POLYLOGARITHMS AND POLY-BERNOULLI POLYNOMIALS

Abdelmejid BAYAD and Yoshinori HAMAHATA

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Abstract. In this paper we investigate special generalized Bernoulli polynomials that generalize classical Bernoulli polynomials and numbers. We call them poly-Bernoulli polynomials. We prove a collection of extremely important and fundamental identities satisfied by our poly-Bernoulli polynomials and numbers. These properties are of arithmetical nature.

1. Introduction and main results

The classical polylogarithmic functions $\text{Li}_s(x)$ are

$$\text{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s}. \quad (1.1)$$

If s is a negative integer, say $s = -r$, then the polylogarithmic function converges for $|x| < 1$ and equals

$$\text{Li}_{-r}(x) = \frac{\sum_{j=0}^r \langle j \rangle^r x^{r-j}}{(1-x)^{r+1}}, \quad (1.2)$$

where the $\langle j \rangle^r$ are the Eulerian numbers. The Eulerian number $\langle j \rangle^r$ is the number of permutations of $\{1, \dots, r\}$ with j permutation ascents. One has

$$\langle j \rangle^r = \sum_{\ell=0}^{j+1} (-1)^\ell \binom{r+1}{\ell} (j-\ell+1)^r. \quad (1.3)$$

We record $\text{Li}_{-r}(x)$ for some r :

$$\begin{aligned} \text{Li}_0(x) &= \frac{x}{1-x}, & \text{Li}_{-1}(x) &= \frac{x}{(1-x)^2}, \\ \text{Li}_{-2}(x) &= \frac{x^2+x}{(1-x)^3}, & \text{Li}_{-3}(x) &= \frac{x^3+4x^2+x}{(1-x)^4}, \\ \text{Li}_{-4}(x) &= \frac{x^4+11x^3+11x^2+x}{(1-x)^5}, & \text{Li}_{-5}(x) &= \frac{x^5+26x^4+66x^3+26x^2+x}{(1-x)^6}. \end{aligned}$$

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From (1.2) we immediately deduce that, when k is a negative integer, $\text{Li}_k(x)$ is a rational function whose denominator is $(1-x)^{|k|+1}$. Precisely, for an integer $k \in \mathbb{Z}$, the formal power series $\text{Li}_k(x)$ is the k th polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. When $k = 1$,

$$\text{Li}_1(x) = -\log(1-x). \quad (1.4)$$

Using $\text{Li}_k(x)$, one can introduce (in the natural way) poly-Bernoulli numbers and polynomials.

For every integer k , we define a sequence of polynomials $(B_n^{(k)}(x))_{n \in \mathbb{N}}$, which we call *poly-Bernoulli polynomials*, by

$$\frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n \geq 0} B_n^{(k)}(x) \frac{t^n}{n!}. \quad (1.5)$$

We remark that $B_n^{(k)}(x)$ are defined in [3] by replacing e^{xt} by e^{-xt} in the left-hand side of (1.5). The numbers $B_n^{(k)} := B_n^{(k)}(0)$ are the poly-Bernoulli numbers. These numbers are introduced by Kaneko [6], and then investigated in [1, 2, 4, 5, 7–9].

On the other hand, from (1.4) and (1.5), it is easy to see that, for any $n \geq 0$,

$$(-1)^n B_n^{(1)}(-x) = B_n(x) \quad (1.6)$$

where $B_n(x)$ are the classical Bernoulli polynomials given by their generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \quad (1.7)$$

In this paper, we establish a series of properties, satisfied by our poly-Bernoulli polynomials and numbers, which are important and fundamental. These properties are of arithmetical nature.

THEOREM 1.1. (Explicit formula) *For $k \in \mathbb{Z}$, $n \geq 0$, we have*

$$B_n^{(k)}(x) = \sum_{m=0}^n \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} (x-j)^n. \quad (1.8)$$

THEOREM 1.2. (Recurrence formula 1) *For all $k \geq 1$, $n \geq 0$, we have*

$$B_n^{(k)}(x) = \sum_{m=0}^n (-1)^m \binom{n}{m} B_{n-m}^{(k-1)} \sum_{l=0}^m \frac{(-1)^l}{n-l+1} \binom{m}{l} B_l(x). \quad (1.9)$$

THEOREM 1.3. (Recurrence formula 2) *For $k \in \mathbb{Z}$, $n \geq 2$, we have*

$$B_n^{(k)}(x) = \frac{1}{n+1} \left\{ B_n^{(k-1)}(x) + x B_0^{(k)}(x) - \sum_{m=1}^{n-1} \left(\binom{n}{m-1} - \binom{n}{m} x \right) B_m^{(k)}(x) \right\}$$

and

$$B_0^{(k)}(x) = 1, \quad B_1^{(k)}(x) = \frac{1}{2} (B_1^{(k-1)}(x) + x B_0^{(k)}(x)).$$

THEOREM 1.4. (Appell sequence) *For $k \in \mathbb{Z}$, $n \geq 0$, we have*

$$\frac{d}{dx} B_{n+1}^{(k)}(x) = (n+1) B_n^{(k)}(x). \quad (1.10)$$

THEOREM 1.5. (Addition formula) *For $k \in \mathbb{Z}$, $n \geq 0$, we have*

$$B_n^{(k)}(x+y) = \sum_{m=0}^n \binom{n}{m} B_m^{(k)}(x) y^{n-m}. \quad (1.11)$$

For $m, n \geq 0$, set

$$C_n^{(-m)}(x, y) = \sum_{k=0}^m \binom{m}{k} B_n^{(-k)}(x) y^{m-k}.$$

Then we have the following result.

THEOREM 1.6. (Symmetric formula) *For m, n and $C_n^{(-m)}(x, y)$ as above, we have*

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n^{(-m)}(x, y) \frac{t^n}{n!} \frac{u^m}{m!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_m^{(-n)}(y, x) \frac{t^n}{n!} \frac{u^m}{m!} \\ &= \frac{e^{xt+yu} e^{t+u}}{e^t + e^u - e^{t+u}}. \end{aligned}$$

As a corollary, we have the following result.

THEOREM 1.7. (Duality) *We have*

$$C_n^{(-m)}(x, y) = C_m^{(-n)}(y, x). \quad (1.12)$$

This is a dual relation for addition formula. In particular, for $x = y = 0$, we obtain

$$B_n^{(-m)} = B_m^{(-n)}, \quad (1.13)$$

the duality property of poly-Bernoulli numbers.

THEOREM 1.8. (Inversion formula) *For $m, n \geq 0$, we have*

$$B_n^{(-m)}(x) = \sum_{k=0}^m \binom{m}{k} C_n^{(-k)}(x, y) (-y)^{m-k}. \quad (1.14)$$

THEOREM 1.9. (Closed formula) *For $m, n \geq 0$, we have*

$$C_n^{(-m)}(x, y) = \sum_{j=0}^{\infty} (j!)^2 \left(\sum_{a=0}^n (x+1)^{n-a} \binom{n}{a} \left\{ \begin{matrix} a \\ j \end{matrix} \right\} \right) \left(\sum_{b=0}^m (y+1)^{m-b} \binom{m}{b} \left\{ \begin{matrix} b \\ j \end{matrix} \right\} \right), \quad (1.15)$$

where

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{(-1)^m}{m!} \sum_{l=0}^m (-1)^l \binom{m}{l} l^n$$

are the so-called Stirling numbers of the second kind.

For $k \in \mathbb{Z}$ we set

$$Z_k(s, x) := \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{-xt} t^{s-1} dt,$$

the Laplace–Mellin integral. It is defined for $\text{Re}(s) > 0$ and $x > 0$ if $k \geq 1$, and for $\text{Re}(s) > 0$ and $x > |k| + 1$ if $k \leq 0$.

THEOREM 1.10. (Interpolation formula) *The function $s \rightarrow Z_k(s, x)$ has analytic continuation to an entire function on the whole complex s -plane and*

$$Z_k(-n, x) = (-1)^n B_n^{(k)}(-x) \quad (1.16)$$

holds for $n \geq 0$ and $x > 0$. Besides, this zeta function can be rewritten as follows. For $k \in \mathbb{Z}$, we have

$$Z_k(s, x) = \sum_{m \geq 0} \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{1}{(x+j)^s}. \quad (1.17)$$

In the case $k \leq 0$, we have a more precise expression for this zeta function. For $k \leq 0$,

$$Z_k(s, x) = \sum_{j=0}^{|k|} \left\langle \begin{matrix} |k| \\ j \end{matrix} \right\rangle \sum_{m=0}^{|k|-j} \frac{\binom{|k|-j}{m} (-1)^m}{(x+m-|k|+1)^s}. \quad (1.18)$$

It should be noted that the first result of the above theorem has been independently obtained by Coppo and Candelpergher [3] and the present authors.

Remark 1.11. We can make the following two remarks.

(1) In particular, for $k = 1$ we have

$$\begin{aligned} Z_1(s, x) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{-xt} t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t}{1 - e^{-t}} e^{-xt} t^{s-1} dt \\ &= \frac{\Gamma(s+1)}{\Gamma(s)} \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{1 - e^{-t}} e^{-xt} dt \\ &= s\zeta(s+1, x), \end{aligned} \quad (1.19)$$

where $\zeta(s, x)$ is the Hurwitz zeta function. Then, for $k = 1, s = -n$, we get from (1.6), (1.10) and (1.19) that

$$Z_1(-n, x) = (-1)^n B_n^{(1)}(-x) = B_n(x)$$

and

$$Z_1(-n, x) = -n\zeta(1-n, x).$$

Hence, we derive the famous classical interpolation relation:

$$B_n(x) = -n\zeta(1-n, x).$$

(2) For $k \leq 0$, from the formula (1.18) we derive another new explicit formula:

$$B_n^{(k)}(x) = \sum_{j=0}^{|k|} \left\langle \begin{matrix} |k| \\ j \end{matrix} \right\rangle \sum_{m=0}^{|k|-j} \binom{|k|-j}{m} (-1)^m (x+m-|k|+1)^n. \quad (1.20)$$

2. Proofs of main results

In this section we will give proofs of the theorems mentioned in the previous section.

Proof of Theorem 1.1. Since

$$\begin{aligned} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} &= \sum_{m=1}^{\infty} \frac{(1 - e^{-t})^{m-1}}{m^k} \\ &= \sum_{m=0}^{\infty} \frac{(1 - e^{-t})^m}{(m+1)^k} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} e^{-jt}, \end{aligned}$$

we have

$$\begin{aligned} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} e^{(x-j)t} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} \sum_{n=0}^{\infty} (x-j)^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} (x-j)^n \frac{t^n}{n!}. \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.2. Since

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \frac{e^t}{e^t - 1} \int_0^t e^{-s} \frac{\text{Li}_{k-1}(1 - e^{-s})}{1 - e^{-s}} ds,$$

we have

$$\begin{aligned} &\sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \\ &= \left(\sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!} \right) \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \right) \left(\sum_{n=0}^{\infty} B_n^{(k-1)} \frac{s^n}{n!} \right) ds \\ &= \left(\sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!} \right) \int_0^t \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^{n-m} \binom{n}{m} B_m^{(k-1)} \frac{s^n}{n!} ds \\ &= \left(\sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!} \right) \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^{n-m} \binom{n}{m} B_m^{(k-1)} \frac{t^{n+1}}{(n+1)!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n B_{n-l}(x) \sum_{m=0}^l (-1)^{l-m} \binom{l}{m} B_m^{(k-1)} \frac{t^n}{(l+1)!(n-l)!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \frac{B_{n-l}(x)}{l+1} \binom{n}{l} \sum_{m=0}^l (-1)^{l-m} \binom{l}{m} B_m^{(k-1)} \frac{t^n}{n!} \\ &\quad \text{by } \binom{n}{l} \binom{l}{m} = \binom{n}{m} \binom{n-m}{n-l} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{m=0}^n B_m^{(k-1)} \binom{n}{m} \sum_{l=m}^n \frac{(-1)^{l-m}}{l+1} \binom{n-m}{n-l} B_{n-l}(x) \frac{t^n}{n!} \\
&\quad \text{put } l' = n - l \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^n B_m^{(k-1)} \binom{n}{m} \sum_{l'=0}^{n-m} \frac{(-1)^{n-l'-m}}{n-l'+1} \binom{n-m}{l'} B_{l'}(x) \frac{t^n}{n!} \\
&\quad \text{put } m' = n - m \\
&= \sum_{n=0}^{\infty} \sum_{m'=0}^n B_{n-m'}^{(k-1)} \binom{n}{m'} \sum_{l'=0}^{m'} \frac{(-1)^{m'+l'}}{n-l'+1} \binom{m'}{l'} B_{l'}(x) \frac{t^n}{n!}.
\end{aligned}$$

The theorem therefore follows. $\square$

Proof of Theorem 1.3. Put $z = 1 - e^{-t}$. Then $t = \text{Li}_1(z)$. From (1.5), we have

$$\text{Li}_k(z) e^{x \text{Li}_1(z)} = z \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{\text{Li}_1(z)^n}{n!}. \quad (2.1)$$

Note that

$$\frac{d}{dz} \text{Li}_k(z) = \begin{cases} (1/z) \text{Li}_{k-1}(z) & (k > 1), \\ 1/(1-z) & (k = 1). \end{cases}$$

Differentiating (2.1) with respect to z , we have

$$\begin{aligned}
&\frac{1}{z} \text{Li}_{k-1}(z) e^{x \text{Li}_1(z)} + \text{Li}_k(z) \frac{x}{1-z} e^{x \text{Li}_1(z)} \\
&= \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{\text{Li}_1(z)^n}{n!} + \frac{z}{1-z} \sum_{n=0}^{\infty} B_{n+1}^{(k)}(x) \frac{\text{Li}_1(z)^n}{n!}.
\end{aligned}$$

Using $z/(1-z) = e^t - 1$, we obtain

$$\sum_{n=0}^{\infty} B_n^{(k-1)}(x) \frac{t^n}{n!} + x(e^t - 1) \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} + (e^t - 1) \sum_{n=0}^{\infty} B_{n+1}^{(k)}(x) \frac{t^n}{n!}.$$

This yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_n^{(k-1)}(x) \frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} x B_m^{(k)}(x) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n+1}^{(k)}(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} B_{m+1}^{(k)}(x).
\end{aligned}$$

Comparing both sides of this identity gives

$$B_n^{(k-1)}(x) - x B_n^{(k)}(x) + \sum_{m=0}^n \binom{n}{m} x B_m^{(k)}(x) = B_n^{(k)}(x) - B_{n+1}^{(k)} + \sum_{m=0}^n \binom{n}{m} B_{m+1}^{(k)}(x).$$

From this, we obtain

$$(n+1) B_n^{(k)}(x) = B_n^{(k-1)}(x) + \sum_{m=0}^{n-1} x \binom{n}{m} B_m^{(k)}(x) - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k)}(x),$$

which leads to the result. $\square$

Proof of Theorem 1.4. Differentiation of both sides of (1.5) with respect to x gives

$$\frac{t \operatorname{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=1}^{\infty} \frac{d}{dx} B_n^{(k)}(x) \frac{t^n}{n!}.$$

From this, we have

$$\frac{\operatorname{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} (n+1) \frac{d}{dx} B_n^{(k)}(x) \frac{t^n}{n!}.$$

This completes the proof. $\square$

Proof of Theorem 1.5. From the definition in (1.5), we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(k)}(x+y) \frac{t^n}{n!} &= \frac{\operatorname{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{(x+y)t} \\ &= \frac{\operatorname{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} e^{yt} \\ &= \left(\sum_{m=0}^{\infty} B_m^{(k)}(x) \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} \frac{y^l t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} B_m^{(k)}(x) y^{n-m} \right) \frac{t^n}{n!}. \end{aligned}$$

The proof follows from this. $\square$

Proof of Theorem 1.6. Using the definition of $C_n^{(-m)}(x, y)$, the left-hand side can be rewritten as

$$\begin{aligned} \text{LHS} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^m B_n^{(-k)}(x) y^{m-k} \frac{t^n}{n!} \frac{u^m}{k!(m-k)!} \\ &\quad \text{put } l = m - k \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} B_n^{(-k)}(x) y^l \frac{t^n}{n!} \frac{u^k}{k!} \frac{u^l}{l!} \\ &= e^{yu} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)}(x) \frac{t^n}{n!} \frac{u^k}{k!} \\ &\quad \text{by definition of } B_n^{(-k)}(x) \\ &= e^{yu} \sum_{k=0}^{\infty} \left(e^{xt} \sum_{n=0}^{\infty} B_n^{(-k)} \frac{t^n}{n!} \right) \frac{u^k}{k!} \\ &= e^{xt+yu} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k)} \frac{t^n}{n!} \frac{u^k}{k!}. \end{aligned}$$

Here, using the expression

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k)} \frac{t^n}{n!} \frac{u^k}{k!} = \frac{e^{t+u}}{e^t + e^u - e^{t+u}}$$

in Kaneko [6], the proof is completed. $\square$

Proof of Theorem 1.8. To prove the theorem, we calculate the generating function $B_n^{(-m)}$.

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{k=0}^m \binom{m}{k} C_n^{(-k)}(x, y) (-y)^{m-k} \right) \frac{t^n}{n!} \frac{u^m}{m!} \\
& \quad \text{put } j = m - k \\
& = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} C_n^{(-k)}(x, y) \frac{(-yu)^j}{j!} \frac{t^n}{n!} \frac{u^k}{k!} \\
& = e^{-yu} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_n^{(-k)}(x, y) \frac{t^n}{n!} \frac{u^k}{k!} \\
& \quad \text{by Theorem 1.6} \\
& = \frac{e^{xt+t+u}}{e^t + e^u - e^{t+u}} \\
& = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_n^{(-m)}(x) \frac{t^n}{n!} \frac{u^m}{m!}.
\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.9. Using Theorem 1.6, we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n^{(-m)}(x, y) \frac{t^n}{n!} \frac{u^m}{m!} \\
& = \frac{e^{xt+yu+t+u}}{e^t + e^u - e^{t+u}} \\
& = \frac{e^{(x+1)t+(y+1)u}}{1 - (e^t - 1)(e^u - 1)} \\
& = e^{(x+1)t+(y+1)u} \sum_{j=0}^{\infty} (e^t - 1)^j (e^u - 1)^j \\
& = \sum_{j=0}^{\infty} e^{(x+1)t} (e^t - 1)^j e^{(y+1)u} (e^u - 1)^j \\
& \quad \text{by } \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{u^n}{n!} = \frac{(e^u - 1)^k}{k} \\
& = \sum_{j=0}^{\infty} \left(j! \sum_{n=0}^{\infty} \frac{(x+1)^n t^n}{n!} \sum_{m=0}^{\infty} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \frac{t^m}{m!} \right) \left(j! \sum_{n=0}^{\infty} \frac{(y+1)^n u^n}{n!} \sum_{m=0}^{\infty} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \frac{u^m}{m!} \right) \\
& = \sum_{j=0}^{\infty} \left(j! \sum_{l=0}^{\infty} \sum_{m=0}^l (x+1)^{l-m} \binom{l}{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \frac{t^l}{l!} \right) \left(j! \sum_{p=0}^{\infty} \sum_{r=0}^p (y+1)^{p-r} \binom{p}{r} \left\{ \begin{matrix} p \\ j \end{matrix} \right\} \frac{u^p}{p!} \right) \\
& = \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \frac{t^l}{l!} \frac{u^p}{p!} \sum_{j=0}^{\infty} (j!)^2 \left(\sum_{m=0}^l (x+1)^{l-m} \binom{l}{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \right) \left(\sum_{r=0}^p (y+1)^{p-r} \binom{p}{r} \left\{ \begin{matrix} p \\ j \end{matrix} \right\} \right),
\end{aligned}$$

which proves the theorem. $\square$

Proof of Theorem 1.10. Since the proof of the first part of the theorem is the same as that of Theorem 2 of Coppo and Candelpergher [3], we omit it.

We now prove (1.17) of Theorem 1.10. We have

$$Z_k(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n=1}^\infty \frac{(1 - e^{-t})^{n-1}}{n^k} e^{-xt} t^{s-1} dt.$$

For $k \geq 1$, it is easy to see that, for all $N \geq 1$ and $t > 0$,

$$\begin{aligned} \sum_{n=N}^\infty \frac{(1 - e^{-t})^{n-1}}{n^k} &\leq \sum_{n=N}^\infty \frac{(1 - e^{-t})^{n-1}}{n} \\ &\leq \sum_{n=1}^\infty \frac{(1 - e^{-t})^{n-1}}{n} \\ &= \frac{t}{1 - e^{-t}}, \end{aligned}$$

which ensures the convergence of the integral absolutely. Then we can invert $\int$ and $\sum$. Hence

$$\begin{aligned} Z_k(s, x) &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \frac{1}{n^k} \int_0^\infty (1 - e^{-t})^{n-1} e^{-xt} t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \frac{1}{(n+1)^k} \int_0^\infty \sum_{j=0}^n \binom{n}{j} (-1)^j e^{-(x+j)t} t^s dt \\ &= \sum_{n=0}^\infty \frac{1}{(n+1)^k} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{\Gamma(s)} \int_0^\infty e^{-(x+j)t} t^{s-1} dt. \end{aligned}$$

We thus obtain (1.17).

We prove the case $k \leq 0$. Since

$$\text{Li}_k(x) = \frac{1}{(1-x)^{|k|+1}} \sum_{j=0}^{|k|} \left\langle \begin{matrix} |k| \\ j \end{matrix} \right\rangle x^{|k|-j},$$

we have

$$\begin{aligned} Z_k(s, x) &= \frac{1}{\Gamma(s)} \sum_{j=0}^{|k|} \int_0^\infty \left\langle \begin{matrix} |k| \\ j \end{matrix} \right\rangle (1 - e^{-t})^{|k|-j} e^{(|k|+1)t} e^{-xt} t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \sum_{j=0}^{|k|} \left\langle \begin{matrix} |k| \\ j \end{matrix} \right\rangle \sum_{m=0}^{|k|-j} (-1)^m \binom{|k|-j}{m} \int_0^\infty e^{-(x+m-|k|+1)t} t^{s-1} dt. \end{aligned}$$

This yields (1.18) for all $k \leq 0$, $\text{Re}(x) > |k| + 1$. □

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Abdelmejid Bayad
Département de Mathématiques
Université d'Évry Val d'Essone
Boulevard F. Mitterrand
91025 Évry Cedex
France
(E-mail: abayad@maths.univ-evry.fr)

Yoshinori Hamahata
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo
153-8914 Japan
(E-mail: hamahata@ms.u-tokyo.ac.jp)