EXTENDING H^p FUNCTIONS FROM SUBMANIFOLDS OF PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

Hong Rae CHO and Yeoung-Tae SEO

(Received 19 November 2009)

Abstract. Let D be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$. Let D_M be a one-dimensional submanifold of D which meets the boundary of D transversally. Then any function in a Hardy space $H^p(D_M)$ extends to the function in the corresponding Hardy space $H^p(D)$.

1. Introduction and statement of the result

Many works have been done on the holomorphic extension of H^p ($1 \leq p < \infty$) functions from a submanifold of a strongly pseudoconvex domain [1, 2, 4, 5, 6, 8]. There are a few results on L^p extensions [12] on convex domains. However, there are no results on H^p extensions on weakly pseudoconvex domains that are non-convex, because we have difficulty in constructing the holomorphic integral kernel for the extension operator in those cases. We show, in this paper, that for pseudoconvex domains of finite type in $\mathbb{C}^2$ the holomorphic extension problem of the H^p -norm is always solvable. In [3], we proved the H^p extension on an analytic polyhedron.

Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type m , with defining function r . Let M be a smooth complex hyperplane in a neighborhood U of $\bar{D}$ of the form $M = \{z \in U : h(z) = 0\}$ with a holomorphic function h on U with $\partial h \neq 0$ everywhere on M and $\partial h \wedge \partial r \neq 0$ at all points of $M \cap bD$. Let $D_M = M \cap D$ and $bD_M = M \cap bD$. We denote $H^p(D)$ or $H^p(D_M)$ by the usual Hardy class on D or D_M , respectively. Then we can identify $H^p(D)$ or $H^p(D_M)$ in the usual way with a subspace of $L^p(bD, d\sigma)$ or $L^p(bD_M, d\sigma_M)$, respectively. Then we have the following result.

THEOREM 1.1. *For $f \in H^p(D_M)$, $1 \leq p < \infty$, there exists $F \in H^p(D)$ such that $F(z) = f(z)$ for $z \in D_M$ and $\|F\|_{H^p(D)} \leq C_p \|f\|_{H^p(D_M)}$.*

In [9], Diederich and Mazzilli introduced a counterexample that implies that the H^∞ extension phenomenon breaks down dramatically on a complex submanifold of a convex domain of finite type in $\mathbb{C}^n$ ($n \geq 3$). However, in [10] Diederich and Mazzilli proved that the H^∞ extension is possible from an affine linear hyperplane D_M to a convex domain of finite type. In our theorem, we can see that in the two-dimensional case the submanifold D_M need not be affine.

2000 Mathematics Subject Classification: Primary 32A26, 32A35.

Keywords: H^p extensions; submanifolds; pseudoconvex domain; finite type.

2. Construction of the holomorphic extension function

Let $p_0 \in bD$. Since $\nabla r(p_0) \neq 0$, we may assume that $\partial r / \partial z_2 \neq 0$ in a neighborhood U_0 of p_0 . Then by a result of Catlin (see [7]) for each $p \in U_0 \cap bD$ there exists a biholomorphic mapping $\phi_p : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $z = \phi_p(\zeta)$, with $\phi_p(0) = p$ such that the function $\rho(\zeta) = r \circ \phi_p(\zeta)$ is of the form

$$\rho(\zeta) = \operatorname{Re} \zeta_2 + \sum_{j+k \leq m, j, k > 0} a_{j,k}(p) \zeta_1^j \bar{\zeta}_1^k + \mathcal{O}(|\zeta_1|^{m+1} + |\zeta_2||\zeta|)$$

and

$$\phi_p(\zeta) = \left(p_1 + \zeta_1, p_2 + d_0(p)\zeta_2 + \sum_{k=1}^m d_k(p)\zeta_1^k \right), \quad (2.1)$$

where $a_{j,k}(p)$, $d_0(p)$ and $d_k(p)$ depend smoothly on p and $d_0(p) \neq 0$ on U_0 . Let $g(p, \cdot)$ denote the second component of the inverse of the biholomorphic mapping $\phi_p : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Since $g(p, z)$ depend smoothly on $p \in U_0$, we can replace p by ζ on a small neighborhood of p in the continuous sense.

Let $D_\epsilon = \{z \in D : r(z) < -\epsilon\}$ and $D_{M,\epsilon} = M \cap D_\epsilon$ for $\epsilon > 0$. If $\epsilon > 0$ is sufficiently small, $r(z) + \epsilon$ is a defining function for the domain D_ϵ , and hence $|r(z)| - \epsilon = -(r(z) + \epsilon) \gtrsim \operatorname{dist}(z, bD_\epsilon)$. Let us introduce $\Gamma_\epsilon(\zeta, z) = \operatorname{dist}(z, bD_\epsilon) + |g(\zeta, z)| + |z - \zeta|^m$.

We can obtain the following results as in [13, pp. 70–71].

PROPOSITION 2.1. *For each $\eta > 0$, there are $a > 0$, $C_\eta < \infty$ and C^∞ functions $w_j^{(\epsilon)}$ on $bD \times D_\epsilon$, $j = 1, 2$, with the following properties:*

- (i) $w_1^{(\epsilon)}(\zeta, z)(z_1 - \zeta_2) + w_2^{(\epsilon)}(\zeta, z)(z_2 - \zeta_2) = 1$;
- (ii) $w_j^{(\epsilon)}(\zeta, \cdot) \in \mathcal{O}(D_\epsilon)$ for $\zeta \in bD$;
- (iii) $|w_j^{(\epsilon)}(\zeta, z)| \leq C_\eta$ for $z \in D_\epsilon$ with $|z - \zeta| \geq a$;
- (iv) $|w_j^{(\epsilon)}(\zeta, z)| \leq C_\eta / [\Gamma_\epsilon(\zeta, z)]^{1+\eta}$ for $z \in D_\epsilon$ with $|z - \zeta| < a$ for all sufficiently small $\epsilon > 0$.

The functions $w_j^{(\epsilon)}$ depend also on η , but the constants C_η are independent of $\epsilon > 0$.

Then $W^{(\epsilon)} = \sum_{j=1}^2 w_j^{(\epsilon)} d\zeta_j$ is a holomorphic generating form on $bD \times D_\epsilon$ in the terminology of [13]. Let h_j be holomorphic functions in $\bar{D} \times \bar{D}$ such that

$$h(\zeta) - h(z) = \sum_{j=1}^2 h_j(\zeta, z)(\zeta_j - z_j).$$

Let

$$\alpha^{(\epsilon)}(\zeta, z) = w_1^{(\epsilon)}(\zeta, z)h_2(\zeta, z) - w_2^{(\epsilon)}(\zeta, z)h_1(\zeta, z)$$

and

$$\beta(\zeta) = \frac{1}{|\partial h(\zeta)|^2} \left(\frac{\partial \bar{h}}{\partial \zeta_2}(\zeta) d\zeta_1 - \frac{\partial \bar{h}}{\partial \zeta_1}(\zeta) d\zeta_2 \right).$$

For $f \in H^1(D_M)$ let f^* be the boundary value function of f . Then for $f \in H^1(D_M)$ we have [11]

$$f(z) = \int_{\zeta \in bD_M} f^*(\zeta) K^{(\epsilon)}(\zeta, z) \quad \text{for } z \in D_{M,\epsilon},$$

where $K^{(\epsilon)}$ is an analog of the Cauchy–Fantappiè kernel for the submanifold D_M defined by

$$K^{(\epsilon)}(\zeta, z) = \frac{1}{2\pi i} \alpha^{(\epsilon)}(\zeta, z) \beta(\zeta).$$

For $z \in D_\epsilon$ and $f \in H^p(D_M)$ ($1 \leq p < \infty$), we define

$$F_\epsilon(z) = \int_{bD_M} f^*(\zeta) K^{(\epsilon)}(\zeta, z) \quad \text{for } z \in D_\epsilon.$$

Then $F_\epsilon \in \mathcal{O}(D_\epsilon)$ and $F_\epsilon|_{D_{M,\epsilon}} = f$.

For the construction of the function $F \in \mathcal{O}(D)$ with $F|_{D_M} = f$, we will show that there is a subsequence of $\{F_\epsilon\}$ which converges uniformly on each compact subset of D . Let $\{\epsilon_\nu\}$ be a decreasing sequence of positive numbers, with $\epsilon_\nu \rightarrow 0$ as $\nu \rightarrow \infty$. Let $F_\nu = F_{\epsilon_\nu}$ and $D_\nu = D_{\epsilon_\nu}$.

Since $\partial h \neq 0$ on M , it follows that

$$|\beta(\zeta)| \lesssim \frac{1}{|\partial h(\zeta)|} \lesssim 1 \quad \text{for } \zeta \in bD_M.$$

Therefore,

$$|K^{(\epsilon)}(\zeta, z)| \lesssim |\alpha^{(\epsilon)}(\zeta, z)| \lesssim \sum_{j=1}^2 |w_j^{(\epsilon)}(\zeta, z)| \quad \text{for } \zeta \in bD_M, z \in D_\epsilon.$$

For $z \in \bar{D}_\mu$ and $\nu > \mu + 1$, by Proposition 2.1, we have

$$\begin{aligned} |F_\nu(z)| &\leq \int_{bD_M} |f^*(\zeta)| |K^{(\epsilon_\nu)}(\zeta, z)| d\sigma_M \\ &\lesssim \sum_{j=1}^2 \left(\int_{bD_M \cap \{|\zeta-z| < a\}} |f^*(\zeta)| |w_j^{(\epsilon)}(\zeta, z)| d\sigma_M \right) \\ &\lesssim \int_{bD_M \cap \{|\zeta-z| < a\}} |f^*(\zeta)| \frac{d\sigma_M}{[\Gamma_{\epsilon_\nu}(\zeta, z)]^{(1+\eta)}} + \int_{bD_M \cap \{|\zeta-z| \geq a\}} |f^*(\zeta)| d\sigma_M \\ &\lesssim \int_{bD_M \cap \{|\zeta-z| < a\}} |f^*(\zeta)| \frac{d\sigma_M}{|\zeta-z|^{m(1+\eta)}} + \int_{bD_M \cap \{|\zeta-z| \geq a\}} |f^*(\zeta)| d\sigma_M \\ &\lesssim \left(1 + \frac{1}{\epsilon_\mu^{m(1+\eta)}} \right) \int_{bD_M} |f^*(\zeta)| d\sigma_M. \end{aligned}$$

Thus if $f \in H^1(D_M)$, then $\{F_\nu : \nu > \mu + 1\}$ is uniformly bounded in $\bar{D}_\mu$. Therefore we can choose a subsequence $\{F_{\nu,\mu}\}$ of $\{F_\nu\}$ which converges uniformly in $\bar{D}_{\mu-1}$. Then $\{F_{\mu,\mu}\}$ converges uniformly on each compact subset of D . If we set

$$F(z) = \lim_{\mu \rightarrow \infty} F_{\mu,\mu}(z) \quad \text{for } z \in D,$$

then $F \in \mathcal{O}(D)$ and satisfies $F|_{D_M} = f$.

3. The H^p estimates for the extension function

LEMMA 3.1. *Fix a z close to p_0 . Then there exist C^∞ real coordinates $t = (t_1, t_2, t_3, t_4)$ for points ζ close to z so that*

$$\begin{aligned} t_1 + it_2 &= r(\zeta) - r(z) + i \operatorname{Im} g(\zeta, z), \\ t_3 + it_4 &= h(\zeta) - h(z). \end{aligned}$$

Proof. We know that [13]

$$dr(p_0) \wedge d_z \operatorname{Im} g(p_0, p_0) \neq 0. \quad (3.1)$$

From (2.1) the biholomorphic map ϕ_p satisfies $\phi_p^{-1}(p) = 0$ for $p \in U_0$. Therefore $\operatorname{grad}_\zeta g(p, p) = -\operatorname{grad}_z g(p, p)$ and so that from (3.1) we have

$$dr(p_0) \wedge d_\zeta \operatorname{Im} g(p_0, p_0) \neq 0.$$

By the transversal condition, we get the result. $\square$

Let $p_0 \in bD_M$. Let B_0 be the ball with center p_0 and radius $a/2$ and $\tilde{B}_0$ the ball with center p_0 and radius a . Let $\delta > 0$ be fixed and $0 < \epsilon < \delta$. We set

$$\tilde{F}_\epsilon(z) = \int_{bD_M \cap \tilde{B}_0} f^*(\zeta) K^{(\epsilon)}(\zeta, z) \quad \text{for } z \in D_\epsilon.$$

By the transversal condition, we can use real coordinates $u = (u_1, u_2, u_3, u_4)$ for z in $D \cap B_0$ such that $u_1 = -r(z)$, $u_2 = \operatorname{Im} g(\zeta, z)$ form real coordinates in $D_M \cap \tilde{B}_0$ and $u(\zeta) = (-r(\zeta), 0, 0, 0)$. Thus it follows that

$$\begin{aligned} \int_{bD_\delta \cap B_0} |\tilde{F}_\epsilon(z)| d\sigma(z) &\lesssim \int_{bD_M \cap \tilde{B}_0} |f^*(\zeta)| \int_{bD_\delta \cap B_0} |K^{(\epsilon)}(\zeta, z)| d\sigma(z) d\sigma_M(\zeta) \\ &\lesssim \int_{bD_M \cap \tilde{B}_0} |f^*(\zeta)| \int_{bD_\delta \cap B_0} \frac{1}{[\Gamma^{(\epsilon)}(\zeta, z)]^{1+\eta}} d\sigma(z) d\sigma_M(\zeta) \\ &\lesssim \int_{bD_M} |f^*(\zeta)| d\sigma_M(\zeta) \int_{|(u_2, u_3, u_4)| \leq c} \frac{du_2 du_3 du_4}{(|u_2| + |(u_3, u_4)|^m)^{1+\eta}} \\ &\lesssim \int_{bD_M} |f^*(\zeta)| d\sigma_M(\zeta) \int_0^c s^{1-m\eta} ds \\ &\lesssim \int_{bD_M} |f^*(\zeta)| d\sigma_M(\zeta), \end{aligned} \quad (3.2)$$

if η is sufficiently small. By Fatou's lemma, we have

$$\sup_{\delta > 0} \int_{bD_\delta} |F(z)| d\sigma(z) \lesssim \int_{bD_M} |f^*(\zeta)| d\sigma_M(\zeta),$$

which shows that $F \in H^1(D)$.

For the case $1 < p < \infty$, let q be a positive number with $1/p + 1/q = 1$. By applying Hölder's inequality, we have

$$\begin{aligned} |\tilde{F}_\epsilon(z)|^p &\lesssim \left(\int_{bD_M \cap \tilde{B}_0} |f^*(\zeta)|^p |K^{(\epsilon)}(\zeta, z)|^{\eta p + 1 - \eta} d\sigma(\zeta) \right) \\ &\quad \times \left(\int_{bD_M \cap \tilde{B}_0} |K^{(\epsilon)}(\zeta, z)|^{1-\eta} d\sigma(\zeta) \right)^{p/q}. \end{aligned}$$

Note that

$$|K^{(\epsilon)}(\zeta, z)|^{\eta p+1-\eta} \lesssim \frac{1}{|\Gamma^{(\epsilon)}(\zeta, z)|^{(1+\eta)(\eta p+1-\eta)}} \lesssim \frac{1}{|\Gamma^{(\epsilon)}(\zeta, z)|^{1+c_\eta}},$$

where $0 < c_\eta = \eta(p + \eta p - \eta) < 1$ for sufficiently small $\eta > 0$. For $\zeta \in bD_M \cap \tilde{B}_0$ if we use the coordinate system t in Lemma 3.1, we have

$$\begin{aligned} \int_{bD_M \cap \tilde{B}_0} |K^{(\epsilon)}(\zeta, z)|^{1-\eta} d\sigma_M(\zeta) &\lesssim \int_{bD_M \cap \tilde{B}_0} \frac{1}{[\Gamma^{(\epsilon)}(\zeta, z)]^{(1+\eta)(1-\eta)}} d\sigma_M(\zeta) \\ &\lesssim \int_0^c \frac{dt_2}{t_2^{1-\eta^2}} \lesssim 1. \end{aligned}$$

Thus it follows that

$$\begin{aligned} \int_{bD_\delta \cap B_0} |\tilde{F}_\epsilon(z)|^p d\sigma(z) &\lesssim \int_{bD_\delta \cap B_0} \int_{bD_M \cap \tilde{B}_0} |f^*(\zeta)|^p |K^{(\epsilon)}(\zeta, z)|^{\eta p+1-\eta} d\sigma_M(\zeta) d\sigma(z) \\ &\lesssim \int_{bD_M \cap \tilde{B}_0} |f^*(\zeta)|^p d\sigma_M(\zeta) \int_{bD_\delta \cap B_0} \frac{1}{[\Gamma^{(\epsilon)}(\zeta, z)]^{1+c_\eta}} d\sigma(z). \end{aligned} \quad (3.3)$$

Here we have

$$\begin{aligned} \int_{bD_\delta \cap B_0} \frac{1}{[\Gamma^{(\epsilon)}(\zeta, z)]^{1+c_\eta}} d\sigma(z) &\lesssim \int_{|(u_2, u_3, u_4)| \leq c} \frac{du_2 du_3 du_4}{(u_2 + |(u_3, u_4)|^m)^{1+c_\eta}} \\ &\lesssim \int_0^c s^{1-mc_\eta} ds \lesssim 1, \end{aligned} \quad (3.4)$$

if η is sufficiently small.

By Fatou's lemma, from (3.3) and (3.4) we get

$$\sup_{\delta > 0} \int_{bD_\delta} |F(z)|^p d\sigma(z) \lesssim \int_{bD_M} |f^*(\zeta)|^p d\sigma_M(\zeta),$$

which shows that $F \in H^p(D)$.

Acknowledgements. This paper was written during a visit of the first named author to the Department of Mathematics, SUNY at Albany, in the academic year 2008–2009. The first author is grateful to Professor Kehe Zhu for the kind invitation and for discussions about the article. This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00036).

REFERENCES

- [1] K. Adachi. H^p extensions of L^p holomorphic functions from submanifolds of strictly pseudoconvex domains with non-smooth boundary. Complex Analysis and its Applications (OCAMI Stud., 2). Osaka Munic. Univ. Press, Osaka, 2007, pp. 113–117.
- [2] K. Adachi. Several Complex Variables and Integral Formulas. World Scientific, Hackensack, NJ, 2007.
- [3] K. Adachi, M. Andersson and H. R. Cho. L^p and H^p extensions of holomorphic functions from subvarieties of analytic polyhedra. Pacific J. Math. **189**(2) (1999), 201–210.

- [4] E. Amar. Extension de fonctions holomorphes et courants. Bull. Sci. Math. 2^e serie **107** (1983), 25–48.
- [5] F. Beatrous. L^p estimates for extensions of holomorphic functions. Michigan Math. J. **32** (1985), 361–380.
- [6] F. Beatrous. Estimates for derivatives of holomorphic functions in pseudoconvex domains. Math. Z. **191** (1986), 91–116.
- [7] D. Catlin. Estimates of invariant metrics on pseudoconvex domains of dimension two. Math. Z. **200** (1989), 429–466.
- [8] A. Cumenge. Extension dans des classes de Hardy de fonctions holomorphes et estimation de type ‘mesures de Carleson’ pour l’équation $\bar{\partial}$. Ann. Inst. Fourier **33** (1983), 59–97.
- [9] K. Diederich and E. Mazzilli. Extension and restriction of holomorphic functions. Ann. Inst. Fourier (Grenoble) **47**(4) (1997), 1079–1099.
- [10] K. Diederich and E. Mazzilli. Extension of bounded holomorphic functions in convex domains. Manuscripta Math. **105** (2001), 1–12.
- [11] T. E. Hatziafratis. An explicit extension formula of bounded holomorphic functions from analytic varieties to strictly convex domains. J. Funct. Anal. **70** (1987), 289–303.
- [12] E. Mazzilli. Extension des fonctions holomorphes dans les pseudo-ellipsoïdes. Math. Z. **227** (1998), 607–622.
- [13] M. Range. Integral kernels and Hölder estimates for $\bar{\partial}$ on pseudoconvex domains of finite type in $\mathbb{C}^2$. Math. Ann. **288** (1990), 63–74

Hong Rae Cho
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
(Current address:

Department of Mathematics and Statistics
State University of New York at Albany
Albany, NY 12222, USA)
(E-mail: chohr@pusan.ac.kr)

Yeoung-Tae Seo
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
(E-mail: ytseo@naver.com)