A NOTE ON NON-NEGATIVITY OF THE \(i\)th \(\Delta\)-GENUS OF QUASI-POLARIZED VARIETIES

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Abstract. Let \((X, L)\) be a quasi-polarized variety of dimension \(n\). In this paper, we will study non-negativity of the \(i\)th \(\Delta\)-genus \(\Delta_i(X, L)\). We will prove the following. Assume that \(X\) is a normal Gorenstein variety such that the irrational locus of \(X\) consists of at most finite points and the dimension of the singular locus of \(X\) is less than or equal to the dimension of the base locus of \(|L|\). If \(i\) is greater than the dimension of the base locus of \(|L|\), then \(\Delta_i(X, L)\) is non-negative. We will also give a lower bound for \(\Delta_i(X, L)\) when \((X, L)\) is a polarized abelian variety.

1. Introduction

Let \(X\) be a projective variety of dimension \(n\) defined over the complex number field and let \(L\) be a line bundle on \(X\). If \(L\) is nef and big (respectively ample), then \((X, L)\) is called a quasi-polarized (respectively polarized) variety. Furthermore, if \(X\) is smooth and \(L\) is nef and big (respectively ample), we say that \((X, L)\) is a quasi-polarized (respectively polarized) manifold. For this \((X, L)\), there are some invariants, for example, the sectional genus \(g(L)\) and the \(\Delta\)-genus \(\Delta(L)\) (see [7]). Fujita studied polarized varieties by using these invariants, and he gave many interesting results (see [7] in detail). On the other hand, in order to study polarized varieties more deeply, the author extended these invariants.

In [8], we defined the \(i\)th sectional geometric genus \(g_i(X, L)\) of \((X, L)\) for every integer \(i\) with \(0 \leq i \leq n\), which is a generalization of the degree \(L^n\) and the sectional genus \(g(L)\) of \((X, L)\). (We note that \(g_0(X, L) = L^n\), \(g_1(X, L) = g(L)\) and \(g_n(X, L) = h^n(\mathcal{O}_X)\).) Some properties of the \(i\)th sectional geometric genus which have been obtained in [8] also show that the \(i\)th sectional geometric genus is a natural generalization of the sectional genus.

On the other hand, in [12], we defined the \(i\)th \(\Delta\)-genus \(\Delta_i(X, L)\) for every integer \(i\) with \(0 \leq i \leq n\). This gives a generalization of the \(\Delta\)-genus. Namely, if \(i = 1\), then \(\Delta_1(X, L)\) is the \(\Delta\)-genus \(\Delta(L)\) of \((X, L)\). Furthermore, in [12] we studied some properties of \(\Delta_i(X, L)\). For example, if \(X\) is smooth and \(\text{Bs}|L| = \emptyset\), then some properties of \(\Delta_i(X, L)\) are similar to those of the \(\Delta\)-genus \(\Delta(L)\) of \((X, L)\) (see [12, Section 3] or Section 3 in this paper). So \(\Delta_i(X, L)\) has some good properties under the assumption that \(X\) is smooth and \(\text{Bs}|L| = \emptyset\).

As the next step, we want to know whether or not the \(i\)th \(\Delta\)-genus for general quasi-polarized varieties has good properties. For example, does the inequality \(\Delta_i(X, L) \geq 0\) hold?
for $2 \leq i \leq n$? Unfortunately, the answer is negative. There exists an example of $(X, L)$ with $\Delta_i(X, L) < 0$ (see [12, Section 4]). Hence it is interesting and important to consider when the $i$th $\Delta$-genus is non-negative. For example, in [13], if $(X, L)$ is a polarized toric variety, then we can show that $\Delta_i(X, L) \geq 0$ (see [13, Theorem 3.3]).

So, in this paper, we study non-negativity of the $i$th $\Delta$-genus for quasi-polarized varieties. First, if $X$ is a normal Gorenstein variety, $L$ is nef and big, $\dim \text{Irr}(X) \leq 0$, $\dim \text{Sing}(X) \leq \dim \text{Bs} |L| \leq n - 1$ and $i \geq \dim \text{Bs} |L| + 1$, then we prove that $\Delta_i(X, L) \geq 0$ (see Theorem 4.1), where $\text{Sing}(X)$ (respectively $\text{Irr}(X)$) denotes the singular locus of $X$ (respectively the irrational locus of $X$ (see Definition 2.2)). Furthermore, we consider the case where $K_X \equiv 0$ (see Proposition 4.1). In particular, we investigate the case where $(X, L)$ is a polarized Abelian variety, and we get a lower bound for $\Delta_i(X, L)$ (see Theorem 4.2). In particular, we see that $\Delta_i(X, L) > 0$ in this case. We will review in Section 3 the $i$th sectional geometric genus and the $i$th $\Delta$-genus for the reader’s convenience.

2. Preliminaries

Definition 2.1. Let $(X, L)$ be a quasi-polarized variety of dimension $n$. Then $L$ has a $k$-ladder if there exists a sequence of irreducible and reduced subvarieties $X \supset X_1 \supset \cdots \supset X_k$ such that $X_i \in |L_{i-1}|$ for $1 \leq i \leq k$, where $X_0 := X$, $L_0 := L$ and $L_i := L|_{X_i}$. Here we note that $\dim X_j = n - j$. Then let $r_{p,q} : H^p(X_q, L_q) \to H^p(X_{q+1}, L_{q+1})$ be the natural map.

Remark 2.1. Let $X$ be a projective variety of dimension $n \geq 3$ and let $L$ be a nef and big line bundle on $X$. Assume that $X$ is normal and Cohen–Macaulay, $\dim \text{Sing}(X) \leq m$ and $\dim \text{Bs} |L| \leq m$, where $m$ is an integer with $m \leq n - 3$. Then there exists a member $X_1 \in |L|$ such that $X_1$ satisfies the following properties:

(2.1.1) $X_1$ is irreducible by Bertini’s theorem (see [16, Theorem 17.16]);
(2.1.2) $X_1$ is Cohen–Macaulay by [7, Section 2 in Chapter 0]. In particular, $X_1$ is reduced;
(2.1.3) $\dim \text{Sing}(X_1) \leq m$ and $\dim \text{Bs} |L_1| \leq m$ by Bertini’s theorem (see [16, Theorem 17.16]). (Here $L_1 := L|_{X_1}$.)

Hence we see that $X_1$ is normal by [3, Section 5.8 in Chapter IV] or [7, (2.8) Fact in Chapter 0]. Therefore, by carrying out this process, we see that $L$ has an $(n - m - 2)$-ladder $X \supset X_1 \supset \cdots \supset X_{n-m-2}$ such that $X_j$ is normal and Cohen–Macaulay for every integer $j$ with $1 \leq j \leq n - m - 2$. Moreover, if $X$ is Gorenstein, then so is each $X_j$.

Definition 2.2. Let $X$ be a normal projective variety of dimension $n$, and let $\pi : \tilde{X} \to X$ be a resolution of singularities of $X$. Then we set

$$\text{Irr}(X) := \bigcup_{i > 0} \text{Supp}(\mathcal{R}^i\pi_*(\mathcal{O}_{\tilde{X}}))$$

and we call this set the irrational locus of $X$. Here we note that $\text{Irr}(X)$ does not depend on the desingularization of $X$.

Theorem 2.1. Let $X$ be a normal projective variety of dimension $n$ and let $L$ be a nef and big line bundle on $X$. Then $h^i(\omega_X \otimes L) = 0$ for $i > \max\{0, \dim \text{Irr}(X)\}$, where $\omega_X$ is the dualizing sheaf of $X$.

Proof. See [22, (0.2.1) Theorem].
3. Review on the $i$th sectional geometric genus and the $i$th $\Delta$-genus of quasi-polarized varieties

Here we are going to review the $i$th sectional geometric genus and the $i$th $\Delta$-genus of quasi-polarized varieties $(X, L)$ for every integer $i$ with $0 \leq i \leq \dim X$. Up until now, many investigations of $(X, L)$ via the sectional genus and the $\Delta$-genus have been given. In order to analyze $(X, L)$ more deeply, we have extended these notions. First, in \cite[Definition 2.1]{8} we gave an invariant called the $i$th sectional geometric genus which is thought to be a generalization of the sectional genus. Here we recall the definition of this invariant.

**Notation 3.1.** Let $(X, L)$ be a quasi-polarized variety of dimension $n$, and let $\chi(tL)$ be the Euler–Poincaré characteristic of $tL$. Then $\chi(tL)$ is a polynomial in $t$ of degree $n$, and we set

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L) \left( t + j - 1 \right).$$

**Definition 3.1.** \cite[Definition 2.1]{8} Let $(X, L)$ be a quasi-polarized variety of dimension $n$. Then, for any integer $i$ with $0 \leq i \leq n$, the $i$th sectional geometric genus $g_i(X, L)$ of $(X, L)$ is defined by the following:

$$g_i(X, L) = (-1)^i \left( \chi_{n-i}(X, L) - \chi(O_X) \right) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(O_X).$$

**Remark 3.1.**

1. If $i = 0$ (respectively $i = 1$), then $g_i(X, L)$ is equal to the degree (respectively the sectional genus) of $(X, L)$.
2. If $i = n$, then $g_n(X, L) = h^n(O_X)$ and $g_n(X, L)$ is independent of $L$.

Then the $i$th sectional geometric genus satisfies the following properties.

**Theorem 3.1.** \cite[Propositions 2.1 and 2.3, and Theorem 2.4]{9} Let $X$ be a projective variety of dimension $n \geq 2$ and let $L$ be a nef and big line bundle on $X$. Assume that $h^t(-sL) = 0$ for every integers $t$ and $s$ with $0 \leq t \leq n - 1$ and $1 \leq s$, and $|L|$ has an $(n - i)$-ladder for an integer $i$ with $1 \leq i \leq n$. Then the $i$th sectional geometric genus has the following properties:

1. $g_i(X, L) = g_i(X_{j+1}, L_{j+1})$ for every integer $j$ with $0 \leq j \leq n - t - 1$ (here we use the notation in Definition 2.1);
2. $g_i(X, L) \geq h^i(O_X)$.

In particular, from Theorem 3.1(1) and Remark 3.1(2) we see that if $(X, L)$ satisfies the assumption in Theorem 3.1, then the $i$th sectional geometric genus is the geometric genus of the $i$-dimensional projective variety $X_{n-i}$. This is the reason why we call this invariant the $i$th sectional geometric genus. From Theorem 3.1 we see that the $i$th sectional geometric genus is expected to have properties similar to those of the geometric genus of $i$-dimensional projective varieties. For other results concerning the $i$th sectional geometric genus see, for example, \cite{8–11}. The following result will be used later.

**Theorem 3.2.** Let $X$ be a projective variety with $\dim X = n$ and let $L$ be a nef and big line bundle on $X$. 

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For any integer $i$ with $0 \leq i \leq n - 1$, we have

$$g_i(X, L) = \sum_{j=0}^{n-i-1} (-1)^{n-i-j} \binom{n-i}{j} \chi(-(n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(O_X).$$

(2) Assume that $X$ is smooth. Then for any integer $i$ with $0 \leq i \leq n - 1$, we have

$$g_i(X, L) = \sum_{j=0}^{n-i-1} (-1)^j \binom{n-i}{j} h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(O_X).$$

**Proof.** (1) By the same argument as in the proof of [8, Theorem 2.2], we obtain

$$\chi_{n-i}(X, L) = \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n-i}{j} \chi(-(n-i-j)L)$$

$$= \sum_{j=0}^{n-i-1} (-1)^{n-i-j} \binom{n-i}{j} \chi(-(n-i-j)L) + \chi(O_X).$$

Hence by Definition 3.1, we get the assertion.

(2) By using the Serre duality and the Kawamata–Viehweg vanishing theorem, we get the assertion from (1).

As the next step, we want to generalize the notion of the $\Delta$-genus. Several generalizations can be considered from various points of view. Here we will give a generalization of the $\Delta$-genus from the following point of view. For the case of $\Delta(X, L)$, the following result has been obtained.

**Theorem 3.3.** (See e.g. [7, §3 in Chapter I] Let $X$ be a projective variety of dimension $n \geq 2$ and let $L$ be a nef and big line bundle on $X$. We use the notation in Definition 2.1. If $|L|$ has an $(n-1)$-ladder and $h^0(L_{n-1}) > 0$, then

$$\Delta(X, L) = \sum_{j=0}^{n-1} \dim \text{Coker}(r_{0,j}).$$

In particular, we have $\Delta(X, L) \geq \Delta(X_1, L_1) \geq \cdots \geq \Delta(X_{n-1}, L_{n-1}) \geq 0$.

Here we want to give the definition of the $i$th $\Delta$-genus which satisfies a generalization of Theorem 3.3. Now we are going to give the definition of the $i$th $\Delta$-genus.

**Definition 3.2.** [12, Definition 2.1] Let $(X, L)$ be a quasi-polarized variety of dimension $n$. For every integer $i$ with $0 \leq i \leq n$, the $i$th $\Delta$-genus $\Delta_i(X, L)$ of $(X, L)$ is defined by the following formula:

$$\Delta_i(X, L) = \begin{cases} 0 & \text{if } i = 0, \\
g_{i-1}(X, L) - \Delta_{i-1}(X, L) + (n-i+1)h^{i-1}(O_X) - h^{i-1}(L) & \text{if } 1 \leq i \leq n. \end{cases}$$

**Remark 3.2.**

(1) If $i = 1$, then $\Delta_1(X, L)$ is equal to the $\Delta$-genus of $(X, L)$. 

(2) If $i = n$, then $\Delta_n(X, L) = h^n(O_X) - h^n(L)$ (see [12, Proposition 2.4]).

(3) For every integer $i$ with $1 \leq i \leq n$, by the definition of the $i$th $\Delta$-genus, we have the following equality which will be used later:

$$\Delta_{i-1}(X, L) = g_{i-1}(X, L) - \Delta_i(X, L) + (n - i + 1)h^{i-1}(O_X) - h^{i-1}(L).$$

Then, for the case of the $i$th $\Delta$-genus, we can prove the following.

**Theorem 3.4.** (See [12, Theorem 2.8 and Corollary 2.9] and [9, Proposition 2.1]) Let $X$ be a projective variety of dimension $n \geq 2$ and let $L$ be a nef and big line bundle on $X$. We use the notation in Definition 2.1. Assume that $h^i(-sL) = 0$ for every integers $t$ and $s$ with $0 \leq t \leq n - 1$ and $1 \leq s$. If $|L|$ has an $(n - i)$-ladder and $h^0(L_{n-i}) > 0$ for an integer $i$ with $1 \leq i \leq n$, then

$$\Delta_i(X, L) = \sum_{j=0}^{n-i} \dim \text{Coker}(r_{i-1,j}).$$

In particular, we have $\Delta_i(X, L) \geq \Delta_1(X_1, L_1) \geq \cdots \geq \Delta_i(X_{n-i}, L_{n-i}) \geq 0$.

The definition of the $i$th $\Delta$-genus is so complicated that many things about the $i$th $\Delta$-genus are unknown. Therefore, it is important to investigate the following problems in order to understand the meaning and properties of the $i$th $\Delta$-genus.

**Problem 3.1.**

(i) Does the $i$th $\Delta$-genus have a property similar to that of the $\Delta$-genus? Concretely:

(i-1) Does $\Delta_i(X, L) \geq 0$ hold?

(i-2) Can we get the $i$th $\Delta$-genus version of the Fujita theory on the $\Delta$-genus?

(ii) Are there any relationships between $g_i(X, L)$ and $\Delta_i(X, L)$?

(iii) Are there any relationships between $\Delta_i(X, L)$ and $\Delta_{i+1}(X, L)$?

(iv) We must classify $(X, L)$ by the value of the $i$th $\Delta$-genus.

(v) What is the geometric meaning of the $i$th $\Delta$-genus?

**Remark 3.3.** If $X$ is smooth and $L$ is nef and big, then the following facts concerning Problem 3.1 are known.

(1) First we consider Problem 3.1(i-1). If $i = 1$, then $\Delta_1(X, L) \geq 0$ (see [7, (4.2) Theorem] and [6, (1.1) Theorem]). Moreover, if $L$ is base point free, then we have $\Delta_i(X, L) \geq 0$ for every integer $i$ with $0 \leq i \leq n$.

(2) Next we consider Problem 3.1(ii). If $i = 1$ and $L$ is merely ample, then it is known that $g_1(X, L) = 0$ if and only if $\Delta_1(X, L) = 0$ (see [7, (12.1) Theorem]). Therefore, we consider the case where $i \geq 2$. Then under the assumption that $Bs|L| = \emptyset$ we see that $g_i(X, L) = 0$ if and only if $\Delta_i(X, L) = 0$ (see [12, Theorem 3.13]).

(3) Next we consider Problem 3.1(iii). Then, for example, we get the following. Assume that $L$ is base point free. If $\Delta_i(X, L) \leq i - 1$, then $\Delta_{i+1}(X, L) = 0$ (see [12, Proposition 3.9]). In particular, if $\Delta_i(X, L) = 0$, then $\Delta_{i+1}(X, L) = 0$. Maybe there will be several relationships between $\Delta_i(X, L)$ and $\Delta_{i+1}(X, L)$ other than this.

(4) For Problem 3.1(iv), we get a classification of $(X, L)$ by the value of $\Delta_2(X, L)$ as follows:

(4.1) a classification of polarized manifolds $(X, L)$ such that $Bs|L| = \emptyset$ and $\Delta_2(X, L) = 0$ (see [12, Theorem 3.13 and Remark 3.13.1]);
(4.2) A classification of polarized manifolds \((X, L)\) such that \(L\) is very ample and \(\Delta_2(X, L) = 1\) (see [12, Theorem 3.17] and [14, Remark 2]).

(5) Problem 3.1(v) seems to be the most difficult problem among the above problems even in the case where \(L\) is base point free or very ample. We also note that we are studying Problem 3.1(i-2) now and we will explain this in a future paper.

As we said in Remark 3.3, under the assumption that \(L\) is base point free, we have some answers for Problem 3.1 above. We believe that the \(i\)th \(\Delta\)-genus has good properties similar to those of the \(\Delta\)-genus and is useful if \(L\) satisfies some special conditions.

Therefore, the main purpose of our investigation for the time being is to consider the following two things.

**Problem 3.2.**

(I) Investigate Problem 3.1 for the case where \(|L|\) has base points.

(II) If \(L\) is very ample or base point free, then find an application using results of the \(i\)th \(\Delta\)-genus such as those mentioned in Remark 3.3. For example, investigate an unsolved problem about \((X, L)\) with very ample \(L\) by using results of the \(i\)th \(\Delta\)-genus.

In this paper, we consider Problem 3.1(i-1) under the assumption that \(|L|\) has base points. As we said above, if \(i = 1\), then \(\Delta_1(X, L)\), which equals the \(\Delta\)-genus, is non-negative for every nef and big line bundle \(L\) on \(X\). This was proved by Fujita. Moreover, Fujita also gave a classification of \((X, L)\) with small \(\Delta(X, L)\) (see [7]). These results are very useful and are used in various problems. So, in order to make the \(i\)th \(\Delta\)-genus useful, it is important to study the non-negativity of the \(i\)th \(\Delta\)-genus for \(2 \leq i \leq n\). As we said in Remark 3.3, if \(\text{Bs } |L| = \emptyset\), then \(\Delta_i(X, L) \geq 0\) for every integer \(i\) with \(1 \leq i \leq n\). However, in general for each integer \(i\) with \(2 \leq i \leq n\) there exists an example of \((X, L)\) with \(\Delta_i(X, L) < 0\) (see [12, Section 4]). So it is interesting and important to know when \(\Delta_i(X, L)\) is non-negative for \(i \geq 2\). This is the theme of this paper.

### 4. Non-negativity of the \(i\)th \(\Delta\)-genus

In this section, we consider non-negativity of \(\Delta_i(X, L)\). First, we consider the case where \(X\) is normal and Gorenstein, \(\dim \text{Irr}(X) \leq 0\) and \(\dim \text{Sing}(X) \leq \dim \text{Bs } |L| \leq \dim X - 1\).

**Theorem 4.1.** Let \((X, L)\) be a quasi-polarized variety of dimension \(n\). Assume that \(X\) is normal and Gorenstein, \(\dim \text{Irr}(X) \leq 0\) and \(\dim \text{Sing}(X) \leq \dim \text{Bs } |L| \leq n - 1\). Then, for every integer \(i\) with \(i \geq \dim \text{Bs } |L| + 1\), \(\Delta_i(X, L) \geq 0\) holds.

**Proof.** Set \(m := \dim \text{Bs } |L|\). If \(m = n - 1\), then we have only to prove that \(\Delta_n(X, L) \geq 0\).

By Remark 3.2(2) and the Serre duality, we have \(\Delta_n(X, L) = h^n(\mathcal{O}_X) - h^n(L) = h^0(\omega_X) - h^0(\omega_X \otimes L^{-1})\). If \(h^0(\omega_X \otimes L^{-1}) = 0\), then we get the assertion. So we may assume that \(h^0(\omega_X \otimes L^{-1}) \neq 0\). However, since \(h^0(L) > 0\), we have \(h^0(\omega_X) - h^0(\omega_X \otimes L^{-1}) \geq h^0(L) - 1 \geq 0\) by [8, Lemma 1.12] or [19, 15.6.2 Lemma]. Therefore, we get \(\Delta_n(X, L) \geq 0\) in this case.

Next we assume that \(m \leq n - 2\). By Remark 2.1, \(L\) has an \((n - m - 2)\)-ladder \(X \supset X_1 \supset \cdots \supset X_{n-m-2}\) such that \(X_j\) is normal and Gorenstein for every integer \(j\) with \(0 \leq j \leq n - m - 2\), where \(X_0 := X\). We set \(L_j := L_{|X_j}\) for \(1 \leq j \leq n - m - 2\) and \(L_0 := L\).
By the exact sequence

\[ 0 \to H^0(\mathcal{O}_X) \to H^0(L_j) \to H^0(L_{j+1}), \]

we see that \( h^0(L_{j+1}) \geq h^0(L_j) - h^0(\mathcal{O}_X) = h^0(L_j) - 1 \) for every integer \( j \) with \( 0 \leq j \leq n - m - 3 \). By assumption, \( h^0(L_0) \geq n - m \) holds (e.g. see [5, (1.7) Lemma]). Hence \( h^0(L_{n-m-2}) \geq h^0(L_{n-m-3}) - 1 \geq \cdots \geq h^0(L) - (n - m - 2) \geq 2 \). Hence, if \( i \geq m + 2 \), then Conditions A1(i) and A2(i) in [12, 2.7.2] are satisfied.

By [17, Chapter III, Corollary 7.7] and Theorem 2.1, we have \( h^i(-sL) = 0 \) for integers \( t \) and \( s \) with \( 0 \leq t \leq n - 1 \) and \( 1 \leq s \). Hence by [9, Proposition 2.1(a)], Condition B(i, i) in [12, 2.7.2] is also satisfied for \( m + 2 \leq i \leq n - 1 \). Therefore, we get \( \Delta_i(X, L) \geq 0 \) for every integer \( i \) with \( i \geq m + 2 \) by [12, Corollary 2.9(2)]. Here we note that if \( i = n \), then we do not need Conditions A1(i) and B(i, i) in [12, 2.7.2] (see the proof of [12, Theorem 2.8(2)]).

Assume that \( i = m + 1 \). As we said above, Condition A1 \((m + 2)\) in [12, 2.7.2] holds. By [9, Proposition 2.1(a)] Condition B \((m + 2, m + 1)\) in [12, 2.7.2] also holds. Hence by [12, Corollary 2.9(1)], we see that

\[ \Delta_{m+1}(X, L) \geq \cdots \geq \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}). \]

Next we calculate \( \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) \). Here we note that by Theorem 3.2(1) we have

\[ g_{m+1}(X_{n-m-2}, L_{n-m-2}) = (-1)^{m+2} \chi(-L_{n-m-2}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+1}(\mathcal{O}_{X_{n-m-2}}). \]

By [9, Claim 2.1.1] we have \((-1)^{m+2} \chi(-L_{n-m-2}) = h^{m+2}(\mathcal{O}_{X_{n-m-2}})\). Hence

\[ g_{m+1}(X_{n-m-2}, L_{n-m-2}) = h^{m+2}(\mathcal{O}_{X_{n-m-2}}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+1}(\mathcal{O}_{X_{n-m-2}}). \]

Since \( \dim \mathcal{B} |L| \leq \dim \mathcal{B} |L| = m = \dim(X_{n-m-2}) - 2 \), by Bertini’s theorem (see [9, Theorem 1.8(3)]) we can take a general member \( X_{n-m-1} \) of \( |L| \) such that \( X_{n-m-1} \) is generically reduced. On the other hand, by [7, (2.4), 2], Chapter 0], \( X_{n-m-1} \) is Cohen–Macaulay. Hence \( X_{n-m-1} \) is reduced. Here we note that \( X_{n-m-1} \) is not always irreducible. Then by using the exact sequence

\[ 0 \to L_{n-m-2} \to \mathcal{O}_{X_{n-m-2}} \to \mathcal{O}_{X_{n-m-1}} \to 0, \]

we see that \( h^{m+2}(\mathcal{O}_{X_{n-m-2}}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+1}(\mathcal{O}_{X_{n-m-1}}) = 0 \), because \( h^{m+1}(\mathcal{O}_{X_{n-m-1}}) = 0 \) by [9, Claim 2.1.1]. Therefore,

\[ g_{m+1}(X_{n-m-2}, L_{n-m-2}) = h^{m+2}(\mathcal{O}_{X_{n-m-2}}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+1}(\mathcal{O}_{X_{n-m-2}}) = h^{m+1}(\mathcal{O}_{X_{n-m-1}}). \]

By Remark 3.2(2),

\[ \Delta_{m+2}(X_{n-m-2}, L_{n-m-2}) = h^{m+2}(\mathcal{O}_{X_{n-m-2}}) - h^{m+2}(L_{n-m-2}). \]

Since, by Remark 3.2(3),

\[ \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) = g_{m+1}(X_{n-m-2}, L_{n-m-2}) - \Delta_{m+2}(X_{n-m-2}, L_{n-m-2}) + h^{m+1}(\mathcal{O}_{X_{n-m-2}}) - h^{m+1}(L_{n-m-2}), \]
Therefore, we get
\[ \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) = h^{m+1}(\mathcal{O}_{X_{n-m-1}}) - (h^{m+2}(\mathcal{O}_{X_{n-m-2}}) - h^{m+2}(L_{n-m-2})) \\
+ h^{m+1}(L_{n-m-1}) - h^{m+1}(L_{n-m-2}). \]

Here we set \( L_{n-m-1} := L|_{X_{n-m-1}} \). Then by using the exact sequence
\[
0 \to \mathcal{O}_{X_{n-m-2}} \to L_{n-m-2} \to L_{n-m-1} \to 0,
\]
we have
\[
h^{m+1}(\mathcal{O}_{X_{n-m-2}}) - h^{m+1}(L_{n-m-2}) \\
+ h^{m+1}(L_{n-m-1}) - h^{m+2}(\mathcal{O}_{X_{n-m-2}}) + h^{m+2}(L_{n-m-2}) \geq 0.
\]
Therefore,
\[
\Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) \geq h^{m+1}(\mathcal{O}_{X_{n-m-1}}) - h^{m+1}(L_{n-m-1}).
\]

Since \( X_{n-m-1} \) is Cohen–Macaulay and equidimensional, by [17, Chapter III, Corollary 7.7], the Serre duality holds, that is,
\[
h^{m+1}(\mathcal{O}_{X_{n-m-1}}) = h^0(\omega_{X_{n-m-1}})
\]
and
\[
h^{m+1}(L_{n-m-1}) = h^0(\omega_{X_{n-m-1}} \otimes (L_{n-m-1})^{-1}).
\]

Here we note that \( \omega_{X_{n-m-1}} \) is a Cartier divisor by [20, Proposition 5.73]. Let \( X_{n-m-1} = \bigcup_j Z_j \), where \( Z_j \) is an irreducible component of \( X_{n-m-1} \) for each \( j \).

Since \( \dim \text{Bs} |L_{n-m-2}| \leq \dim \text{Bs} |L| = m = \dim(X_{n-m-2}) - 2 \), there exists an element \( s \in H^0(L_{n-m-2}) \) such that every \( Z_j \) is an irreducible component of the divisor \( \text{div}(s) \) of zeros of the global section \( s \). Let \( \delta : H^0(L_{n-m-2}) \to H^0(L_{n-m-1}) \). Then \( \delta(s) \in H^0(L_{n-m-1}) \) defines a non-zero homomorphism
\[
\rho : \mathcal{O}_{X_{n-m-1}} \to L_{n-m-1}.
\]

**Claim 4.1.** \( \ker(\rho) = 0 \).

**Proof.** Since \( \delta(s) \) does not vanish identically on any \( Z_j \), we see that \( \dim \text{Supp}(\ker(\rho)) < \dim X_{n-m-1} \). However, since \( X_{n-m-1} \) is Cohen–Macaulay, \( X_{n-m-1} \) is locally 1-Macaulay. (For the definition of locally 1-Macaulay, see [7, (2.4) Definition in Chapter 0].) Therefore, by using [4, (1.12) Corollary] we have \( \ker(\rho) = 0 \). \( \square \)

Hence \( \rho \) is injective, and \( \omega_{X_{n-m-1}} \otimes (L_{n-m-1})^{-1} \to \omega_{X_{n-m-1}} \) is injective because \( \omega_{X_{n-m-1}} \) and \( (L_{n-m-1})^{-1} \) are invertible. So we have
\[
h^0(\omega_{X_{n-m-1}}) - h^0(\omega_{X_{n-m-1}} \otimes (L_{n-m-1})^{-1}) \geq 0.
\]

Hence \( \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) \geq h^0(\omega_{X_{n-m-1}}) - h^0(\omega_{X_{n-m-1}} \otimes (L_{n-m-1})^{-1}) \geq 0 \). Therefore, \( \Delta_{m+1}(X, L) \geq \Delta_{m+1}(X_{n-m-2}, L_{n-m-2}) \geq 0 \). This completes the proof. \( \square \)

By Theorem 4.1 we can get the following corollary.
COROLLARY 4.1. Let \((X, L)\) be a quasi-polarized manifold of dimension \(n\). Assume that \(n \geq 2\) and \(\dim Bs |L| \leq n - 1\). Then \(\Delta_i(X, L) \geq 0\) for every integer \(i\) with \(i \geq \dim Bs |L| + 1\).

Moreover, we obtain the following corollaries.

COROLLARY 4.2. Let \((X, L)\) be a polarized variety of dimension \(n\). Assume that \(X\) is normal and Gorenstein, \(\dim \text{Irr}(X) \leq 0\) and \(\dim \text{Sing}(X) \leq \dim Bs|L| \leq n - 1\). If \(i \geq \Delta(X, L)\), then \(\Delta_i(X, L) \geq 0\).

Proof. Since \(i \geq \Delta(X, L)\), we have \(i \geq \dim Bs |L| + 1\) by the \(\Delta\)-genus inequality of Fujita [7, (4.2) Theorem]. By Theorem 4.1, we get the assertion.

COROLLARY 4.3. Let \((X, L)\) be a polarized manifold of dimension \(n \geq 2\). Assume that \(\Delta(X, L) \leq 2\). Then \(\Delta_i(X, L) \geq 0\) for every integer \(i\) with \(2 \leq i \leq n\).

Proof. If \(h^0(L) = 0\), then \(\Delta(X, L) = n + L^n \geq n + 1 \geq 3\). But then this contradicts the assumption. Hence \(h^0(L) \geq 1\), that is, \(\dim Bs |L| \leq n - 1\). By assumption, we have \(i \geq 2 \geq \Delta(X, L)\). Hence by Corollary 4.2 we get \(\Delta_i(X, L) \geq 0\) for every \(i\) with \(i \geq 2\). Therefore, we get the assertion.

The following examples show that Corollary 4.1 is the best possible.

Example 4.1.

(1) Let \(Y\) be a smooth projective variety of dimension \(m \geq 2\) such that \(K_Y\) is ample with \(h^0(K_Y) \neq 0\). (There exists such a \(Y\) for every \(m \geq 2\).) Let \(A_1, \ldots, A_{n-m}\) be ample divisors on \(Y\) such that \(\dim Bs |A_i| = 0\) and \(h^m(A_i) = 0\) for any \(i\), where \(n > m\). We set \(\mathcal{E} := K_Y \oplus A_1 \oplus \cdots \oplus A_{n-m}\) and \(X := \mathbb{P}(\mathcal{E})\). Let \(\pi : \mathbb{P}(\mathcal{E}) \rightarrow Y\) be its projection. For \(1 \leq i \leq n - m\), let \(E_i := K_Y \oplus A_1 \oplus \cdots \oplus A_{i-1} \oplus A_{i+1} \oplus \cdots \oplus A_{n-m}\) and \(D_i := \mathbb{P}(E_i)\). Then we see that \(D_i \in |H(E) - \pi^*A_i|\) and \(D_1 \cap \cdots \cap D_{n-m} = \mathbb{P}(K_Y)\). We set \(L := \mathcal{H}(E)\). Then \(L\) is ample and \(D_i + \pi^*A_i \in |L|\) for each \(i\). Since \(D_1 \cap \cdots \cap D_{n-m} \cong \mathbb{P}(K_Y)\) and \(A_i\) is spanned by its global sections, we have \(\dim Bs |L| \leq m\). On the other hand, by [8, Example 2.10(8)] and [12, Lemma 2.12.1] we have \(g_m(X, L) = h^m(\mathcal{O}_X)\) and \(\Delta_{m+1}(X, L) = 0\). Hence by Remark 3.2(3) we see that

\[
\Delta_m(X, L) = g_m(X, L) - \Delta_{m+1}(X, L) + (n - m)h^m(\mathcal{O}_X) - h^m(L)
\]

\[
= (n - m + 1)h^m(\mathcal{O}_X) - h^m(L).
\]

Here we note that \(h^m(\mathcal{O}_X) = h^m(\mathcal{O}_Y) = h^0(K_Y) = 0\) and

\[
h^m(L) = h^m(\pi_*(L))
\]

\[
= h^m(\mathcal{E})
\]

\[
= h^m(K_Y \oplus A_1 \oplus \cdots \oplus A_{n-m})
\]

\[
= h^m(K_Y) = 1.
\]

Hence \(\Delta_m(X, L) = -1 < 0\) and by Theorem 4.1 we see that \(\dim Bs |L| \geq m\). Therefore, \(\dim Bs |L| = m\). This \((X, L)\) is an example with \(\dim Bs |L| = m\) and \(\Delta_m(X, L) < 0\).

(2) By [1, Theorem 1.1 and Section 2], there exists a Calabi-Yau 3-fold \(X\) and an ample line bundle \(L\) on \(X\) such that \(h^0(L) = 1\). Then we note that \(\mathcal{O}(K_X) \sim \mathcal{O}_X, h^1(\mathcal{O}_X) = 0\).
\[ h^2(\mathcal{O}_X) = 0, \ h^3(\mathcal{O}_X) = 1 \] and \( h^j(L) = 0 \) for every \( j \geq 1 \). Hence we have \( \Delta_3(X, L) = h^3(\mathcal{O}_X) - h^3(L) = 1 \). On the other hand, since \( h^0(K_X + L) = h^0(L) = 1 \), we see that \( g_2(X, L) = 0 \) by Theorem 3.2(2). Therefore, by Remark 3.2(3)

\[ \Delta_2(X, L) = g_2(X, L) - \Delta_3(X, L) + h^2(\mathcal{O}_X) - h^2(L) \]

\[ = 0 - 1 + 0 - 0 \]

\[ = -1. \]

Since \( \dim \, \text{Bs} \, |L| = 2 \), we infer that this \( (X, L) \) is an example with \( i = \dim \, \text{Bs} \, |L| \) and \( \Delta_i(X, L) < 0 \).

Next we consider the case where \( (X, L) \) is a quasi-polarized manifold with \( \dim X = 3 \) and \( K_X \equiv 0 \).

**Proposition 4.1.** Let \( (X, L) \) be a quasi-polarized manifold of dimension three. Assume that \( K_X \equiv 0 \). Then the following hold:

1. \( \Delta_3(X, L) \geq 0 \);
2. (i) if \( \mathcal{O}(K_X) \not\sim \mathcal{O}_X \), then \( \Delta_2(X, L) \geq 1 \);
   (ii) if \( \mathcal{O}(K_X) \sim \mathcal{O}_X \), then \( \Delta_2(X, L) \geq -1 \).

**Proof.** Here we note that \( L - K_X \) is nef and big by assumption. Therefore, by the Kawamata–Viehweg vanishing theorem, we have \( h^j(L) = 0 \) for every \( j \geq 1 \).

1. By Remark 3.2(2), we have \( \Delta_3(X, L) = h^3(\mathcal{O}_X) - h^3(L) = h^3(\mathcal{O}_X) \geq 0 \).
2. By Theorem 3.2(2) and Remark 3.2(2) and (3), we get

\[ \Delta_2(X, L) = g_2(X, L) - \Delta_3(X, L) + h^2(\mathcal{O}_X) - h^2(L) \]

\[ = g_2(X, L) - h^3(\mathcal{O}_X) + h^2(\mathcal{O}_X) \]

\[ = h^0(K_X + L) - 2h^3(\mathcal{O}_X) + 2h^2(\mathcal{O}_X). \]

Since \( K_X \) is nef, we see that \( c_2(X)L \geq 0 \) by a theorem of Miyaoka’s [21, Theorem 6.6]. Hence, by the Kawamata–Viehweg vanishing theorem, the Serre duality and the Hirzebruch–Riemann–Roch theorem (see [18] or [15, Example 15.2.5]), we have

\[ h^0(K_X + L) = -\chi(-L) \]

\[ = \frac{1}{6}L^3 + \frac{1}{3}K_X L^2 + \frac{1}{12}(K_X^2 + c_2(X))L + \frac{1}{24}c_2(X)K_X \]

\[ \geq \frac{1}{6}L^3 > 0. \]

(i) If \( \mathcal{O}(K_X) \not\sim \mathcal{O}_X \), then \( h^3(\mathcal{O}_X) = 0 \). Hence \( \Delta_2(X, L) = h^0(K_X + L) - 2h^3(\mathcal{O}_X) + 2h^2(\mathcal{O}_X) \geq 1 \).

(ii) If \( \mathcal{O}(K_X) \sim \mathcal{O}_X \), then \( h^3(\mathcal{O}_X) = 1 \). Hence \( \Delta_2(X, L) = h^0(K_X + L) - 2h^3(\mathcal{O}_X) + 2h^2(\mathcal{O}_X) \geq -1 \). This completes the proof. \( \square \)

**Remark 4.1.** The inequality of Proposition 4.1(2)(ii) is the best possible. See Example 4.1(2).

Next, as a special case of \( K_X \equiv 0 \), we consider the case where \( (X, L) \) is a polarized Abelian variety. Here we note that if \( X \) is an Abelian variety, then a line bundle \( L \) on \( X \) is ample if and only if \( L \) is nef and big (see, e.g., [2, Proposition 4.5.2]).
Let $X$ be an Abelian variety of dimension $n$ and let $L$ be an ample line bundle on $X$. Here we study a lower bound for the $i$th $\Delta$-genus of $(X, L)$. If $i = n$, then by Remark 3.2(2) and the Kodaira vanishing theorem we have $\Delta_n(X, L) = h^n(O_X) = 1$. So we consider the case where $1 \leq i \leq n - 1$.

**Theorem 4.2.** Let $(X, L)$ be a polarized Abelian variety of dimension $n \geq 2$, and let $i$ be an integer with $1 \leq i \leq n - 1$. Then

$$\Delta_i(X, L) \geq \begin{cases} \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 2, \\ \frac{L^n}{n!} + n & \text{if } i = 1. \end{cases}$$

**Proof.** In this case $h^i(L) = 0$ for every integer $i$ with $i > 0$, and $h^j(O_X) = \binom{n}{j}$ for every integer $j$ with $j \geq 0$.

By the definition of the $i$th $\Delta$-genus, we see that, for every integer $i$ with $i \geq 2$,

$$\Delta_i(X, L) = g_{i-1}(X, L) - \Delta_{i-1}(X, L) + (n - i + 1)h^{i-1}(O_X) - h^{i-1}(L)$$

$$= g_{i-1}(X, L) - \Delta_{i-1}(X, L) + (n - i + 1)\binom{n}{i-1}.$$ 

Hence we obtain

$$\Delta_i(X, L) = g_{i-1}(X, L) - \Delta_{i-1}(X, L) + (n - i + 1)\binom{n}{i-1}.$$ 

$$= (g_{i-1}(X, L) - g_{i-2}(X, L)) + \Delta_{i-2}(X, L)$$ 

$$- (n - i + 2)h^{i-2}(O_X) + (n - i + 1)\binom{n}{i-1}$$ 

$$= \sum_{k=0}^{i-2} (-1)^k g_{i-1-k}(X, L) + \Delta_{i-2}(X, L) + \sum_{j=0}^{i-1} (-1)^j (n - i + 1 + j)\binom{n}{i-1-j}$$ 

$$= \cdots$$ 

$$= \sum_{k=0}^{i-2} (-1)^k g_{i-1-k}(X, L) + (-1)^{i-1} \Delta_1(X, L)$$ 

$$+ \sum_{j=0}^{i-2} (-1)^j (n - i + 1 + j)\binom{n}{i-1-j}$$ 

$$= \sum_{k=0}^{i-1} (-1)^k g_{i-1-k}(X, L) + (-1)^i h^0(L) + \sum_{j=0}^{i-1} (-1)^j (n - i + 1 + j)\binom{n}{i-1-j}$$ 

$$= \sum_{k=0}^{i-1} (-1)^k \left( g_{i-1-k}(X, L) + (n - i + 1 + k)\binom{n}{n-i+1+k} \right) + (-1)^i h^0(L).$$
By [9, Claim 3.A.4.1], we have
\[
\sum_{j=0}^{(i-1-k)-1} (-1)^{(i-1-k)-j} h^{j} \left( \mathcal{O}_{X} \right) = \sum_{j=0}^{(i-1-k)-1} (-1)^{(i-1-k)-j} \binom{n}{j} = \binom{n-1}{(i-1-k)-1} = \binom{n-1}{n-i+1+k}.
\]

Therefore, by [9, Remark 3.A.3.1 (1)], we have
\[
g_{i-1-k}(X, L) = (n-i+1+k)!S(n, n-i+1+k) \frac{L^{n}}{n!} + \binom{n-1}{n-i+1+k}.
\]

Here \( S(n, n-i+1+k) \) denotes the Stirling number of the second kind with the type \((n, n-i+1+k)\) (see [9, Definition 3.A.1]). Hence
\[
\Delta_{i}(X, L) = \sum_{k=0}^{i-1} (-1)^{k} \left( g_{i-1-k}(X, L) + (n-i+1+k) \binom{n}{n-i+1+k} \right) + (-1)^{i-1} h^{0}(L)
\]
\[
= \sum_{k=0}^{i-1} (-1)^{k} (n-i+1+k)!S(n, n-i+1+k) \frac{L^{n}}{n!} + \sum_{k=0}^{i-1} (-1)^{k} \binom{n-1}{n-i+1+k}
\]
\[
+ \sum_{k=0}^{i-1} (-1)^{k} (n-i+1+k) \binom{n}{n-i+1+k} + (-1)^{i-1} h^{0}(L).
\]

First, we calculate \( \sum_{k=0}^{i-1} (-1)^{k} (n-i+1+k) \binom{n}{n-i+1+k} \). Then
\[
\sum_{k=0}^{i-1} (-1)^{k} (n-i+1+k) \binom{n}{n-i+1+k} = \sum_{k=0}^{i-1} (-1)^{k} (i-k) \binom{n}{i-k}
\]
\[
= \sum_{i=1}^{i} (-1)^{i-1} \binom{n}{i}
\]
\[
= \sum_{i=1}^{i} (-1)^{i-1} \binom{n-1}{i-1}
\]
\[
= (-1)^{i} n \sum_{i=1}^{i} (-1)^{i} \binom{n-1}{i-1}
\]
\[
= (-1)^{i} n \sum_{p=0}^{i-1} (-1)^{p+1} \binom{n-1}{p}
\]
\[
= n \sum_{p=0}^{i-1} (-1)^{i-p-1} \binom{n-1}{p}
\]
\[
= n \binom{n-2}{i-1}.
\]
(In the last step, we have used [9, Claim 3.A.4.1].)

Next we calculate \[ \sum_{k=0}^{i-1} (-1)^k \binom{n-1}{n-i+1+k}. \] If \( i \geq 2 \), then
\[
\sum_{k=0}^{i-1} (-1)^k \binom{n-1}{n-i+1+k} = \sum_{k=0}^{i-2} (-1)^k \binom{n-1}{i-2-k} = \sum_{t=0}^{i-2} (-1)^{i-2-t} \binom{n-1}{t} = \binom{n-2}{i-2}.
\]

(In the last step, we have used [9, Claim 3.A.4.1].) If \( i = 1 \), then
\[
\sum_{k=0}^{i-1} (-1)^k \binom{n-1}{n-i+1+k} = 0.
\]

Finally, we calculate \[ \sum_{k=0}^{i-1} (-1)^k (n - i + 1 + k)!S(n, n - i + 1 + k)(L^n/n!) + (-1)^i h^0(L) \]. We note that
\[
\sum_{k=0}^{i-1} (-1)^k (n - i + 1 + k)!S(n, n - i + 1 + k)
= \sum_{t=0}^{i-1} (-1)^{i-1-t} (n - t)!S(n, n - t)
= (-1)^{n+i-1} \sum_{t=0}^{i-1} (-1)^{n-t} (n - t)!S(n, n - t)
= (-1)^{n+i-1} \sum_{k=n-i+1}^{n} (-1)^k k!S(n, k).
\]

By [23, (24d) on p. 34], we have
\[ x^n = \sum_{k=0}^{n} S(n, k)[x]_k, \tag{1} \]
where
\[ [x]_k := \begin{cases} x(x-1) \cdots (x-k+1) & \text{if } k \geq 1, \\ 1 & \text{if } k = 0. \end{cases} \]

So we have
\[
(-1)^n = \sum_{k=1}^{n} (-1)^k k!S(n, k)
= \sum_{k=1}^{n-i} (-1)^k k!S(n, k) + \sum_{k=n-i+1}^{n} (-1)^k k!S(n, k).
\]
Because $h^0(L) = L^n/n!$, we get

$$(-1)^{n+i-1} \sum_{k=n-i+1}^{n} (-1)^k k! S(n, k) \frac{L^n}{n!} + (-1)^i (h^0(L))$$

$$= (-1)^i \frac{L^n}{n!} - (-1)^{n+i-1} \sum_{k=1}^{n-i} (-1)^k k! S(n, k) \frac{L^n}{n!} + (-1)^{i-1} (-h^0(L))$$

$$= (-1)^{n-i} \sum_{k=1}^{n-i} (-1)^k k! S(n, k) \frac{L^n}{n!}.$$

Therefore, we get

$$\Delta_i(X, L) = \begin{cases} (-1)^{n-i} \sum_{k=1}^{n-i} (-1)^k k! S(n, k) \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 2, \\ (-1)^{n-1} \sum_{k=1}^{n-i} (-1)^k k! S(n, k) \frac{L^n}{n!} + n & \text{if } i = 1. \end{cases}$$

Here we prove the following.

**Lemma 4.1.** For every integer $p$ with $p \leq n - 1$, the following holds:

$$(-1)^{n-p} \sum_{k=1}^{n-p} (-1)^k k! S(n, k) = \sum_{t=1}^{n-p} (-1)^{n-p-t} \binom{n-p+1}{t} t^n.$$ 

**Proof.** By the Stirling formula [23, (24a) on p. 34], we get

$$k! S(n, k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n. \quad (2)$$

Hence

$$\sum_{k=1}^{n-p} (-1)^k k! S(n, k) = \sum_{k=1}^{n-p} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n$$

$$= (-1)^1 \left( \binom{1}{1} + \binom{2}{1} + \cdots + \binom{n-p}{1} \right) 1^n$$

$$+ (-1)^2 \left( \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n-p}{2} \right) 2^n$$

$$+ \cdots + (-1)^{n-p} \binom{n-p}{n-p} (n-p)^n$$

$$= \sum_{t=2}^{n-p+1} (-1)^{t-1} \left( \sum_{k=t-1}^{n-p} \binom{k}{t-1} (t-1)^n \right). \quad (3)$$

Here we prove the following.
CLAIM 4.2. Let $s$ and $t$ be integers with $0 \leq s \leq t$. Then
\[
\sum_{k=s}^{t} \binom{k}{s} = \binom{t + 1}{s + 1}.
\]

Proof. If $t - s = 0$, then this is true.
Assume that this equality holds for $0 \leq t - s \leq p$. We consider the case where $t - s = p + 1$. Then
\[
\sum_{k=s}^{t} \binom{k}{s} = \sum_{k=s}^{s+p+1} \binom{k}{s} = \binom{s+p}{s} + \binom{s+p+1}{s} = \binom{s+p+1}{s+1}.
\]
This completes the proof of Claim 4.2. \hfill \Box

By Claim 4.2 and (3), we get
\[
\sum_{k=1}^{n-p} (-1)^k k! S(n, k) = \sum_{i=2}^{n-p+1} (-1)^{i-1} \binom{n-p+1}{t}(t-1)^n
\]
\[
= \sum_{i=1}^{n-p} (-1)^i \binom{n-p+1}{t+1} t^n.
\]
Therefore, we get the assertion of Lemma 4.1. \hfill \Box

Here, for any integer $r$ with $r \geq 1$, we set
\[
a_r = \sum_{i=1}^{r} (-1)^{r-i} \binom{r+1}{t+1} t^n.
\]
Then, by Lemma 4.1,
\[
\Delta_i(X, L) = \begin{cases} 
    a_{n-i} - \frac{L^n}{n!} + n \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \geq 2, \\
    a_{n-1} - \frac{L^n}{n!} + n & \text{if } i = 1.
\end{cases}
\]

Here we note that $tL$ is ample and spanned by its global sections for any large $t$, and $\Delta_i(X, tL) \geq 0$ by [12, Corollary 3.3]. Since $L^n > 0$, we obtain
\[
a_{n-i} \geq 0. \tag{4}
\]

LEMMA 4.2. $a_{n-i} \geq 1$.

Proof. First we prove the following.
CLAIM 4.3. \( a_p + a_{p+1} = (p + 1)!S(n, p + 1) \).

Proof. By using (2) we have

\[
a_{p+1} = \sum_{t=1}^{p+1} (-1)^{p+1-t} \binom{p+2}{t+1} t^n
\]

\[
= \sum_{t=1}^{p+1} (-1)^{p+1-t} \left( \binom{p+1}{t+1} + \binom{p+1}{t} \right) t^n
\]

\[
= \sum_{t=1}^{p+1} (-1)^{p+1-t} \binom{p+1}{t+1} t^n + (p + 1)!S(n, p + 1)
\]

\[
= \sum_{t=1}^{p} (-1)^{p-t} \binom{p+1}{t+1} t^n + (p + 1)!S(n, p + 1)
\]

\[
= - \sum_{t=1}^{p} (-1)^{p-t} \binom{p+1}{t+1} t^n + (p + 1)!S(n, p + 1)
\]

\[
= -a_p + (p + 1)!S(n, p + 1).
\]

This completes the proof of Claim 4.3.

CLAIM 4.4. \( a_{n-k} - (-1)^k \) is divisible by \( (n - k + 1)! \) for every integer \( k \) with \( 1 \leq k \leq n \).

Proof. (a) By Claim 4.3, we see that

\[
a_n = \sum_{t=1}^{n} (-1)^{n-t} \binom{n+1}{t+1} t^n
\]

\[
= (-1)^n \sum_{t=1}^{n} (-1)^t t!S(n, t)
\]

\[
= (-1)^n (1) = 1.
\]

Hence we get \( a_{n-1} - (-1)^1 = n! \) and the assertion is true for \( k = 1 \).

(b) Assume that \( a_{n-t} - (-1)^t \) is divisible by \( (n - t + 1)! \). We set

\[
a_{n-t} - (-1)^t = (n - t + 1)!b_{n-t}.
\]

Then, by Claim 4.3, we have \( a_{n-t} + a_{n-t-1} = (n - t)!S(n, n - t) \). Hence

\[
a_{n-t-1} = (n - t)!S(n, n - t) - a_{n-t}
\]

\[
= (n - t)!S(n, n - t) - (n - t + 1)!b_{n-t} - (-1)^t
\]

\[
= (n - t)!S(n, n - t) - (n - t + 1)b_{n-t} - (-1)^t+1.
\]

Hence \( a_{n-t-1} - (-1)^{t+1} \) is divisible by \( (n - t)! \) and we get the assertion of Claim 4.4.

Here we go back to the proof of Lemma 4.2. We use the notation in the proof of Claim 4.4. Namely, we set

\[
b_{n-i} := \frac{a_{n-i} - (-1)^i}{(n - i + 1)!}.
\]
Then $b_{n-i}$ is an integer by Claim 4.4. Assume that $1 \leq i \leq n - 1$. If $b_{n-i} \leq -1$, then $a_{n-i} < 0$ since $1 \leq i \leq n - 1$ and $n \geq 2$. So this is impossible by (4). Hence $b_{n-i} \geq 0$.

(i) Assume that $i$ is even. Then $a_{n-i} \geq 1$ because $b_{n-i} \geq 0$ and $(-1)^i = 1$.

(ii) Assume that $i$ is odd. Then $a_{n-i} = (n-i+1)!b_{n-i} - 1$. If $b_{n-i} = 0$, then $a_{n-i} < 0$ and this is impossible by (4). Hence $b_{n-i} > 0$ and $a_{n-i} = (n-i+1)!b_{n-i} - 1 \geq 2 - 1 = 1$ because $i \leq n - 1$.

Therefore, we get the assertion of Lemma 4.2.

By Lemma 4.2, we get

$$
\Delta_i(X, L) = \begin{cases} 
  a_{n-i} \frac{L^n}{n!} + n \left( \frac{n-2}{i-1} \right) + \left( \frac{n-2}{i-2} \right) & \text{if } i \geq 2, \\
  a_{n-1} \frac{L^n}{n!} + n & \text{if } i = 1,
\end{cases}
$$

Hence we get the assertion of Theorem 4.2.

**Remark 4.2.**

(1) Let $(X, L)$ be a polarized Abelian variety of dimension $n \geq 2$. Assume that $i = n - 1$. Then, by Theorem 3.2(2) and Remark 3.2(2) and (3),

$$
\Delta_{n-1}(X, L) = g_{n-1}(X, L) - \Delta_0(X, L) = h^{n-1}(\mathcal{O}_X) - h^{n-1}(L)
$$

$$
= (h^0(K_X + L) - h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_X)) - h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_X)
$$

$$
= h^0(L) + 2h^{n-1}(\mathcal{O}_X) - 2h^n(\mathcal{O}_X)
$$

$$
= h^0(L) + 2n - 2
$$

$$
= \frac{L^n}{n!} + 2n - 2
$$

$$
= \begin{cases} 
  \frac{L^n}{n!} + n \left( \frac{n-2}{(n-1)-1} \right) + \left( \frac{n-2}{(n-1)-2} \right) & \text{if } n \geq 3, \\
  \frac{L^n}{n!} + n & \text{if } n = 2.
\end{cases}
$$

Therefore, the inequality in Theorem 4.2 is the best possible.

(2) If $i$ is odd with $1 \leq i \leq n - 1$ and

$$
\Delta_i(X, L) = \begin{cases} 
  \frac{L^n}{n!} + n \left( \frac{n-2}{i-1} \right) + \left( \frac{n-2}{i-2} \right) & \text{if } i \geq 3, \\
  \frac{L^n}{n!} + n & \text{if } i = 1,
\end{cases}
$$

then by the proof of Theorem 4.2 we can prove that $i = n - 1$ (see (ii) in the proof of Theorem 4.2).
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REFERENCES


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