ON THE CLASSIFICATION OF IRREDUCIBLE $\mathfrak{osp}_{2,2}$ REPRESENTATIONS

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(Received 17 January 2011)

Abstract. The orthosymplectic Lie superalgebra $\mathfrak{osp}_{2,2}$ is a classical simple Lie superalgebra. Its Lie part forms a copy of the direct sum of $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$. In this paper, we first study its structure in detail. Based on the well-known description of irreducible representations of $\mathfrak{sl}_2$, we obtain the classification of the irreducible representations of $\mathfrak{osp}_{2,2}$ subject to some technical restrictions. These irreducible representations actually appear in the natural correspondence of $(O_{p,q}, \mathfrak{osp}_{2,2})$ acting on the space $\mathcal{S}(\mathbb{R}^n, \Lambda^*((\mathbb{R}^n)^*))$ of Schwartz class differential forms on $\mathbb{R}^n$.

1. Introduction

The notion of dual pairs formulated by Howe [Ho1] describes certain phenomena that had been observed in invariant theory [Ho1], and in the theory of automorphic forms [Wa, Ge]. Somewhat later, it was observed also to have several connections with physics [Ho2], specifically, with several of the classical linear partial differential equations of physics. In particular, Laplace’s equation and Maxwell’s equations are both connected to dual pairs, in analogous ways.

The notion of dual pair is very general. In the context of groups, a dual pair is a pair of subgroups (G, G') of the metaplectic group $\widetilde{Sp}_{2n}(\mathbb{R})$, the two-fold covering of the symplectic group $Sp_{2n}(\mathbb{R})$, such that G' is the centralizer of G in $\widetilde{Sp}_{2n}(\mathbb{R})$, and *vice versa*. The metaplectic group has a canonical minimal representation (actually, two mutually contragredient ones) on $L^2(\mathbb{R}^n)$. The main phenomenon of interest has been that, when this representation is restricted to the members of a dual pair (G, G') in $\widetilde{Sp}_{2n}(\mathbb{R})$, a canonical bijection results between the representations of G and of G' [Ho3]. These ‘local θ -correspondences’ (so-called because of their relevance in the theory of theta series [Ho3]) have manifold consequences, including the close connection to Laplace’s equation mentioned above.

The dual pair related to Maxwell’s equations is more complicated than the one for the Laplacian. It involves not Lie algebras or groups but Lie superalgebras and supergroups [Ka, Kap]. Specifically, to deal with the operators involved in Maxwell’s equations, one needs to consider dual pairs $(O_{p,q}, \widetilde{OSp}_{2,2})$ inside the orthosymplectic Lie supergroup $\widetilde{OSp}_{2n,2n}$, where $n = p + q$ (or, at the infinitesimal level, the dual pair $(\mathfrak{so}_{p,q}, \mathfrak{osp}_{2,2}) \subset \mathfrak{osp}_{2n,2n}$ of

2000 Mathematics Subject Classification: Primary 17B10.

Keywords: Lie algebras; superalgebras; Casimir operator; irreducible representations; weight; classification; eigenvalue.

associated Lie superalgebras). Also it is desirable to extend the canonical bijection to this case. For these purposes, it is necessary to study the structure of representations of both $O_{p,q}$ and $\mathfrak{osp}_{2,2}$. The representations of $O_{p,q}$ have been studied widely in the literature. However, despite the simple structure of $\mathfrak{osp}_{2,2}$ as a Lie superalgebra, to our knowledge the complete classification of its irreducible representations does not appear explicitly in the literature. In this paper, we give an explicit description of its irreducible representations and also discuss their structure when restricted to the Lie part $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$.

Here is an outline of the paper. Section 2 is devoted to studying the structure of this classical simple Lie superalgebra $\mathfrak{osp}_{2,2}$. In Section 3, we briefly recall the well-known result of the classification of irreducible representations of $\mathfrak{sl}_2$. Section 4 is devoted to describing the classification of irreducible representations of $\mathfrak{osp}_{2,2}$ subject to some technical restrictions. The main strategy there is to explain the structure of the irreducible representations of $\mathfrak{osp}_{2,2}$ in terms of the description of representations of its Lie subalgebra $\mathfrak{sl}_2$ in Section 3. A key role is played by the Casimir operator for $\mathfrak{osp}_{2,2}$ —the generator of the center of its enveloping algebra. It is shown that the structure of $\mathfrak{osp}_{2,2}$ representations is exceptional when the action of the Casimir operator vanishes.

The results of this paper are essentially contained in the author's doctoral dissertation at Yale [Lu]. The author expresses his thanks to Professor Howe for his suggestions and guidance.

2. Structure of the orthosymplectic Lie superalgebra $\mathfrak{osp}_{2,2}$

In this section, we look at $\mathfrak{osp}_{2,2}$ in detail. We give a basis for $\mathfrak{osp}_{2,2}$ in the form of matrices and discuss the (anti)commutation relations for the basis elements. We also find out the generator of the center of its enveloping algebra.

From [Ka], we see that a matrix in $\mathfrak{osp}_{2,2}$ is of the form

$$\begin{aligned} & \left(\begin{array}{cc|cc} k & 0 & x & x_1 \\ 0 & -k & y & y_1 \\ \hline y_1 & x_1 & a & b \\ -y & -x & c & -a \end{array} \right) \\ \mathfrak{osp}_{2,2}^0 &= \left\{ \left(\begin{array}{cc|cc} k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & -a \end{array} \right) \cong \mathfrak{sl}_2 \oplus \mathfrak{so}_2 \right\} \\ \mathfrak{osp}_{2,2}^1 &= \left\{ \left(\begin{array}{cc|cc} 0 & 0 & x & x_1 \\ 0 & 0 & y & y_1 \\ \hline y_1 & x_1 & 0 & 0 \\ -y & -x & 0 & 0 \end{array} \right) \right\}, \end{aligned} \quad (2.1)$$

where $\mathfrak{osp}_{2,2}^0$ is its Lie part, and $\mathfrak{osp}_{2,2}^1$ is its super part.

Again we let

$$e^+ = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad e^- = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right), \quad h = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right). \quad (2.2)$$

It is easy to check that $\{e^+, e^-, h\}$ is a standard basis for $\mathfrak{sl}_2$. Additionally, the element

$$h' = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (2.3)$$

is a basis for $\mathfrak{so}_2$.

Define

$$\begin{aligned} d &= \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right), & d^* &= \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right), \\ \delta &= \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), & \delta^* &= \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \end{aligned} \quad (2.4)$$

After simple calculations, we have the following commutation relations. Since, for $x, y \in \mathfrak{osp}_{2,2}^1$, $[x, y] = xy + (-1)^{1 \times 1}yx = xy + yx = \{x, y\}$, we use $\{\cdot, \cdot\}$ to indicate the operation for $x, y \in \mathfrak{osp}_{2,2}^1$.

For the Lie subalgebra $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$,

$$[h, e^+] = 2e^+, \quad [h, e^-] = -2e^-, \quad [e^+, e^-] = h, \quad [h', x] = 0, \quad (2.5)$$

where x is anything in $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$.

Between the Lie algebra and the super part,

$$\begin{aligned} [e^+, d] &= \delta^*, & [e^+, \delta^*] &= 0, & [e^+, d^*] &= \delta, & [e^+, \delta] &= 0, \\ [e^-, d] &= 0, & [e^-, \delta^*] &= d, & [e^-, d^*] &= 0, & [e^-, \delta] &= d^*, \\ [h, d] &= -d, & [h, \delta^*] &= \delta^*, & [h, d^*] &= -d^*, & [h, \delta] &= \delta, \\ [h', d] &= d, & [h', \delta^*] &= \delta^*, & [h', d^*] &= -d^*, & [h', \delta] &= -\delta. \end{aligned} \quad (2.6)$$

Anticommutation relations for the super part, are

$$\begin{aligned} d^2 &= \{d, \delta^*\} = (\delta^*)^2 = 0, & (d^*)^2 &= \{d^*, \delta\} = \delta^2 = 0, \\ \{d, d^*\} &= -2e^-, & \{d, \delta\} &= -h - h', & \{d^*, \delta^*\} &= -h + h', & \{\delta, \delta^*\} &= 2e^+. \end{aligned} \quad (2.7)$$

From the above commutation relations, we obtain the following theorem.

THEOREM 2.1. $\mathfrak{osp}_{2,2}$ is a classical Lie superalgebra. The pairs $\{d, \delta^*\}$ and $\{d^*, \delta\}$ each span a two-dimensional irreducible module for $\mathfrak{sl}_2$ with eigenvalues $+1$ and -1 for h' , respectively.

For the classification of irreducible representations of $\mathfrak{osp}_{2,2}$, it is useful to know the center of $\mathfrak{U}(\mathfrak{osp}_{2,2})$, where $\mathfrak{U}(\mathfrak{osp}_{2,2})$ is the universal enveloping algebra of $\mathfrak{osp}_{2,2}$.

THEOREM 2.2. *Define*

$$\Omega_{\mathfrak{osp}} = \Omega_{\mathfrak{sl}_2} - h'^2 - ([d, \delta] + [d^*, \delta^*])$$

in $\mathfrak{U}(\mathfrak{osp}_{2,2})$. Then $\Omega_{\mathfrak{osp}}$ is in the center of $\mathfrak{U}(\mathfrak{osp}_{2,2})$; in other words, $\Omega_{\mathfrak{osp}}$ commutes with every element in $\mathfrak{U}(\mathfrak{osp}_{2,2})$. Moreover, $\Omega_{\mathfrak{osp}}$ generates the center of $\mathfrak{U}(\mathfrak{osp}_{2,2})$ as an (associative) algebra, i.e. $Z(\mathfrak{U}(\mathfrak{osp}_{2,2})) = \mathbb{C}[\Omega_{\mathfrak{osp}}]$.

Proof. From the (anti)commutation relations of the basis elements in (2.5)–(2.7), a simple computation shows that $\Omega_{\mathfrak{osp}}$ commutes with every basis element in $\mathfrak{osp}_{2,2}$. (See [Lu] for detailed arguments.) Thus we have proved that $\Omega_{\mathfrak{osp}}$ is a central element inside $\mathfrak{U}(\mathfrak{osp}_{2,2})$. From [Se], we know that the center of $\mathfrak{U}(\mathfrak{osp}_{2,2})$ has only one generator, hence up to a multiple, $\Omega_{\mathfrak{osp}}$ is the generator of $Z(\mathfrak{U}(\mathfrak{osp}_{2,2}))$. $\square$

Remark 2.3. For later use, we also have the following alternative form for $\Omega_{\mathfrak{osp}}$:

$$\begin{aligned} \Omega_{\mathfrak{osp}} &= \Omega_{\mathfrak{sl}_2} - h'^2 - 2h' - 2d\delta + 2\delta^*d^* \\ &= \Omega_{\mathfrak{sl}_2} - h'^2 + 2h + 2\delta d + 2\delta^*d^*. \end{aligned} \quad (2.8)$$

3. Irreducible representations of $\mathfrak{sl}_2$

Let $\{e^+, e^-, h\}$ be the standard basis of $\mathfrak{sl}_2$ and (ρ, V) be a representation of $\mathfrak{sl}_2$. For $\lambda \in \mathbb{C}$, the generalized h eigenspace of V corresponding to λ is $V(\lambda)$, where

$$V(\lambda) = \{v \in V \mid (h - \lambda I)^n v = 0 \text{ for some } n \in \mathbb{Z}_+\},$$

so $V(\lambda) \cap V(\mu) = \{0\}$ if $\lambda \neq \mu$. We say a representation (ρ, V) is *h-admissible* if

$$V = \sum_{\lambda \in \mathbb{C}} V(\lambda) \quad (3.1)$$

and each $V(\lambda)$ is finite-dimensional. If the decomposition (3.1) holds and each $V(\lambda)$ is an eigenspace, that is, $V(\lambda) = \{v \in V \mid hv = \lambda v\}$, then we say that (ρ, V) is *h-semisimple*. A representation (ρ, V) is *quasisimple* if the Casimir element $\Omega_{\mathfrak{sl}_2}$ acts via a multiple of the identity on V , i.e. $\Omega_{\mathfrak{sl}_2} v = cv$ for any $v \in V$. We will describe all irreducible h -semisimple, h -admissible and quasisimple $\mathfrak{sl}_2$ -modules in this section. This is a classical result which has been known for a long time [HT].

(1) *Lowest weight modules* $V_\lambda, \lambda \in \mathbb{C}$. V_λ has a basis of h eigenvectors $\{v_j \mid j = 0, 1, 2, \dots\}$ where

$$\begin{aligned} hv_j &= (\lambda + 2j)v_j, \quad j \in \mathbb{Z}_+, \\ e^+ v_j &= v_{j+1}, \\ e^- v_j &= -j(\lambda + j - 1)v_{j-1}, \\ e^- v_0 &= 0, \\ \Omega_{\mathfrak{sl}_2} v &= (\lambda^2 - 2\lambda)v. \end{aligned} \quad (3.2)$$

The vector v_0 is called the *lowest weight vector* and λ is called the *lowest weight*. Moreover, this $\mathfrak{sl}_2$ -module is irreducible if and only if $\lambda \notin \mathbb{Z}^-$.

(2) *Highest weight modules* $\bar{V}_\lambda$, $\lambda \in \mathbb{C}$. $\bar{V}_\lambda$ has a basis of h eigenvectors $\{\bar{v}_j \mid j = 0, 1, 2, \dots\}$ where

$$\begin{aligned} h v_j &= (\lambda - 2j) \bar{v}_j, \\ e^+ \bar{v}_0 &= 0, \\ e^- \bar{v}_j &= \bar{v}_{j+1}, \\ e^+ \bar{v}_j &= j(\lambda - j + 1) \bar{v}_{j-1}, \\ \Omega_{\mathfrak{sl}_2} v &= (\lambda^2 + 2\lambda)v, \quad v \in \bar{V}_\lambda. \end{aligned} \tag{3.3}$$

The vector $\bar{v}_0$ is called the *highest weight vector* and λ is the *highest weight* of $\bar{V}_\lambda$. Moreover this $\mathfrak{sl}_2$ -module is irreducible if and only if $\lambda \notin \mathbb{Z}^+$.

Remark 3.1. If we let $e_1^+ = e^-$, $e_1^- = e^+$ and $h_1 = -h$, then $\{e_1^+, e_1^-, h_1\}$ is again a standard basis. Under this basis, a highest weight module becomes a lowest weight module, thus it is not necessary for us to distinguish these two classes of modules in many later discussions.

(3) *Finite-dimensional modules* $F_{-\lambda}$ for $\lambda \in \mathbb{Z}^+$. $F_{-\lambda}$ has a basis $\{v_0, \dots, v_\lambda\}$ such that

$$\begin{aligned} h v_j &= (-\lambda + 2j) v_j, \\ e^+ v_i &= v_{j+1}, \quad 0 \leq j < \lambda; \quad e^+ v_\lambda = 0, \\ e^- v_0 &= 0, \\ e^- v_j &= j(\lambda + 1 - j) v_{j-1}, \quad 0 \leq j \leq \lambda, \\ \Omega_{\mathfrak{sl}_2} v_j &= (\lambda^2 + 2\lambda) v_j. \end{aligned} \tag{3.4}$$

(4) $U(v^+, v^-)$, $v^+, v^- \in \mathbb{C}$. $U(v^+, v^-)$ has a linear basis of h eigenvectors $\{v_j \mid j \in \mathbb{Z}\}$ where

$$\begin{aligned} h v_j &= (v^+ - v^- + 2j) v_j, \\ e^+ v_j &= (v^+ + j) v_{j+1}, \\ e^- v_j &= (v^- - j) v_{j-1}, \\ \Omega_{\mathfrak{sl}_2} v &= (v^+ + v^-)(v^+ + v^- - 2)v, \quad v \in U(v^+, v^-). \end{aligned} \tag{3.5}$$

Note that $U(v^+, v^-) \cong U(v^+ + 1, v^- - 1)$ and $U(v^+, v^-)$ is irreducible if and only if neither of v^+ or v^- is an integer.

THEOREM 3.2. [HT] *All h -admissible, h -semisimple quasisimple irreducible $\mathfrak{sl}_2$ -modules are classified into the four classes defined above:*

- (1) V_λ , $\lambda \notin \mathbb{Z}_-$;
- (2) $\bar{V}_\lambda$, $\lambda \notin \mathbb{Z}^+$;
- (3) $F_{-\lambda}$, $\lambda \in \mathbb{Z}^+$;
- (4) $U(v^+, v^-)$, $v^+, v^- \notin \mathbb{Z}$.

4. Classification of irreducible representations of $\mathfrak{osp}_{2,2}$

Having dealt with the structure of $\mathfrak{osp}_{2,2}$ and given the commutation relations for the basis vectors of $\mathfrak{osp}_{2,2}$, in this part we will classify its h -admissible, h, h' -semisimple $\Omega_{\mathfrak{osp}}$ -quasisimple irreducible representations.

Since the Lie part is a direct sum of $\mathfrak{sl}_2$ and $\mathfrak{so}_2$, we classify the irreducible $\mathfrak{osp}_{2,2}$ -modules in terms of their structures considered as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ -modules. From Section 3, we know that the h -semisimple, h -admissible quasisimple irreducible representations of $\mathfrak{sl}_2$ are classified into four different classes. Since $\mathfrak{so}_2$ is an abelian Lie algebra, its irreducible representation can be identified with $\mathbb{C}$. Also since $\mathfrak{so}_2$ commutes with $\mathfrak{sl}_2$, each h, h' -semisimple irreducible $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ -module has only one eigenvalue for h' . We use $V_{\lambda, \mu}$, $\bar{V}_{\lambda, \mu}$, $F_{\lambda, \mu}$ and $U(v^+, v^-, \mu)$ to denote the irreducible $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ -modules of the same $\mathfrak{sl}_2$ -types as in Section 3 with h' -eigenvalue μ . We will analyze h -admissible, h, h' -semisimple, $\Omega_{\mathfrak{osp}}$ -quasisimple $\mathfrak{osp}_{2,2}$ -modules which contain $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ -modules from these four different classes. We choose the basis for $\mathfrak{osp}_{2,2}$ as in Section 2.

Definition 4.1. Let $\Gamma = \Gamma_0 \oplus \Gamma_1$ be a Lie superalgebra, let $\mathfrak{U}(\Gamma)$ be its universal enveloping algebra and let Γ' be a Lie subsuperalgebra. Suppose that V is a Γ' -module. Define $Y = \text{Ind}_{\Gamma'}^{\Gamma}(V) = \mathfrak{U}(\Gamma) \otimes_{\Gamma'} V$ [Ja], which is a Γ -module by the left multiplication of $\mathfrak{U}(\Gamma)$.

For $\mathfrak{osp}_{2,2}$, we consider $\mathfrak{q} = \mathfrak{sl}_2 \oplus \mathfrak{so}_2 \oplus E$ where E is the span of d^* and δ . It is easy to check that $\mathfrak{q}$ is a Lie subsuperalgebra, and actually we can get the following theorem which is helpful for us to classify irreducible h -admissible, h, h' -semisimple, $\Omega_{\mathfrak{osp}}$ -quasisimple $\mathfrak{osp}_{2,2}$ -modules.

THEOREM 4.2. *Suppose V is an h, h' -semisimple, h -admissible, $\Omega_{\mathfrak{osp}}$ -quasisimple, irreducible $\mathfrak{osp}_{2,2}$ -module.*

- (a) *Set $W_1 = \ker d^* \cap \ker \delta$. Then W_1 is invariant under $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$, and is a non-zero, irreducible, h, h' -semisimple, quasisimple $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ -module.*
- (b) *Set $W = \text{Ind}_{\mathfrak{q}}^{\mathfrak{osp}_{2,2}} W_1$. Then W is an indecomposable $\mathfrak{osp}(2, 2)$ module, and V is a quotient of W .*

Proof. First, we show that $W_1 \neq \{0\}$. We want to find a non-zero vector in W_1 . Let v be any non-zero vector in V . If $d^*v = 0 = \delta v$, then v is the vector we are looking for. Suppose, however, that $d^*v \neq 0$. Then consider d^*v . Since $(d^*)^2 = 0$, we must have $d^*(d^*v) = 0$. If $\delta(d^*v) = 0$, then d^*v is the vector we are looking for. Suppose, however, that $\delta(d^*v) = v_1$ is also non-zero. Then since $\delta^2 = 0$, we have $\delta(v_1) = \delta^2(d^*v) = 0$. Furthermore, $d^*(v_1) = d^*(\delta d^*v) = (d^*\delta)(d^*v) = -(\delta d^*)(d^*v) = -\delta(d^*)^2v = 0$. Thus, v_1 is the vector we are looking for. Since this covers all cases, we conclude that W_1 is non-trivial, as claimed.

Since the span of d^* and δ is invariant under commutators with $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$, an easy calculation shows that W_1 will be an $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ submodule of V . It will then also necessarily be h and h' -semisimple. We will now show that it must also be irreducible.

Let W_2 be any $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ submodule of W_1 . Consider

$$U = W_2 + d(W_2) + \delta^*(W_2) + d\delta^*(W_2).$$

We claim that U is an $\mathfrak{osp}_{2,2}$ submodule of V . Since $d^2 = (\delta^*)^2 = \{d, \delta^*\} = 0$, it is easy to check that U is invariant under d and δ^* . Since $\mathfrak{osp}_{2,2}$ is generated by d, δ^*, d^* and δ , it is enough to check that U is also invariant under d^* and δ . We compute

$$d^*(d(W_2)) = (d^*d)(W_2) = (\{d^*, d\} - dd^*)(W_2) = -2e^-(W_2) \subset W_2 \subset U.$$

Similarly,

$$d^*(\delta^*(W_2)) = (d^*\delta^*)(W_2) = (\{d^*, \delta^*\} - \delta^*d^*)(W_2) = (-h + h')(W_2) \subset W_2 \subset U.$$

Finally,

$$\begin{aligned} d^*(d\delta^*(W_2)) &= (d^*d\delta^*)(W_2) = (\{d^*, d\}\delta^* - dd^*\delta^*)(W_2) \subset -2e^-(\delta^*(W_2) + d(W_2)) \\ &\subset \delta^*(W_2) + d(W_2) \subset U. \end{aligned}$$

Hence, U is invariant under d^* . Similar computations show that U is also invariant under δ , establishing the claim.

We will now show that W_1 must be irreducible as an $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module. First we will show that W_1 is a single eigenspace for h' . If not, let μ be one eigenvalue for h' acting on W_1 , and let W_2 be the corresponding h' -eigenspace. Since $\mathfrak{sl}_2$ commutes with $\mathfrak{so}_2$, W_2 will be an $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ submodule of W_1 . Applying the construction just above, we construct a corresponding $\mathfrak{osp}_{2,2}$ submodule U of V .

An easy computation using the commutation relations (2.6) shows that $d(W_2) + \delta^*(W_2)$ is an eigenspace for h' with eigenvalue $\mu + 1$, and $d\delta^*(W_2)$ is an eigenspace for h' with eigenvalue $\mu + 2$. Hence, the only possible eigenvalues for h' on U are just $\mu, \mu + 1$ and $\mu + 2$.

Since V is irreducible, the submodule U must be all of V . Hence, the eigenvalues of h' on V will also be just $\mu, \mu + 1$ and $\mu + 2$. If W_1 is not an eigenspace for h' , then it will contain some h' -eigenvectors with eigenvalues $\mu + 1$ or $\mu + 2$. Let W_3 be the h' eigenspace of W_1 with eigenvalue $\mu + 2$, if this is non-zero, or otherwise with eigenvalue $\mu + 1$. Then W_3 will be an $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ submodule of W_1 . Repeating the construction above with W_3 will produce an $\mathfrak{osp}(2, 2)$ submodule U' of V with h' -eigenvalues contained in the set $\{\mu + 1, \mu + 2, \mu + 3, \mu + 4\}$. Since U' must also be all of V , this would mean that μ is not actually an h' -eigenvalue for V . This contradiction implies that W_1 is exactly the μ -eigenspace for h' .

Now we will show that W_1 is irreducible under $\mathfrak{sl}_2$. Consider any non-trivial $\mathfrak{sl}_2$ submodule $W_2 \subset W_1$. Applying our construction to W_2 , we obtain the $\mathfrak{osp}_{2,2}$ submodule U . According to our h' -eigenvalue analysis, W_2 will be exactly the μ -eigenspace for h' in U . Since U must be all of V by irreducibility of V , it follows that W_2 must be the μ -eigenspace for h' on V . Hence $W_2 = W_1$. That is, W_1 has no proper non-trivial $\mathfrak{sl}_2$ submodules, so it is irreducible for $\mathfrak{sl}_2$.

Since W_1 is a $\mathfrak{q}$ submodule of V , the universal property of induction [Hu] implies that there is an $\mathfrak{osp}_{2,2}$ -morphism from W to V . Since V is irreducible, this must be onto.

Finally, let us show that W is indecomposable. If $W = U_1 \oplus U_2$ is a decomposition of W into $\mathfrak{osp}_{2,2}$ submodules, then we must have $W_1 = W_1 \cap U_1 \oplus W_1 \cap U_2$, since W_1 is the full μ -eigenspace for h' in W . Both spaces $W_1 \cap U_j$ must be invariant under $\mathfrak{sl}_2$. Since we know that W_1 is irreducible under the action of $\mathfrak{sl}_2$, it follows that one of the $W_1 \cap U_j$ must be all

of W_1 , that is, $W_1 \subset U_j$. Since W_1 generates W , this implies that $U_j = W$, which means that W is indeed indecomposable. This concludes the proof of the theorem. $\square$

THEOREM 4.3. *Suppose V is an h, h' -semisimple, h -admissible, $\Omega_{\mathfrak{osp}}$ -quasisimple, irreducible $\mathfrak{osp}_{2,2}$ module and $W_1 = \ker d^* \cap \ker \delta$. Let $W = \text{Ind}_q^{\mathfrak{osp}_{2,2}} W_1$. Then W is irreducible if and only if $\Omega_{\mathfrak{osp}} W \neq 0$. In this case, V is isomorphic to W .*

Proof. In the following we just write $a \otimes v$ simply as av for $a \in \mathfrak{U}(\mathfrak{osp}_{2,2})$, $v \in W_1$. From Theorem 4.2, $W = \text{span}\{W_1, dW_1, \delta^* W_1, d\delta^* W_1\}$. If we let $\tilde{W}_1 = d\delta^* W_1$, then

$$\tilde{W} = \tilde{W}_1 + d^* \tilde{W}_1 + \delta \tilde{W}_1 + d^* \delta \tilde{W}_1$$

is an $\mathfrak{osp}_{2,2}$ submodule of V by the reasoning of Theorem 4.2. By comparing h' -eigenvalues, we see that $d^* \delta \tilde{W}_1 \subset W_1$. Thus we know that W is irreducible if and only if $d^* \delta \tilde{W}_1 \neq 0$, that is, if and only if $d^* \delta d\delta^* W_1 \neq 0$. We compute that

$$\begin{aligned} d^* \delta d\delta^* &= d^* (\{d, \delta\} - d\delta) \delta^* = -d^* (h + h') \delta^* - d^* d\delta\delta^* \\ &= -2d^* \delta^* - d^* \delta^* (h + h') - (\{d^*, d\} - dd^*) (\{\delta, \delta^*\} - \delta^* \delta) \\ &= -2(\{d^*, \delta^*\} - \delta^* d^*) - (\{d^*, \delta^*\} - \delta^* d^*) (h + h') - (-2e^- - dd^*) (2e^+ - \delta^* \delta) \\ &= -2(-h + h') - (-h + h') (h + h') + 4e^- e^+ + 4\delta^* d^* \\ &\quad - 2d\delta + \delta^* (h + h') d^* + 2de^+ d^* - 2e^- \delta^* \delta - dd^* \delta^* \delta \\ &= (h^2 + 2h + 4e^- e^+) - h'^2 - 2h' + 2\delta^* d^* - 2d\delta + 2\delta^* d^* \\ &\quad + \delta^* (h + h') d^* + 2de^+ d^* - 2e^- \delta^* \delta - dd^* \delta^* \delta \\ &= \Omega_{\mathfrak{osp}} + 2\delta^* d^* + \delta^* (h + h') d^* + 2de^+ d^* - 2e^- \delta^* \delta - dd^* \delta^* \delta \quad (\text{by (2.8)}) \end{aligned}$$

For any $v \in W_1$, v is annihilated by d^* and by δ , and thus $d^* \delta d\delta^* v = \Omega_{\mathfrak{osp}} v$. We get $d^* \delta d\delta^* W_1 \neq 0$ if and only if $\Omega_{\mathfrak{osp}} v \neq 0$. When W is irreducible, Theorem 4.2 implies the isomorphism of V and W . This completes the proof. $\square$

LEMMA 4.4. *Suppose $V = \text{Ind}_q^{\mathfrak{osp}_{2,2}} (V_{\lambda+1, \mu-1})$ where $V_{\lambda+1, \mu-1} = \ker d^* \cap \ker \delta$ is a lowest weight irreducible module with lowest weight $\lambda + 1$ ($\lambda \in \mathbb{Z}^+$) and h' -eigenvalue $\mu - 1$ restricted to $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$. Then V is an indecomposable $\mathfrak{osp}_{2,2}$ module. Furthermore,*

- (1) when $\mu \neq \pm\lambda$, V is irreducible and $V|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{\lambda, \mu} \oplus V_{\lambda+1, \mu-1} \oplus V_{\lambda+1, \mu+1} \oplus V_{\lambda+2, \mu}$;
- (2) when $\lambda \neq 0$,
 - (a) $\mu = -\lambda$, V has a submodule U_1 with $U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{\lambda, -\lambda} \oplus V_{\lambda+1, -\lambda+1}$ and a quotient U_2 with $U_2|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{\lambda+1, -(\lambda+1)} \oplus V_{\lambda+2, -\lambda}$;
 - (b) $\mu = \lambda$, V has a submodule U_1 with $U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{\lambda+1, \lambda+1} \oplus V_{\lambda+2, \lambda}$ and a quotient U_2 with $U_2|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{\lambda, \lambda} \oplus V_{\lambda+1, \lambda-1}$;
- (3) when $\lambda = \mu = 0$, V has a composition series $0 \subset U_1 \subset U_2 \subset U_3 = V$ with $U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{1,1} \oplus V_{2,0}$, $U_2/U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{1,-1} \oplus V_{2,0}$ and U_3/U_2 a trivial module.

Proof. In the proof, we just write $a \otimes v$ simply as av for $a \in \mathfrak{U}(\mathfrak{osp}_{2,2})$, $v \in V_{\lambda+1, \mu-1}$.

Since d and δ^* generate a four-dimensional subalgebra inside $\mathfrak{U}(\mathfrak{osp}_{2,2})$, which is isomorphic to $F_{0,0} \oplus F_{-1,1} \oplus F_{0,2}$ as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module. Therefore, as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module,

FIGURE 1. Lowest weight vectors in V . Generically, they generate an irreducible $\mathfrak{osp}_{2,2}$ module.

V is isomorphic to $V_{\lambda+1,\mu-1} \otimes (F_{0,0} \oplus F_{-1,1} \oplus F_{0,2}) \cong V_{\lambda+1,\mu-1} \oplus V_{\lambda+1,\mu-1} \otimes F_{-1,1} \oplus V_{\lambda+1,\mu+1}$. From [HT], $V_{\lambda+1,\mu-1} \otimes F_{-1,1} \cong V_{\lambda,\mu} \oplus V_{\lambda+2,\mu}$. Thus as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module, V is generated by four e^- -null vectors. Let v_0 be the lowest weight vector in $V_{\lambda+1,\mu-1}$ with h, h' -eigenvalues $\lambda + 1$ and $\mu - 1$ respectively. It is easy to check that dv_0 , $d\delta^*v_0$ and $(\delta^*h + de^+)v_0$ are the e^- -null vectors. See Figure 1.

We compute that

$$\begin{aligned} hdv_0 &= \lambda dv_0, & h'(dv_0) &= \mu dv_0, \\ h d\delta^*v_0 &= (\lambda + 1)d\delta^*v_0, & h' d\delta^*v_0 &= (\mu + 1)d\delta^*v_0, \\ h(\delta^*h + de^+)v_0 &= (\lambda + 2)(\delta^*h v_0 + de^+v_0), & h'(\delta^*h + de^+)v_0 &= \mu(\delta^*h v_0 + de^+v_0). \end{aligned}$$

We find that $v_0, dv_0, (\delta^*h + de^+)v_0, d\delta^*v_0$ generate $V_{\lambda+1,\mu-1}, V_{\lambda,\mu}, V_{\lambda+2,\mu}, V_{\lambda,\mu+1}$ as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ modules.

The indecomposability is implied by Theorem 4.2. Now we want to discuss whether these four spaces can be mapped back and forth by the four operators from the super part. That is, we want to determine the irreducibility of V and also its composition factors.

From Theorem 4.3, we need to calculate the eigenvalue of $\Omega_{\mathfrak{osp}}$. Let $v \in V_{\lambda+1,\mu-1}$ and we use (2.8) for $\Omega_{\mathfrak{osp}}$,

$$\begin{aligned} \Omega_{\mathfrak{osp}}v &= (\Omega_{\mathfrak{sl}_2} - h'^2 - 2h' - 2d\delta + 2\delta^*d^*)v \\ &= (\Omega_{\mathfrak{sl}_2} - h'^2 - 2h')v \quad \text{since } d^*v = \delta v = 0 \\ &= ((\lambda + 1)^2 - 2(\lambda + 1) - (\mu - 1)^2 - 2(\mu - 1))v = (\lambda + \mu)(\lambda - \mu)v = 0. \end{aligned}$$

Hence, we see that $\Omega_{\mathfrak{osp}} = 0$ exactly when $\mu = \pm\lambda$.

By Theorem 4.3, we find that when $\mu \neq \pm\lambda$ the four spaces can be mapped back and forth by the operators from the super part, which indicates the irreducibility of V . This gives part (1).

Now considering part (2) $\lambda \neq 0$.

(a) When $\mu = -\lambda$, $\delta dv_0 = (-h - h')v_0 - d\delta v_0 = (-\lambda - 1 - (\mu - 1))v_0 = 0$. Since $\delta dv_0 = 0$ and the basis vectors inside $V_{\lambda,\mu}$ consist of $(e^+)^k dv_0$, but e^+ commutes with δ , we have $\delta(V_{\lambda,-\lambda}) = 0$. Since $\delta(V_{\lambda,-\lambda}) = 0$, $d^*(e^+)^k dv_0 = (e^+d^* - \delta)(e^+)^{k-1} dv_0 = e^+d^*(e^+)^{k-1} dv_0 = (e^+)^k d^*dv_0 = 0$. This indicates that $V_{\lambda,-\lambda}$ could not be mapped to $V_{\lambda+1,-(\lambda+1)}$.

Under this condition $\mu = -\lambda$, and we find that

$$\begin{aligned} d^*(d\delta^*v_0) &= (-2e^- - dd^*)\delta^*v_0 = -2dv_0 - d(-\lambda - 1 + \mu - 1)v_0 = (\lambda - \mu)dv_0, \\ \delta(d\delta^*v_0) &= (-h - h' - \delta)\delta^*v_0 = -2\delta^*v_0 - (\lambda + \mu)\delta^*v_0 - 2(e^+d - \delta^*)v_0 = -2e^+(dv_0). \end{aligned}$$

Thus we find that $V_{\lambda, -\lambda} \oplus V_{\lambda+1, -\lambda+1}$ is a submodule inside V and $V_{\lambda+1, -(\lambda+1)} \oplus V_{\lambda+2, -\lambda}$ is a quotient.

(b) When $\mu = \lambda$, we calculate that

$$\begin{aligned} \delta(\delta^*h + de^+)v_0 &= 2e^+hv_0 - \delta^*\delta hv_0 + (-h - h')e^+v_0 - d\delta e^+v_0 \\ &= 2e^+hv_0 - \delta^*(h\delta - \delta)v_0 - (h + h')e^+v_0 - d\delta e^+v_0 \\ &= 2(\lambda + 1 - (\lambda + 3 + \mu - 1))e^+v_0 \quad \text{since } \delta v_0 = 0 \\ &= (\lambda - \mu)e^+v_0 = 0, \\ d^*(\delta^*h + de^+)v_0 &= (-h + h')hv_0 - \delta^*d^*hv_0 - 2e^-e^+v_0 - dd^*e^+v_0 \\ &= -(h - h')hv_0 - \delta^*(hd^* + d^*)v_0 - 2e^+e^-v_0 + 2hv_0 \\ &\quad - d\delta v_0 - de^+d^*v_0 \\ &= -(\lambda + 1 - \mu + 1)(\lambda + 1)v_0 + 2(\lambda + 1)v_0 \\ &\quad \text{since } d^*v_0 = \delta v_0 = e^-v_0 = 0 \\ &= -(\lambda + 1)(\lambda - \mu)v_0 = 0. \end{aligned}$$

Thus, when $\mu = \lambda$, $V_{\lambda+2, \lambda}$ could not be mapped to $V_{\lambda+1, \lambda-1}$.

Similarly, we can check that under this condition $V_{\lambda+1, \lambda+1} \oplus V_{\lambda+2, \lambda}$ is a submodule inside V and $V_{\lambda, \lambda} \oplus V_{\lambda+1, \lambda-1}$ is a quotient.

Finally, we consider part (3) $\lambda = \mu = 0$. In this case, as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module, $V \cong V_{1, -1} \oplus V_{0, 0} \oplus V_{2, 0} \oplus V_{1, 1}$ and $V_{0, 0}$ is not irreducible with a trivial quotient $F_{0, 0}$ and a submodule isomorphic to $V_{2, 0}$ denoted by $V'_{2, 0}$.

By similar calculations as for (2), we get that $U_1 = V_{1, 1} \oplus V_{2, 0}$ is an irreducible submodule. Also when we consider the action of $\mathfrak{osp}_{2, 2}$ on $V_{1, -1} \oplus V'_{2, 0}$, a simple calculation shows that $V_{1, -1} \oplus V'_{2, 0}$ is mapped to $U_1 \oplus V_{1, -1} \oplus V'_{2, 0}$, which indicates that $U_2 = U_1 \oplus V_{1, -1} \oplus V'_{2, 0}$ has an irreducible quotient U_2/U_1 isomorphic to $V_{1, -1} \oplus V'_{2, 0}$ as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module.

Finally, dv_0 is mapped by the elements in $\mathfrak{osp}_{2, 2}$ into U_2 , and thus V/U_2 is a trivial $\mathfrak{osp}_{2, 2}$ module. Therefore, when $\lambda = \mu = 0$, V has a composition series $0 \subset U_1 \subset U_2 \subset U_3 = V$ with $U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{1, 1} \oplus V_{2, 0}$, $U_2/U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong V_{1, -1} \oplus V'_{2, 0}$, $U_3/U_2|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{0, 0}$. This gives part (3). $\square$

Remark 4.5. The analysis for an h, h' -semisimple, h -admissible, $\Omega_{\mathfrak{osp}}$ -quasisimple, $\mathfrak{osp}_{2, 2}$ module induced from an irreducible $\mathfrak{q}$ module $\bar{V}_{\lambda-1, \mu-1}$ annihilated by d^* and δ is the same as for Lemma 4.4. We will just state this result in our classification.

Now we are able to describe the structures of the h, h' -semisimple, h -admissible, $\Omega_{\mathfrak{osp}}$ -quasisimple irreducible lowest and highest weight $\mathfrak{osp}_{2, 2}$ modules.

(1) Irreducible lowest weight modules $W(\lambda, \mu)$ ($\lambda \notin \mathbb{Z}^-$).

(i) $\mu \neq \pm\lambda$,

(ii) $\mu = \lambda$ ($\lambda \neq 0$),

(iii) $\mu = -\lambda$ ($\lambda \neq 0$),

(2) Irreducible highest weight modules $\bar{W}(\lambda, \mu)$ ($\lambda \notin \mathbb{Z}^+$).

(i) $\mu \neq \pm\lambda$,

(ii) $\mu = \lambda$ ($\lambda \neq 0$),

(iii) $\mu = -\lambda$ ($\lambda \neq 0$),

LEMMA 4.6. Suppose $V = \text{Ind}_{\mathfrak{q}}^{\mathfrak{osp}_{2,2}}(F_{\lambda+1, \mu-1})$ where $F_{\lambda+1, \mu-1} = \ker d^* \cap \ker \delta$ is a finite-dimensional module with lowest h -eigenvalue $\lambda + 1$ ($\lambda \leq -1$, $\lambda \in \mathbb{Z}$) and h' -eigenvalue $\mu - 1$ restricted to $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$. Then V is an indecomposable $\mathfrak{osp}_{2,2}$ module. Furthermore,

- (1) for $\lambda \leq -2$, $\lambda \in \mathbb{Z}$,
- (a) when $\mu \neq \pm\lambda$, V is irreducible and $V|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{\lambda+1, \mu-1} \oplus F_{\lambda, \mu} \oplus F_{\lambda+2, \mu} \oplus F_{\lambda+1, \mu+1}$,
 - (b) when $\mu = -\lambda$, V has a submodule U_1 with $U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{\lambda, -\lambda} \oplus F_{\lambda+1, -(\lambda-1)}$ and a quotient U_2 with $U_2|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{\lambda+1, -(\lambda+1)} \oplus F_{\lambda+2, -\lambda}$,
 - (c) when $\mu = \lambda$, V has a submodule U_1 with $U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{\lambda+1, \lambda+1} \oplus F_{\lambda+2, \lambda}$ and a quotient U_2 with $U_2|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{\lambda, \lambda} \oplus F_{\lambda+1, \lambda-1}$;
- (2) for $\lambda = -1$,
- (a) when $\mu \neq \pm 1$, V is irreducible and $V|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{-1, \mu} \oplus F_{0, \mu-1} \oplus F_{0, \mu+1}$,
 - (b) when $\mu = 1$, V has a submodule U_1 with $U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{-1, 1} \oplus F_{0, 2}$ and a trivial quotient U_2 ,
 - (c) when $\mu = -1$, V has a trivial submodule U_1 and a quotient U_2 with $U_2|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{-1, -1} \oplus F_{0, -2}$.

Proof. When $\lambda \neq -1$, that is, $F_{\lambda+1, \mu-1}$ is not the trivial $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module, the idea used to prove this lemma is the same as that in Lemma 4.4. The only difference is that in this case the $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ modules generated by e^- -null vectors are finite-dimensional.

When $\lambda = -1$, that is, $F_{0, \mu-1}$ is the trivial module, the induced $\mathfrak{osp}_{2,2}$ module V is four-dimensional and $V|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} = F_{-1, \mu} \oplus F_{0, \mu-1} \oplus F_{0, \mu+1}$. Similar calculations as in Lemma 4.4 show that when $\mu \neq \pm 1$ the eigenvalue of the Casimir operator is non-zero, and thus V is irreducible. When $\mu = \pm 1$, we can also get the composition factors similar to as in Lemma 4.4. $\square$

By Lemma 4.6, we can easily classify the irreducible finite-dimensional $\mathfrak{osp}_{2,2}$ modules.

- (3) Irreducible finite-dimensional modules $F_{\mathfrak{osp}}(\lambda, \mu) (\lambda \in \mathbb{Z}^-)$.
- (i) Trivial module $F_{\mathfrak{osp}}(0, 0)$.
 - (ii) $\mu \neq \pm\lambda$,
 - (a) $\lambda \neq -1$,

$$\begin{array}{ccc}
 & F_{\lambda+1, \mu+1} & \\
 & \swarrow \quad \searrow & \\
 F_{\lambda, \mu} & & F_{\lambda+2, \mu} \\
 & \swarrow \quad \searrow & \\
 & F_{\lambda+1, \mu-1} &
 \end{array}$$

- (b) $\lambda = -1$,

$$F_{\mathfrak{osp}}(-1, \mu)|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{-1, \mu} \oplus F_{0, \mu-1} \oplus F_{0, \mu+1},$$

- (iii) $\mu = -\lambda (\lambda \neq 0)$,

$$\begin{array}{c}
 F_{\mathfrak{osp}}(\lambda, -\lambda) = \begin{array}{c} F_{\lambda+1, -\lambda+1} \\ \swarrow \\ F_{\lambda, -\lambda} \end{array}
 \end{array}$$

$$(iv) \quad \mu = \lambda(\lambda \neq 0),$$

$$F_{\mathfrak{osp}(\lambda, \lambda)} = \begin{array}{c} F_{\lambda, \lambda} \\ \searrow \\ F_{\lambda+1, \lambda-1} \end{array}$$

LEMMA 4.7. Suppose $V = \text{Ind}_{\mathfrak{q}}^{\mathfrak{osp}_{2,2}}(U)$ where $U = \ker d^* \cap \ker \delta$ is an irreducible $\mathfrak{q}$ module isomorphic to $U(v^+, v^- + 1, \mu - 1)(v^+, v^- \notin \mathbb{Z}, \mu \in \mathbb{C})$ restricted to $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$. Then V is an indecomposable $\mathfrak{osp}_{2,2}$ module. Furthermore,

(1) when $\mu \neq \pm(v^+ + v^-)$, V is irreducible and

$$V|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(v^+, v^-, \mu) \oplus U(v^+, v^- + 1, \mu - 1) \oplus U(v^+, v^- + 1, \mu + 1) \\ \oplus U(v^+ + 1, v^- + 1, \mu);$$

(2) when $\mu = v^+ + v^-$, V has a submodule U_1 and a quotient U_2 with

$$U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(v^+ + 1, v^- + 1, \mu) \oplus U(v^+, v^- + 1, \mu + 1), \\ U_2|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(v^+, v^- + 1, \mu - 1) \oplus U(v^+, v^-, \mu);$$

(3) when $\mu = -(v^+ + v^-)$, V has a submodule U_1 and a quotient U_2 with

$$U_1|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(v^+, v^-, \mu) \oplus U(v^+, v^- + 1, \mu + 1), \\ U_2|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(v^+, v^- + 1, \mu - 1) \oplus U(v^+ + 1, v^- + 1, \mu).$$

Proof. Let $\{v_j\}$ be a standard basis of U as in (3.5). In the following we just write $a \otimes v$ simply as av for $a \in \mathfrak{U}(\mathfrak{osp}_{2,2})$, $v \in U$. Since U is annihilated by d^* , δ and d , δ^* generates a four-dimensional subalgebra inside $\mathfrak{U}(\mathfrak{osp}_{2,2})$, and we know that

$$W_1 = U, \\ W_2 = \text{span}\{-\delta^* e^- v_{j+1} + d e^+ v_j\}, \\ W_3 = \text{span}\{\delta^* v_j + d v_{j+1}\}, \\ W_4 = \text{span}\{d \delta^* v_j\} \text{ span the whole } V.$$

We first check that $\{-\delta^* e^- v_{j+1} + d e^+ v_j\}$ is a basis for $U(v^+, v^-, \mu)$:

$$h'(-\delta^* e^- v_{j+1} + d e^+ v_j) = -\delta^* e^- v_{j+1} - \delta^* h' e^- v_{j+1} + d e^+ v_j + d h' e^+ v_j \\ = \mu(-\delta^* e^- v_{j+1} + d e^+ v_j), \\ h(-\delta^* e^- v_{j+1} + d e^+ v_j) = -\delta^* e^- v_{j+1} - \delta^* h e^- v_{j+1} - d e^+ v_j + d h e^+ v_j \\ = -\delta^* e^- v_{j+1} - (v^+ - v^- - 1 + 2j)\delta^* e^- v_{j+1} \\ - d e^+ v_j + (v^+ - v^- + 2j + 1)d e^+ v_j \\ = (v^+ - v^- + 2j)(-\delta^* v_{j+1} + d e^+ v_j),$$

$$\begin{aligned}
e^+(-\delta^* e^- v_{j+1} + de^+ v_j) &= -\delta^* h v_{j+1} - \delta^* e^- e^+ v_{j+1} + (v^+ + j) \delta^* v_{j+1} \\
&\quad + (v^+ + j) de^+ v_{j+1} \\
&= (v^- - j - 1) \delta^* v_{j+1} - (v^+ + j + 1) \delta^* e^- v_{j+2} \\
&\quad + (v^+ + j) de^+ v_{j+1} \\
&= \delta^* e^- v_{j+2} - (v^+ + j + 1) \delta^* e^- v_{j+2} + (v^+ + j) de^+ v_{j+1} \\
&= (v^+ + j) (-\delta^* e^- v_{j+2} + de^+ v_{j+1}), \\
e^-(-\delta^* e^- v_{j+1} + de^+ v_j) &= -de^- v_{j+1} - \delta^* (e^-)^2 v_{j+1} + d(e^+ e^- - h) v_j \\
&= -(v^- - j) dv_j - (v^- - j) \delta^* e^- v_j \\
&\quad + (v^- + 1 - j) de^+ v_{j-1} - (v^+ - v^- - 1 - 2j) dv_j \\
&= -(v^+ + j - 1) dv_j - (v^- - j) \delta^* e^- v_j \\
&\quad + (v^- + 1 - j) de^+ v_{j-1} \\
&= -de^+ v_{j-1} - (v^- - j) \delta^* e^- v_j + (v^- + 1 - j) de^+ v_{j-1} \\
&= (v^- - j) (-\delta^* e^- v_j + de^+ v_{j-1}).
\end{aligned}$$

Also

$$\begin{aligned}
h'(\delta^* v_j + dv_{j+1}) &= \delta^* v_j + \delta^* h' v_j + dv_{j+1} + dh' v_{j+1} = \mu(\delta^* v_j + dv_{j+1}), \\
h(\delta^* v_j + dv_{j+1}) &= (v^+ - v^- - 1 + 2j + 1) \delta^* v_j + (v^+ - v^- - 1 + 2j + 1) dv_{j+1} \\
&= (v^+ + 1 - (v^- + 1) + 2j) (\delta^* v_j + dv_{j+1}), \\
e^+(\delta^* v_j + dv_{j+1}) &= (v^+ + j) \delta^* v_{j+1} + (v^+ + j + 1) dv_{j+2} + \delta^* v_{j+1} \\
&= (v^+ + 1 + j) (\delta^* v_{j+1} + dv_{j+2}), \\
e^-(\delta^* v_j + dv_{j+1}) &= dv_j + (v^- + 1 - j) \delta^* v_{j-1} + (v^- + 1 - j - 1) dv_j \\
&= (v^- + 1 - j) (\delta^* v_{j-1} + dv_j).
\end{aligned}$$

Hence $\{\delta^* v_j + dv_{j+1}\}$ is a basis for $U(v^+ + 1, v^- + 1, \mu)$.

Finally,

$$\begin{aligned}
h' d \delta^* v_j &= (\mu + 1) d \delta^* v_j, \\
h d \delta^* v_j &= -d \delta^* v_j + d \delta^* v_j + (v^+ - v^- - 1 + 2j) v_j = (v^+ - v^- - 1 + 2j) d \delta^* v_j, \\
e^+ d \delta^* v_j &= d \delta^* e^+ v_j = (v^+ + j) d \delta^* v_{j+1}, \\
e^- d \delta^* v_j &= d \delta^* e^- v_j = (v^- + 1 - j) d \delta^* v_{j-1}.
\end{aligned}$$

Thus $\{d \delta^* v_j\}$ is a basis for $U(v^+, v^- + 1, \mu + 1)$.

By Theorem 4.3, we need to calculate the eigenvalue of $\Omega_{\mathfrak{osp}}$ to see the irreducibility and we use the alternative form (2.8) for $\Omega_{\mathfrak{osp}}$. Let $v \in U \cong U(v^+, v^- + 1, \mu - 1)$:

$$\begin{aligned}
\Omega_{\mathfrak{osp}} v &= (\Omega_{\mathfrak{sl}_2} - h'^2 - 2h' - 2d\delta + 2\delta^* d^*) v \\
&= (v^+ + v^- + 1)(v^+ + v^- + 1 - 2)v - (\mu - 1)^2 v - 2(\mu - 1)v
\end{aligned}$$

$$\begin{aligned}
&= (v^+ + v^- + 1)(v^+ + v^- - 1)v - (\mu - 1)^2v - 2(\mu - 1)v \\
&= (v^+ + v^- - \mu)(v^+ + v^- + \mu)v.
\end{aligned}$$

Thus when $\mu \neq \pm(v^+ + v^-)$, V is irreducible.

When $\mu = v^+ + v^-$, first we consider $W_3 = \text{span}\{\delta^*v_j + dv_{j+1}\}$:

$$\begin{aligned}
d^*(\delta^*v_j + dv_{j+1}) &= ((-h + h') - \delta^*\delta^*)v_j - 2e^-v_{j+1} - d\delta^*v_{j+1} \\
&= (-(v^+ - v^- - 1 + 2j) + \mu - 1)v_j - 2(v^- + 1 - j - 1)v_j \\
&= (\mu - v^+ - v^-)v_j = 0, \\
\delta(\delta^*v_j + dv_{j+1}) &= (2e^+ - \delta^*\delta)v_j + [(-h - h') - d\delta]v_{j+1} \\
&= 2(v^+ + j)v_{j+1} - (v^+ - v^- - 1 + 2j + 2 + \mu - 1)v_{j+1} \\
&= (v^+ + v^- - \mu)v_j = 0.
\end{aligned}$$

Thus when $\mu = v^+ + v^-$, W_3 is annihilated by d^* and δ . Under this condition,

$$\begin{aligned}
d^*(d\delta^*v_j) &= (-2e^- - dd^*)\delta^*v_j = -2dv_j - 2(v^- + 1 - j)\delta^*v_{j-1} + d(h - h')v_j \\
&= -2dv_j - 2(v^- + 1 - j)\delta^*v_{j-1} + d(-2v^- + 2j)v_j \\
&= -2(v^- + 1 - j)(\delta^*v_{j-1} + dv_j), \\
\delta(d\delta^*v_j) &= (-h - h')\delta^*v_j - d\delta\delta^*v_j \\
&= -(2v^+ + 2j - 2)\delta^*v_j - 2\delta^*v_j - d(2e^+ - \delta^*\delta)v_j \\
&= -2(v^+ + j)(\delta^*v_j + dv_{j+1}).
\end{aligned}$$

Therefore, $W_3 \oplus W_4$ is a submodule and $W_1 \oplus W_2$ is a quotient.

Similarly it can be checked when $\mu = -(v^+ + v^-)$, $d^*(W_2) = \delta(W_2) = 0$ and $d^*(W_4) = \delta(W_4) = W_2$. In this case $W_2 \oplus W_4$ is a submodule and $W_1 \oplus W_3$ is a quotient. This finishes the proof. $\square$

(4) Irreducible modules $W(v^+, v^-, \mu)$, $v^+, v^- \notin \mathbb{Z}$, $\mu \in \mathbb{C}$.

(i) $\mu \neq \pm(v^+ + v^-)$,

$$\begin{array}{ccc}
& U(v^+, v^- + 1, \mu + 1) & \\
& \swarrow \quad \searrow & \\
U(v^+, v^-, \mu) & & U(v^+ + 1, v^- + 1, \mu) \\
& \swarrow \quad \searrow & \\
& U(v^+, v^- + 1, \mu - 1) &
\end{array}$$

(ii) $\mu = v^+ + v^-$,

$$W|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(v^+, v^-, \mu) \oplus U(v^+, v^- + 1, \mu - 1).$$

(iii) $\mu = -(v^+ + v^-)$,

$$W|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(v^+, v^-, \mu) \oplus U(v^+ + 1, v^-, \mu + 1).$$

THEOREM 4.8. *All h, h' -semisimple, h -admissible, $\Omega_{\mathfrak{osp}}$ -quasisimple irreducible representations of $\mathfrak{osp}_{2,2}$ are classified into four groups listed in the lemmas above:*

- (1) $W(\lambda, \mu), \lambda \in \mathbb{C}, \lambda \notin \mathbb{Z}^-, \mu \in \mathbb{C};$
- (2) $\bar{W}(\lambda, \mu), \lambda \notin \mathbb{Z}^+, \mu \in \mathbb{C};$
- (3) $F_{\mathfrak{osp}}(\lambda, \mu), \lambda \in \mathbb{Z}^-;$
- (4) $W(v^+, v^-, \mu), v^+, v^- \notin \mathbb{Z}, \mu \in \mathbb{C}.$

Remark 4.9. In this list, when $\mu \neq \pm\lambda$ in (1)–(3) and $\mu \neq \pm(v^+ + v^-)$ in (4), the irreducible $\mathfrak{osp}_{2,2}$ module is isomorphic to $W \otimes E$ as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module where $W = \ker d^* \cap \ker \delta$, and E is the four-dimensional subalgebra generated by d and δ^* . When $\mu = \pm\lambda$ in (1)–(3), and $\mu = \pm(v^+ + v^-)$ in (4), the irreducible $\mathfrak{osp}_{2,2}$ module is composed of two irreducible $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ modules.

Proof. In Theorem 4.2, we proved that each h -admissible, h, h' -semisimple, $\Omega_{\mathfrak{osp}}$ -quasisimple irreducible $\mathfrak{osp}_{2,2}$ module V is a quotient of an induced module from an irreducible $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ module which is annihilated by d^* and δ . Moreover, we have discussed the structures of these modules induced from four different classes of $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$ modules in detail and found out their irreducible quotients. Thus V must be isomorphic to one of the $\mathfrak{osp}_{2,2}$ modules listed in the theorem. $\square$

From Lemmas 4.4–4.7, we also obtained the eigenvalues of the Casimir operator for the irreducible $\mathfrak{osp}_{2,2}$ modules summarized in Theorem 4.8.

THEOREM 4.10. *The Casimir operator $\Omega_{\mathfrak{osp}}$ of $\mathfrak{osp}_{2,2}$ has the following eigenvalues in the irreducible $\mathfrak{osp}_{2,2}$ representations:*

- (1) $\Omega_{\mathfrak{osp}}v = (\lambda^2 - \mu^2)v, v \in W(\lambda, \mu), \lambda \notin \mathbb{Z}^-, \mu \in \mathbb{C};$
- (2) $\Omega_{\mathfrak{osp}}v = (\lambda^2 - \mu^2)v, v \in \bar{W}(\lambda, \mu), \lambda \notin \mathbb{Z}^+, \mu \in \mathbb{C};$
- (3) $\Omega_{\mathfrak{osp}}v = (\lambda^2 - \mu^2)v, v \in F_{\mathfrak{osp}}(\lambda, \mu), \lambda \in \mathbb{Z}^-, \mu \in \mathbb{C};$
- (4) $\Omega_{\mathfrak{osp}}v = ((v^+ + v^-)^2 - \mu^2)v, v \in W(v^+, v^-, \mu), v^+, v^- \notin \mathbb{Z}, \mu \in \mathbb{C}.$

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