SCHWARZ IMAGES OF ALGEBRAIC ${}_3F_2$

Mitsuo KATO

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Abstract. There are infinitely many generalized hypergeometric differential equations ${}_3E_2(a_0, a_1, a_2; b_1, b_2; x)$ having the same finite irreducible primitive monodromy group. Their Schwarz images are different plane curves. In this paper, we study the degrees of these curves. We also give the defining equations of Schwarz images of the minimal degrees.

1. Introduction

The generalized hypergeometric function ${}_{n+1}F_n$ defined by

$${}_{n+1}F_n(a_0, a_1, \dots, a_n; b_1, \dots, b_n; x) = \sum_{m=0}^{\infty} \frac{\prod_{j=0}^n (a_j, m)}{\prod_{j=1}^n (b_j, m)} \frac{z^m}{(1, m)}, \quad (a, m) = \frac{\Gamma(a+m)}{\Gamma(a)}$$

satisfies a Fuchsian differential equation

$${}_{n+1}E_n(x) = {}_{n+1}E_n(a; b; x) = {}_{n+1}E_n(a_0, a_1, \dots, a_n; b_1, \dots, b_n; x)$$

of rank $n+1$ with singularities at $x = 0, 1, \infty$. The local exponents of ${}_{n+1}E_n(x)$ at these singular points are

$$\begin{array}{ll} 0, 1-b_1, 1-b_2, \dots, 1-b_n & \text{at } x=0, \\ 0, 1, \dots, n-1, \sum b_j - \sum a_j & \text{at } x=1, \\ a_0, a_1, \dots, a_n & \text{at } x=\infty. \end{array}$$

If the monodromy group of ${}_{n+1}E_n$ is finite, then the image of a Schwarz map defined by the ratio of linearly independent $n+1$ solutions is an algebraic curve in $\mathbf{P}^n$.

Assume ${}_3E_2(a; b; x) = {}_3E_2(a_0, a_1, a_2; b_1, b_2; x)$ has finite irreducible monodromy group. Note that a_j and b_j are rational numbers in this case. Then the projective monodromy group does not change under the following three transformations of parameters a_j, b_j :

Transformation 1. (Contiguous transformations.) Adding integers to each a_j and b_j .

Transformation 2. Multiplying all a_j and b_j by a non-zero integer which is relatively prime to all denominators of a_j and b_j .

Transformation 3. (Projective transformations.) Replacing a_j and b_j by a'_j and b'_j , respectively, when ${}_3E_2(a_0, a_1, a_2; b_1, b_2)$ is projectively equivalent to ${}_3E_2(a'_0, a'_1, a'_2; b'_1, b'_2)$; that is, when the equality ${}_3E_2(a_0, a_1, a_2; b_1, b_2; x) = x^r {}_3E_2(a'_0, a'_1, a'_2; b'_1, b'_2; \sigma(x))$ holds, where $\sigma(x) = x$ or $1/x$ and r is a rational number.

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Beukers and Heckman [BH] proved that all ${}_3E_2$ with finite irreducible primitive monodromy groups are classified into eleven classes by the three transformations above; that is, the class of Nos 2–12 of Table 8.3 in [BH]. Transformations 1–3 generate a transitive permutation group acting on the set of ${}_3E_2$ in the class of No. k for each k .

For $2 \leq k \leq 12$, we denote by $\mathcal{E}_k$ the set of all ${}_3E_2$ in the class of No. k whose Schwarz map has a single-valued inverse map. For example, $E(x) = {}_3E_2(-\frac{1}{14}, \frac{1}{14}, \frac{9}{14}; \frac{6}{7}, \frac{2}{7}; x)$ is in the class of No. 4. But, since it satisfies the equality

$$E(x) = (1-x)^{1/14} {}_3E_2\left(-\frac{1}{28}, \frac{13}{28}, \frac{3}{14}; \frac{6}{7}, \frac{2}{7}; -\frac{4x}{(1-x)^2}\right),$$

$E(x)$ is not in $\mathcal{E}_4$. For ${}_3E_2$ in $\mathcal{E}_k$, there is a simple formula to give the degree n of the Schwarz image of ${}_3E_2$ (see equation (2.1) in Lemma 2.2). But, in general, it is not so easy to determine whether an equation ${}_3E_2$ in the class of No. k is in $\mathcal{E}_k$ or not.

If $E(x)$ and $E'(x)$ are in $\mathcal{E}_k$ for some k , then the Schwarz images of $E(x)$ and $E'(x)$ are birational (Lemma 2.13), and hence have the same genus, for example. But the degrees of these two curves are not equal in general. For $2 \leq k \leq 12$, we denote by $\mathcal{N}_k$ the set of degrees of the Schwarz images of ${}_3E_2$ in $\mathcal{E}_k$ for $2 \leq k \leq 12$:

$$\mathcal{N}_k = \{\text{the degree of the Schwarz image of } {}_3E_2 \mid {}_3E_2 \in \mathcal{E}_k\}.$$

In Table 1 in [Kt2], we gave the minimal value of $\mathcal{N}_k$, and the corresponding ${}_3E_2 \in \mathcal{E}_k$.

One of the main results of this paper is the following table of $\mathcal{N}_k$ (Table 1). Every ${}_3E_2$ in the class of No. k is obtained from any one in the class by a finite combination of Transformations 1–3 and so we give a representative ${}_3E_2$ in the table. We also give the order g' of the projective monodromy group of ${}_3E_2$ of No. k for each k because it plays an important role in determining the set $\mathcal{N}_k$ (see equation (2.1)). Derivation of this table is given in Sections 2–4.

In Section 5, we give another main result. That is, for k with $2 \leq k \leq 11$, we give the concrete forms of the defining equations of Schwarz images of ${}_3E_2 \in \mathcal{E}_k$ giving the minimal value of $\mathcal{N}_k$. The forms for $k = 12$ were given in [Kt4].

2. Preliminary lemmas

In this section we give several lemmas. In Table 1, we gave a representative set of parameters of ${}_3E_2$ for each class of Nos 2–12. All other ${}_3E_2$ are derived from the representative one by Transformations 1–3. In order to find the set $\mathcal{N}_k$ for each class of No. k , we give the complete list of the set of parameters of ${}_3E_2$ up to Transformation 3.

LEMMA 2.1. *All the parameters $(a_0, a_1, a_2; b_1, b_2)$ of ${}_3E_2$ in the class of Nos 2–12 are derived from one in Table 2 by a projective transformation, where l_0, l_1, l_2, m_1, m_2 run over all integers.*

Proof: Case No. 2. Put $E_1 = {}_3E_2(-\frac{3}{14}, \frac{1}{14}, \frac{9}{14}; \frac{2}{3}, \frac{1}{3})$ and $E_2 = {}_3E_2(-\frac{1}{14}, \frac{3}{14}, \frac{5}{14}; \frac{2}{3}, \frac{1}{3})$. Note E_1 is equal to the representative ${}_3E_2$ of No. 2 in Table 1 up to contiguous transformations, Transformation 1. By the multiplication $a_i \rightarrow 5a_i, b_i \rightarrow 5b_i$ applied to E_1 , we get E_2 up to Transformation 1. By the multiplication $a_i \rightarrow 11a_i, b_i \rightarrow 11b_i$, we get again

TABLE 1. The degrees of the Schwarz images.

No.	a_0	a_1	a_2	b_1	b_2	g'	$\mathcal{N}_k$
2	$\frac{9}{14}$	$\frac{15}{14}$	$\frac{11}{14}$	$\frac{4}{3}$	$\frac{5}{3}$	168	$\{\frac{168}{42}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{7}, j \neq 2, 4, 6\}$
3	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{5}{4}$	$\frac{3}{4}$	168	$\{\frac{168}{28}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{7}, j \neq 4\}$
4	$\frac{-3}{14}$	$\frac{-1}{14}$	$\frac{3}{14}$	$\frac{4}{7}$	$\frac{-1}{7}$	168	$\{\frac{168}{14}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{7}, j \neq 1, 2\}$
5	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{7}{10}$	$\frac{4}{3}$	$\frac{2}{3}$	60	$\{\frac{60}{30}j \mid j \in \mathbb{N}, j \neq 2, 4, 7\}$
6	$\frac{5}{10}$	$\frac{7}{10}$	$\frac{13}{10}$	$\frac{8}{5}$	$\frac{7}{5}$	60	$\{\frac{60}{10}j \mid j \in \mathbb{N}\}$
7	$\frac{-1}{30}$	$\frac{5}{30}$	$\frac{11}{30}$	$\frac{4}{5}$	$\frac{1}{5}$	360	$\{\frac{360}{30}j \mid j \in \mathbb{N}, j \equiv 1 \pmod{3}, j \neq 4\}$
8	$\frac{7}{30}$	$\frac{13}{30}$	$\frac{25}{30}$	$\frac{3}{4}$	$\frac{5}{4}$	360	$\{\frac{360}{60}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{3}, j \neq 4, 8\}$
9	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{5}{4}$	$\frac{3}{4}$	216	$\{\frac{216}{12}j \mid j \in \mathbb{N}\}$
10	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{7}{9}$	$\frac{7}{6}$	$\frac{3}{6}$	216	$\{\frac{216}{18}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{3}\}$
11	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{8}{9}$	$\frac{5}{4}$	$\frac{3}{4}$	216	$\{\frac{216}{36}j \mid j \in \mathbb{N}, j \equiv \pm 1, \pm 4, \pm 5 \pmod{12}\}$
12	$\frac{-1}{12}$	$\frac{2}{12}$	$\frac{5}{12}$	$\frac{3}{4}$	$\frac{1}{4}$	36	$\{\frac{36}{12}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{3}\}$

No. of the registration number in Table 8.3 of [BH]; a_j, b_j are parameters of a representative ${}_3E_2$; g' is the order of the projective monodromy group.

TABLE 2. Parameters up to projective transformations.

No.	
2	$(-\frac{3}{14} + l_0, \frac{1}{14} + l_1, \frac{9}{14} + l_2; \frac{2}{3} - m_1, \frac{1}{3} - m_2), (-\frac{1}{14} + l_0, \frac{3}{14} + l_1, \frac{5}{14} + l_2; \frac{2}{3} - m_1, \frac{1}{3} - m_2)$
3	$(-\frac{3}{14} + l_0, \frac{1}{14} + l_1, \frac{9}{14} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2), (-\frac{1}{14} + l_0, \frac{3}{14} + l_1, \frac{5}{14} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2)$
4	$(-\frac{1}{14} + l_0, \frac{1}{14} + l_1, \frac{9}{14} + l_2; \frac{6}{7} - m_1, \frac{2}{7} - m_2), (-\frac{1}{14} + l_0, \frac{3}{14} + l_1, \frac{11}{14} + l_2; \frac{6}{7} - m_1, \frac{4}{7} - m_2)$
5	$(-\frac{1}{10} + l_0, \frac{1}{10} + l_1, \frac{1}{2} + l_2; \frac{2}{5} - m_1, \frac{1}{5} - m_2), (-\frac{3}{10} + l_0, \frac{3}{10} + l_1, \frac{1}{2} + l_2; \frac{2}{5} - m_1, \frac{1}{5} - m_2)$
6	$(-\frac{3}{10} + l_0, \frac{3}{10} + l_1, \frac{1}{2} + l_2; \frac{3}{5} - m_1, \frac{2}{5} - m_2)$
7	$(-\frac{1}{30} + l_0, \frac{1}{6} + l_1, \frac{11}{30} + l_2; \frac{4}{5} - m_1, \frac{1}{5} - m_2), (-\frac{7}{30} + l_0, \frac{1}{6} + l_1, \frac{17}{30} + l_2; \frac{3}{5} - m_1, \frac{2}{5} - m_2)$
8	$(-\frac{1}{30} + l_0, \frac{1}{6} + l_1, \frac{11}{30} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2), (-\frac{7}{30} + l_0, \frac{1}{6} + l_1, \frac{17}{30} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2),$ $(-\frac{1}{6} + l_0, \frac{7}{30} + l_1, \frac{13}{30} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2), (-\frac{1}{6} + l_0, \frac{1}{30} + l_1, \frac{19}{30} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2)$
9	$(-\frac{1}{6} + l_0, \frac{1}{6} + l_1, \frac{2}{3} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2), (-\frac{1}{6} + l_0, \frac{1}{6} + l_1, \frac{1}{3} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2)$
10	$(-\frac{1}{9} + l_0, \frac{2}{9} + l_1, \frac{5}{9} + l_2; \frac{5}{6} - m_1, \frac{1}{2} - m_2), (-\frac{2}{9} + l_0, \frac{1}{9} + l_1, \frac{4}{9} + l_2; \frac{1}{2} - m_1, \frac{1}{6} - m_2)$
11	$(-\frac{1}{9} + l_0, \frac{2}{9} + l_1, \frac{5}{9} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2), (-\frac{2}{9} + l_0, \frac{1}{9} + l_1, \frac{4}{9} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2)$
12	$(-\frac{1}{12} + l_0, \frac{1}{6} + l_1, \frac{5}{12} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2), (-\frac{1}{6} + l_0, \frac{1}{12} + l_1, \frac{7}{12} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2)$

E_1 up to Transformation 1. By the multiplication $a_i \rightarrow 13a_i$, $b_i \rightarrow 13b_i$ applied to E_1 , we get E_2 up to Transformation 1. This proves that all ${}_3E_2$ in the class of No. 2 are obtained from E_1 and E_2 by Transformation 1 and Transformation 3. This proves the lemma for the case of No. 2. The proofs for Nos 3–12 are completely analogous. $\square$

We proved the following lemma [Kt2, Theorem 3.6] which gives the degree of a Schwarz image.

LEMMA 2.2. *Assume that $E(x) = {}_3E_2(a; b; x)$ has a finite irreducible monodromy group and that its Schwarz map has a single-valued inverse map. Let e_α be the minimal local exponent of $E(x)$ at α ($\alpha \in \{0, 1, \infty\}$) and let g' be the order of the projective monodromy group of $E(x)$. Then the degree n of the Schwarz image of $E(x)$ is given by*

$$n = -g'(e_0 + e_1 + e_\infty). \quad (2.1)$$

Let $e_\alpha(a; b; l; m)$ be the minimal local exponent of

$${}_3E_2(a + l; b - m; x) = {}_3E_2(a_0 + l_0, a_1 + l_1, a_2 + l_2; b_1 - m_1, b_2 - m_2; x)$$

at $x = \alpha$ ($\alpha \in \{0, 1, \infty\}$). Let

$$\mathcal{Z} = \{(l; m) = (l_0, l_1, l_2; m_1, m_2) \mid l_i, m_i \in \mathbb{Z}\} \simeq \mathbb{Z}^5,$$

and put

$$m_0 = 0, \quad s(l; m) = \sum (l_j + m_j).$$

We put

$$\mathcal{Z}_0^{(l)} = \{(l; m) \in \mathcal{Z} \mid l_0 \leq l_1, l_2\}, \quad \mathcal{Z}_1^{(l)} = \{(l; m) \in \mathcal{Z} \mid l_1 \leq l_0 - 1, l_2\},$$

$$\mathcal{Z}_2^{(l)} = \{(l; m) \in \mathcal{Z} \mid l_2 \leq l_0 - 1, l_1 - 1\},$$

$$\mathcal{Z}_0^{(m)} = \{(l; m) \in \mathcal{Z} \mid m_0 \leq m_1, m_2\}, \quad \mathcal{Z}_1^{(m)} = \{(l; m) \in \mathcal{Z} \mid m_1 \leq m_0 - 1, m_2\},$$

$$\mathcal{Z}_2^{(m)} = \{(l; m) \in \mathcal{Z} \mid m_2 \leq m_0 - 1, m_1 - 1\},$$

$$\mathcal{Z}_0^{(s)} = \{(l; m) \in \mathcal{Z} \mid s(l; m) \leq 0\}, \quad \mathcal{Z}_2^{(s)} = \{(l; m) \in \mathcal{Z} \mid s(l; m) \geq 1\},$$

and

$$\mathcal{Z}_{p,q,r} = \mathcal{Z}_p^{(l)} \cap \mathcal{Z}_q^{(m)} \cap \mathcal{Z}_r^{(s)},$$

for p, q, r with $0 \leq p, q \leq 2$, $r \in \{0, 2\}$. Note that $\mathcal{Z}$ is the disjoint union of these $\mathcal{Z}_{p,q,r}$. For p, q, r with $0 \leq p, q \leq 2$, $r \in \{0, 2\}$, put

$$f_{p,q,r}(l; m) = \begin{cases} l_p + m_q & \text{if } r = 0, \\ l_p + m_q - s(l; m) & \text{if } r = 2. \end{cases}$$

Put $b_0 = 1$ and put

$$e_{p,q,r}(a; b; l; m) = \begin{cases} a_p + (1 - b_q) + f_{p,q,0}(l; m) & \text{if } r = 0, \\ a_p + (1 - b_q) + \sum_{j=1}^2 b_j - \sum_{j=0}^2 a_j + f_{p,q,2}(l; m) & \text{if } r = 2. \end{cases} \quad (2.2)$$

From the definition of $e_{p,q,r}$, we get the following lemma immediately.

LEMMA 2.3. *The equality $e_{p,q,r}(a; b; l; m) = e_{p,q,r}(a; b; l'; m')$ holds if and only if $f_{p,q,r}(l; m) = f_{p,q,r}(l'; m')$.*

LEMMA 2.4. *Assume that the parameters of ${}_3E_2(a; b; x) = {}_3E_2(a_0, a_1, a_2; b_1, b_2; x)$ satisfy the inequalities*

$$a_0 < a_1 < a_2 < a_0 + 1, \quad 0 < 1 - b_1 < 1 - b_2 < 1, \quad 0 < \sum b_j - \sum a_j < 1. \quad (2.3)$$

If $(l; m) = (l_0, l_1, l_2; m_1, m_2) \in \mathcal{Z}_{p,q,r}$, then

$$e_0(a; b; l; m) + e_1(a; b; l; m) + e_\infty(a; b; l; m) = e_{p,q,r}(a; b; l; m).$$

Proof. Since the local exponents of ${}_3E_2(a + l; b - m; x)$ are

$$\begin{array}{ll} 0, 1 - b_1 + m_1, 1 - b_2 + m_2 & \text{at } x = 0, \\ 0, 1, \sum b_j - \sum a_j - s(l; m) & \text{at } x = 1, \\ a_0 + l_0, a_1 + l_1, a_2 + l_2 & \text{at } x = \infty, \end{array}$$

the lemma holds. $\square$

Note that all the equations ${}_3E_2$ with $l_i = 0, m_i = 0$ in Table 2 satisfy the inequalities in (2.3).

Let $\mathcal{R}_{p,q,r}$ be the closed set in $\mathbb{R}^5 = \{(l; m) = (l_0, l_1, l_2; m_1, m_2) \mid l_i, m_i \in \mathbb{R}\}$ defined by the same inequalities as $\mathcal{Z}_{p,q,r}$. For example, $\mathcal{R}_{0,0,0} = \{(l; m) \in \mathbb{R}^5 \mid l_0 \leq l_1, l_2; m_1, m_2 \geq 0; s(l; m) \leq 0\}$. By definition, $\mathcal{R}_{p,q,r}$ is a closed set bonded by five hyperplanes, which we call walls of $\mathcal{R}_{p,q,r}$. As is verified easily, these walls intersect transversally, and hence have only one common point. We call it the vertex of $\mathcal{R}_{p,q,r}$. Thus we may call $\mathcal{R}_{p,q,r}$ a closed cone in $\mathbb{R}^5$. The following lemma follows from direct calculations.

LEMMA 2.5. *For each p, q, r , the vertex of the cone $\mathcal{R}_{p,q,r}$ is in $\mathcal{Z}$ if and only if $p + q + r \equiv 0 \pmod{3}$; that is, (p, q, r) is one of*

$$(0, 0, 0), (2, 1, 0), (1, 2, 0), (1, 0, 2), (0, 1, 2), (2, 2, 2).$$

The corresponding vertexes are

$$\begin{aligned} &(0, 0, 0; 0, 0), (1, 1, 0; -1, -1), (1, 0, 0; 0, -1), (1, 0, 0; 0, 0), (1, 1, 1; -1, -1), \\ &(1, 1, 0; 0, -1). \end{aligned}$$

LEMMA 2.6. (1) *For each p, q, r , the maximal value $\max f_{p,q,r}$ of the function $f_{p,q,r}(l; m)$ on $\mathcal{Z}_{p,q,r}$ is given by*

$$\max f_{p,q,r} = [-(p + q + r)/3],$$

where $[a]$ denotes the biggest integer n satisfying $n \leq a$.

(2) *If the vertex of $\mathcal{R}_{p,q,r}$ is in $\mathcal{Z}$, $f_{p,q,r}(l; m)$ takes the maximal value $\max f_{p,q,r}$ only at the vertex.*

Proof. Let $(l; m) = (l_0, l_1, l_2; m_1, m_2)$ be a point in $\mathcal{Z}_{p,q,r}$. Recall that $m_0 = 0$ and $s(l, m) = \sum(l_j + m_j)$.

Assume $r = 0$. Then

$$f_{p,q,r}(l; m) = (l_p + m_q) \leq \frac{1}{3}(3(l_p + m_q) - s(l; m)) \leq \frac{1}{3}(-p - q) = -\frac{1}{3}(p + q + r),$$

TABLE 3. Representative points in $\mathcal{Z}_{p,q,r} \cap f_{p,q,r}^{-1}(j)$.

(p, q, r)	$m_{p,q,r}$	$\mathcal{Z}_{p,q,r} \cap f_{p,q,r}^{-1}(j), j \leq m_{p,q,r}$	Vertex
$(0, 0, 0)$	0	$(j, 0, -j; 0, 0), j \leq 0$	$(0, 0, 0; 0, 0)$
$(1, 0, 0)$	-1	$(0, j, -j; 0, 0), j \leq -1$	
$(2, 0, 0)$	-1	$(-j, 0, j; 0, 0), j \leq -1$	
$(0, 1, 0)$	-1	$(j+1, 0, -j; -1, 0), j \leq -1$	
$(1, 1, 0)$	-1	$(1, j+1, -j; -1, -1), j \leq -1$	
$(2, 1, 0)$	-1	$(1, -j, j+1; -1, -1), j \leq -1$	$(1, 1, 0; -1, -1)$
$(0, 2, 0)$	-1	$(j+1, 0, -j; 0, -1), j \leq -1$	
$(1, 2, 0)$	-1	$(-2j-1, j+1, j+1; 0, -1), j \leq -1$	$(1, 0, 0; 0, -1)$
$(2, 2, 0)$	-2	$(0, -j, j+1; 0, -1), j \leq -2$	
$(0, 0, 2)$	-1	$(j+1, 0, -j; 0, 0), j \leq -1$	
$(1, 0, 2)$	-1	$(-j, j+1, 0; 0, 0), j \leq -1$	$(1, 0, 0; 0, 0)$
$(2, 0, 2)$	-2	$(0, -j, j+1; 0, 0), j \leq -2$	
$(0, 1, 2)$	-1	$(j+2, 1, -j; -1, -1), j \leq -1$	$(1, 1, 1; -1, -1)$
$(1, 1, 2)$	-2	$(1, j+2, -j; -1, -1), j \leq -2$	
$(2, 1, 2)$	-2	$(-j, 1, j+2; -1, -1), j \leq -2$	
$(0, 2, 2)$	-2	$(j+2, 0, -j; 0, -1), j \leq -2$	
$(1, 2, 2)$	-2	$(1, j+2, -j-1; 0, -1), j \leq -2$	
$(2, 2, 2)$	-2	$(-j-1, 1, j+2; 0, -1), j \leq -2$	$(1, 1, 0; 0, -1)$

where the equality holds if and only if the point $(l; m)$ is the vertex of the cone $\mathcal{R}_{p,q,r}$.

Assume $r = 2$. Then

$$\begin{aligned} f_{p,q,r}(l; m) &= (l_p + m_q) - s(l; m) \leq \frac{1}{3}(3(l_p + m_q) - s(l; m) - 2s(l; m)) \\ &\leq \frac{1}{3}(-p - q - 2) = -\frac{1}{3}(p + q + r), \end{aligned}$$

where the equality holds if and only if the point $(l; m)$ is the vertex of the cone $\mathcal{R}_{p,q,r}$. This proves the lemma. $\square$

Table 3 implies that the set $\mathcal{Z}_{p,q,r} \cap f_{p,q,r}^{-1}(j)$ is not empty for any p, q, r and any $j \leq [-(p + q + r)/3]$. In the table, $m_{p,q,r}$ denotes $\max f_{p,q,r}$ on $\mathcal{Z}_{p,q,r}$, and the point $(l_0, l_1, l_2; m_1, m_2)$ in the third column is a representative point in $\mathcal{Z}_{p,q,r} \cap f_{p,q,r}^{-1}(j)$ for each $j \leq m_{p,q,r}$.

We denote the set of all the points $(l; m)$ in Table 3 by $\mathcal{Z}[\text{Table 3}]$:

$$\begin{aligned} \mathcal{Z}[\text{Table 3}] &= \{(j, 0, -j; 0, 0) \mid j \leq 0\} \cup \{(0, j, -j; 0, 0) \mid j \leq -1\} \cup \dots \\ &\quad \dots \cup \{(-j-1, 1, j+2; 0, -1) \mid j \leq -2\}. \end{aligned}$$

The following lemma follows from Lemma 2.5 and Table 3 immediately.

LEMMA 2.7. *All the vertexes in $\mathcal{Z}$ are in $\mathcal{Z}[\text{Table 3}]$.*

We now give several lemmas and corollaries in order to determine whether an equation ${}_3E_2$ in the class of No. k is in $\mathcal{E}_k$ or not (see Corollary 2.12).

Let U_3 be the vector space $\{u = (u_1, u_2, u_3) \mid u_j \in \mathbb{C}\}$ and let G be a finite irreducible complex reflection group in $\text{GL}(U_3)$. Let $E(\pi_R, H_m; z)$ be the generating system for G , where $\pi_R(u) = [R_1(u) : R_2(u) : 1]$ is a rational G -quotient map of $\mathbf{P}(U_3)$

to a projective plane Z defined by G -invariant homogeneous rational functions $R_1(u)$ and $R_2(u)$ with degree 0, and $H_m(u) = H_1(u)/H_2(u)$ is a rational function defined by a pair of G -invariant homogeneous polynomials $H_1(u)$ and $H_2(u)$ such that $\deg H_1 - \deg H_2 = m := |(\text{the center of } G)|$ (see [HK]). We denote by $\mathbf{P}(G) = G/(\text{the center of } G)$ the projectivization of G .

LEMMA 2.8. *Let the notation be as above. Let $E(x)$ be a third-order Fuchsian differential equation on a projective line X such that:*

- (1) *its monodromy group is finite and irreducible;*
- (2) *its projective monodromy group is a subgroup of $\mathbf{P}(G)$; and*
- (3) *its Schwarz image C is not contained in the zeros of $H_1(u)H_2(u)$, and the restriction of π_R to C is defined and not constant.*

Then there is a rational map $z = z(x)$ of X to Z , a rational function $\theta(x)$ on X and an integer $k (\neq 0)$ such that the algebraic transformation

$$E(x) = \theta(x)^{1/k} E(\pi_R, H_m; z(x)) \quad (2.4)$$

holds.

Proof. Since a Schwarz map $x \rightarrow [u_1(x) : u_2(x) : u_3(x)]$ of $E(x)$ is defined by linearly independent solutions $u_j(x)$, its image C is not a line. Consequently, from [HK, Lemma 2.4], except for a finite number of points $[u]$ of C , it holds that $|\mathbf{P}(G) \cdot [u]| = |\mathbf{P}(G)|$. This implies that the restriction $\pi_R|_C$ of π_R to C is a $|\mathbf{P}(G)| : 1$ map, that is, a rational G -quotient map of C . Let N' be the order of the monodromy group and put

$$z(x) = \pi_R(u(x)), \quad \theta(x) = H_m(u(x))^{N'}.$$

Then, from [HK, Lemma 2.3], $\theta(x)$ is invariant under the action of the monodromy group of $E(x)$, and hence a rational function on X . By [HK, Lemma 2.7], $x \mapsto z(x)$ is a rational map of X to Z . Under this notation, we have

$$u(x) = \theta(x)^{1/(mN')} \cdot \frac{u(x)}{\sqrt[m]{H_m(u(x))}} = \theta(x)^{1/(mN')} \cdot \frac{\pi_R^{-1}(z(x))}{\sqrt[m]{H_m(\pi_R^{-1}(z(x)))}}.$$

If we put

$$\frac{\pi_R^{-1}(z)}{\sqrt[m]{H_m(\pi_R^{-1}(z))}} = (\varphi_1(z), \varphi_2(z), \varphi_3(z)), \quad z \in Z,$$

then, by definition of $E(\pi_R, H_m; z)$, $\varphi_j(z)$ ($1 \leq j \leq 3$) form a fundamental system of solutions of $E(\pi_R, H_m; z)$. This proves equality (2.4) if we put $k = mN'$. $\square$

COROLLARY 2.9. *Let $E(x)$ be the same as in Lemma 2.8. Then there is a Fuchsian differential equation $E'(y)$ on $Y \simeq \mathbf{P}^1$ with a finite irreducible monodromy group such that:*

- (1) *the Schwarz map of $E'(y)$ has a single-valued inverse map; and that*
- (2) *there is an algebraic transformation*

$$E(x) = \theta(x)^{1/k} E'(\sigma(x)), \quad (2.5)$$

with some rational functions $\theta(x)$ and $\sigma(x)$ and an integer $k (\neq 0)$.

Proof. Let C be the image of a Schwarz map of $E(x)$. Put $C' = \pi_R(C)$. Then C' is a rational curve in Z . Consequently there is a birational map $\sigma'(z)$ of C' to Y . Let $E'(y)$ be the image by σ' of the restriction $E(\pi_R, H_m; z)|_{C'}$ of $E(\pi_R, H_m; z)$ to C' . Then equation (2.4) implies equation (2.5). $\square$

There are five types of finite irreducible primitive complex reflection groups; that is, G_{120} , G_{336} , G_{648} , G_{1296} and G_{2160} . Here the sub-indexes denote the order of the groups. Each group G has a set of generators $F_{d_j}(u) = F_{d_j}(u_1, u_2, u_3)$, $j = 1, 2, 3$, of the ring of G -invariant polynomials. Here the index d_j denotes the degree of $F_{d_j}(u)$. The set $\{d_1, d_2, d_3\}$ of degrees for the reflection groups G_{120} , G_{336} , G_{648} , G_{1296} , G_{2160} are given by

$$G_{120} : \{2, 6, 10\}, \quad G_{336} : \{4, 6, 14\}, \quad G_{648} : \{6, 9, 12\}, \quad G_{1296} : \{6, 12, 18\}, \\ G_{2160} : \{6, 12, 30\}.$$

Note that the projective monodromy groups of $E(x)$ in the class of Nos 2, 3 and 4 are $\mathbf{P}(G_{336})$, those of Nos 5 and 6 are $\mathbf{P}(G_{120})$, those of Nos 7 and 8 are $\mathbf{P}(G_{2160})$, and those of Nos 9–11 are $\mathbf{P}(G_{648}) \simeq \mathbf{P}(G_{1296})$.

LEMMA 2.10. *Assume that $E(x) = {}_3E_2(a; b; x)$ are in the class of Nos 2–12. Then there is a Fuchsian differential equation $E'(y)$ on $Y \simeq \mathbf{P}^1$ with a finite irreducible monodromy group such that in Corollary 2.9(1) and (2) hold.*

Proof. We first prove the lemma for $E(x)$ in the class of Nos 2–11. Let G denote the complex reflection group G_{336} or G_{120} or G_{2160} or G_{648} depending on if $E(x)$ is of Nos 2, 3 and 4 or Nos 5 and 6 or Nos 7 and 8 or Nos 9–11, respectively. It is obvious that $E(x)$ satisfies properties (1) and (2) of Lemma 2.8.

Let C be the Schwarz image of $E(x)$ by a Schwarz map $x \mapsto [u_1(x) : u_2(x) : u_3(x)]$ defined by linearly independent solutions $u_j(x)$ of $E(x)$. If we prove the existence of a rational quotient map $\pi_R(u)$ and a G -invariant function $H_m(u)$ satisfying (3) of Lemma 2.8, then the lemma is proved by Corollary 2.9. So we prove this existence.

(I) Let $k = 2, 3$ or 4 . In this case, the projective monodromy group of $E(x)$ is $\mathbf{P}(G_{336})$. The number $m = |G_{336}|/|\mathbf{P}(G_{336})|$ is 2. Let $F_j(u_1, u_2, u_3)$, $j = 4, 6, 14$, be generators of G_{336} -invariant polynomials.

(I.1) Assume C is not in the zeros of $F_4(u)F_6(u)$. We define $\pi_R(u)$ and $H_m(u)$ by $\pi_R = [F_{14}(u)/(F_4^2(u)F_6(u)) : F_6(u)^2/F_4(u)^3 : 1]$ and $H_m(u) = F_6(u)/F_4(u)$ as in [HK]. Then (3) of Lemma 2.8 holds.

(I.2) Assume C is equal to the zeros of $F_4(u)$. Put

$$\pi_R = [z_1 : z_2 : 1] := \left[\frac{F_{14}(u)^3}{F_6(u)^7} : \frac{F_4(u)F_{14}(u)}{F_6(u)^3} : 1 \right] \quad \text{and} \quad H_m(u) = \frac{F_{14}(u)}{F_6(u)^2}.$$

Since

$$\frac{F_6(u)^2}{F_4(u)^3} = \frac{F_{14}(u)^3}{F_6(u)^7} \left(\frac{F_4(u)F_{14}(u)}{F_6(u)^3} \right)^{-3} \quad \text{and} \quad \frac{F_{14}(u)}{F_4(u)^2 F_6(u)} = \frac{F_{14}(u)^3}{F_6(u)^7} \left(\frac{F_4(u)F_{14}(u)}{F_6(u)^3} \right)^{-2},$$

it holds that

$$\pi_R^{-1}([z_1 : z_2 : 1]) = \left\{ [u] \in \mathbf{P}(U_3) \mid \frac{F_6(u)^2}{F_4(u)^3} = \frac{z_1}{z_2^3}, \frac{F_{14}(u)}{F_4(u)^2 F_6(u)} = \frac{z_1}{z_2^2} \right\},$$

for generic z_1, z_2 . This implies that π_R defines a rational G -quotient map. It is clear that (3) of Lemma 2.8 holds for these π_R and H_m .

(I.3) Assume C is equal to the zeros of $F_6(u)$. Put

$$\pi_R = [z_1 : z_2 : 1] := \left[\frac{F_{14}(u)^2}{F_4(u)^7} : \frac{F_4(u)^2 F_6(u)}{F_{14}(u)} : 1 \right] \quad \text{and} \quad H_m(u) = \frac{F_{14}(u)}{F_4(u)^3},$$

in this case. Since

$$\frac{F_6(u)^2}{F_4(u)^3} = \frac{F_{14}(u)^2}{F_4(u)^7} \left(\frac{F_4(u)^2 F_6(u)}{F_{14}(u)} \right)^2,$$

then

$$\pi_R^{-1}([z_1 : z_2 : 1]) = \left\{ [u] \in \mathbf{P}(U_3) \mid \frac{F_6(u)^2}{F_4(u)^3} = z_1 z_2^2, \frac{F_4(u)^2 F_6(u)}{F_{14}(u)} = z_2 \right\},$$

for generic z_1, z_2 . This implies that π_R defines a rational G -quotient map. It is clear that (3) of Lemma 2.8 holds for these π_R and H_m .

(II) Let $k = 5$ or 6 . In this case, the projective monodromy group of $E(x)$ is $\mathbf{P}(G_{120})$, where G_{120} is the complex reflection group of order 120. The number $m = |G_{120}|/|\mathbf{P}(G_{120})|$ is 2. Let $F_j(u_1, u_2, u_3)$, $j = 2, 6, 10$, be generators of G_{120} -invariant polynomials.

(II.1) Assume C is not the zero set of $F_2(u)$. We define $\pi_R(u)$ and $H_m(u)$ by $\pi_R = [F_{10}(u)/F_2^5(u) : F_6(u)/F_2(u)^3 : 1]$ and $H_m(u) = F_2(u)$ as in [HK]. Then (3) of Lemma 2.8 holds.

(II.2) Assume C is equal to the zeros of $F_2(u)$. Put

$$\pi_R = [z_1 : z_2 : 1] := \left[\frac{F_{10}(u)^3}{F_6(u)^5} : \frac{F_2(u) F_{10}(u)}{F_6(u)^2} : 1 \right] \quad \text{and} \quad H_m(u) = \frac{F_6(u)^2}{F_{10}(u)}.$$

Since

$$\frac{F_{10}(u)}{F_2(u)^5} = \left(\frac{F_{10}(u)^3}{F_6(u)^5} \right)^{1/3} \left(\frac{F_2(u) F_{10}(u)}{F_6(u)^2} \right)^{-5/3} \quad \text{and} \quad \frac{F_6(u)}{F_2(u)^3} = \frac{F_{10}(u)^3}{F_6(u)^5} \left(\frac{F_2(u) F_{10}(u)}{F_6(u)^2} \right)^{-3},$$

then

$$\pi_R^{-1}([z_1 : z_2 : 1]) = \left\{ [u] \in \mathbf{P}(U_3) \mid \frac{F_{10}(u)}{F_2(u)^5} = \frac{z_1^2}{z_2^5}, \frac{F_6(u)}{F_2(u)^3} = \frac{z_1}{z_2^3} \right\},$$

for generic z_1, z_2 . This implies that π_R defines a rational G -quotient map. It is clear that (3) of Lemma 2.8 holds for these π_R and H_m .

(III) Let $k = 7$ or 8 . In this case, the projective monodromy group of $E(x)$ is $\mathbf{P}(G_{2160})$, where G_{2160} is the complex reflection group of order 2160. Let $F_j(u_1, u_2, u_3)$, $j = 6, 12, 30$, be generators of G_{2160} -invariant polynomials. $m = 6$ in this case.

(III.1) Assume C is not the zero set of $F_6(u)$. We define $\pi_R(u)$ and $H_m(u)$ by $\pi_R = [F_{30}(u)/F_6^5(u) : F_{12}(u)/F_6(u)^2 : 1]$ and $H_m(u) = F_6(u)$ as in [HK]. Then (3) of Lemma 2.8 holds.

(III.2) Assume C is equal to the zeros of $F_6(u)$. Put

$$\pi_R = [z_1 : z_2 : 1] := \left[\frac{F_{30}(u)^2}{F_{12}(u)^5} : \frac{F_6(u) F_{12}(u)^2}{F_{30}(u)} : 1 \right] \quad \text{and} \quad H_m(u) = \frac{F_{30}(u)}{F_{12}(u)^2}.$$

Since

$$\frac{F_6(u)^5}{F_{30}(u)} = \left(\frac{F_{30}(u)^2}{F_{12}(u)^5} \right)^2 \left(\frac{F_6(u)F_{12}(u)^2}{F_{30}(u)} \right)^5 \quad \text{and} \quad \frac{F_6(u)^2}{F_{12}(u)} = \frac{F_{30}(u)^2}{F_{12}(u)^5} \left(\frac{F_6(u)F_{12}(u)^2}{F_{30}(u)} \right)^2,$$

then

$$\pi_R^{-1}([z_1 : z_2 : 1]) = \left\{ [u] \in \mathbf{P}(U_3) \mid \frac{F_6(u)^5}{F_{30}(u)} = z_1^2 z_2^5, \frac{F_6(u)^2}{F_{12}(u)} = z_1 z_2^2 \right\},$$

for generic z_1, z_2 . This implies that π_R defines a rational G -quotient map. It is clear that (3) of Lemma 2.8 holds for these π_R and H_m .

(IV) Let $k = 9$ or 10 or 11 . In this case, the projective monodromy group of $E(x)$ is $\mathbf{P}(G_{648})$, where G_{648} is the complex reflection group of order 648. Let $F_j(u_1, u_2, u_3)$, $j = 6, 9, 12$, be generators of G_{648} -invariant polynomials ($m = 3$ in this case). Because $F_9(u)$ is a product of nine linear functions (see, for example, [HK]), C is not in the zero set of $F_9(u)$.

(IV.1) Assume C is not the zero set of $F_6(u)$. We define $\pi_R(u)$ and $H_m(u)$ by $\pi_R = [F_9(u)^2/F_6^3(u) : F_{12}(u)/F_6(u)^2 : 1]$ and $H_m(u) = F_6(u)^2/F_9(u)$ as in [HK]. Then (3) of Lemma 2.8 holds.

(IV.2) Assume C is equal to the zeros of $F_6(u)$. Then put

$$\pi_R = [z_1 : z_2 : 1] := \left[\frac{F_{12}(u)^3}{F_9(u)^4} : \frac{F_6(u)F_{12}(u)}{F_9(u)^2} : 1 \right] \quad \text{and} \quad H_m(u) = \frac{F_{12}(u)}{F_9(u)}.$$

Since

$$\frac{F_{12}(u)}{F_6(u)^2} = \frac{F_{12}(u)^3}{F_9(u)^4} \left(\frac{F_6(u)F_{12}(u)}{F_9(u)^2} \right)^{-2},$$

then

$$\pi_R^{-1}([z_1 : z_2 : 1]) = \left\{ [u] \in \mathbf{P}(U_3) \mid \frac{F_{12}(u)^3}{F_9(u)^4} = z_1, \frac{F_{12}(u)}{F_6(u)^2} = \frac{z_1}{z_2^2} \right\},$$

for generic z_1, z_2 . This implies that π_R defines a rational G -quotient map. It is clear that (3) of Lemma 2.8 holds for these π_R and H_m .

We now prove the lemma for $E(x)$ in the class of No. 12. Let G be the monodromy group of ${}_3E_2(\frac{-1}{12}, \frac{2}{12}, \frac{5}{12}; \frac{3}{4}, \frac{1}{4})$ in the class of No. 12. Relating to this group, we recall two facts from [Kt4]:

- (i) G is generated by a normal subgroup G_r of order 54 and a cyclic subgroup $\langle L_0 \rangle$ of order 4 satisfying $G_r \cap \langle L_0 \rangle = \{1\}$. G_r is a complex reflection group of type $G(3, 3, 3)$. The order of G is 216 and that of the center of G is 6.
- (ii) There are basic G_r -invariant polynomials $I_3(u)$, $I'_3(u)$, $I_6(u)$ satisfying $L_0 \cdot I_3(u) = iI_3(u)$, $L_0 \cdot I'_3(u) = -iI'_3(u)$, $L_0 \cdot I_6(u) = I_6(u)$, which imply that $I_3(u)I'_3(u)$, $I_6(u)$, $(I'_3(u)/I_3(u))^2$ are G -invariant.

Now we return to the proof.

(V.1) Assume C is not contained in the zero set of $I_3(u)I'_3(u)$. We define $\pi_R(u)$ and $H_m(u)$ by $\pi_R = [I_6(u)/(I_3(u)I'_3(u)) : (I'_3(u)/I_3(u))^2 : 1]$ and $H_m(u) = I_3(u)I'_3(u)$. Then we can define a generating system $E(\pi_R, H_m; z)$ for G by using the method from [HK]. Since C satisfies (3) of Lemma 2.8, we can prove that $E(x)$ satisfies the algebraic equation (2.4) for $\theta(x) = H_m(u(x))^{36}$, $z(x) = \pi_R(u(x))$. Consequently, Corollary 2.9 can also be proved for this $E(x)$.

(V.2) Assume C is equal to the zeros of $I'_3(u)$. In this case, we define the $\mathbf{P}(G)$ -quotient map π of C to a projective line Y by

$$\pi : [u] \in C \mapsto [I_6(u)^2 : I_3(u)^4] \in Y$$

as in [Kt4]. Let $E'(y)$ be the third-order differential equation satisfied by $(u_j I_6(u)^{-1/6}) \circ \pi^{-1}$, $j = 1, 2, 3$. Then, by definition, the following facts hold:

- (iii) G is the monodromy group of $E'(y)$;
- (iv) π is a single-valued inverse map of a Schwarz map of $E'(y)$;
- (v) equality (2.5) holds for

$$\theta(x) = I_6(u_1(x), u_2(x), u_3(x)), \quad k = 6, \quad \sigma(x) = I_6(u(x))^2 / I_3(u(x))^4.$$

Since $I_6(u_1(x), u_2(x), u_3(x))$ and $I_3(u(x))^4$ are rational functions of x , (1) and (2) of Corollary 2.9 hold. Consequently the lemma holds.

(V.3) In the case when C is equal to the zeros of $I_3(u)$, the lemma is proved the same way as case (V.2).

This completes the proof. $\square$

LEMMA 2.11. *Let $E(x) = {}_3E_2(a; b; x)$ and $E'(y)$ be the same as in Lemma 2.10. If the degree of the rational map $x \mapsto y = \sigma(x)$ in the algebraic transformation (2.5) is one, then (2.5) is a projective transformation in Transformation 3, otherwise, the transformation (2.5) is one of algebraic transformations (2.1)–(2.5) of [Kt1] or a Clausen type transformation [Kt1, Theorem 2.2].*

Proof. [Kt1, Theorem 2.2] proves the lemma. $\square$

COROLLARY 2.12. *Let $E(x) = {}_3E_2(a; b; x)$ be in the class of No. k , $2 \leq k \leq 12$. Assume any $E'(x) = {}_3E_2(a'; b'; x)$ obtained from $E(x)$ by Transformation 3 is not equal to any of the following ${}_3E_2$:*

- (1) ${}_3E_2(\alpha, \alpha + \lambda, \alpha + \mu; 1 - \lambda, 1 - \mu; x)$;
- (2) ${}_3E_2(\alpha, \alpha + 2\lambda, \alpha + 2\mu; 1 - \lambda, 1 - \mu; x)$;
- (3) ${}_3E_2(\alpha, \alpha + 1/3, \alpha + \mu; 2/3, 1/3; x)$;
- (4) ${}_3E_2(2\alpha, \alpha + \beta, 2\beta; \alpha + \beta + 1/2, 2\alpha + 2\beta; x)$.

Then $E(x)$ is in $\mathcal{E}_k$.

We call $E(x)$ a Clausen-type ${}_3E_2$ if $E(x)$ is equal to ${}_3E_2(2\alpha, \alpha + \beta, 2\beta; \alpha + \beta + 1/2, 2\alpha + 2\beta; x)$ for some α and β .

LEMMA 2.13. *If $E(x)$ and $E'(x)$ are in $\mathcal{E}_k$ for some k , then the Schwarz images of $E(x)$ and $E'(x)$ are birational.*

Proof. Let $E(x) = {}_3E_2(a; b; x) = {}_3E_2(a_0, a_1, a_2; b_1, b_2; x)$ and $E'(x) = {}_3E_2(a'; b'; x) = {}_3E_2(a'_0, a'_1, a'_2; b'_1, b'_2; x)$. The lemma is obvious if $E'(x)$ is obtained from $E(x)$ by Transformation 3. We assume that $E'(x)$ is obtained from $E(x)$ by Transformation 1 or 2, and prove the lemma.

Let x_0 be a small positive number, and let $u_j(x)$, $u'_j(x)$, $j = 1, 2, 3$, be solutions of $E(x)$, $E'(x)$ at x_0 of the form (1.2) of [Kt4]. For a closed curve $\gamma \in \pi_1(\mathbf{C} \setminus \{0, 1\}, x_0)$, we denote by $M_u(\gamma)$ the monodromy matrix such that the analytic continuation $\gamma_*(u_1, u_2, u_3)$

of (u_1, u_2, u_3) along γ is equal to $(u_1, u_2, u_3)M_u(\gamma)$. Recall that (see [Kt4]) all the elements of $M_u(\gamma)$ are polynomials in $e(a_j)$, $e(b_k)$ with the coefficients in $\mathbf{Q}$, where $e(x) = e^{2\pi\sqrt{-1}x}$.

Let C and C' be the Schwarz images of $E(x)$ and $E'(x)$ defined by $x \mapsto [u_1(x) : u_2(x) : u_3(x)]$ and $x \mapsto [u'_1(x) : u'_2(x) : u'_3(x)]$, respectively. Let π and π' be the single-valued inverse maps of the Schwarz maps $x \mapsto [u_1(x) : u_2(x) : u_3(x)]$ and $x \mapsto [u'_1(x) : u'_2(x) : u'_3(x)]$, respectively. Choose arbitrary points P_0 and P'_0 in $\pi^{-1}(x_0)$ and $(\pi')^{-1}(x_0)$, respectively. Let P be a generic point in C , and let Γ be a curve in C starting at P_0 and ending at P . Put $\gamma = \pi(\Gamma)$. Let Γ' be the curve in C' over γ starting at P'_0 . Let P' be the end point of Γ' . We define a map φ of C to C' by $\varphi : P \mapsto P'$. We prove that φ is well defined and birational.

First assume $P = P_0$. This implies that the monodromy matrix $M_u(\gamma)$ is a scalar matrix. That is, $(M_u(\gamma))_{i,j} = 0$ for $i \neq j$, and $(M_u(\gamma))_{i,i} = (M_u(\gamma))_{1,1}$ for $i = 2, 3$. These equalities hold if a_j, b_k are changed to a'_j, b'_k by Transformation 1 or 2. This implies that $P' = P'_0$. This proves that $\varphi : P \mapsto P'$ is well defined. Since (1) φ is defined on a Zariski open set U of C ; (2) φ is one-to-one and (3) φ is locally biholomorphic at every point of U , we conclude that φ is birational. $\square$

COROLLARY 2.14. *All the Schwarz images of ${}_3E_2 \in \mathcal{E}_k$ have the same genus for each k .*

In [Kt3, Table 1], we gave the genus of the Schwarz images of ${}_3E_2 \in \mathcal{E}_k$.

3. Differential equations ${}_3E_2$ in $\mathcal{E}_k$

LEMMA 3.1. *Assume $E(x)$ is a Clausen-type ${}_3E_2$, that is, $E(x) = {}_3E_2(2\alpha, \alpha + \beta, 2\beta; \alpha + \beta + 1/2, 2\alpha + 2\beta; x)$ for some α and β . Then the projective monodromy group of $E(x)$ is isomorphic to that of ${}_2E_1(\alpha, \beta; \alpha + \beta + 1/2; x)$.*

Proof. Put $E'(x) = {}_2E_1(\alpha, \beta; \alpha + \beta + 1/2; x)$. Then, from Clausen's formula, it holds that $E(x) = E'(x)^2$. This implies that if $\varphi_1(x), \varphi_2(x)$ are linearly independent solutions of $E'(x)$, then $\varphi_1(x)^2, \varphi_1(x)\varphi_2(x), \varphi_2(x)^2$ are those of $E(x)$. This proves the lemma. $\square$

COROLLARY 3.2. *Let $E(x) = {}_3E_2(a; b; x)$ be in the class of No. k , $2 \leq k \leq 12$, $k \neq 5, 6$. Then any $E'(x) = {}_3E_2(a'; b'; x)$ obtained from $E(x)$ by any projective transformation is not equal to ${}_3E_2$ in Corollary 2.12(4).*

Proof. Because the projective monodromy group of $E'(x)$ is not isomorphic to any of cyclic, dihedral, tetrahedral, octahedral or icosahedral group, $E'(x)$ cannot be a Clausen-type ${}_3E_2$. This proves the corollary. $\square$

3.1. ${}_3E_2$ in $\mathcal{E}_k$, $k = 2, 3, 8, 9, 10, 11$

LEMMA 3.3. *For $k = 2, 3, 8, 9, 10, 11$, all ${}_3E_2$ in the class of No. k are in $\mathcal{E}_k$.*

Proof. Let $E(x) = E(a; b; x)$ be one in the class of No. 2 in Table 2, and let $E'(x) = {}_3E_2(a'; b'; x)$ be an equation obtained from $E(x)$ by any projective transformation, Transformation 3. Then it holds that $a'_i - a'_j = p/14$ with $(p, 14) = 1$, $1 - b_j = q/3$ with $(q, 3) = 1$, or that $a'_i - a'_j = p/3$ with $(p, 3) = 1$, $1 - b_j = q/14$ with $(q, 14) = 1$. In any case, $E'(x)$ is not equal to ${}_3E_2$ in Corollary 2.12(1)–(3). Since Corollary 3.2 states that $E'(x)$

is not equal to ${}_3E_2$ in Corollary 2.12(4), the corollary proves the lemma for $k = 2$. For other k , the lemma is proved in the same manner. $\square$

3.2. ${}_3E_2$ in $\mathcal{E}_4$

Let $E_1(x) = {}_3E_2(-\frac{1}{14} + l_0, \frac{1}{14} + l_1, \frac{9}{14} + l_2; \frac{6}{7} - m_1, \frac{2}{7} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{1}{14} + l_0, \frac{3}{14} + l_1, \frac{11}{14} + l_2; \frac{6}{7} - m_1, \frac{4}{7} - m_2)$ be equations in Table 2.

LEMMA 3.4. $E_1(x)$ or its projective transform is equal to ${}_3E_2$ in Corollary 2.12(1) if and only if $m_1 = l_1 - l_0$, $m_2 = l_2 - l_0$ and to ${}_3E_2$ in (2) if and only if $2m_1 = l_0 - l_2 - 1$, $2m_2 = l_1 - l_2 - 2$, or $m_1 = 2(l_2 - l_1) + 1$, $m_2 = 2(l_0 - l_1) - 1$.

$E_2(x)$ or its projective transform is equal to ${}_3E_2$ in Corollary 2.12(1) if and only if $m_1 = l_0 - l_2 - 1$, $m_2 = l_1 - l_2 - 1$ and to ${}_3E_2$ in (2) if and only if $2m_1 = l_1 - l_0$, $2m_2 = l_2 - l_0$, or $m_1 = 2(l_2 - l_1) + 1$, $m_2 = 2(l_0 - l_1) - 1$.

COROLLARY 3.5. Assume $(l; m) \in \mathcal{Z}[\text{Table 3}]$. Then $E_1(x)$ is in $\mathcal{E}_4$ if and only if $(l; m)$ is not the vertex points $(0, 1, -1; 0, 0)$ of $\mathcal{Z}_{0,0,0}$, $(1, 1, 0; -1, -1)$ of $\mathcal{Z}_{2,1,0}$, $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$, $(1, 1, 0; 0, -1)$ of $\mathcal{Z}_{2,2,2}$. $E_2(x)$ is in $\mathcal{E}_4$ if and only if $(l; m)$ is not the vertex points $(0, 1, -1; 0, 0)$ of $\mathcal{Z}_{0,0,0}$, $(1, 1, 0; -1, -1)$ of $\mathcal{Z}_{2,1,0}$, $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$, $(1, 1, 1; -1, -1)$ of $\mathcal{Z}_{0,1,2}$.

3.3. ${}_3E_2$ in $\mathcal{E}_5$

Let $E_1(x) = {}_3E_2(-\frac{1}{10} + l_0, \frac{1}{10} + l_1, \frac{1}{2} + l_2; \frac{2}{3} - m_1, \frac{1}{3} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{3}{10} + l_0, \frac{3}{10} + l_1, \frac{1}{2} + l_2; \frac{2}{3} - m_1, \frac{1}{3} - m_2; x)$ be equations in Table 2. It is proved in the same manner as the proof of Lemma 3.3 that any $E'(x)$ obtained from E_1 or E_2 by any projective transformation is not equal to ${}_3E_2$ in Corollary 2.12(1)–(3).

LEMMA 3.6. $E_1(x)$ or its projective transform is equal to a Clausen-type ${}_3E_2$ if and only if $l_0 + l_1 = 1$, $l_2 = 0$ and $m_1 + m_2 = -1$. The same property also holds for $E_2(x)$.

Proof. The set $\{-\frac{1}{10} + l_0, \frac{1}{10} + l_1, \frac{1}{2} + l_2\}$ of the local exponents of $E_1(x)$ at $x = \infty$ forms an arithmetical progression if and only if

$$l_0 + l_1 = 1 + 2l_2. \quad (3.1)$$

By the same consideration at $x = 0$, we have

$$m_1 = 2m_2 + 1 \quad \text{or} \quad m_2 = 2m_1 \quad \text{or} \quad m_1 + m_2 = -1, \quad (3.2)$$

and at $x = 1$,

$$l_0 + l_1 + l_2 + m_1 + m_2 = 0. \quad (3.3)$$

These three equations prove the lemma. $\square$

COROLLARY 3.7. Assume $(l; m) \in \mathcal{Z}[\text{Table 3}]$. Then $E_2(x)$ is always in $\mathcal{E}_5$. $E_1(x)$ is in $\mathcal{E}_5$ if and only if $(l; m)$ is not the vertex points $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$.

Proof. The point $(l; m) \in \mathcal{Z}[\text{Table 3}]$ satisfies $l_0 + l_1 = 1$, $l_2 = 0$ and $m_1 + m_2 = -1$ if and only if $(l; m) = (1, 0, 0; 0, -1)$. In this case, $E_2(x)$ is equal to ${}_3E_2(\frac{7}{10}, \frac{3}{10}, \frac{1}{2}; \frac{2}{3}, \frac{4}{3}; x)$ whose Schwarz map has a single-valued inverse map (see [Kt2]). On the other hand, $E_1(x)$ is equal to ${}_3E_2(\frac{9}{10}, \frac{1}{10}, \frac{1}{2}; \frac{2}{3}, \frac{4}{3}; x)$, which is equal to

$$\begin{aligned} x^{-1/3} {}_3E_2\left(-\frac{7}{30}, \frac{1}{6}, \frac{17}{30}; \frac{1}{3}, \frac{2}{3}; x\right) &= x^{-1/3} {}_2E_1\left(-\frac{7}{60}, \frac{17}{60}; \frac{2}{3}; x\right)^2 \\ &= x^{-1/3} \theta(x)^{1/6} {}_2E_1\left(-\frac{1}{60}, \frac{11}{60}; \frac{2}{3}; \sigma(x)\right)^2 \\ &= x^{-1/3} \theta(x)^{1/6} {}_3E_2\left(-\frac{1}{30}, \frac{1}{6}, \frac{11}{30}; \frac{1}{3}, \frac{2}{3}; \sigma(x)\right), \end{aligned}$$

where $\theta(x) = 64 - 189x$, $\sigma(x) = -x(3584 + 2457x - 2916x^2)^3 / (64 - 189x)^5$. Hence $E_1(x)$ is not in $\mathcal{E}_5$. $\square$

3.4. ${}_3E_2$ in $\mathcal{E}_6$

Let $E(x) = {}_3E_2(-\frac{3}{10} + l_0, \frac{3}{10} + l_1, \frac{1}{2} + l_2; \frac{3}{5} - m_1, \frac{2}{5} - m_2; x)$ be equations in Table 2. It is clear that any $E'(x)$ obtained from $E(x)$ by any projective transformation is not equal to ${}_3E_2$ in Corollary 2.12(1) or (3). The following lemma and its corollary are proved in the same manner as the previous case of No. 5.

LEMMA 3.8. *$E(x)$ or its projective transform is equal to ${}_3E_2$ in Corollary 2.12(2) if and only if $m_1 = 2(l_0 - l_2) - 2$ and $m_2 = 2(l_1 - l_2) - 1$, and is equal to ${}_3E_2$ in Corollary 2.12(4) if and only if $l_0 + l_1 = 1$, $l_2 = 0$ and $m_1 + m_2 = -1$. If $(l; m) \in \mathcal{Z}[\text{Table 3}] \setminus \{(1, 0, 0; 0, -1)\}$, then $(l; m)$ does not satisfy either of the above two conditions. If $(l; m) = (1, 0, 0; 0, -1)$, then $(l; m)$ satisfies both of the above conditions.*

COROLLARY 3.9. *Assume $(l; m) \in \mathcal{Z}[\text{Table 3}]$. Then $E(x)$ is in $\mathcal{E}_6$ if and only if $(l; m)$ is not the vertex points $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$.*

3.5. ${}_3E_2$ in $\mathcal{E}_7$

Let $E_1(x) = {}_3E_2(-\frac{1}{30} + l_0, \frac{1}{6} + l_1, \frac{11}{30} + l_2; \frac{4}{5} - m_1, \frac{1}{5} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{7}{30} + l_0, \frac{1}{6} + l_1, \frac{17}{30} + l_2; \frac{3}{5} - m_1, \frac{2}{5} - m_2; x)$ be equations in Table 2.

LEMMA 3.10. *$E_1(x)$, $E_2(x)$ or their projective transforms are equal to ${}_3E_2$ in Corollary 2.12(1) if and only if $m_1 = l_2 - l_1$, $m_2 = l_0 - l_1 - 1$. They cannot be equal to ${}_3E_2$ in Corollary 2.12(2) or (3).*

COROLLARY 3.11. *Assume $(l; m) \in \mathcal{Z}[\text{Table 3}]$. Then, for each $j = 1, 2$, $E_j(x)$ is in $\mathcal{E}_7$ if and only if $(l; m)$ is not the vertex points $(1, 1, 0; -1, -1)$ of $\mathcal{Z}_{2,1,0}$ or $(1, 0, 0; 0, 0)$ of $\mathcal{Z}_{1,0,2}$.*

3.6. ${}_3E_2$ in $\mathcal{E}_{12}$

Let $E_1(x) = {}_3E_2(-\frac{1}{12} + l_0, \frac{1}{6} + l_1, \frac{5}{12} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$, and $E_2(x) = {}_3E_2(-\frac{1}{6} + l_0, \frac{1}{12} + l_1, \frac{7}{12} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ be equations in Table 2.

LEMMA 3.12. $E_1(x)$ or its projective transform is equal to ${}_3E_2$ in Corollary 2.12(1) if and only if $m_1 = l_2 - l_1$, $m_2 = l_0 - l_1 - 1$. $E_2(x)$ or its projective transform is equal to ${}_3E_2$ in Corollary 2.12(1) if and only if $m_1 = l_1 - l_0$, $m_2 = l_2 - l_0$. They are never equal to ${}_3E_2$ in Corollary 2.12(2) or (3).

COROLLARY 3.13. Assume $(l; m) \in \mathcal{Z}[\text{Table 3}]$. Then $E_1(x)$ is in $\mathcal{E}_{12}$ if and only if $(l; m)$ is not the vertex points $(1, 1, 0; -1, -1)$ of $\mathcal{Z}_{2,1,0}$ or $(1, 0, 0; 0, 0)$ of $\mathcal{Z}_{1,0,2}$, and $E_2(x)$ is in $\mathcal{E}_{12}$ if and only if $(l; m)$ is not the vertex points $(0, 0, 0; 0, 0)$ of $\mathcal{Z}_{0,0,0}$ or $(1, 1, 0; 0, -1)$ of $\mathcal{Z}_{2,2,2}$.

Lemma 3.3 and Corollaries 3.5, 3.7, 3.9, 3.11 and 3.13 prove the following lemma.

LEMMA 3.14. Let $E(x) = {}_3E_2(a + l; b - m; x)$ be a ${}_3E_2$ of No. k in Table 2. If $(l; m)$ is in $\mathcal{Z}[\text{Table 3}]$ and is not a vertex, then $E(x)$ is in $\mathcal{E}_k$.

4. The set $\mathcal{N}_k$

Recall that $\mathcal{N}_k$ is the set of the degrees $n = -g'(e_0 + e_1 + e_\infty)$ of the Schwarz images of ${}_3E_2$ in $\mathcal{E}_k$. Let $\mathcal{N}_k[\text{Table 2}] \cdot [\text{Table 3}]$ be the set of the degrees $n = -g'(e_0 + e_1 + e_\infty)$ of the Schwarz images of ${}_3E_2$ in $\mathcal{E}_k$ and in Table 2 with $(l; m)$ in $\mathcal{Z}[\text{Table 3}]$. By definition, it holds that

$$\mathcal{N}_k[\text{Table 2}] \cdot [\text{Table 3}] \subset \mathcal{N}_k.$$

The converse inclusion also holds.

LEMMA 4.1. It holds that $\mathcal{N}_k[\text{Table 2}] \cdot [\text{Table 3}] = \mathcal{N}_k$ for each k .

Proof. Let $n \in \mathcal{N}_k$, which is the degree of the Schwarz image of $E(x) = {}_3E_2(a_0 + l_0, a_1 + l_1, a_2 + l_2, ; b_1 - m_1, b_2 - m_2; x)$ in Table 2 and in $\mathcal{E}_k$. Let $(l, m) \in \mathcal{Z}_{p,q,r}$. Then $n = -g'e_{p,q,r}(a; b; l; m)$ from Lemma 2.4. If $(l, m) \in \mathcal{Z}[\text{Table 3}]$, then $n \in \mathcal{N}_k[\text{Table 2}] \cdot [\text{Table 3}]$.

Assume $(l, m) \notin \mathcal{Z}[\text{Table 3}]$. Let $f_{p,q,r}(l; m) = j$. Because $j \leq \max f_{p,q,r}$, there is a point $(l'; m')$ in $\mathcal{Z}_{p,q,r} \cap f_{p,q,r}^{-1}(j) \cap \mathcal{Z}[\text{Table 3}]$. Then Lemma 2.6(2) implies that $(l'; m')$ is not the vertex of $\mathcal{Z}_{p,q,r}$. Hence the corresponding equation ${}_3E_2(a + l'; b - m'; x)$ is in $\mathcal{E}_k$ from Lemma 3.14. Then the degree n' of the Schwarz image of $E_j(a + l'; b - m'; x)$ is in $\mathcal{N}_k[\text{Table 2}] \cdot [\text{Table 3}]$, and is equal to $-g'e_{p,q,r}(a; b; l'; m')$, which is equal to $-g'e_{p,q,r}(a; b; l; m) (= n)$ from Lemma 2.3. This implies $n \in \mathcal{N}_k[\text{Table 2}] \cdot [\text{Table 3}]$. $\square$

4.1. $\mathcal{N}_2$

Let $E_1(x) = {}_3E_2(-\frac{3}{14} + l_0, \frac{1}{14} + l_1, \frac{9}{14} + l_2; \frac{2}{3} - m_1, \frac{1}{3} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{1}{14} + l_0, \frac{3}{14} + l_1, \frac{5}{14} + l_2; \frac{2}{3} - m_1, \frac{1}{3} - m_2; x)$ be equations in Table 2.

From Lemma 3.3, $E_1(x)$ and $E_2(x)$ are in $\mathcal{E}_2$ for all $(l; m) \in \mathcal{Z}$. If $(l; m) \in \mathcal{Z}_{p,q,r}$, then $e_0 + e_1 + e_\infty = e_{p,q,r}$ from Lemma 2.4. Since $e_{p,q,r}$ is given by (2.2), and since the maximal value $\max f_{p,q,r}$ on $\mathcal{Z}_{p,q,r}$ is given in Table 3, we easily find the range of the value $-e_{p,q,r}$ of $E_1(x)$ and $E_2(x)$ for each (p, q, r) . Thus we have the following proposition.

PROPOSITION 4.2. The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 4, where j runs over all integers ≥ 0 .

TABLE 4. The set of $-e_{p,q,r}$ for $\mathcal{E}_2$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
(0, 0, 0)	$\frac{9}{42} + j$	$\frac{3}{42} + j$	(0, 0, 2)	$\frac{30}{42} + j$	$\frac{24}{42} + j$
(1, 0, 0)	$\frac{39}{42} + j$	$\frac{33}{42} + j$	(1, 0, 2)	$\frac{18}{42} + j$	$\frac{12}{42} + j$
(2, 0, 0)	$\frac{15}{42} + j$	$\frac{27}{42} + j$	(2, 0, 2)	$\frac{36}{42} + j$	$\frac{48}{42} + j$
(0, 1, 0)	$\frac{37}{42} + j$	$\frac{31}{42} + j$	(0, 1, 2)	$\frac{16}{42} + j$	$\frac{10}{42} + j$
(1, 1, 0)	$\frac{25}{42} + j$	$\frac{19}{42} + j$	(1, 1, 2)	$\frac{46}{42} + j$	$\frac{40}{42} + j$
(2, 1, 0)	$\frac{1}{42} + j$	$\frac{13}{42} + j$	(2, 1, 2)	$\frac{22}{42} + j$	$\frac{34}{42} + j$
(0, 2, 0)	$\frac{23}{42} + j$	$\frac{17}{42} + j$	(0, 2, 2)	$\frac{44}{42} + j$	$\frac{38}{42} + j$
(1, 2, 0)	$\frac{11}{42} + j$	$\frac{5}{42} + j$	(1, 2, 2)	$\frac{32}{42} + j$	$\frac{26}{42} + j$
(2, 2, 0)	$\frac{29}{42} + j$	$\frac{41}{42} + j$	(2, 2, 2)	$\frac{8}{42} + j$	$\frac{20}{42} + j$

COROLLARY 4.3. The set $\mathcal{N}_2$ is equal to the set $\{\frac{168}{42}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{7}, j \neq 2, 4, 6\}$.

Remark 4.1. The value $-e_{p,q,r}$ of $E_1(x)$ attains the minimal value $1/42$ when $(l; m)$ is the vertex $(1, 1, 0; -1, -1)$ of $\mathcal{Z}_{2,1,0}$. The corresponding ${}_3E_2(\frac{11}{14}, \frac{15}{14}, \frac{9}{14}; \frac{5}{3}, \frac{4}{3}; x)$ is equal to Min.2 in [Kt2, Table 1].

4.2. $\mathcal{N}_3$

Let $E_1(x) = {}_3E_2(-\frac{3}{14} + l_0, \frac{1}{14} + l_1, \frac{9}{14} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{1}{14} + l_0, \frac{3}{14} + l_1, \frac{5}{14} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ be equations in Table 2.

From Lemma 3.3, $E_1(x)$ and $E_2(x)$ are in $\mathcal{E}_3$ for all $(l; m) \in \mathcal{Z}$.

PROPOSITION 4.4. The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 5, where j runs over all integers ≥ 0 .

COROLLARY 4.5. The set $\mathcal{N}_3$ is equal to the set $\{\frac{168}{28}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{7}, j \neq 4\}$.

Remark 4.2. The value $-e_{p,q,r}$ of $E_2(x)$ attains the minimal value $1/28$ when $(l; m)$ is the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. The corresponding ${}_3E_2(\frac{13}{14}, \frac{3}{14}, \frac{5}{14}; \frac{3}{4}, \frac{5}{4}; x)$ is equal to Min.3 in [Kt2, Table 1].

4.3. $\mathcal{N}_4$

Let $E_1(x) = {}_3E_2(-\frac{1}{14} + l_0, \frac{1}{14} + l_1, \frac{9}{14} + l_2; \frac{6}{7} - m_1, \frac{2}{7} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{1}{14} + l_0, \frac{3}{14} + l_1, \frac{11}{14} + l_2; \frac{6}{7} - m_1, \frac{4}{7} - m_2; x)$ be equations in Table 2.

From Corollary 3.5, we obtain the following proposition.

PROPOSITION 4.6. The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 6, where j runs over all integers ≥ 0 .

COROLLARY 4.7. The set $\mathcal{N}_4$ is equal to the set $\{\frac{168}{14}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{7}, j \neq 1, 2\}$.

TABLE 5. The set of $-e_{p,q,r}$ for $\mathcal{E}_3$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
$(0, 0, 0)$	$\frac{6}{28} + j$	$\frac{2}{28} + j$	$(0, 0, 2)$	$\frac{20}{28} + j$	$\frac{16}{28} + j$
$(1, 0, 0)$	$\frac{26}{28} + j$	$\frac{22}{28} + j$	$(1, 0, 2)$	$\frac{12}{28} + j$	$\frac{8}{28} + j$
$(2, 0, 0)$	$\frac{10}{28} + j$	$\frac{18}{28} + j$	$(2, 0, 2)$	$\frac{24}{28} + j$	$\frac{32}{28} + j$
$(0, 1, 0)$	$\frac{27}{28} + j$	$\frac{23}{28} + j$	$(0, 1, 2)$	$\frac{13}{28} + j$	$\frac{9}{28} + j$
$(1, 1, 0)$	$\frac{19}{28} + j$	$\frac{15}{28} + j$	$(1, 1, 2)$	$\frac{33}{28} + j$	$\frac{29}{28} + j$
$(2, 1, 0)$	$\frac{3}{28} + j$	$\frac{11}{28} + j$	$(2, 1, 2)$	$\frac{17}{28} + j$	$\frac{25}{28} + j$
$(0, 2, 0)$	$\frac{13}{28} + j$	$\frac{9}{28} + j$	$(0, 2, 2)$	$\frac{27}{28} + j$	$\frac{23}{28} + j$
$(1, 2, 0)$	$\frac{5}{28} + j$	$\frac{1}{28} + j$	$(1, 2, 2)$	$\frac{19}{28} + j$	$\frac{15}{28} + j$
$(2, 2, 0)$	$\frac{17}{28} + j$	$\frac{25}{28} + j$	$(2, 2, 2)$	$\frac{3}{28} + j$	$\frac{11}{28} + j$

TABLE 6. The set of $-e_{p,q,r}$ for $\mathcal{E}_4$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
$(0, 0, 0)$	$\frac{15}{14} + j$	$\frac{15}{14} + j$	$(0, 0, 2)$	$\frac{8}{14} + j$	$\frac{8}{14} + j$
$(1, 0, 0)$	$\frac{13}{14} + j$	$\frac{11}{14} + j$	$(1, 0, 2)$	$\frac{6}{14} + j$	$\frac{4}{14} + j$
$(2, 0, 0)$	$\frac{5}{14} + j$	$\frac{3}{14} + j$	$(2, 0, 2)$	$\frac{12}{14} + j$	$\frac{10}{14} + j$
$(0, 1, 0)$	$\frac{13}{14} + j$	$\frac{13}{14} + j$	$(0, 1, 2)$	$\frac{6}{14} + j$	$\frac{20}{14} + j$
$(1, 1, 0)$	$\frac{11}{14} + j$	$\frac{9}{14} + j$	$(1, 1, 2)$	$\frac{18}{14} + j$	$\frac{16}{14} + j$
$(2, 1, 0)$	$\frac{17}{14} + j$	$\frac{15}{14} + j$	$(2, 1, 2)$	$\frac{10}{14} + j$	$\frac{8}{14} + j$
$(0, 2, 0)$	$\frac{5}{14} + j$	$\frac{9}{14} + j$	$(0, 2, 2)$	$\frac{12}{14} + j$	$\frac{16}{14} + j$
$(1, 2, 0)$	$\frac{17}{14} + j$	$\frac{19}{14} + j$	$(1, 2, 2)$	$\frac{10}{14} + j$	$\frac{12}{14} + j$
$(2, 2, 0)$	$\frac{9}{14} + j$	$\frac{11}{14} + j$	$(2, 2, 2)$	$\frac{16}{14} + j$	$\frac{4}{14} + j$

Remark 4.3. The set $\mathcal{Z}_{2,0,0} \cap f_{2,0,0}^{-1}(-1)$ consists of four points: $P^{(1)} = (1, 0, -1; 0, 0)$, $P^{(2)} = (0, 1, -1; 0, 0)$, $P^{(3)} = (0, 0, -1; 1, 0)$ and $P^{(4)} = (0, 0, -1; 0, 1)$. The values of $-e_{2,0,0}$ of the equation $E_2(x)$ with these $(l; m)$ are all $3/14$. The corresponding differential equations are

$$\begin{aligned}
 E^{(1)} &= {}_3E_2\left(\frac{13}{14}, \frac{3}{14}, -\frac{3}{14}; \frac{6}{7}, \frac{4}{7}\right), & E^{(2)} &= {}_3E_2\left(-\frac{1}{14}, \frac{17}{14}, -\frac{3}{14}; \frac{6}{7}, \frac{4}{7}\right), \\
 E^{(3)} &= {}_3E_2\left(-\frac{1}{14}, \frac{3}{14}, -\frac{3}{14}; -\frac{1}{7}, \frac{4}{7}\right), & E^{(4)} &= {}_3E_2\left(-\frac{1}{14}, \frac{3}{14}, -\frac{3}{14}; \frac{6}{7}, -\frac{3}{7}\right).
 \end{aligned}$$

$E^{(3)}$ and $E^{(4)}$ are equal to Min.4.1 and Min.4.2 in [Kt2, Table 1], respectively. $E^{(1)}$ and $E^{(2)}$ are equal to $E^{(3)}$ and $E^{(4)}$ up to projective transformations, respectively.

TABLE 7. The set of $-e_{p,q,r}$ for $\mathcal{E}_5$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
$(0, 0, 0)$	$\frac{3}{30} + j$	$\frac{9}{30} + j$	$(0, 0, 2)$	$\frac{18}{30} + j$	$\frac{24}{30} + j$
$(1, 0, 0)$	$\frac{27}{30} + j$	$\frac{21}{30} + j$	$(1, 0, 2)$	$\frac{12}{30} + j$	$\frac{6}{30} + j$
$(2, 0, 0)$	$\frac{15}{30} + j$	$\frac{15}{30} + j$	$(2, 0, 2)$	$\frac{30}{30} + j$	$\frac{30}{30} + j$
$(0, 1, 0)$	$\frac{23}{30} + j$	$\frac{29}{30} + j$	$(0, 1, 2)$	$\frac{8}{30} + j$	$\frac{14}{30} + j$
$(1, 1, 0)$	$\frac{17}{30} + j$	$\frac{11}{30} + j$	$(1, 1, 2)$	$\frac{32}{30} + j$	$\frac{26}{30} + j$
$(2, 1, 0)$	$\frac{5}{30} + j$	$\frac{5}{30} + j$	$(2, 1, 2)$	$\frac{20}{30} + j$	$\frac{20}{30} + j$
$(0, 2, 0)$	$\frac{13}{30} + j$	$\frac{19}{30} + j$	$(0, 2, 2)$	$\frac{28}{30} + j$	$\frac{34}{30} + j$
$(1, 2, 0)$	$\frac{37}{30} + j$	$\frac{1}{30} + j$	$(1, 2, 2)$	$\frac{22}{30} + j$	$\frac{16}{30} + j$
$(2, 2, 0)$	$\frac{25}{30} + j$	$\frac{25}{30} + j$	$(2, 2, 2)$	$\frac{10}{30} + j$	$\frac{10}{30} + j$

4.4. $\mathcal{N}_5$

Let $E_1(x) = {}_3E_2(-\frac{1}{10} + l_0, \frac{1}{10} + l_1, \frac{1}{2} + l_2; \frac{2}{3} - m_1, \frac{1}{3} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{3}{10} + l_0, \frac{3}{10} + l_1, \frac{1}{2} + l_2; \frac{2}{3} - m_1, \frac{1}{3} - m_2; x)$ be equations in Table 2.

From Corollary 3.7, we obtain the following proposition.

PROPOSITION 4.8. *The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 7, where j runs over all integers ≥ 0 .*

COROLLARY 4.9. *The set $\mathcal{N}_5$ is equal to the set $\{\frac{60}{30}j \mid j \in \mathbb{N}, j \neq 2, 4, 7\}$.*

Remark 4.4. The value $-e_{p,q,r}$ of $E_2(x)$ attains the minimal value $1/30$ when $(l; m)$ is the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. The corresponding ${}_3E_2(\frac{7}{10}, \frac{3}{10}, \frac{1}{2}; \frac{2}{3}, \frac{4}{3}; x)$ is equal to Min.5 in [Kt2, Table 1].

4.5. $\mathcal{N}_6$

Let $E(x) = {}_3E_2(-\frac{3}{10} + l_0, \frac{3}{10} + l_1, \frac{1}{2} + l_2; \frac{3}{5} - m_1, \frac{2}{5} - m_2; x)$ be the equation in Table 2.

From Corollary 3.9, we obtain the following proposition.

PROPOSITION 4.10. *The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 8, where j runs over all integers ≥ 0 .*

COROLLARY 4.11. *The set $\mathcal{N}_6$ is equal to the set $\{\frac{60}{10}j \mid j \in \mathbb{N}\}$.*

Remark 4.5. The value $-e_{p,q,r}$ of $E(x)$ attains the minimal value $1/10$ when $(l; m)$ is the vertex $(1, 1, 0; -1, -1)$ of $\mathcal{Z}_{2,1,0}$. The corresponding ${}_3E_2(\frac{7}{10}, \frac{13}{10}, \frac{1}{2}; \frac{8}{5}, \frac{7}{5}; x)$ is equal to Min.6 in [Kt2, Table 1].

TABLE 8. The set of $-e_{p,q,r}$ for $\mathcal{E}_6$.

(p, q, r)	$-e_{p,q,r}$ of E	(p, q, r)	$-e_{p,q,r}$ of E
(0, 0, 0)	$\frac{3}{10} + j$	(0, 0, 2)	$\frac{8}{10} + j$
(1, 0, 0)	$\frac{7}{10} + j$	(1, 0, 2)	$\frac{2}{10} + j$
(2, 0, 0)	$\frac{5}{10} + j$	(2, 0, 2)	$\frac{10}{10} + j$
(0, 1, 0)	$\frac{9}{10} + j$	(0, 1, 2)	$\frac{4}{10} + j$
(1, 1, 0)	$\frac{3}{10} + j$	(1, 1, 2)	$\frac{8}{10} + j$
(2, 1, 0)	$\frac{1}{10} + j$	(2, 1, 2)	$\frac{6}{10} + j$
(0, 2, 0)	$\frac{7}{10} + j$	(0, 2, 2)	$\frac{12}{10} + j$
(1, 2, 0)	$\frac{11}{10} + j$	(1, 2, 2)	$\frac{6}{10} + j$
(2, 2, 0)	$\frac{9}{10} + j$	(2, 2, 2)	$\frac{4}{10} + j$

TABLE 9. The set of $-e_{p,q,r}$ for $\mathcal{E}_7$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
(0, 0, 0)	$\frac{1}{30} + j$	$\frac{7}{30} + j$	(0, 0, 2)	$\frac{16}{30} + j$	$\frac{22}{30} + j$
(1, 0, 0)	$\frac{25}{30} + j$	$\frac{25}{30} + j$	(1, 0, 2)	$\frac{40}{30} + j$	$\frac{40}{30} + j$
(2, 0, 0)	$\frac{19}{30} + j$	$\frac{13}{30} + j$	(2, 0, 2)	$\frac{34}{30} + j$	$\frac{28}{30} + j$
(0, 1, 0)	$\frac{25}{30} + j$	$\frac{25}{30} + j$	(0, 1, 2)	$\frac{10}{30} + j$	$\frac{10}{30} + j$
(1, 1, 0)	$\frac{19}{30} + j$	$\frac{13}{30} + j$	(1, 1, 2)	$\frac{34}{30} + j$	$\frac{28}{30} + j$
(2, 1, 0)	$\frac{43}{30} + j$	$\frac{31}{30} + j$	(2, 1, 2)	$\frac{28}{30} + j$	$\frac{16}{30} + j$
(0, 2, 0)	$\frac{7}{30} + j$	$\frac{19}{30} + j$	(0, 2, 2)	$\frac{22}{30} + j$	$\frac{34}{30} + j$
(1, 2, 0)	$\frac{1}{30} + j$	$\frac{7}{30} + j$	(1, 2, 2)	$\frac{16}{30} + j$	$\frac{22}{30} + j$
(2, 2, 0)	$\frac{25}{30} + j$	$\frac{25}{30} + j$	(2, 2, 2)	$\frac{10}{30} + j$	$\frac{10}{30} + j$

4.6. $\mathcal{N}_7$

Let $E_1(x) = {}_3E_2(-\frac{1}{30} + l_0, \frac{1}{6} + l_1, \frac{11}{30} + l_2; \frac{4}{5} - m_1, \frac{1}{5} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{7}{30} + l_0, \frac{1}{6} + l_1, \frac{17}{30} + l_2; \frac{3}{5} - m_1, \frac{2}{5} - m_2; x)$ be equations in Table 2.

From Corollary 3.11, we obtain the following proposition.

PROPOSITION 4.12. *The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 9, where j runs over all integers ≥ 0 .*

COROLLARY 4.13. *The set $\mathcal{N}_7$ is equal to the set $\{\frac{360}{30}j \mid j \in \mathbb{N}, j \equiv 1 \pmod{3}, j \neq 4\}$.*

Remark 4.6. The value $-e_{p,q,r}$ of $E_1(x)$ is $1/30$ only when $(l; m)$ is the vertex $(0, 0, 0; 0, 0)$ of $\mathcal{Z}_{0,0,0}$ or the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. The corresponding two

TABLE 10. The set of $-e_{p,q,r}$ for $\mathcal{E}_8$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	$-e_{p,q,r}$ of E_3	$-e_{p,q,r}$ of E_4
(0, 0, 0)	$\frac{2}{60} + j$	$\frac{14}{60} + j$	$\frac{10}{60} + j$	$\frac{10}{60} + j$
(1, 0, 0)	$\frac{50}{60} + j$	$\frac{50}{60} + j$	$\frac{46}{60} + j$	$\frac{58}{60} + j$
(2, 0, 0)	$\frac{38}{60} + j$	$\frac{26}{60} + j$	$\frac{34}{60} + j$	$\frac{22}{60} + j$
(0, 1, 0)	$\frac{47}{60} + j$	$\frac{59}{60} + j$	$\frac{55}{60} + j$	$\frac{55}{60} + j$
(1, 1, 0)	$\frac{35}{60} + j$	$\frac{35}{60} + j$	$\frac{31}{60} + j$	$\frac{43}{60} + j$
(2, 1, 0)	$\frac{23}{60} + j$	$\frac{11}{60} + j$	$\frac{19}{60} + j$	$\frac{7}{60} + j$
(0, 2, 0)	$\frac{17}{60} + j$	$\frac{29}{60} + j$	$\frac{25}{60} + j$	$\frac{25}{60} + j$
(1, 2, 0)	$\frac{5}{60} + j$	$\frac{5}{60} + j$	$\frac{1}{60} + j$	$\frac{13}{60} + j$
(2, 2, 0)	$\frac{53}{60} + j$	$\frac{41}{60} + j$	$\frac{49}{60} + j$	$\frac{37}{60} + j$
(0, 0, 2)	$\frac{32}{60} + j$	$\frac{44}{60} + j$	$\frac{40}{60} + j$	$\frac{40}{60} + j$
(1, 0, 2)	$\frac{20}{60} + j$	$\frac{20}{60} + j$	$\frac{16}{60} + j$	$\frac{28}{60} + j$
(2, 0, 2)	$\frac{68}{60} + j$	$\frac{56}{60} + j$	$\frac{64}{60} + j$	$\frac{52}{60} + j$
(0, 1, 2)	$\frac{17}{60} + j$	$\frac{29}{60} + j$	$\frac{25}{60} + j$	$\frac{25}{60} + j$
(1, 1, 2)	$\frac{65}{60} + j$	$\frac{65}{60} + j$	$\frac{61}{60} + j$	$\frac{73}{60} + j$
(2, 1, 2)	$\frac{53}{60} + j$	$\frac{41}{60} + j$	$\frac{49}{60} + j$	$\frac{37}{60} + j$
(0, 2, 2)	$\frac{47}{60} + j$	$\frac{59}{60} + j$	$\frac{55}{60} + j$	$\frac{55}{60} + j$
(1, 2, 2)	$\frac{35}{60} + j$	$\frac{35}{60} + j$	$\frac{31}{60} + j$	$\frac{43}{60} + j$
(2, 2, 2)	$\frac{23}{60} + j$	$\frac{11}{60} + j$	$\frac{19}{60} + j$	$\frac{7}{60} + j$

equations ${}_3E_2(-\frac{1}{30}, \frac{1}{6}, \frac{11}{30}; \frac{4}{5}, \frac{1}{5}; x)$ and ${}_3E_2(\frac{29}{30}, \frac{1}{6}, \frac{11}{30}; \frac{4}{5}, \frac{6}{5}; x)$ are equal up to a projective transformation. The first one is equal to Min.7 in [Kt2, Table 1].

4.7. $\mathcal{N}_8$

Let $E_1(x) = {}_3E_2(-\frac{1}{30} + l_0, \frac{1}{6} + l_1, \frac{11}{30} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$, $E_2(x) = {}_3E_2(-\frac{7}{30} + l_0, \frac{1}{6} + l_1, \frac{17}{30} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$, $E_3(x) = {}_3E_2(-\frac{1}{6} + l_0, \frac{7}{30} + l_1, \frac{13}{30} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ and $E_4(x) = {}_3E_2(-\frac{1}{6} + l_0, \frac{1}{30} + l_1, \frac{19}{30} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ be equations in Table 2.

From Lemma 3.3, $E_j(x)$, $1 \leq j \leq 4$, are in $\mathcal{E}_8$ for all $(l; m) \in \mathcal{Z}$, and we obtain the following proposition.

PROPOSITION 4.14. *The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 10, where j runs over all integers ≥ 0 .*

COROLLARY 4.15. *The set $\mathcal{N}_8$ is equal to the set $\{\frac{360}{60}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{3}, j \neq 4, 8\}$.*

TABLE 11. The set of $-e_{p,q,r}$ for $\mathcal{E}_9$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
$(0, 0, 0)$	$\frac{2}{12} + j$	$\frac{2}{12} + j$	$(0, 0, 2)$	$\frac{10}{12} + j$	$\frac{6}{12} + j$
$(1, 0, 0)$	$\frac{10}{12} + j$	$\frac{10}{12} + j$	$(1, 0, 2)$	$\frac{6}{12} + j$	$\frac{2}{12} + j$
$(2, 0, 0)$	$\frac{4}{12} + j$	$\frac{8}{12} + j$	$(2, 0, 2)$	$\frac{12}{12} + j$	$\frac{12}{12} + j$
$(0, 1, 0)$	$\frac{11}{12} + j$	$\frac{11}{12} + j$	$(0, 1, 2)$	$\frac{7}{12} + j$	$\frac{3}{12} + j$
$(1, 1, 0)$	$\frac{7}{12} + j$	$\frac{7}{12} + j$	$(1, 1, 2)$	$\frac{15}{12} + j$	$\frac{11}{12} + j$
$(2, 1, 0)$	$\frac{1}{12} + j$	$\frac{5}{12} + j$	$(2, 1, 2)$	$\frac{9}{12} + j$	$\frac{9}{12} + j$
$(0, 2, 0)$	$\frac{5}{12} + j$	$\frac{5}{12} + j$	$(0, 2, 2)$	$\frac{13}{12} + j$	$\frac{9}{12} + j$
$(1, 2, 0)$	$\frac{1}{12} + j$	$\frac{1}{12} + j$	$(1, 2, 2)$	$\frac{9}{12} + j$	$\frac{5}{12} + j$
$(2, 2, 0)$	$\frac{7}{12} + j$	$\frac{11}{12} + j$	$(2, 2, 2)$	$\frac{3}{12} + j$	$\frac{3}{12} + j$

Remark 4.7. The value $-e_{p,q,r}$ of $E_3(x)$ attains the minimal value $1/60$ when $(l; m)$ is the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. The corresponding ${}_3E_2(\frac{5}{6}, \frac{7}{30}, \frac{13}{30}; \frac{3}{4}, \frac{5}{4}; x)$ is equal to Min.8 in [Kt2, Table 1].

4.8. $\mathcal{N}_9$

Let $E_1(x) = {}_3E_2(-\frac{1}{6} + l_0, \frac{1}{6} + l_1, \frac{2}{3} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{1}{6} + l_0, \frac{1}{6} + l_1, \frac{1}{3} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ be equations in Table 2.

From Lemma 3.3, $E_1(x)$, $E_2(x)$ are in $\mathcal{E}_9$ for all $(l; m) \in \mathcal{Z}$, and we obtain the following proposition.

PROPOSITION 4.16. *The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 11, where j runs over all integers ≥ 0 .*

COROLLARY 4.17. *The set $\mathcal{N}_9$ is equal to the set $\{\frac{216}{12}j \mid j \in \mathbb{N}\}$.*

Remark 4.8. The value $-e_{p,q,r}$ of $E_1(x)$ is $1/12$ when $(l; m)$ is the vertex $(1, 1, 0; -1, -1)$ of $\mathcal{Z}_{2,1,0}$ or the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. That of $E_2(x)$ is $1/12$ when $(l; m)$ is the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. The corresponding three equations are ${}_3E_2(\frac{5}{6}, \frac{7}{6}, \frac{2}{3}; \frac{7}{4}, \frac{5}{4}; x)$, ${}_3E_2(\frac{5}{6}, \frac{1}{6}, \frac{2}{3}; \frac{3}{4}, \frac{5}{4}; x)$ and ${}_3E_2(\frac{5}{6}, \frac{1}{6}, \frac{1}{3}; \frac{3}{4}, \frac{5}{4}; x)$. The first one is equal to Min.9.2, the second one to Min.9.1, and the third one to Min.9.3 in [Kt2, Table 1].

4.9. $\mathcal{N}_{10}$

Let $E_1(x) = {}_3E_2(-\frac{1}{9} + l_0, \frac{2}{9} + l_1, \frac{5}{9} + l_2; \frac{5}{6} - m_1, \frac{1}{2} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{2}{9} + l_0, \frac{1}{9} + l_1, \frac{4}{9} + l_2; \frac{1}{2} - m_1, \frac{1}{6} - m_2; x)$ be equations in Table 2.

From Lemma 3.3, $E_1(x)$, $E_2(x)$ are in $\mathcal{E}_{10}$ for all $(l; m) \in \mathcal{Z}$, and we obtain the following proposition.

PROPOSITION 4.18. *The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 12, where j runs over all integers ≥ 0 .*

TABLE 12. The set of $-e_{p,q,r}$ for $\mathcal{E}_{10}$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
(0, 0, 0)	$\frac{2}{18} + j$	$\frac{4}{18} + j$	(0, 0, 2)	$\frac{8}{18} + j$	$\frac{16}{18} + j$
(1, 0, 0)	$\frac{14}{18} + j$	$\frac{16}{18} + j$	(1, 0, 2)	$\frac{2}{18} + j$	$\frac{10}{18} + j$
(2, 0, 0)	$\frac{8}{18} + j$	$\frac{10}{18} + j$	(2, 0, 2)	$\frac{14}{18} + j$	$\frac{22}{18} + j$
(0, 1, 0)	$\frac{17}{18} + j$	$\frac{13}{18} + j$	(0, 1, 2)	$\frac{5}{18} + j$	$\frac{7}{18} + j$
(1, 1, 0)	$\frac{11}{18} + j$	$\frac{7}{18} + j$	(1, 1, 2)	$\frac{17}{18} + j$	$\frac{19}{18} + j$
(2, 1, 0)	$\frac{5}{18} + j$	$\frac{1}{18} + j$	(2, 1, 2)	$\frac{11}{18} + j$	$\frac{13}{18} + j$
(0, 2, 0)	$\frac{11}{18} + j$	$\frac{7}{18} + j$	(0, 2, 2)	$\frac{17}{18} + j$	$\frac{19}{18} + j$
(1, 2, 0)	$\frac{5}{18} + j$	$\frac{1}{18} + j$	(1, 2, 2)	$\frac{11}{18} + j$	$\frac{13}{18} + j$
(2, 2, 0)	$\frac{17}{18} + j$	$\frac{13}{18} + j$	(2, 2, 2)	$\frac{5}{18} + j$	$\frac{7}{18} + j$

COROLLARY 4.19. *The set $\mathcal{N}_{10}$ is equal to the set $\{\frac{216}{18}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{3}\}$.*

Remark 4.9. The value $-e_{p,q,r}$ of $E_2(x)$ is $1/18$ when $(l; m)$ is the vertex $(1, 1, 0; -1, -1)$ of $\mathcal{Z}_{2,1,0}$ or the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. The corresponding two equations ${}_3E_2(\frac{7}{9}, \frac{10}{9}, \frac{4}{9}; \frac{3}{2}, \frac{7}{6}; x)$ and ${}_3E_2(\frac{7}{9}, \frac{1}{9}, \frac{4}{9}; \frac{1}{2}, \frac{7}{6}; x)$ are equal to Min.10.2 and Min.10.1 in [Kt2, Table 1].

4.10. $\mathcal{N}_{11}$

Let $E_1(x) = {}_3E_2(-\frac{1}{9} + l_0, \frac{2}{9} + l_1, \frac{5}{9} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{2}{9} + l_0, \frac{1}{9} + l_1, \frac{4}{9} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ be equations in Table 2.

From Lemma 3.3, $E_1(x), E_2(x)$ are in $\mathcal{E}_{11}$ for all $(l; m) \in \mathcal{Z}$, and we obtain the following proposition.

PROPOSITION 4.20. *The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 13, where j runs over all integers ≥ 0 .*

COROLLARY 4.21. *The set $\mathcal{N}_{11}$ is equal to the set $\{\frac{216}{36}j \mid j \in \mathbb{N}, j \equiv \pm 1, \pm 4, \pm 5 \pmod{12}\}$.*

Remark 4.10. The value $-e_{p,q,r}$ of $E_1(x)$ is $1/36$ when $(l; m)$ is the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. The corresponding equation ${}_3E_2(\frac{8}{9}, \frac{2}{9}, \frac{5}{9}; \frac{3}{4}, \frac{5}{4}; x)$ is equal to Min.11.1 in [Kt2, Table 1].

4.11. $\mathcal{N}_{12}$

Let $E_1(x) = {}_3E_2(-\frac{1}{12} + l_0, \frac{1}{6} + l_1, \frac{5}{12} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ and $E_2(x) = {}_3E_2(-\frac{1}{6} + l_0, \frac{1}{12} + l_1, \frac{7}{12} + l_2; \frac{3}{4} - m_1, \frac{1}{4} - m_2; x)$ be equations in Table 2.

From Corollary 3.13, we obtain the following proposition.

PROPOSITION 4.22. *The set of $-e_{p,q,r}$ for $(l; m) \in \mathcal{Z}_{p,q,r}$ is given in Table 14, where j runs over all integers ≥ 0 .*

TABLE 13. The set of $-e_{p,q,r}$ for $\mathcal{E}_{11}$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
$(0, 0, 0)$	$\frac{4}{36} + j$	$\frac{8}{36} + j$	$(0, 0, 2)$	$\frac{28}{36} + j$	$\frac{20}{36} + j$
$(1, 0, 0)$	$\frac{28}{36} + j$	$\frac{32}{36} + j$	$(1, 0, 2)$	$\frac{16}{36} + j$	$\frac{8}{36} + j$
$(2, 0, 0)$	$\frac{16}{36} + j$	$\frac{20}{36} + j$	$(2, 0, 2)$	$\frac{40}{36} + j$	$\frac{32}{36} + j$
$(0, 1, 0)$	$\frac{31}{36} + j$	$\frac{35}{36} + j$	$(0, 1, 2)$	$\frac{19}{36} + j$	$\frac{11}{36} + j$
$(1, 1, 0)$	$\frac{19}{36} + j$	$\frac{23}{36} + j$	$(1, 1, 2)$	$\frac{43}{36} + j$	$\frac{35}{36} + j$
$(2, 1, 0)$	$\frac{7}{36} + j$	$\frac{11}{36} + j$	$(2, 1, 2)$	$\frac{31}{36} + j$	$\frac{23}{36} + j$
$(0, 2, 0)$	$\frac{13}{36} + j$	$\frac{17}{36} + j$	$(0, 2, 2)$	$\frac{37}{36} + j$	$\frac{29}{36} + j$
$(1, 2, 0)$	$\frac{1}{36} + j$	$\frac{5}{36} + j$	$(1, 2, 2)$	$\frac{25}{36} + j$	$\frac{17}{36} + j$
$(2, 2, 0)$	$\frac{25}{36} + j$	$\frac{29}{36} + j$	$(2, 2, 2)$	$\frac{13}{36} + j$	$\frac{5}{36} + j$

TABLE 14. The set of $-e_{p,q,r}$ for $\mathcal{E}_{12}$.

(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2	(p, q, r)	$-e_{p,q,r}$ of E_1	$-e_{p,q,r}$ of E_2
$(0, 0, 0)$	$\frac{1}{12} + j$	$\frac{14}{12} + j$	$(0, 0, 2)$	$\frac{7}{12} + j$	$\frac{8}{12} + j$
$(1, 0, 0)$	$\frac{10}{12} + j$	$\frac{11}{12} + j$	$(1, 0, 2)$	$\frac{16}{12} + j$	$\frac{5}{12} + j$
$(2, 0, 0)$	$\frac{7}{12} + j$	$\frac{5}{12} + j$	$(2, 0, 2)$	$\frac{13}{12} + j$	$\frac{11}{12} + j$
$(0, 1, 0)$	$\frac{10}{12} + j$	$\frac{11}{12} + j$	$(0, 1, 2)$	$\frac{4}{12} + j$	$\frac{5}{12} + j$
$(1, 1, 0)$	$\frac{7}{12} + j$	$\frac{8}{12} + j$	$(1, 1, 2)$	$\frac{13}{12} + j$	$\frac{14}{12} + j$
$(2, 1, 0)$	$\frac{16}{12} + j$	$\frac{2}{12} + j$	$(2, 1, 2)$	$\frac{10}{12} + j$	$\frac{8}{12} + j$
$(0, 2, 0)$	$\frac{4}{12} + j$	$\frac{5}{12} + j$	$(0, 2, 2)$	$\frac{10}{12} + j$	$\frac{11}{12} + j$
$(1, 2, 0)$	$\frac{1}{12} + j$	$\frac{2}{12} + j$	$(1, 2, 2)$	$\frac{7}{12} + j$	$\frac{8}{12} + j$
$(2, 2, 0)$	$\frac{10}{12} + j$	$\frac{8}{12} + j$	$(2, 2, 2)$	$\frac{4}{12} + j$	$\frac{14}{12} + j$

COROLLARY 4.23. The set $\mathcal{N}_{12}$ is equal to the set $\{\frac{36}{12}j \mid j \in \mathbb{N}, j \not\equiv 0 \pmod{3}\}$.

Remark 4.11. The value $-e_{p,q,r}$ of $E_1(x)$ is $1/12$ only when $(l; m)$ is the vertex $(0, 0, 0; 0, 0)$ of $\mathcal{Z}_{0,0,0}$ or the vertex $(1, 0, 0; 0, -1)$ of $\mathcal{Z}_{1,2,0}$. The corresponding two equations ${}_3E_2(-\frac{1}{12}, \frac{1}{6}, \frac{5}{12}; \frac{3}{4}, \frac{1}{4}; x)$ and ${}_3E_2(\frac{11}{12}, \frac{1}{6}, \frac{5}{12}; \frac{3}{4}, \frac{5}{4}; x)$ are equal up to a projective transformation. The first one is equal to Min.12 in [Kt2, Table 1].

5. The defining equations of the minimal Schwarz images

In [Kt2] and in the previous sections, we found the equations ${}_3E_2 \in \mathcal{E}_k$ which give the minimal values of $\mathcal{N}_k$. In this section, we give the defining equations of these minimal Schwarz images for $2 \leq k \leq 11$. That for $k = 12$ was given in [Kt4]. The process of finding

the defining equation is similar for each ${}_3E_2$, hence so we give the process only for $k = 2, 3, 4$.

5.1. Nos 2, 3 and 4

Let $G = G_{336}$ be the complex reflection group of order 336. All the equations ${}_3E_2$ in the class of Nos 2, 3 and 4 have the projective monodromy group $\mathbf{P}(G)$. As is shown in [K11], we have the following basic set of G -invariant polynomials:

$$\begin{aligned} F_4(u) &= u_1^3 u_2 + u_2^3 u_3 + u_3^3 u_1, \\ F_6(u) &= \frac{1}{54} \det(\partial^2 F_4 / \partial u_j \partial u_k) = 5(u_1 u_2 u_3)^2 - u_1^5 u_3 - u_3^5 u_2 - u_2^5 u_1, \\ F_{14}(u) &= \frac{1}{9} \det \begin{pmatrix} \partial^2 F_4 / \partial u_1 \partial u_1 & \partial^2 F_4 / \partial u_1 \partial u_2 & \partial^2 F_4 / \partial u_1 \partial u_3 & \partial F_6 / \partial u_1 \\ \partial^2 F_4 / \partial u_2 \partial u_1 & \partial^2 F_4 / \partial u_2 \partial u_2 & \partial^2 F_4 / \partial u_2 \partial u_3 & \partial F_6 / \partial u_2 \\ \partial^2 F_4 / \partial u_3 \partial u_1 & \partial^2 F_4 / \partial u_3 \partial u_2 & \partial^2 F_4 / \partial u_3 \partial u_3 & \partial F_6 / \partial u_3 \\ \partial F_6 / \partial u_1 & \partial F_6 / \partial u_2 & \partial F_6 / \partial u_3 & 0 \end{pmatrix} \\ &= u_1^{14} + u_2^{14} + u_3^{14} + \cdots. \end{aligned}$$

5.1.1. No. 2. The Schwarz image of ${}_3E_2(\frac{9}{14}, \frac{11}{14}, \frac{15}{14}; \frac{4}{3}, \frac{5}{3})$ is defined by $F_4(u_1, u_2, u_3) = 0$. This means that if we choose a suitable system $u_1(x), u_2(x), u_3(x)$ of solutions, then $F_4(u_1(x), u_2(x), u_3(x))$ vanishes identically. For example,

$$\begin{aligned} u_1(x) &= (12^3 x)^{-9/14} {}_3F_2\left(\frac{9}{14}, \frac{13}{42}, -\frac{1}{42}; \frac{6}{7}, \frac{4}{7}; \frac{1}{x}\right), \\ u_2(x) &= -(12^3 x)^{-15/14} {}_3F_2\left(\frac{15}{14}, \frac{31}{42}, \frac{17}{42}; \frac{10}{7}, \frac{9}{7}; \frac{1}{x}\right), \\ u_3(x) &= (12^3 x)^{-11/14} {}_3F_2\left(\frac{11}{14}, \frac{19}{42}, \frac{5}{42}; \frac{8}{7}, \frac{5}{7}; \frac{1}{x}\right) \end{aligned}$$

form such a system.

5.1.2. No. 3. The Schwarz image of ${}_3E_2(\frac{3}{14}, \frac{5}{14}, \frac{13}{14}; \frac{5}{4}, \frac{3}{4})$ is defined by $F_6(u_2, u_3, v_3) = 0$. For example, the system

$$\begin{aligned} u_1(x) &= (2^8 x)^{-3/14} {}_3F_2\left(\frac{3}{14}, -\frac{1}{28}, \frac{13}{28}; \frac{6}{7}, \frac{2}{7}; \frac{1}{x}\right), \\ u_2(x) &= (2^8 x)^{-5/14} {}_3F_2\left(\frac{5}{14}, \frac{3}{28}, \frac{17}{28}; \frac{8}{7}, \frac{3}{7}; \frac{1}{x}\right), \\ u_3(x) &= (2^8 x)^{-13/14} {}_3F_2\left(\frac{13}{14}, \frac{19}{28}, \frac{33}{28}; \frac{11}{7}, \frac{12}{7}; \frac{1}{x}\right) \end{aligned}$$

satisfies the equation $F_6(u) = 0$.

5.1.3. No. 4. We said in Remark 4.3 (or in [Kt2]), that the minimal Schwarz images are given by $E_1 := {}_3E_2(-\frac{3}{14}, \frac{3}{14}, -\frac{1}{14}; \frac{4}{7}, -\frac{1}{7})$ and $E_2 := {}_3E_2(-\frac{3}{14}, \frac{3}{14}, -\frac{1}{14}; -\frac{3}{7}, \frac{6}{7})$.

The local exponents of $E_1(x)$ at $x = \infty$ are $-\frac{3}{14}, \frac{3}{14}, -\frac{1}{14}$. Let $r_0 = -\frac{3}{14}, r_1 = \frac{3}{14}, r_2 = -\frac{1}{14}$, and

$$y_j(1/x) = x^{-r_j}(1 + c_j^1 x^{-1} + c_j^2 x^{-2} + c_j^3 x^{-3} + \dots)$$

be the solutions of $E_1(x)$ at $x = \infty$. The coefficients c_j^k are obtained by the Frobenius method.

As stated in [Kt2, Section 4], we want to find solutions $v_j(x)$, $j = 0, 1, 2$, such that $F_d(x^{1/7}u_1(x), x^{1/7}u_2(x), x^{1/7}u_3(x))$, $d = 4, 6, 14$, are single-valued (hence rational) functions in x . This is achieved if we put

$$u_1(x) = 2^{9/14}y_0(x), \quad u_2(x) = 2^{-27/14} \cdot 5y_1(x), \quad u_3(x) = 2^{-3/14} \cdot 3y_2(x).$$

Then $F_k(u_1(x), u_2(x), u_3(x))$, $k = 4, 6, 14$, are found to be

$$F_4(u_1(x), u_2(x), u_3(x)) = 32x^{3/7}, \quad F_6(u_1(x), u_2(x), u_3(x)) = -24x^{8/7},$$

$$F_{14}(u_1(x), u_2(x), u_3(x)) = 512(x^3 - 1008x^2 + 9216x - 16384),$$

which implies

$$F_6(u(x))^2/F_4(u(x))^3 = (9x)/512,$$

$$F_{14}(u(x))/F_4(u(x))^2 F_6(u(x)) = (-x^3 + 1008x^2 - 9216x + 16384)/(48x^2).$$

Eliminating x from these two equations we obtain the defining equation

$$F_{14}(u)F_6(u)^3 F_4(u) - \frac{27}{256}F_4(u)^9 + \frac{8}{27}F_4(u)^6 F_6(u)^2 - 21F_4(u)^3 F_6(u)^4 + \frac{32}{27}F_6(u)^6 = 0$$

of the Schwarz image of $E_1(x)$.

The local exponents of $E_2(x)$ at $x = \infty$ are the same as $E_1(x)$. In the same manner as for $E_1(x)$ we find solutions $u_1(x), u_2(x), u_3(x)$ of $E_2(x)$ such that

$$F_4(u_1(x), u_2(x), u_3(x)) = -2^{31/7}x^{3/7}, \quad F_6(u_1(x), u_2(x), u_3(x)) = 2^{-6/7}x^{1/7}(5x - 128),$$

$$F_{14}(u_1(x), u_2(x), u_3(x)) = x^3 - 2^5 \cdot 457x^2 + 2^{11} \cdot 17^2x - 2^{14},$$

which implies

$$\frac{F_6(u(x))^2}{F_4(u(x))^3} = -\frac{(5x - 128)^2}{2^{15}x},$$

$$\frac{F_{14}(u(x))}{F_4(u(x))^2 F_6(u(x))} = \frac{x^3 - 2^5 \cdot 457x^2 + 2^{11} \cdot 17x - 2^{14}}{256x(5x - 128)}.$$

Eliminating x from these two equations we get the defining equation

$$\begin{aligned} & 2^{24} \cdot 5^3 F_{14}(u)^2 F_4(u)^2 + 2^{16} \cdot 5 \cdot 101 \cdot 1861 F_{14}(u) F_6(u) F_4(u)^4 \\ & + 2^{24} \cdot 11 \cdot 23 \cdot F_{14}(u) F_6(u)^3 F_4(u) + 3^2 \cdot 7^4 \cdot 17^4 F_4(u)^9 \\ & + 2^{12} \cdot 104760977 F_6(u)^2 F_4(u)^6 + 2^{21} \cdot 17 \cdot 1753 F_6(u)^4 F_4(u)^3 + 2^{31} F_6(u)^6 = 0 \end{aligned}$$

of the Schwarz image of $E_2(x)$.

The Schwarz image of $E_1(x)$ has 38 (3, 8)-cusps while that of $E_2(x)$ has no cusps (it has only flexes). Consequently, the two Schwarz images are not projectively equivalent to each other.

5.2. Nos 5 and 6

Let $G = G_{120}$ be the complex reflection group of order 120. All the equations ${}_3E_2$ in the class of Nos 5 and 6 have the projective monodromy group $\mathbf{P}(G)$. As is shown in [HK], we have the following basic set of G -invariant polynomials:

$$\begin{aligned} F_2(u) &= u_2^2 - u_1 u_3, \\ F_6(u) &= u_2(u_1^5 - u_3^5 + u_2^5 + 5u_1 u_2 u_3(u_2^2 + u_1 u_3)), \\ F_{10}(u) &= -u_1^{10} - u_3^{10} + (u_1^5 - u_3^5)u_2^3(108u_2^2 + 120u_1 u_3) - 14u_2^{10} - 90u_2^8 u_1 u_3 \\ &\quad - 180u_2^6(u_1 u_3)^2 - 120u_2^4(u_1 u_3)^3 - 90u_2^2(u_1 u_3)^4. \end{aligned}$$

5.2.1. No. 5. The Schwarz image of ${}_3E_2(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}; \frac{4}{3}, \frac{2}{3})$ is defined by $F_2(u_1, u_2, u_3) = 0$.

5.2.2. No. 6. The Schwarz image of ${}_3E_2(\frac{13}{10}, \frac{5}{10}, \frac{7}{10}; \frac{8}{5}, \frac{7}{5})$ is defined by $F_6(u_1, u_2, u_3) - F_2(u_1, u_2, u_3)^3 = 0$.

5.3. Nos 7 and 8

Let $G = G_{2160}$ be the complex reflection group of order 120. All the equation ${}_3E_2$ in the class of Nos 7 and 8 have the projective monodromy group $\mathbf{P}(G)$. As is shown in [HK], we have the following basic set of G -invariant polynomials:

$$\begin{aligned} F_6(u) &= 10u_1^3 u_2^3 + 9u_3(u_1^5 + u_2^5) - 45(u_1 u_2 u_3)^2 - 135u_1 u_2 u_3^4 + 27u_3^6, \\ F_{12}(u) &= \frac{1}{121 \cdot 500} \det(\partial^2 F_6 / \partial u_i \partial u_j) + \frac{1}{6} F_6^2 = -u_1 u_2 (u_1^{10} + u_2^{10}) + \cdots, \\ F_{30}(u) &= \frac{1}{675} \begin{vmatrix} \partial^2 F_6 / \partial u_1 \partial u_1 & \partial^2 F_6 / \partial u_1 \partial u_2 & \partial^2 F_6 / \partial u_1 \partial u_3 & \partial F_{12} / \partial u_1 \\ \partial^2 F_6 / \partial u_2 \partial u_1 & \partial^2 F_6 / \partial u_2 \partial u_2 & \partial^2 F_6 / \partial u_2 \partial u_3 & \partial F_{12} / \partial u_2 \\ \partial^2 F_6 / \partial u_3 \partial u_1 & \partial^2 F_6 / \partial u_3 \partial u_2 & \partial^2 F_6 / \partial u_3 \partial u_3 & \partial F_{12} / \partial u_3 \\ \partial F_{12} / \partial u_1 & \partial F_{12} / \partial u_2 & \partial F_{12} / \partial u_3 & 0 \end{vmatrix} \\ &= 3(u_1^{30} + u_2^{30}) + \cdots. \end{aligned}$$

5.3.1. No. 7. The Schwarz image of ${}_3E_2(-\frac{1}{30}, \frac{5}{30}, \frac{11}{30}; \frac{4}{5}, \frac{1}{5})$ is defined by $F_{12}(u_1, u_2, u_3) = 0$.

5.3.2. No. 8. The Schwarz image of ${}_3E_2(-\frac{1}{30}, \frac{5}{30}, \frac{11}{30}; \frac{3}{4}, \frac{5}{4})$ is defined by $F_6(u_1, u_2, u_3) = 0$.

5.4. Nos 9, 10 and 11

Let $G = G_{648}$ be the complex reflection group of order 648. All the equations ${}_3E_2$ in the class of Nos 9, 10 and 11 have the projective monodromy group $\mathbf{P}(G)$. As is shown in [HK],

we have the following basic set of G -invariant polynomials:

$$F_6(u) = (u_1^3 + u_2^3 + u_3^3)^2 - 12((u_1u_2)^3 + (u_2u_3)^3 + (u_3u_1)^3),$$

$$F_9(u) = (u_2^3 - u_3^3)(u_3^3 - u_1^3)(u_1^3 - u_2^3),$$

$$F_{12}(u) = (u_1^3 + u_2^3 + u_3^3)((u_1^3 + u_2^3 + u_3^3)^3 + 216(u_1u_2u_3)^3).$$

5.4.1. No. 9. The Schwarz images of

$${}_3E_2(\frac{1}{6}, \frac{4}{6}, \frac{5}{6}; \frac{5}{4}, \frac{3}{4}), \quad {}_3E_2(\frac{7}{6}, \frac{4}{6}, \frac{5}{6}; \frac{5}{4}, \frac{7}{4}), \quad {}_3E_2(\frac{1}{6}, \frac{2}{6}, \frac{5}{6}; \frac{5}{4}, \frac{3}{4})$$

are defined by

$$F_6(u)^3 - 384F_9(u)^2 - 4F_6(u)F_{12}(u) = 0, \quad F_9(u)^2 - 432F_6(u)^3 = 0,$$

$$F_6(u)^3 - 48F_9(u)^2 - F_6(u)F_{12}(u) = 0,$$

respectively. The first curve has 36 (3, 4)-cusps over $x = \infty$ and no other cusps. The second curve has 54 (2, 3)-cusps over $x = 0$ and no other cusps. The third curve has 72 (2, 3)-cusps over $x = 1$ and no other cusps. Consequently, these curves are not projectively equivalent to one another.

5.4.2. No. 10. The Schwarz images of

$${}_3E_2(\frac{1}{9}, \frac{4}{9}, \frac{7}{9}; \frac{7}{6}, \frac{3}{6}), \quad {}_3E_2(\frac{10}{9}, \frac{4}{9}, \frac{7}{9}; \frac{7}{6}, \frac{9}{6})$$

are defined by

$$F_6(u)^2 - F_{12}(u) = 0, \quad F_6(u)^2 - 4F_{12}(u) = 0,$$

respectively. The two curves are not projectively equivalent because the latter curve has 36 (2, 3)-cusps over $x = 0$ while the former curve has no cusps.

5.4.3. No. 11. The Schwarz image of ${}_3E_2(\frac{2}{9}, \frac{5}{9}, \frac{8}{9}; \frac{5}{4}, \frac{3}{4})$ is defined by $F_6(u) = 0$.

A. Appendix

Let $G = G_{336}$ be the complex reflection group of order 336 and $F_k(u_1, u_2, u_3)$, $k = 4, 6, 14$, are the G -invariant polynomials given in the previous section.

As we referred in [Kt3], Fricke [Fr] studied a pencil of $\mathbf{P}(G)$ -invariant plane curves $C_{12}(\mu) := \{[u] = [u_1 : u_2 : u_3] \in \mathbf{P}^2 \mid F_6(u)^2 + \mu F_4(u)^3 = 0\}$ of degree 12. He showed that any $C_{12}(\mu)$ with $\mu \neq 0$ is irreducible and that its genus is 3 if $\mu = -4/27$, is 10 if $\mu = 4$, and is 31 otherwise. In [Kt3, Proposition 2.3], we showed that $C_{12}(-4/27)$ is the Schwarz image of ${}_3E_2(-\frac{1}{14}, \frac{3}{14}, \frac{5}{14}; \frac{2}{3}, \frac{1}{3})$ and $C_{12}(4)$ is that of ${}_3E_2(-\frac{1}{14}, \frac{3}{14}, \frac{5}{14}; \frac{3}{4}, \frac{1}{4})$.

In Subsections A.1 and A.2 below, we study the pencil

$$C_{14}(\mu) = \{[u] \in \mathbf{P}^2 \mid F_{14}(u) + \mu F_4(u)^2 F_6(u) = 0\}$$

and the net

$$C_{18}(\mu_1, \mu_2) = \{[u] \in \mathbf{P}^2 \mid (F_{14}F_4 + \mu_1 F_6^3 + \mu_2 F_6 F_4^3)(u) = 0\}$$

of $\mathbf{P}(G)$ -invariant curves of degree 14 and 18 respectively.

Before the study we give a lemma concerning $\mathbf{P}(G)$ -orbits on $\mathbf{P}^2$. Let C_{21} be the zeros of the Jacobian $F_{21}(u)$ of $F_4(u)$, $F_6(u)$ and $F_{14}(u)$. It is the union of 21 mirrors of complex reflections in G . Since $F_{21}(u)^2$ is G -invariant, there is a polynomial $P_J(x_4, x_6, x_{14})$ such that $F_{21}(u)^2 = P_J(F_4(u), F_6(u), F_{14}(u))$ holds (see [Kt3, equation (1.1)]).

The following lemma is Lemma 1.1 in [Kt3], already known from [Kl1, Fr, Adl].

LEMMA A.1. *There exist only five G -orbits with cardinality less than 84, that is, p_3, p_4, p_7, p'_3 and p'_4 with $\sharp p_3 = 56, \sharp p_4 = 42, \sharp p_7 = 24, \sharp p'_3 = 28$ and $\sharp p'_4 = 21$. The sets p_3 and p_7 do not intersect with C_{21} while p'_3, p'_4 and p_4 are contained in C_{21} . For $v \in C_{21} - p'_3 \cup p'_4 \cup p_4$, the orbit $G \cdot v$ has cardinality 84. For v not in C_{21} or p_j ($j = 3, 4, 7$) or p'_j ($j = 3, 4$), the orbit $G \cdot v$ consists of 168 points.*

The following lemma is Proposition 2.4 in [Kt3].

LEMMA A.2. *As subsets of $\mathbf{P}^2$, we have*

$$\begin{aligned} p_3 &= \{F_4 = 0, F_{14} = 0\}, & p_4 &= \{F_6 = 0, F_{14} = 0\}, & p_7 &= \{F_4 = 0, F_6 = 0\}, \\ p'_3 &= \{F_6^2 - \frac{4}{27}F_4^3 = 0, F_{14} + \frac{8}{3}F_4^2F_6 = 0\}, \\ p'_4 &= \{F_6^2 + 4F_4^3 = 0, F_{14} - 72F_4^2F_6 = 0\}. \end{aligned}$$

A.1. *The pencil $C_{14}(\mu)$*

LEMMA A.3. *If $\mu^2 + 131\mu + 4376 = 0$, that is, if*

$$\mu = -\frac{131 \pm \sqrt{-7}}{2}, \quad (\text{A.1})$$

then $C_{14}(\mu)$ is a union of seven quadrics, otherwise it is irreducible.

Proof. As is shown in [Kl2, (12)], if (A.1) holds then we have

$$\begin{aligned} &F_{14}(u) + \mu F_4(u)^2 F_6(u) \\ &= \prod_{j=0}^6 \left[\zeta^{2j} u_1^2 + \zeta^j u_2^2 + \zeta^{4j} u_3^2 - \frac{1 \pm \sqrt{-7}}{2} (\zeta^{5j} u_1 u_2 + \zeta^{6j} u_2 u_3 + \zeta^{3j} u_3 u_1) \right], \end{aligned}$$

where $\zeta = e^{2\pi\sqrt{-1}/7}$. If (A.1) does not hold, then the irreducibility of $C_{14}(\mu)$ is shown in the same way as Fricke proved the irreducibility of $C_{12}(\mu)$ ($\mu \neq 0$) in [Fr]. $\square$

We define a rational map π_μ of $C_{14}(\mu)$ to $\mathbf{P}^1$ by

$$z = \pi_\mu(u) := \frac{F_6(u)^2}{F_4(u)^3}. \quad (\text{A.2})$$

Since $F_j, j = 4, 6, 14$, are algebraically independent, π_μ is defined at every point of $C_{14}(\mu)$. The proof of the following lemma is the same as that of [Kt3, Lemma 3.2].

LEMMA A.4. *π_μ is a G -quotient map of $C_{14}(\mu)$ and we have*

$$\pi_\mu^{-1}(0) = p_4 \quad \text{and} \quad \pi_\mu^{-1}(\infty) = p_3.$$

Substituting $F_{14}(u) = -\mu F_4(u)^2 F_6(u)$ and $F_6(u) = \sqrt{z F_4(u)^3}$ in the equality $F_{21}(u)^2 = P_J(F_4(u), F_6(u), F_{14}(u))$, we have $F_{21}(u)^2 / (F_4(u)^9 F_6(u)) = \tilde{P}(\mu, \pi_\mu(u))$ if $[u] \in C_{14}(\mu) - p_3 \cup p_4$, where $\tilde{P}(\mu, z)$ is the following cubic polynomial:

$$\tilde{P}(\mu, z) = 1728z^3 - 112z^2(9\mu + 536) - z(\mu^3 + 88\mu^2 + 1088\mu - 22016) + 256(\mu - 8).$$

LEMMA A.5. *The discriminant of $\tilde{P}(\mu, z)$ is*

$$(\mu^2 + 131\mu + 4376)(3\mu - 8)^3(\mu + 72)^4$$

up to a (non-zero) constant multiplication.

- If $\mu^2 + 131\mu + 4376 = 0$ then $C_{14}(\mu)$ is reducible (see Lemma A.2).
- If $\mu = 8/3$ then $\tilde{P}(\mu, z)$ has a double zero $4/27$ and a simple zero 36 .
- If $\mu = -72$ then $\tilde{P}(\mu, z)$ has a double zero -4 and a simple zero $20/27$.
- $\tilde{P}(\mu, 0) = 0$ if and only if $\mu = 8$ and we have

$$\tilde{P}(8, z) = z(z - 4(133 + 50\sqrt{7})/27)(z - 4(133 - 50\sqrt{7})/27).$$

THEOREM A.6. *Assume $\mu^2 + 131\mu + 4376 \neq 0$. Then the following hold.*

- (0) $\pi_\mu^{-1}(\infty) = p_3$, each point of which is smooth in $C_{14}(\mu)$. If $\mu \neq 8$, then $\pi_\mu^{-1}(0) = p_4$, each point of which is smooth in $C_{14}(\mu)$.
- (1) Assume $\mu = 8/3$. Then $\pi_\mu^{-1}(4/27) = p'_3$ and each point of p'_3 is a node in $C_{14}(\mu)$. $C_{14}(\mu)$ has no other singular points. $\sharp\pi_\mu^{-1}(36) = 84$. If $z \neq 0, 4/27, 36, \infty$, then $\sharp\pi_\mu^{-1}(z) = 168$. The genus of $C_{14}(\mu)$ is 50.
- (2) Assume $\mu = -72$. Then $\pi_\mu^{-1}(-4) = p'_4$ and each point of p'_4 is a node in $C_{14}(\mu)$. $C_{14}(\mu)$ has no other singular points. $\sharp\pi_\mu^{-1}(20/27) = 84$. If $z \neq 0, -4, 20/27, \infty$, then $\sharp\pi_\mu^{-1}(z) = 168$. The genus of $C_{14}(\mu)$ is 57.
- (3) Assume $\mu = 8$. Then $\tilde{P}(\mu, z)$ has three zeros $0, z_+, z_-$, where $z_\pm = 4(133 \pm 50\sqrt{7})/27$. $\pi_\mu^{-1}(0) = p_4$, each point of which is a $(2, 3)$ -cusp in $C_{14}(\mu)$. $C_{14}(\mu)$ has no other singular points. $\sharp\pi_\mu^{-1}(z_\pm) = 84$. If $z \neq 0, z_\pm, \infty$, then $\sharp\pi_\mu^{-1}(z) = 168$. The genus of $C_{14}(\mu)$ is 36.
- (4) If $\mu \neq -(131 \pm \sqrt{-7})/2, 8/3, -72, 8$, then $C_{14}(\mu)$ is smooth. $\tilde{P}(\mu, z)$ has three distinct zeros. $\sharp\pi_\mu^{-1}(z) = 84$ if and only if $\tilde{P}(\mu, z) = 0$. The genus of $C_{14}(\mu)$ is 78.

Proof of (0). Adler [Adl, Lemma 11.7] gave the homogeneous coordinates of representative points in p_j , $j = 3, 4, 7$, and p'_j , $j = 3, 4$. Let $[\alpha_0 : \alpha_1 : \alpha_2]$ be that of p_3 or p_4 . Let

$$u_1 = \alpha_0, \quad u_2 = \alpha_1 + x, \quad u_3 = \alpha_2 + y$$

be non-homogeneous coordinates at this point. Taking the Taylor expansion of $(F_{14} + \mu F_4^2 F_6)(u)$ at $(x, y) = (0, 0)$, we obtain the desired result.

Proof of (1). Let $\mu = 8/3$. The fact that $\pi_\mu^{-1}(4/27) = p'_3$ follows from Lemma A.2. The fact that $C_{14}(\mu)$ has a node at p'_3 is proved in the same way as the proof of (0). Since $\pi_\mu^{-1}(36)$ is not p_j in the same way as the or p'_j from Lemma A.2, it has cardinality 84 from Lemma A.1. Then $C_{14}(\mu)$ must be smooth at $\pi_\mu^{-1}(36)$, otherwise the genus of $C_{14}(\mu)$ turns out to be negative by a genus formula. If $z \neq 0, 4/27, 36, \infty$, then, from Lemma A.1, $\pi_\mu^{-1}(z)$ consists of 168 smooth points. Consequently, $C_{14}(\mu)$ has singularities only at p'_3 , whence its genus is known to be $50 = (13 \cdot 12)/2 - 28$.

Proof of (2), (3) and (4). The proof is completely the same as that of (1). $\square$

A.2. The net $C_{18}(\mu_1, \mu_2)$

LEMMA A.7. $C_{18}(\mu_1, \mu_2)$ is irreducible if $\mu_1 \neq 0$.

Proof. The following proof is the same as that Fricke used to prove the irreducibility of $C_{12}(\mu)$ in [Fr].

Let V be an irreducible component of $C = C_{18}(\mu_1, \mu_2)$ of degree d with $d < 18$. For any $g \in G'$, $g(V)$ is again an irreducible component of C . Let r be the index $(G' : G'_V)$, where $G'_V = \{g \in G' \mid g(V) = V\}$. Then $G \cdot V = \bigcup_{g \in G'} g(V)$ is a G -invariant curve of degree rd contained in C . Since $rd \leq 18$, and $r \in \{1, 7, 8, 14, 21, \dots, 168\}$ (see [Adl, Lemma 11.1] or [K11, p. 431]), r is one of 1, 7, 8, 14.

Assume $r = 1$. Then V is itself G -invariant and hence $C_{18}(\mu_1, \mu_2) - V$ is also a G -invariant curve of degree $18 - d$. Since d is one of 4, 6, 12, 14, C has a G -invariant irreducible component of degree 4 or 6. This cannot happen because $\mu_1 \neq 0$.

Assume $r = 7$. If $d = 1$, then $G \cdot V$ is a G -invariant curve of degree 7. This cannot happen. If $d = 2$, then a G -invariant quartic $C - G \cdot V$ is equal to $\{F_4 = 0\}$. This cannot happen because $\mu_1 \neq 0$.

Assume $r = 8$. If $d = 1$, then $G \cdot V$ is a G -invariant (reduced) curve of degree 8. This cannot happen. If $d = 2$, then we have a G -invariant quadric $C - G \cdot V$. This cannot happen.

Assume $r = 14$. Then we have $d = 1$. This cannot happen as stated in the case of $r = 7, d = 2$.

This completes the proof. $\square$

We assume, from now on, that $\mu_1 \neq 0$.

LEMMA A.8.

- (1) We have $p_4, p_7 \subset C_{18}(\mu_1, \mu_2)$ and $p_3 \cap C_{18}(\mu_1, \mu_2) = \emptyset$.
- (2) $C_{18}(\mu_1, \mu_2)$ has (1, 3)-flex at each point of p_7 .
- (3) $C_{18}(\mu_1, \mu_2)$ is singular at p_4 if and only if $\mu_2 = 8$. In this case, $C_{18}(\mu_1, \mu_2)$ has a (2, 5)-cusp at each point of p_4 if $\mu_1 = -28$ and a (2, 3)-cusp there otherwise.

Proof. (1) follows from Lemma A.2.

At each point $\alpha = [\alpha_0 : \alpha_1 : \alpha_2] \in p_7$, we have $F_4(\alpha) = 0$, $F_6(\alpha) = 0$, $F_{14}(\alpha) \neq 0$. Assume $\alpha_0 \neq 0$. Because the intersection $\{F_4 = 0\} \cap \{f_6 = 0\}$ is transversal at all 24 points, $F_4(u_2/u_1, u_3/u_1)$ and $F_6(u_2/u_1, u_3/u_1)$ form a set of local coordinate functions there. Thus the defining function of $C_{18}(\mu_1, \mu_2)$ is an equation of a (1,3)-flex. This proves (2).

As in the proof of (0) of Theorem A.6, we obtain a Taylor expansion of $F_{14}F_4 + \mu_1 F_6^3 + \mu_2 F_6 F_4^3$ at a point in p_4 . (3) is derived from this expansion. $\square$

We define a rational map $\pi_{(\mu_1, \mu_2)}$ of $C_{18}(\mu_1, \mu_2)$ to $\mathbf{P}^1$ by

$$\pi_{(\mu_1, \mu_2)}(u) = \frac{F_6(u)^2}{F_4(u)^3}. \quad (\text{A.3})$$

We denote $C_{18}(\mu_1, \mu_2)$ by C and $\pi_{(\mu_1, \mu_2)}$ by π , for short. Recall here the defining equation of C :

$$F_{14}(u)F_4(u) + \mu_1 F_6(u)^3 + \mu_2 F_6(u)F_4(u)^3 = 0. \quad (\text{A.4})$$

LEMMA A.9. π is defined at all points of C and defines a G -quotient map. We have $\pi^{-1}(0) = p_4$ and $\pi^{-1}(\infty) = p_7$.

Proof. If $\alpha \in C$ is a zero of F_4 , then it is also a zero of F_6 from (A.4). From the same equation, it holds that $F_4(u) = O(F_6(u)^3)$ as $[u] \in C \longrightarrow \alpha$, which means $\pi(\alpha) = \infty$. This proves both that π is defined at all points of C and that $\pi^{-1}(\infty) = p_7$. We have $\pi^{-1}(0) = p_4$ by the definition (A.3) of π .

For generic $z \neq 0, \infty$, we have

$$\begin{aligned}\pi^{-1}(z) &= \{[u] \mid F_6(u)^2 - zF_4(u)^3 = 0, F_{14}(u)F_4(u) + \mu_1 F_6(u)^3 + \mu_2 F_6(u)F_4(u)^3 = 0\} \\ &= \{F_6(u)^2 - zF_4(u)^3 = 0, F_{14}(u)F_4(u) + (\mu_1 z + \mu_2)F_6(u)F_4(u)^3 = 0\} \\ &= \{F_6(u)^2 - zF_4(u)^3 = 0, F_{14}(u) + (\mu_1 z + \mu_2)F_6(u)F_4(u)^2 = 0\},\end{aligned}$$

which is the set of 168 points. This proves that π is a G -quotient map. $\square$

Substituting $F_{14} = -(\mu_1 F_6^3 + \mu_2 F_6 F_4^3)/F_4$ and $F_6 = \sqrt{z F_4^3}$ in $P_J(F_4(u), F_6(u), F_{14}(u))$, we have

$$\begin{aligned}&P_J(F_4(u), F_6(u), F_{14}(u))/(-\mu_1^3 F_4(u)^9 F_6(u)) \\ &= z^4 + z^3(3\mu_1^2 \mu_2 + 88\mu_1^2 + 1008\mu_1 - 1728)/\mu_1^3 \\ &\quad + z^2(3\mu_1 \mu_2^2 + 176\mu_1 \mu_2 + 1088\mu_1 + 1008\mu_2 + 60\,032)/\mu_1^3 \\ &\quad + z(-256\mu_1 + \mu_2^3 + 88\mu_2^2 + 1088\mu_2 - 22\,016)/\mu_1^3 + 256(-\mu_2 + 8)/\mu_1^3,\end{aligned}$$

which we denote by $P(\mu_1, \mu_2, z)$.

LEMMA A.10. $P(\mu_1, \mu_2, 0) = 0$ if and only if $\mu_2 = 8$. If $z \in \mathbb{C}$ is not a zero of $zP(\mu_1, \mu_2, z)$, then $\sharp \pi^{-1}(z) = 168$ and π does not ramify at z , whence, in particular, C is smooth at $\pi^{-1}(z)$.

Proof. The first statement is verified directly. Let $[u] \in C_{18}(\mu_1, \mu_2)$. From Lemma A.1, we know that $\sharp G \cdot [u] < 168$ if and only if $[u] \in p_7 \cup p_4 \cup C_{21}$ because $p_3 \cap C_{18}(\mu_1, \mu_2) = \emptyset$. This proves the lemma. $\square$

The discriminant of $P(\mu_1, \mu_2, z)$ has the form

$$4096\mu_1^3 \delta_1(\mu_1, \mu_2)^4 \delta_2(\mu_1, \mu_2)^3 \delta_3(\mu_1, \mu_2),$$

where

$$\begin{aligned}\delta_1(\mu_1, \mu_2) &= 4\mu_1 - \mu_2 - 72, \quad \delta_2(\mu_1, \mu_2) = 4\mu_1 + 27\mu_2 - 72, \\ \delta_3(\mu_1, \mu_2) &= 1728\mu_1^3 + \mu_1^2 \mu_2^3 + 144\mu_1^2 \mu_2^2 + 5904\mu_1^2 \mu_2 + 38\,016\mu_1^2 \\ &\quad + 224\mu_1 \mu_2^2 + 29\,568\mu_1 \mu_2 + 1\,003\,520\mu_1 - 256\mu_2^2 - 33\,536\mu_2 - 1\,120\,256.\end{aligned}$$

Let $M = \{(\mu_1, \mu_2) \in \mathbb{C}^2 \mid \mu_1 \neq 0\}$. In this section we denote the point (μ_1, μ_2) in M by μ and $\delta_j(\mu_1, \mu_2)$ by $\delta_j(\mu)$ and so on. Let $\delta_0(\mu) = \mu_2 - 8$ and $D_j = \{\mu \in M \mid \delta_j(\mu) = 0\}$ for $j = 0, 1, 2, 3$. Then, by direct calculations, we have the following lemma.

LEMMA A.11.

$$\begin{aligned} D_0 \cap D_1 &= \{(20, 8)\}, & D_0 \cap D_2 &= \{(-36, 8)\}, & D_1 \cap D_2 &= \{(18, 0)\}, \\ D_0 \cap D_3 &= \{(-28, 8), (28/27, 8)\}, & D_1 \cap D_3 &= \{(4, -56), (2, -64)\}, \\ D_2 \cap D_3 &= \{(-66, 112/9), (36(17 \pm 12\sqrt{2}), -8(11 \pm 8\sqrt{2}))\}. \end{aligned}$$

LEMMA A.12. (1) $p'_4 \subset C_{18}(\mu)$ if and only if $\delta_1(\mu) = 0$. Assume this case happens. We have $p'_4 = \pi^{-1}(-4)$. If $\mu \neq (4, -56)$, then $C_{18}(\mu)$ has a node at each point of p'_4 . If $\mu = (4, -56)$, then it has a singular point of multiplicity 4 with four distinct tangent lines at each point of p'_4 .

(2) $p'_3 \subset C_{18}(\mu)$ if and only if $\delta_2(\mu) = 0$. Assume this case happens. We have $p'_3 = \pi^{-1}(4/27)$. If $\mu \neq (-66, 112/9)$, then $C_{18}(\mu)$ has a node at each point of p'_3 . If $\mu = (-66, 112/9)$, then it has a singular point of multiplicity 3 with three distinct tangent lines at each point of p'_3 .

Proof. We first prove (1). Let $\alpha = [\alpha_0 : \alpha_1 : \alpha_2] = [\zeta^5 + \zeta^2 - 1 : \zeta^4 + \zeta^3 : \zeta^6 + \zeta + 2]$, where $\zeta = e^{2\pi\sqrt{-1}/7}$. This is a point of p'_4 . (This is a little bit different from that in [Adl, Lemma 11.7].) It is verified directly that $\alpha \in C_{18}(\mu_1, \mu_2)$ if and only if $\delta_1(\mu_1, \mu_2) = 0$.

If $p'_4 \subset C$, then, from Lemma A.2, we have $p'_4 = \pi^{-1}(-4)$.

Taking the Taylor expansion of $F_{14}F_4 + \mu_1 F_6^3 + \mu_2 F_6 F_4^3$ at the point α , we can determine the type of the singularity of this point. This proves (1).

(2) is proved in the same way. □

As for the multiple zeros of $P(\mu, z)$, we have the following lemma.

LEMMA A.13.

- (1) If $\mu = (4, -56)$, then $P(\mu, z) = (z + 4)^4$.
- (2) If $\mu = (-66, 112/9)$, then $P(\mu, z) = (z - 4/27)^3(z - 1620/1331)$.
- (3) If $\mu = (\frac{2}{9}(9 \pm \sqrt{-3}), -2(33 \pm \sqrt{-3}))$, then $P(\mu, z) = (z - z_1)^3(z - z_2)$ for some $z_j \neq 0, -4, 4/27, z_1 \neq z_2$.
- (4) If $\mu = (18, 0)$, then $P(\mu, z) = (z + 4)^2(z - 4/27)^2$.
- (5) If $\mu = (2, -64)$, then $P(\mu, z) = (z + 4)^2(z - 12)^2$.
- (6) If $\mu = (36(17 \pm 12\sqrt{2}), -8(11 \pm 8\sqrt{2}))$, then $P(\mu, z) = (z - 4/27)^2(z - 2(-58 \pm 41\sqrt{2}))^2$.
- (7) Otherwise, $P(\mu, z)$ has at least three distinct zeros.

THEOREM A.14. (M) For $\mu \in M - \cup_{0 \leq j \leq 3} D_j$, $C_{18}(\mu)$ is smooth and has genus 136.

(D₀) For $\mu \in D_0 - D_1 \cup D_2 \cup D_3$, $C_{18}(\mu)$ has a (2, 3)-cusp at each point of p_4 and has no other singularities. It has genus 94.

(D₁) For $\mu \in D_1 - D_0 \cup D_2 \cup D_3$, $C_{18}(\mu)$ has a node at each point of p'_4 and has no other singularities. It has genus 115.

(D₂) For $\mu \in D_2 - D_0 \cup D_1 \cup D_3$, $C_{18}(\mu)$ has a node at each point of p'_3 and has no other singularities. It has genus 108.

(D₃) For $\mu \in D_3 - D_0 \cup D_1 \cup D_2$, $C_{18}(\mu)$ has a singular point at each point of a G -orbit of cardinality 84 and has no other singularities. The singularity is a (2, 3)-cusp if $\mu = (2(9 \pm \sqrt{-3})/9, -2(33 \pm \sqrt{-3}))$ and a node otherwise. In any case, $C_{18}(\mu)$ has genus 52.

($D_0 \cap D_1$) For $\mu \in D_0 \cap D_1$, $C_{18}(\mu)$ has a (2, 3)-cusp at each point of p_4 and a node at each point of p'_4 . It has no other singular points. It has genus 73.

($D_0 \cap D_2$) For $\mu \in D_0 \cap D_2$, $C_{18}(\mu)$ has a (2, 3)-cusp at each point of p_4 and a node at each point of p'_3 . It has no other singular points. It has genus 66.

($D_1 \cap D_2$) For $\mu \in D_1 \cap D_2$, $C_{18}(\mu)$ has a node at each point of $p'_3 \cup p'_4$. It has no other singular points. It has genus 87.

($D_0 \cap D_3$) If $\mu = (-28, 8)$, $C_{18}(\mu)$ has a (2, 5)-cusp at each point of p_4 and has no other singularities. It has genus 52.

If $\mu = (28/27, 8)$, $C_{18}(\mu)$ has a (2, 3)-cusp at each point of p_4 and a node at each point of a G -orbit of cardinality 84. It has no other singular points. It has genus 10.

($D_1 \cap D_3$) If $\mu = (2, -64)$, $C_{18}(\mu)$ has a node at each point of p'_4 and a G -orbit of cardinality 84. It has no other singularities. It has genus 31.

If $\mu = (4, -56)$, $C_{18}(\mu)$ has a singular point with multiplicity 4 with four distinct tangent lines at each point of p'_4 . It has no other singular points. It has genus 10.

($D_2 \cap D_3$) If $\mu = (-66, 112/9)$, $C_{18}(\mu)$ has a singular point with multiplicity 3 with three distinct tangent lines at each point of p'_3 . It has no other singular points. It has genus 52.

If $\mu = (36(17 \pm 12\sqrt{2}), -8(11 \pm 8\sqrt{2}))$, $C_{18}(\mu)$ has a node at each point of p'_3 and a G -orbit of cardinality 84. It has no other singularities. It has genus 24.

Proof of (M). In this case $P(\mu, z)$ has distinct zeros z_j , $1 \leq j \leq 4$ with $z_j \neq 0$. From Lemma A.10, we know that $\pi^{-1}(z)$ consists of 168 smooth points if $z \neq 0, z_j, \infty$. From Lemma A.8, C is smooth at each point of $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$. Since p'_3 and p'_4 are not contained in C from Lemma A.11, we have $\sharp\pi^{-1}(z_j) = 84$ from Lemma A.1. Then we know, from [Kt3, Lemma 1.2], that the ramification index r_j of π at each point of $\pi^{-1}(z_j)$ is 1 or 2. The genus $p(C)$ of C is given by

$$p(C) = 1 + 84 \left(-2 + (1 - 1/7) + (1 - 1/4) + \sum (1 - 1/r_j) \right) \quad (\text{A.5})$$

from the Riemann–Hurwitz formula. If $r_j = 1$, then, from [Kt3, Lemma 1.2], C has two local irreducible components at each point of $\pi^{-1}(z_j)$, in particular, C has 84 singular points there. Hence there is at most one j with $r_j = 1$. If $r_1 = 1$ and $r_j = 2$, $j = 2, 3, 4$, say, then $p(C) \leq (17 \cdot 16)/2 - 84$ by a genus formula, at the same time, $p(C) = 94$ from (A.5). Hence we must have $r_j = 2$ for all j and $p(C) = 136$ from (A.5) and C must be smooth by a genus formula. This proves the case of (M).

Proof of (D_0). Since $\mu_2 = 8$ and the discriminant of $P(\mu, z)$ does not vanish, $P(\mu, z)$ has distinct zeros $0, z_1, z_2, z_3$, say (see Lemma A.10). Since $\mu_1 \neq -28$, C has a (2, 3)-cusp at each point of $p_4 = \pi^{-1}(0)$ (see Lemma A.8(3)). As in the proof of (M), we know that $\sharp\pi^{-1}(z_j) = 84$ and that C is smooth at $\pi^{-1}(z_j)$, $j = 1, 2, 3$. This proves the case of (D_0).

Proof of (D_1). From Lemmas A.11–A.13, $P(\mu, z)$ has three distinct zeros $-4, z_1, z_2$ with $z_1, z_2 \neq 0, 4/27$. C has a node at each point of $p'_4 = \pi^{-1}(-4)$ and $\sharp\pi^{-1}(z_j) = 84$. As in the proof of (M), we know that C is smooth at $\pi^{-1}(z_j)$, $j = 1, 2, 3$. This proves the case of (D_1).

Proof of (D_2). From Lemmas A.11–A.13, $P(\mu, z)$ has three distinct zeros $4/27, z_1, z_2$ with $z_1, z_2 \neq 0, 4/27$. C has a node at each point of $p'_4 = \pi^{-1}(-4)$ and $\sharp\pi^{-1}(z_j) = 84$. The proof is the same as the case of (D_1).

Proof of (D_3) . Assume first that $\mu \neq (\frac{2}{9}(9 \pm \sqrt{-3}), -2(33 \pm \sqrt{-3}))$. Then from Lemma A.13, $P(\mu, z)$ has three distinct zeros z_1, z_2, z_3 with $z_1, z_2, z_3 \neq 0, -4, 4/27$. We have $\sharp\pi^{-1}(z_j) = 84$ for $j = 1, 2, 3$. Let r_j be the ramification index of π at $z = z_j$. Then $r_j = 1$ or 2 . As in the proof of (M) , by renumbering j , if necessary, we know that $r_1 = 1, r_2 = r_3 = 2$, C has a node at each point of $\pi^{-1}(z_1)$ and C is smooth elsewhere.

Assume next that $\mu = (\frac{2}{9}(9 \pm \sqrt{-3}), -2(33 \pm \sqrt{-3}))$. Then $P(\mu, z)$ has two distinct zeros z_1, z_2 , say. By two genus formulas as in the proof of (M) , C has a $(2, 3)$ -cusp at each point of $\pi^{-1}(z_1)$, say, and C is smooth at $\pi^{-1}(z_2)$. This proves the case of (D_3) .

Proof of $(D_0 \cap D_1)$. From Lemmas A.8–A.12, $P(\mu, z)$ has three distinct zeros $0, -4$ and $z_1 (\neq 4/27)$, say. C has $(2, 3)$ -cusps at $p_4 = \pi^{-1}(0)$ and nodes at $p'_4 = \pi^{-1}(-4)$. Since $p'_3 \cap C = \emptyset$, we have $\sharp\pi^{-1}(z_1) = 84$. As in the proof of (M) , we know that C is smooth at $\pi^{-1}(z_1)$. This proves the case of $(D_0 \cap D_1)$.

Proof of $(D_0 \cap D_2)$ and $(D_1 \cap D_2)$. The proof is just the same as that of $(D_0 \cap D_1)$.

Proof of $(D_0 \cap D_3)$. From Lemmas A.10–A.13, $P(\mu, z)$ has three distinct zeros 0 and $z_1, z_2 (\neq 4/27, -4)$, say. We have $\sharp\pi^{-1}(z_j) = 84$. If $\mu = (-28, 8)$, then C has $(2, 5)$ -cusps at $p_4 = \pi^{-1}(0)$. As in the proof of (M) , C is smooth at $\pi^{-1}(z_j)$. If $\mu = (28/27, 8)$, then C has $(2, 3)$ -cusps at $p_4 = \pi^{-1}(0)$. As in the proof of (M) , we know the following facts: the ramification index r_j is 1 at one point z_1 , say and 2 at z_2 ; C has nodes at $\pi^{-1}(z_1)$ and is smooth at $\pi^{-1}(z_2)$. This proves the case of $(D_0 \cap D_3)$.

Proof of $(D_1 \cap D_3)$ and $(D_2 \cap D_3)$. The proof is just the same as that of $(D_0 \cap D_3)$. □

Remark A.1. There are two curves $C_{18}(28/27, 8)$ and $C_{18}(4, -56)$ of genus 10 . Adler referred to these curves in [Adl]. As is shown in Theorem A.14, $C_{18}(28/27, 8)$ has $(2, 3)$ -cusps, while $C_{18}(4, -56)$ does not. In Table 5 of Proposition 4.4 we find two cases of degree $n = 18$, that is, E_1 with $(l_0, l_1, l_2; m_1, m_2) = (1, 1, 0; -1, -1)$ and $(l_0, l_1, l_2; m_1, m_2) = (1, 1, 0; 0, -1)$. They are ${}_3E_2(\frac{9}{14}, \frac{15}{14}, \frac{11}{14}; \frac{5}{4}, \frac{7}{4})$ and ${}_3E_2(\frac{9}{14}, \frac{15}{14}, \frac{11}{14}; \frac{5}{4}, \frac{3}{4})$, respectively. The former equation has the local exponents $0, -1/4, -3/4$ at $x = 0$. Consequently, its Schwarz image has 42 $(2, 3)$ -cusps over $x = 0$. This implies that the former Schwarz image is $C_{18}(28/27, 8)$, and the latter one is $C_{18}(4, -56)$.

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Mitsuo Kato
Department of Mathematics
College of Education
University of the Ryukyus
Nishihara-cho, Okinawa 903–0213
Japan
(E-mail: mkato@edu.u-ryukyu.ac.jp)