $h^0(L_{n-1}) > 0$, then*

$$\Delta(X, L) = \sum_{j=0}^{n-1} \dim \operatorname{Coker}(r_{0,j}).$$

In particular, we have $\Delta(X, L) \geq \Delta(X_1, L_1) \geq \cdots \geq \Delta(X_{n-1}, L_{n-1}) \geq 0$.

Here we want to give the definition of the i th Δ -genus which satisfies a generalization of Theorem 3.3. Now we are going to give the definition of the i th Δ -genus.

Definition 3.2. [12, Definition 2.1] *Let (X, L) be a quasi-polarized variety of dimension n . For every integer i with $0 \leq i \leq n$, the i th Δ -genus $\Delta_i(X, L)$ of (X, L) is defined by the following formula:*

$$\Delta_i(X, L) = \begin{cases} 0 & \text{if } i = 0, \\ g_{i-1}(X, L) - \Delta_{i-1}(X, L) \\ \quad + (n-i+1)h^{i-1}(\mathcal{O}_X) - h^{i-1}(L) & \text{if } 1 \leq i \leq n. \end{cases}$$

Remark 3.2.

(1) If $i = 1$, then $\Delta_1(X, L)$ is equal to the Δ -genus of (X, L) .

- (2) If $i = n$, then $\Delta_n(X, L) = h^n(\mathcal{O}_X) - h^n(L)$ (see [12, Proposition 2.4]).
- (3) For every integer i with $1 \leq i \leq n$, by the definition of the i th Δ -genus, we have the following equality which will be used later:

$$\Delta_{i-1}(X, L) = g_{i-1}(X, L) - \Delta_i(X, L) + (n - i + 1)h^{i-1}(\mathcal{O}_X) - h^{i-1}(L).$$

Then, for the case of the i th Δ -genus, we can prove the following.

THEOREM 3.4. (See [12, Theorem 2.8 and Corollary 2.9] and [9, Proposition 2.1]) *Let X be a projective variety of dimension $n \geq 2$ and let L be a nef and big line bundle on X . We use the notation in Definition 2.1. Assume that $h^t(-sL) = 0$ for every integers t and s with $0 \leq t \leq n - 1$ and $1 \leq s$. If $|L|$ has an $(n - i)$ -ladder and $h^0(L_{n-i}) > 0$ for an integer i with $1 \leq i \leq n$, then*

$$\Delta_i(X, L) = \sum_{j=0}^{n-i} \dim \operatorname{Coker}(r_{i-1,j}).$$

In particular, we have $\Delta_i(X, L) \geq \Delta_i(X_1, L_1) \geq \cdots \geq \Delta_i(X_{n-i}, L_{n-i}) \geq 0$.

The definition of the i th Δ -genus is so complicated that many things about the i th Δ -genus are unknown. Therefore, it is important to investigate the following problems in order to understand the meaning and properties of the i th Δ -genus.

Problem 3.1.

- (i) Does the i th Δ -genus have a property similar to that of the Δ -genus? Concretely:
 - (i-1) Does $\Delta_i(X, L) \geq 0$ hold?
 - (i-2) Can we get the i th Δ -genus version of the Fujita theory on the Δ -genus?
- (ii) Are there any relationships between $g_i(X, L)$ and $\Delta_i(X, L)$?
- (iii) Are there any relationships between $\Delta_i(X, L)$ and $\Delta_{i+1}(X, L)$?
- (iv) We must classify (X, L) by the value of the i th Δ -genus.
- (v) What is the geometric meaning of the i th Δ -genus?

Remark 3.3. If X is smooth and L is nef and big, then the following facts concerning Problem 3.1 are known.

- (1) First we consider Problem 3.1(i-1). If $i = 1$, then $\Delta_1(X, L) \geq 0$ (see [7, (4.2) Theorem] and [6, (1.1) Theorem]). Moreover, if L is base point free, then we have $\Delta_i(X, L) \geq 0$ for every integer i with $0 \leq i \leq n$.
- (2) Next we consider Problem 3.1(ii). If $i = 1$ and L is merely ample, then it is known that $g_1(X, L) = 0$ if and only if $\Delta_1(X, L) = 0$ (see [7, (12.1) Theorem]). Therefore, we consider the case where $i \geq 2$. Then under the assumption that $\operatorname{Bs} |L| = \emptyset$ we see that $g_i(X, L) = 0$ if and only if $\Delta_i(X, L) = 0$ (see [12, Theorem 3.13]).
- (3) Next we consider Problem 3.1(iii). Then, for example, we get the following. Assume that L is base point free. If $\Delta_i(X, L) \leq i - 1$, then $\Delta_{i+1}(X, L) = 0$ (see [12, Proposition 3.9]). In particular, if $\Delta_i(X, L) = 0$, then $\Delta_{i+1}(X, L) = 0$. Maybe there will be several relationships between $\Delta_i(X, L)$ and $\Delta_{i+1}(X, L)$ other than this.
- (4) For Problem 3.1(iv), we get a classification of (X, L) by the value of $\Delta_2(X, L)$ as follows:
 - (4.1) a classification of polarized manifolds (X, L) such that $\operatorname{Bs} |L| = \emptyset$ and $\Delta_2(X, L) = 0$ (see [12, Theorem 3.13 and Remark 3.13.1]);

- (4.2) a classification of polarized manifolds (X, L) such that L is very ample and $\Delta_2(X, L) = 1$ (see [12, Theorem 3.17] and [14, Remark 2]).
- (5) Problem 3.1(v) seems to be the most difficult problem among the above problems even in the case where L is base point free or very ample. We also note that we are studying Problem 3.1(i-2) now and we will explain this in a future paper.

As we said in Remark 3.3, under the assumption that L is base point free, we have some answers for Problem 3.1 above. We believe that the i th Δ -genus has good properties similar to those of the Δ -genus and is useful if L satisfies some special conditions.

Therefore, the main purpose of our investigation for the time being is to consider the following two things.

Problem 3.2.

- (I) Investigate Problem 3.1 for the case where $|L|$ has base points.
- (II) If L is very ample or base point free, then find an application using results of the i th Δ -genus such as those mentioned in Remark 3.3. For example, investigate an unsolved problem about (X, L) with very ample L by using results of the i th Δ -genus.

In this paper, we consider Problem 3.1(i-1) under the assumption that $|L|$ has base points. As we said above, if $i = 1$, then $\Delta_1(X, L)$, which equals the Δ -genus, is non-negative for every nef and big line bundle L on X . This was proved by Fujita. Moreover, Fujita also gave a classification of (X, L) with small $\Delta(X, L)$ (see [7]). These results are very useful and are used in various problems. So, in order to make the i th Δ -genus useful, it is important to study the non-negativity of the i th Δ -genus for $2 \leq i \leq n$. As we said in Remark 3.3, if $\text{Bs } |L| = \emptyset$, then $\Delta_i(X, L) \geq 0$ for every integer i with $1 \leq i \leq n$. However, in general for each integer i with $2 \leq i \leq n$ there exists an example of (X, L) with $\Delta_i(X, L) < 0$ (see [12, Section 4]). So it is interesting and important to know when $\Delta_i(X, L)$ is non-negative for $i \geq 2$. This is the theme of this paper.

4. Non-negativity of the i th Δ -genus

In this section, we consider non-negativity of $\Delta_i(X, L)$. First, we consider the case where X is normal and Gorenstein, $\dim \text{Irr}(X) \leq 0$ and $\dim \text{Sing}(X) \leq \dim \text{Bs } |L| \leq \dim X - 1$.

THEOREM 4.1. *Let (X, L) be a quasi-polarized variety of dimension n . Assume that X is normal and Gorenstein, $\dim \text{Irr}(X) \leq 0$ and $\dim \text{Sing}(X) \leq \dim \text{Bs } |L| \leq n - 1$. Then, for every integer i with $i \geq \dim \text{Bs } |L| + 1$, $\Delta_i(X, L) \geq 0$ holds.*

Proof. Set $m := \dim \text{Bs } |L|$. If $m = n - 1$, then we have only to prove that $\Delta_n(X, L) \geq 0$. By Remark 3.2(2) and the Serre duality, we have $\Delta_n(X, L) = h^n(\mathcal{O}_X) - h^n(L) = h^0(\omega_X) - h^0(\omega_X \otimes L^{-1})$. If $h^0(\omega_X \otimes L^{-1}) = 0$, then we get the assertion. So we may assume that $h^0(\omega_X \otimes L^{-1}) \neq 0$. However, since $h^0(L) > 0$, we have $h^0(\omega_X) - h^0(\omega_X \otimes L^{-1}) \geq h^0(L) - 1 \geq 0$ by [8, Lemma 1.12] or [19, 15.6.2 Lemma]. Therefore, we get $\Delta_n(X, L) \geq 0$ in this case.

Next we assume that $m \leq n - 2$. By Remark 2.1, L has an $(n - m - 2)$ -ladder $X \supset X_1 \supset \cdots \supset X_{n-m-2}$ such that X_j is normal and Gorenstein for every integer j with $0 \leq j \leq n - m - 2$, where $X_0 := X$. We set $L_j := L|_{X_j}$ for $1 \leq j \leq n - m - 2$ and $L_0 := L$.

By the exact sequence

$$0 \rightarrow H^0(\mathcal{O}_{X_j}) \rightarrow H^0(L_j) \rightarrow H^0(L_{j+1}),$$

we see that $h^0(L_{j+1}) \geq h^0(L_j) - h^0(\mathcal{O}_{X_j}) = h^0(L_j) - 1$ for every integer j with $0 \leq j \leq n - m - 3$. By assumption, $h^0(L_0) \geq n - m$ holds (e.g. see [5, (1.7) Lemma]). Hence $h^0(L_{n-m-2}) \geq h^0(L_{n-m-3}) - 1 \geq \dots \geq h^0(L) - (n - m - 2) \geq 2$. Hence, if $i \geq m + 2$, then Conditions $A_1(i)$ and $A_2(i)$ in [12, 2.7.2] are satisfied.

By [17, Chapter III, Corollary 7.7] and Theorem 2.1, we have $h^t(-sL) = 0$ for integers t and s with $0 \leq t \leq n - 1$ and $1 \leq s$. Hence by [9, Proposition 2.1(a)], Condition $B(i, i)$ in [12, 2.7.2] is also satisfied for $m + 2 \leq i \leq n - 1$. Therefore, we get $\Delta_i(X, L) \geq 0$ for every integer i with $i \geq m + 2$ by [12, Corollary 2.9(2)]. Here we note that if $i = n$, then we do not need Conditions $A_1(i)$ and $B(i, i)$ in [12, 2.7.2] (see the proof of [12, Theorem 2.8(2)]).

Assume that $i = m + 1$. As we said above, Condition $A_1(m + 2)$ in [12, 2.7.2] holds. By [9, Proposition 2.1(a)] Condition $B(m + 2, m + 1)$ in [12, 2.7.2] also holds. Hence by [12, Corollary 2.9(1)], we see that

$$\Delta_{m+1}(X, L) \geq \dots \geq \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}).$$

Next we calculate $\Delta_{m+1}(X_{n-m-2}, L_{n-m-2})$. Here we note that by Theorem 3.2(1) we have

$$g_{m+1}(X_{n-m-2}, L_{n-m-2}) = (-1)^{m+2} \chi(-L_{n-m-2}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+1}(\mathcal{O}_{X_{n-m-2}}).$$

By [9, Claim 2.1.1] we have $(-1)^{m+2} \chi(-L_{n-m-2}) = h^{m+2}(-L_{n-m-2})$. Hence

$$g_{m+1}(X_{n-m-2}, L_{n-m-2}) = h^{m+2}(-L_{n-m-2}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+1}(\mathcal{O}_{X_{n-m-2}}).$$

Since $\dim \text{Bs } |L_{n-m-2}| \leq \dim \text{Bs } |L| = m = \dim(X_{n-m-2}) - 2$, by Bertini's theorem (see [9, Theorem 1.8(3)]) we can take a general member X_{n-m-1} of $|L_{n-m-2}|$ such that X_{n-m-1} is generically reduced. On the other hand, by [7, (2.4), 2), Chapter 0], X_{n-m-1} is Cohen-Macaulay. Hence X_{n-m-1} is reduced. Here we note that X_{n-m-1} is not always irreducible. Then by using the exact sequence

$$0 \rightarrow -L_{n-m-2} \rightarrow \mathcal{O}_{X_{n-m-2}} \rightarrow \mathcal{O}_{X_{n-m-1}} \rightarrow 0,$$

we see that $h^{m+2}(-L_{n-m-2}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+1}(\mathcal{O}_{X_{n-m-2}}) - h^{m+1}(\mathcal{O}_{X_{n-m-1}}) = 0$ because $h^{m+1}(-L_{n-m-2}) = 0$ by [9, Claim 2.1.1]. Therefore,

$$\begin{aligned} g_{m+1}(X_{n-m-2}, L_{n-m-2}) &= h^{m+2}(-L_{n-m-2}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+1}(\mathcal{O}_{X_{n-m-2}}) \\ &= h^{m+1}(\mathcal{O}_{X_{n-m-1}}). \end{aligned}$$

By Remark 3.2(2),

$$\Delta_{m+2}(X_{n-m-2}, L_{n-m-2}) = h^{m+2}(\mathcal{O}_{X_{n-m-2}}) - h^{m+2}(L_{n-m-2}).$$

Since, by Remark 3.2(3),

$$\begin{aligned} \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) &= g_{m+1}(X_{n-m-2}, L_{n-m-2}) - \Delta_{m+2}(X_{n-m-2}, L_{n-m-2}) \\ &\quad + h^{m+1}(\mathcal{O}_{X_{n-m-2}}) - h^{m+1}(L_{n-m-2}), \end{aligned}$$

we get

$$\begin{aligned}\Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) &= h^{m+1}(\mathcal{O}_{X_{n-m-1}}) - (h^{m+2}(\mathcal{O}_{X_{n-m-2}}) - h^{m+2}(L_{n-m-2})) \\ &\quad + h^{m+1}(\mathcal{O}_{X_{n-m-2}}) - h^{m+1}(L_{n-m-2}).\end{aligned}$$

Here we set $L_{n-m-1} := L|_{X_{n-m-1}}$. Then by using the exact sequence

$$0 \rightarrow \mathcal{O}_{X_{n-m-2}} \rightarrow L_{n-m-2} \rightarrow L_{n-m-1} \rightarrow 0,$$

we have

$$\begin{aligned}h^{m+1}(\mathcal{O}_{X_{n-m-2}}) - h^{m+1}(L_{n-m-2}) \\ + h^{m+1}(L_{n-m-1}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+2}(L_{n-m-2}) \geq 0.\end{aligned}$$

Therefore,

$$\Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) \geq h^{m+1}(\mathcal{O}_{X_{n-m-1}}) - h^{m+1}(L_{n-m-1}).$$

Since X_{n-m-1} is Cohen–Macaulay and equidimensional, by [17, Chapter III, Corollary 7.7], the Serre duality holds, that is,

$$h^{m+1}(\mathcal{O}_{X_{n-m-1}}) = h^0(\omega_{X_{n-m-1}})$$

and

$$h^{m+1}(L_{n-m-1}) = h^0(\omega_{X_{n-m-1}} \otimes (L_{n-m-1})^{-1}).$$

Here we note that $\omega_{X_{n-m-1}}$ is a Cartier divisor by [20, Proposition 5.73].

Let $X_{n-m-1} = \bigcup_j Z_j$, where Z_j is an irreducible component of X_{n-m-1} for each j . Since $\dim \text{Bs } |L_{n-m-2}| \leq \dim \text{Bs } |L| = m = \dim(X_{n-m-2}) - 2$, there exists an element $s \in H^0(L_{n-m-2})$ such that every Z_j is not an irreducible component of the divisor $\text{div}(s)$ of zeros of the global section s . Let $\delta : H^0(L_{n-m-2}) \rightarrow H^0(L_{n-m-1})$. Then $\delta(s) \in H^0(L_{n-m-1})$ defines a non-zero homomorphism

$$\rho : \mathcal{O}_{X_{n-m-1}} \rightarrow L_{n-m-1}.$$

CLAIM 4.1. $\text{Ker}(\rho) = 0$.

Proof. Since $\delta(s)$ does not vanish identically on any Z_j , we see that $\dim \text{Supp}(\text{Ker}(\rho)) < \dim X_{n-m-1}$. However, since X_{n-m-1} is Cohen–Macaulay, X_{n-m-1} is locally 1-Macaulay. (For the definition of locally 1-Macaulay, see [7, (2.4) Definition in Chapter 0].) Therefore, by using [4, (1.12) Corollary] we have $\text{Ker}(\rho) = 0$. $\square$

Hence ρ is injective, and $\omega_{X_{n-m-1}} \otimes (L_{n-m-1})^{-1} \rightarrow \omega_{X_{n-m-1}}$ is injective because $\omega_{X_{n-m-1}}$ and $(L_{n-m-1})^{-1}$ are invertible. So we have

$$h^0(\omega_{X_{n-m-1}}) - h^0(\omega_{X_{n-m-1}} \otimes (L_{n-m-1})^{-1}) \geq 0.$$

Hence $\Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) \geq h^0(\omega_{X_{n-m-1}}) - h^0(\omega_{X_{n-m-1}} \otimes (L_{n-m-1})^{-1}) \geq 0$. Therefore, $\Delta_{m+1}(X, L) \geq \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) \geq 0$. This completes the proof. $\square$

By Theorem 4.1 we can get the following corollary.

COROLLARY 4.1. *Let (X, L) be a quasi-polarized manifold of dimension n . Assume that $n \geq 2$ and $\dim \text{Bs } |L| \leq n - 1$. Then $\Delta_i(X, L) \geq 0$ for every integer i with $i \geq \dim \text{Bs } |L| + 1$.*

Moreover, we obtain the following corollaries.

COROLLARY 4.2. *Let (X, L) be a polarized variety of dimension n . Assume that X is normal and Gorenstein, $\dim \text{Irr}(X) \leq 0$ and $\dim \text{Sing}(X) \leq \dim \text{Bs } |L| \leq n - 1$. If $i \geq \Delta(X, L)$, then $\Delta_i(X, L) \geq 0$.*

Proof. Since $i \geq \Delta(X, L)$, we have $i \geq \dim \text{Bs } |L| + 1$ by the Δ -genus inequality of Fujita [7, (4.2) Theorem]. By Theorem 4.1, we get the assertion. $\square$

COROLLARY 4.3. *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Assume that $\Delta(X, L) \leq 2$. Then $\Delta_i(X, L) \geq 0$ for every integer i with $2 \leq i \leq n$.*

Proof. If $h^0(L) = 0$, then $\Delta(X, L) = n + L^n \geq n + 1 \geq 3$. But then this contradicts the assumption. Hence $h^0(L) \geq 1$, that is, $\dim \text{Bs } |L| \leq n - 1$. By assumption, we have $i \geq 2 \geq \Delta(X, L)$. Hence by Corollary 4.2 we get $\Delta_i(X, L) \geq 0$ for every i with $i \geq 2$. Therefore, we get the assertion. $\square$

The following examples show that Corollary 4.1 is the best possible.

Example 4.1.

(1) Let Y be a smooth projective variety of dimension $m \geq 2$ such that K_Y is ample with $h^0(K_Y) = 0$. (There exists such a Y for every $m \geq 2$.) Let $A_1, \dots, A_{n-m}$ be ample divisors on Y such that $\text{Bs } |A_i| = \emptyset$ and $h^m(A_i) = 0$ for any i , where $n > m$. We set $\mathcal{E} := K_Y \oplus A_1 \oplus \dots \oplus A_{n-m}$ and $X := \mathbb{P}_Y(\mathcal{E})$. Let $\pi : \mathbb{P}_Y(\mathcal{E}) \rightarrow Y$ be its projection. For $1 \leq i \leq n - m$, let $\mathcal{E}_i := K_Y \oplus A_1 \oplus \dots \oplus A_{i-1} \oplus A_{i+1} \oplus \dots \oplus A_{n-m}$ and $D_i := \mathbb{P}_Y(\mathcal{E}_i)$. Then we see that $D_i \in |H(\mathcal{E}) - \pi^* A_i|$ and $D_1 \cap \dots \cap D_{n-m} = \mathbb{P}(K_Y)$. We set $L := H(\mathcal{E})$. Then L is ample and $D_i + \pi^* A_i \in |L|$ for each i . Since $D_1 \cap \dots \cap D_{n-m} = \mathbb{P}(K_Y)$ and A_i is spanned by its global sections, we have $\dim \text{Bs } |L| \leq m$. On the other hand, by [8, Example 2.10(8)] and [12, Lemma 2.12.1] we have $g_m(X, L) = h^m(\mathcal{O}_X)$ and $\Delta_{m+1}(X, L) = 0$. Hence by Remark 3.2(3) we see that

$$\begin{aligned} \Delta_m(X, L) &= g_m(X, L) - \Delta_{m+1}(X, L) + (n - m)h^m(\mathcal{O}_X) - h^m(L) \\ &= (n - m + 1)h^m(\mathcal{O}_X) - h^m(L). \end{aligned}$$

Here we note that $h^m(\mathcal{O}_X) = h^m(\mathcal{O}_Y) = h^0(K_Y) = 0$ and

$$\begin{aligned} h^m(L) &= h^m(\pi_*(L)) \\ &= h^m(\mathcal{E}) \\ &= h^m(K_Y \oplus A_1 \oplus \dots \oplus A_{n-m}) \\ &= h^m(K_Y) = 1. \end{aligned}$$

Hence $\Delta_m(X, L) = -1 < 0$ and by Theorem 4.1 we see that $\dim \text{Bs } |L| \geq m$. Therefore, $\dim \text{Bs } |L| = m$. This (X, L) is an example with $\dim \text{Bs } |L| = m$ and $\Delta_m(X, L) < 0$.

(2) By [1, Theorem 1.1 and Section 2], there exists a Calabi-Yau 3-fold X and an ample line bundle L on X such that $h^0(L) = 1$. Then we note that $\mathcal{O}(K_X) \sim \mathcal{O}_X$, $h^1(\mathcal{O}_X) =$

$h^2(\mathcal{O}_X) = 0$, $h^3(\mathcal{O}_X) = 1$ and $h^j(L) = 0$ for every $j \geq 1$. Hence we have $\Delta_3(X, L) = h^3(\mathcal{O}_X) - h^3(L) = 1$. On the other hand, since $h^0(K_X + L) = h^0(L) = 1$, we see that $g_2(X, L) = 0$ by Theorem 3.2(2). Therefore, by Remark 3.2(3)

$$\begin{aligned}\Delta_2(X, L) &= g_2(X, L) - \Delta_3(X, L) + h^2(\mathcal{O}_X) - h^2(L) \\ &= 0 - 1 + 0 - 0 \\ &= -1.\end{aligned}$$

Since $\dim \text{Bs } |L| = 2$, we infer that this (X, L) is an example with $i = \dim \text{Bs } |L|$ and $\Delta_i(X, L) < 0$.

Next we consider the case where (X, L) is a quasi-polarized manifold with $\dim X = 3$ and $K_X \equiv 0$.

PROPOSITION 4.1. *Let (X, L) be a quasi-polarized manifold of dimension three. Assume that $K_X \equiv 0$. Then the following hold:*

- (1) $\Delta_3(X, L) \geq 0$;
- (2) (i) if $\mathcal{O}(K_X) \not\sim \mathcal{O}_X$, then $\Delta_2(X, L) \geq 1$;
- (ii) if $\mathcal{O}(K_X) \sim \mathcal{O}_X$, then $\Delta_2(X, L) \geq -1$.

Proof. Here we note that $L - K_X$ is nef and big by assumption. Therefore, by the Kawamata–Viehweg vanishing theorem, we have $h^j(L) = 0$ for every $j \geq 1$.

(1) By Remark 3.2(2), we have $\Delta_3(X, L) = h^3(\mathcal{O}_X) - h^3(L) = h^3(\mathcal{O}_X) \geq 0$.

(2) By Theorem 3.2(2) and Remark 3.2(2) and (3), we get

$$\begin{aligned}\Delta_2(X, L) &= g_2(X, L) - \Delta_3(X, L) + h^2(\mathcal{O}_X) - h^2(L) \\ &= g_2(X, L) - h^3(\mathcal{O}_X) + h^2(\mathcal{O}_X) \\ &= h^0(K_X + L) - 2h^3(\mathcal{O}_X) + 2h^2(\mathcal{O}_X).\end{aligned}$$

Since K_X is nef, we see that $c_2(X)L \geq 0$ by a theorem of Miyaoka's [21, Theorem 6.6]. Hence, by the Kawamata–Viehweg vanishing theorem, the Serre duality and the Hirzebruch–Riemann–Roch theorem (see [18] or [15, Example 15.2.5]), we have

$$\begin{aligned}h^0(K_X + L) &= -\chi(-L) \\ &= \frac{1}{6}L^3 + \frac{1}{4}K_X L^2 + \frac{1}{12}(K_X^2 + c_2(X))L + \frac{1}{24}c_2(X)K_X \\ &\geq \frac{1}{6}L^3 > 0.\end{aligned}$$

(i) If $\mathcal{O}(K_X) \not\sim \mathcal{O}_X$, then $h^3(\mathcal{O}_X) = 0$. Hence $\Delta_2(X, L) = h^0(K_X + L) - 2h^3(\mathcal{O}_X) + 2h^2(\mathcal{O}_X) \geq 1$.

(ii) If $\mathcal{O}(K_X) \sim \mathcal{O}_X$, then $h^3(\mathcal{O}_X) = 1$. Hence $\Delta_2(X, L) = h^0(K_X + L) - 2h^3(\mathcal{O}_X) + 2h^2(\mathcal{O}_X) \geq -1$. This completes the proof. $\square$

Remark 4.1. The inequality of Proposition 4.1(2)(ii) is the best possible. See Example 4.1(2).

Next, as a special case of $K_X \equiv 0$, we consider the case where (X, L) is a polarized Abelian variety. Here we note that if X is an Abelian variety, then a line bundle L on X is ample if and only if L is nef and big (see, e.g., [2, Proposition 4.5.2]).

Let X be an Abelian variety of dimension n and let L be an ample line bundle on X . Here we study a lower bound for the i th Δ -genus of (X, L) . If $i = n$, then by Remark 3.2(2) and the Kodaira vanishing theorem we have $\Delta_n(X, L) = h^n(\mathcal{O}_X) = 1$. So we consider the case where $1 \leq i \leq n - 1$.

THEOREM 4.2. *Let (X, L) be a polarized Abelian variety of dimension $n \geq 2$, and let i be an integer with $1 \leq i \leq n - 1$. Then*

$$\Delta_i(X, L) \geq \begin{cases} \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 2, \\ \frac{L^n}{n!} + n & \text{if } i = 1. \end{cases}$$

Proof. In this case $h^i(L) = 0$ for every integer i with $i > 0$, and $h^j(\mathcal{O}_X) = \binom{n}{j}$ for every integer j with $j \geq 0$.

By the definition of the i th Δ -genus, we see that, for every integer i with $i \geq 2$,

$$\begin{aligned} \Delta_i(X, L) &= g_{i-1}(X, L) - \Delta_{i-1}(X, L) + (n - i + 1)h^{i-1}(\mathcal{O}_X) - h^{i-1}(L) \\ &= g_{i-1}(X, L) - \Delta_{i-1}(X, L) + (n - i + 1)\binom{n}{i-1}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \Delta_i(X, L) &= g_{i-1}(X, L) - \Delta_{i-1}(X, L) + (n - i + 1)\binom{n}{i-1} \\ &= (g_{i-1}(X, L) - g_{i-2}(X, L)) + \Delta_{i-2}(X, L) \\ &\quad - (n - i + 2)h^{i-2}(\mathcal{O}_X) + (n - i + 1)\binom{n}{i-1} \\ &= \sum_{k=0}^1 (-1)^k g_{i-1-k}(X, L) + \Delta_{i-2}(X, L) + \sum_{j=0}^1 (-1)^j (n - i + 1 + j) \binom{n}{i-1-j} \\ &= \dots \\ &= \sum_{k=0}^{i-2} (-1)^k g_{i-1-k}(X, L) + (-1)^{i-1} \Delta_1(X, L) \\ &\quad + \sum_{j=0}^{i-2} (-1)^j (n - i + 1 + j) \binom{n}{i-1-j} \\ &= \sum_{k=0}^{i-1} (-1)^k g_{i-1-k}(X, L) + (-1)^i h^0(L) + \sum_{j=0}^{i-1} (-1)^j (n - i + 1 + j) \binom{n}{i-1-j} \\ &= \sum_{k=0}^{i-1} (-1)^k \left(g_{i-1-k}(X, L) + (n - i + 1 + k) \binom{n}{n-i+1+k} \right) + (-1)^i h^0(L). \end{aligned}$$

By [9, Claim 3.A.4.1], we have

$$\begin{aligned} \sum_{j=0}^{(i-1-k)-1} (-1)^{(i-1-k)-1-j} h^j(\mathcal{O}_X) &= \sum_{j=0}^{(i-1-k)-1} (-1)^{(i-1-k)-1-j} \binom{n}{j} \\ &= \binom{n-1}{(i-1-k)-1} = \binom{n-1}{n-i+1+k}. \end{aligned}$$

Therefore, by [9, Remark 3.A.3.1 (1)], we have

$$g_{i-1-k}(X, L) = (n-i+1+k)! S(n, n-i+1+k) \frac{L^n}{n!} + \binom{n-1}{n-i+1+k}.$$

Here $S(n, n-i+1+k)$ denotes the Stirling number of the second kind with the type $(n, n-i+1+k)$ (see [9, Definition 3.A.1]). Hence

$$\begin{aligned} \Delta_i(X, L) &= \sum_{k=0}^{i-1} (-1)^k \left(g_{i-1-k}(X, L) + (n-i+1+k) \binom{n}{n-i+1+k} \right) + (-1)^i h^0(L) \\ &= \sum_{k=0}^{i-1} (-1)^k (n-i+1+k)! S(n, n-i+1+k) \frac{L^n}{n!} + \sum_{k=0}^{i-1} (-1)^k \binom{n-1}{n-i+1+k} \\ &\quad + \sum_{k=0}^{i-1} (-1)^k (n-i+1+k) \binom{n}{n-i+1+k} + (-1)^i h^0(L). \end{aligned}$$

First, we calculate $\sum_{k=0}^{i-1} (-1)^k (n-i+1+k) \binom{n}{n-i+1+k}$. Then

$$\begin{aligned} \sum_{k=0}^{i-1} (-1)^k (n-i+1+k) \binom{n}{n-i+1+k} &= \sum_{k=0}^{i-1} (-1)^k (i-k) \binom{n}{i-k} \\ &= \sum_{t=1}^i (-1)^{i-t} t \binom{n}{t} \\ &= \sum_{t=1}^i (-1)^{i-t} n \binom{n-1}{t-1} \\ &= (-1)^i n \sum_{t=1}^i (-1)^t \binom{n-1}{t-1} \\ &= (-1)^i n \sum_{p=0}^{i-1} (-1)^{p+1} \binom{n-1}{p} \\ &= n \sum_{p=0}^{i-1} (-1)^{i-p-1} \binom{n-1}{p} \\ &= n \binom{n-2}{i-1}. \end{aligned}$$

(In the last step, we have used [9, Claim 3.A.4.1].)

Next we calculate $\sum_{k=0}^{i-1} (-1)^k \binom{n-1}{n-i+1+k}$. If $i \geq 2$, then

$$\begin{aligned} \sum_{k=0}^{i-1} (-1)^k \binom{n-1}{n-i+1+k} &= \sum_{k=0}^{i-2} (-1)^k \binom{n-1}{n-i+1+k} \\ &= \sum_{k=0}^{i-2} (-1)^k \binom{n-1}{i-2-k} \\ &= \sum_{t=0}^{i-2} (-1)^{i-2-t} \binom{n-1}{t} \\ &= \binom{n-2}{i-2}. \end{aligned}$$

(In the last step, we have used [9, Claim 3.A.4.1].) If $i = 1$, then

$$\sum_{k=0}^{i-1} (-1)^k \binom{n-1}{n-i+1+k} = 0.$$

Finally, we calculate $\sum_{k=0}^{i-1} (-1)^k (n-i+1+k)! S(n, n-i+1+k) (L^n/n!) + (-1)^i h^0(L)$. We note that

$$\begin{aligned} &\sum_{k=0}^{i-1} (-1)^k (n-i+1+k)! S(n, n-i+1+k) \\ &= \sum_{t=0}^{i-1} (-1)^{i-1-t} (n-t)! S(n, n-t) \\ &= (-1)^{n+i-1} \sum_{t=0}^{i-1} (-1)^{n-t} (n-t)! S(n, n-t) \\ &= (-1)^{n+i-1} \sum_{k=n-i+1}^n (-1)^k k! S(n, k). \end{aligned}$$

By [23, (24d) on p. 34], we have

$$x^n = \sum_{k=0}^n S(n, k) [x]_k, \quad (1)$$

where

$$[x]_k := \begin{cases} x(x-1) \cdots (x-k+1) & \text{if } k \geq 1, \\ 1 & \text{if } k = 0. \end{cases}$$

So we have

$$\begin{aligned} (-1)^n &= \sum_{k=1}^n (-1)^k k! S(n, k) \\ &= \sum_{k=1}^{n-i} (-1)^k k! S(n, k) + \sum_{k=n-i+1}^n (-1)^k k! S(n, k). \end{aligned}$$

Because $h^0(L) = L^n/n!$, we get

$$\begin{aligned}
& (-1)^{n+i-1} \left\{ \sum_{k=n-i+1}^n (-1)^k k! S(n, k) \right\} \frac{L^n}{n!} + (-1)^i (h^0(L)) \\
&= (-1)^{i-1} \frac{L^n}{n!} - (-1)^{n+i-1} \sum_{k=1}^{n-i} (-1)^k k! S(n, k) \frac{L^n}{n!} + (-1)^{i-1} (-h^0(L)) \\
&= (-1)^{n-i} \sum_{k=1}^{n-i} (-1)^k k! S(n, k) \frac{L^n}{n!}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
& \Delta_i(X, L) \\
&= \begin{cases} (-1)^{n-i} \left\{ \sum_{k=1}^{n-i} (-1)^k k! S(n, k) \right\} \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 2, \\ (-1)^{n-1} \left\{ \sum_{k=1}^{n-1} (-1)^k k! S(n, k) \right\} \frac{L^n}{n!} + n & \text{if } i = 1. \end{cases}
\end{aligned}$$

Here we prove the following.

LEMMA 4.1. *For every integer p with $p \leq n-1$, the following holds:*

$$(-1)^{n-p} \sum_{k=1}^{n-p} (-1)^k k! S(n, k) = \sum_{t=1}^{n-p} (-1)^{n-p-t} \binom{n-p+1}{t+1} t^n.$$

Proof. By the Stirling formula [23, (24a) on p. 34], we get

$$k! S(n, k) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n. \quad (2)$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n-p} (-1)^k k! S(n, k) &= \sum_{k=1}^{n-p} \sum_{j=0}^k (-1)^j \binom{k}{j} j^n \\
&= (-1)^1 \left(\binom{1}{1} + \binom{2}{1} + \cdots + \binom{n-p}{1} \right) 1^n \\
&\quad + (-1)^2 \left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n-p}{2} \right) 2^n \\
&\quad + \cdots + (-1)^{n-p} \binom{n-p}{n-p} (n-p)^n \\
&= \sum_{t=2}^{n-p+1} (-1)^{t-1} \left(\sum_{k=t-1}^{n-p} \binom{k}{t-1} (t-1)^n \right). \quad (3)
\end{aligned}$$

Here we prove the following.

CLAIM 4.2. *Let s and t be integers with $0 \leq s \leq t$. Then*

$$\sum_{k=s}^t \binom{k}{s} = \binom{t+1}{s+1}.$$

Proof. If $t - s = 0$, then this is true.

Assume that this equality holds for $0 \leq t - s \leq p$. We consider the case where $t - s = p + 1$. Then

$$\begin{aligned} \sum_{k=s}^t \binom{k}{s} &= \sum_{k=s}^{s+p+1} \binom{k}{s} \\ &= \sum_{k=s}^{s+p} \binom{k}{s} + \binom{s+p+1}{s} \\ &= \binom{s+p+1}{s+1} + \binom{s+p+1}{s} \\ &= \binom{s+p+2}{s+1}. \end{aligned}$$

This completes the proof of Claim 4.2. □

By Claim 4.2 and (3), we get

$$\begin{aligned} \sum_{k=1}^{n-p} (-1)^k k! S(n, k) &= \sum_{t=2}^{n-p+1} (-1)^{t-1} \binom{n-p+1}{t} (t-1)^n \\ &= \sum_{t=1}^{n-p} (-1)^t \binom{n-p+1}{t+1} t^n. \end{aligned}$$

Therefore, we get the assertion of Lemma 4.1. □

Here, for any integer r with $r \geq 1$, we set

$$a_r = \sum_{t=1}^r (-1)^{r-t} \binom{r+1}{t+1} t^n.$$

Then, by Lemma 4.1,

$$\Delta_i(X, L) = \begin{cases} a_{n-i} \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 2, \\ a_{n-1} \frac{L^n}{n!} + n & \text{if } i = 1. \end{cases}$$

Here we note that tL is ample and spanned by its global sections for any large t , and $\Delta_i(X, tL) \geq 0$ by [12, Corollary 3.3]. Since $L^n > 0$, we obtain

$$a_{n-i} \geq 0. \tag{4}$$

LEMMA 4.2. $a_{n-i} \geq 1$.

Proof. First we prove the following.

CLAIM 4.3. $a_p + a_{p+1} = (p+1)!S(n, p+1)$.

Proof. By using (2) we have

$$\begin{aligned}
 a_{p+1} &= \sum_{t=1}^{p+1} (-1)^{p+1-t} \binom{p+2}{t+1} t^n \\
 &= \sum_{t=1}^{p+1} (-1)^{p+1-t} \left(\binom{p+1}{t+1} + \binom{p+1}{t} \right) t^n \\
 &= \sum_{t=1}^{p+1} (-1)^{p+1-t} \binom{p+1}{t+1} t^n + (p+1)!S(n, p+1) \\
 &= \sum_{t=1}^p (-1)^{p+1-t} \binom{p+1}{t+1} t^n + (p+1)!S(n, p+1) \\
 &= - \sum_{t=1}^p (-1)^{p-t} \binom{p+1}{t+1} t^n + (p+1)!S(n, p+1) \\
 &= -a_p + (p+1)!S(n, p+1).
 \end{aligned}$$

This completes the proof of Claim 4.3. $\square$

CLAIM 4.4. $a_{n-k} - (-1)^k$ is divisible by $(n-k+1)!$ for every integer k with $1 \leq k \leq n$.

Proof. (a) By Claim 4.3, we see that $a_n + a_{n-1} = n!$ holds. On the other hand, by Lemma 4.1 and (1) we have

$$\begin{aligned}
 a_n &= \sum_{t=1}^n (-1)^{n-t} \binom{n+1}{t+1} t^n \\
 &= (-1)^n \sum_{t=1}^n (-1)^t t! S(n, t) \\
 &= (-1)^n (-1)^n = 1.
 \end{aligned}$$

Hence we get $a_{n-1} - (-1)^1 = n!$ and the assertion is true for $k = 1$.

(b) Assume that $a_{n-t} - (-1)^t$ is divisible by $(n-t+1)!$. We set

$$a_{n-t} - (-1)^t = (n-t+1)!b_{n-t}.$$

Then, by Claim 4.3, we have $a_{n-t} + a_{n-t-1} = (n-t)!S(n, n-t)$. Hence

$$\begin{aligned}
 a_{n-t-1} &= (n-t)!S(n, n-t) - a_{n-t} \\
 &= (n-t)!S(n, n-t) - (n-t+1)!b_{n-t} - (-1)^t \\
 &= (n-t)!(S(n, n-t) - (n-t+1)b_{n-t}) + (-1)^{t+1}.
 \end{aligned}$$

Hence $a_{n-t-1} - (-1)^{t+1}$ is divisible by $(n-t)!$ and we get the assertion of Claim 4.4. $\square$

Here we go back to the proof of Lemma 4.2. We use the notation in the proof of Claim 4.4. Namely, we set

$$b_{n-i} := \frac{a_{n-i} - (-1)^i}{(n-i+1)!}.$$

Then b_{n-i} is an integer by Claim 4.4. Assume that $1 \leq i \leq n-1$. If $b_{n-i} \leq -1$, then $a_{n-i} < 0$ since $1 \leq i \leq n-1$ and $n \geq 2$. So this is impossible by (4). Hence $b_{n-i} \geq 0$.

(i) Assume that i is even. Then $a_{n-i} \geq 1$ because $b_{n-i} \geq 0$ and $(-1)^i = 1$.

(ii) Assume that i is odd. Then $a_{n-i} = (n-i+1)!b_{n-i} - 1$. If $b_{n-i} = 0$, then $a_{n-i} < 0$ and this is impossible by (4). Hence $b_{n-i} > 0$ and $a_{n-i} = (n-i+1)!b_{n-i} - 1 \geq 2 - 1 = 1$ because $i \leq n-1$.

Therefore, we get the assertion of Lemma 4.2. $\square$

By Lemma 4.2, we get

$$\begin{aligned} \Delta_i(X, L) &= \begin{cases} a_{n-i} \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 2, \\ a_{n-1} \frac{L^n}{n!} + n & \text{if } i = 1, \end{cases} \\ &\geq \begin{cases} \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 2, \\ \frac{L^n}{n!} + n & \text{if } i = 1. \end{cases} \end{aligned}$$

Hence we get the assertion of Theorem 4.2. $\square$

Remark 4.2.

(1) Let (X, L) be a polarized Abelian variety of dimension $n \geq 2$. Assume that $i = n-1$. Then, by Theorem 3.2(2) and Remark 3.2(2) and (3),

$$\begin{aligned} \Delta_{n-1}(X, L) &= g_{n-1}(X, L) - \Delta_n(X, L) + h^{n-1}(\mathcal{O}_X) - h^{n-1}(L) \\ &= (h^0(K_X + L) - h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_X)) - h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_X) \\ &= h^0(L) + 2h^{n-1}(\mathcal{O}_X) - 2h^n(\mathcal{O}_X) \\ &= h^0(L) + 2n - 2 \\ &= \frac{L^n}{n!} + 2n - 2 \\ &= \begin{cases} \frac{L^n}{n!} + n \binom{n-2}{(n-1)-1} + \binom{n-2}{(n-1)-2} & \text{if } n \geq 3, \\ \frac{L^n}{n!} + n & \text{if } n = 2. \end{cases} \end{aligned}$$

Therefore, the inequality in Theorem 4.2 is the best possible.

(2) If i is odd with $1 \leq i \leq n-1$ and

$$\Delta_i(X, L) = \begin{cases} \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 3, \\ \frac{L^n}{n!} + n & \text{if } i = 1, \end{cases}$$

then by the proof of Theorem 4.2 we can prove that $i = n-1$ (see (ii) in the proof of Theorem 4.2).

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