h TRANSFORM OF ONE-DIMENSIONAL GENERALIZED DIFFUSION OPERATORS

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Abstract. We are concerned with two types of h transform of one-dimensional generalized diffusion operators treated by Maeno (2006) and by Tomisaki (2007). We show that these two types of h transform are in inverse relation to each other in some sense. Further, we show that a recurrent one-dimensional generalized diffusion operator cannot be represented as an h transform of another one-dimensional generalized diffusion operator different from the original one. We also consider a spectral representation of elementary solutions corresponding to h transformed one-dimensional generalized diffusion operators.

1. Introduction

Let $\mathcal{G}_{s,m,k}$ be a one-dimensional generalized diffusion operator on an interval I with scale function s , speed measure m , and killing measure k (ODGDO with (s, m, k) for brevity). Let $\mathcal{M}_{s,m}$ be the set of all positive functions h satisfying $\mathcal{G}_{s,m,0}h \leq 0$, where $\mathcal{G}_{s,m,0}$ is an ODGDO with $(s, m, 0)$ and 0 denotes the null killing measure. For $\beta \geq 0$, let $\mathcal{H}_{s,m,k,\beta}$ be the set of all positive functions satisfying $\mathcal{G}_{s,m,k}h = \beta h$. In this paper, we are concerned with two types of h transform based on $h \in \mathcal{M}_{s,m}$ and $h \in \mathcal{H}_{s,m,k,\beta}$.

It is well known that the generator of a one-dimensional generalized diffusion process (ODGDP) is represented as an ODGDO. In Section 4.3 of [3], Itô and McKean used a method to derive such a general form $\mathcal{G}_{s,m,k}$ for the generator of ODGDP. Their method is an h transform based on the probability which the sample paths hit one of the end points of the state interval. Following their idea, Maeno [4] considered h transforms of ODGDPs based on $h \in \mathcal{M}_{s,m}$ and studied some properties corresponding to h transformed ODGDP. On the other hand, Tomisaki [8] considered h transforms of ODGDOs based on $h \in \mathcal{H}_{s,m,k,\beta}$ and obtained some results for h transformed ODGDOs. As we will see in Proposition 1.1 below, the sets $\mathcal{M}_{s,m}$ and $\mathcal{H}_{s,m,k,\beta}$ are usually disjoint to each other. Therefore the results of [4] and [8] are not necessarily derived from each other.

PROPOSITION 1.1.

- (i) *The set $\mathcal{H}_{s,m,0,0}$ coincides with the set $\{h \in \mathcal{M}_{s,m}; D_s h(x) \text{ is a constant function on } I\}$.*
- (ii) *If k is not a null measure or $\beta > 0$, then $\mathcal{M}_{s,m} \cap \mathcal{H}_{s,m,k,\beta} = \emptyset$.*

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Here $D_s h$ is the right derivative of h with respect to the scale function s . We show this proposition in Section 3.

Let $p(t, x, y)$ be the elementary solution of the equation

$$\frac{\partial}{\partial t} p(t, x, y) = \mathcal{G}_{s,m,k} p(t, x, y), \quad t > 0, x, y \in I, \quad (1.1)$$

in the sense of McKean [5], where $\mathcal{G}_{s,m,k}$ is applied to x or y .

Let $p^o(t, x, y)$ be the elementary solution of equation (1.1) with $\mathcal{G}_{s,m,k}$ replaced by $\mathcal{G}_{s,m,0}$. For $h \in \mathcal{M}_{s,m}$, we set

$$p_h^o(t, x, y) = p^o(t, x, y) / h(x)h(y), \quad t > 0, x, y \in I, \quad (1.2)$$

$$s_h(x) = \int_{(c_o, x]} h(y)^{-2} ds(y), \quad (1.3)$$

$$m_h(x) = \int_{(c_o, x]} h(y)^2 dm(y), \quad (1.4)$$

$$k_h(x) = - \int_{(c_o, x]} h(y) dD_s h(y), \quad (1.5)$$

where c_o is a point of I fixed arbitrarily. Maeno [4] showed that $p_h^o(t, x, y)$ is the elementary solution of equation (1.1) with $\mathcal{G}_{s_h, m_h, k_h}$ in place of $\mathcal{G}_{s,m,k}$. Further, Maeno studied the asymptotic behavior near the boundaries of I for sample paths of ODGDP with generator $\mathcal{G}_{s_h, m_h, k_h}$, and gave a precise classification of the states of the boundaries by means of s , m and h . We call $\mathcal{G}_{s_h, m_h, k_h}$ the h transform of $\mathcal{G}_{s,m,0}$ with $h \in \mathcal{M}_{s,m}$.

We next turn to an h transform treated in [8]. Let $p(t, x, y)$ be the elementary solution of equation (1.1). For $h \in \mathcal{H}_{s,m,k,\beta}$ set

$$p_h^*(t, x, y) = e^{-\beta t} p(t, x, y) / h(x)h(y), \quad t > 0, x, y \in I. \quad (1.6)$$

Then $p_h^*(t, x, y)$ is the elementary solution of equation (1.1) with $\mathcal{G}_{s_h, m_h, 0}$ in place of $\mathcal{G}_{s,m,k}$, where s_h and m_h are given by (1.3) and (1.4), respectively. We note that Takemura [7] studied some asymptotic properties of sample paths near the boundaries of I for ODGDP with generator $\mathcal{G}_{s_h, m_h, 0}$, and obtained a precise classification of the states of the boundaries by means of s , m , k and h . We call $\mathcal{G}_{s_h, m_h, 0}$ the h transform of $\mathcal{G}_{s,m,k}$ with $h \in \mathcal{H}_{s,m,k,\beta}$.

For $h \in \mathcal{M}_{s,m}$ (respectively $h \in \mathcal{H}_{s,m,k,\beta}$), we set $H_h^o \mathcal{G}_{s,m,0} u = h^{-1} \mathcal{G}_{s,m,0}(hu)$ (respectively $H_h^* \mathcal{G}_{s,m,k+\beta m} u = h^{-1} \mathcal{G}_{s,m,k+\beta m}(hu)$) for u satisfying $hu \in D(\mathcal{G}_{s,m,0})$ (respectively $hu \in D(\mathcal{G}_{s,m,k+\beta m})$). In the case that $\mathcal{G}_{s,m,0}$ and $\mathcal{G}_{s,m,k}$ are differential operators of second order, it is easy to see that $H_h^o \mathcal{G}_{s,m,0}$ and $H_h^* \mathcal{G}_{s,m,k+\beta m}$ coincide with $\mathcal{G}_{s_h, m_h, k_h}$ and $\mathcal{G}_{s_h, m_h, 0}$, respectively. We show that this is also true for ODGDOs.

THEOREM 1.2. *Let u be a measurable function on I .*

- (i) *Let $h \in \mathcal{M}_{s,m}$. Then u belongs to $D(\mathcal{G}_{s_h, m_h, k_h})$ if and only if hu belongs to $D(\mathcal{G}_{s,m,0})$. Further, $\mathcal{G}_{s_h, m_h, k_h} u = H_h^o \mathcal{G}_{s,m,0} u$ holds true for $u \in D(\mathcal{G}_{s_h, m_h, k_h})$.*
- (ii) *Let $h \in \mathcal{H}_{s,m,k,\beta}$. Then u belongs to $D(\mathcal{G}_{s_h, m_h, 0})$ if and only if hu belongs to $D(\mathcal{G}_{s,m,k})$. Further, $\mathcal{G}_{s_h, m_h, 0} u = H_h^* \mathcal{G}_{s,m,k+\beta m} u$ holds true for $u \in D(\mathcal{G}_{s_h, m_h, 0})$.*

COROLLARY 1.3.

- (i) *Let $h \in \mathcal{H}_{s,m,0,0}$. Then $H_h^o \mathcal{G}_{s,m,0} = H_h^* \mathcal{G}_{s,m,0} = \mathcal{G}_{s_h, m_h, 0}$.*

- (ii) Let $h \in \mathcal{M}_{s,m}$. Then $H_h^o \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0}$ if and only if h is a positive constant function.
- (iii) Let $h \in \mathcal{H}_{s,m,k,\beta}$. Then $H_h^* \mathcal{G}_{s,m,k+\beta m} = \mathcal{G}_{s,m,k+\beta m}$ if and only if k is the null measure, $\beta = 0$, and h is a positive constant function.

We next show that H_h^o and H_h^* are in inverse relation to each other in some sense.

THEOREM 1.4.

- (i) Let $h \in \mathcal{M}_{s,m}$. Then h^{-1} belongs to $\mathcal{H}_{s_h, m_h, k_h, 0}$ and $H_{h^{-1}}^* H_h^o \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0}$.
- (ii) Let $h \in \mathcal{H}_{s,m,k,\beta}$. Then h^{-1} belongs to $\mathcal{M}_{s_h, m_h}$ and $H_{h^{-1}}^o H_h^* \mathcal{G}_{s,m,k+\beta m} = \mathcal{G}_{s,m,k+\beta m}$.

We should note that Theorem 1.4 does not necessarily ensure one-to-one correspondence of an h transform. Indeed, some examples are given in Section 6, which show that an h transform of an ODGDO is not a one-to-one correspondence.

Now Theorem 1.2 implies that an ODGDO is represented as an h transform of another ODGDO, but it does not always imply that an ODGDO is represented as an h transform of another ODGDO different from the original one. As we will see in the following theorem, this problem is related to a global property of ODGDOs. We denote by Φ the set of all ODGDOs on I . Let Φ^R be the set of all recurrent ODGDOs on I , that is,

$$\Phi^R = \{\mathcal{G}_{s,m,k} \in \Phi : s(l_1) = -\infty, s(l_2) = \infty, \text{ and } k \text{ is the null measure}\}.$$

We set $\Phi^T = \Phi \setminus \Phi^R$, which is the set of all transient ODGDOs on I .

THEOREM 1.5.

- (i) Let $\mathcal{G}_{s,m,0} \in \Phi^R$. If $H_h^o \mathcal{G}_{s,m,0} \in \Phi^R$ for some $h \in \mathcal{M}_{s,m}$, then h is a positive constant function. If $H_h^* \mathcal{G}_{s,m,\beta m} \in \Phi^R$ for some $h \in \mathcal{H}_{s,m,0,\beta}$, then h is a positive constant function and $\beta = 0$.
- (ii) Let $\mathcal{G}_{s,m,0} \in \Phi^R$. If $H_h^o \mathcal{G}_{s,m,0} \in \Phi^T$ for some $h \in \mathcal{M}_{s,m}$, then h is not a positive constant function. If $H_h^* \mathcal{G}_{s,m,\beta m} \in \Phi^T$ for some $h \in \mathcal{H}_{s,m,0,\beta}$, then h is not a positive constant function or $\beta > 0$. Conversely, if $h \in \mathcal{M}_{s,m}$ and h is not a positive constant function, then $H_h^o \mathcal{G}_{s,m,0} \in \Phi^T$. If $h \in \mathcal{H}_{s,m,0,\beta}$ and either h is not a positive constant function or $\beta > 0$, then $H_h^* \mathcal{G}_{s,m,\beta m} \in \Phi^T$.
- (iii) If $\mathcal{G}_{s,m,0} \in \Phi^T$, then $H_h^o \mathcal{G}_{s,m,0} \in \Phi^T$ for any $h \in \mathcal{M}_{s,m}$. If $\mathcal{G}_{s,m,k} \in \Phi^T$, then $H_h^* \mathcal{G}_{s,m,k+\beta m} \in \Phi^T$ for any $h \in \mathcal{H}_{s,m,k,\beta}$.

We will show this theorem in Section 4.

We finally consider spectral representations of elementary solutions. We assume that $\text{supp}[m] = I$ and l_1 is not (s, m, k) -natural. Further, we assume that $p(t, x, y)$ is represented as

$$p(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \psi(x, \lambda) \psi(y, \lambda) d\sigma(\lambda), \quad t > 0, x, y \in I, \quad (1.7)$$

where $d\sigma(\lambda)$ is a Borel measure on $[0, \infty)$ such that $\sigma(\{0\}) = 0$ if l_1 is (s, m, k) -regular or -exit. Further, $\psi(x, \lambda)$, $x \in I$, $\lambda \geq 0$, is the solution of the integral equation (5.1) or (5.2) below. Let $h \in \mathcal{H}_{s,m,k,\beta}$ and $p_h^*(t, x, y)$ be the elementary solution of equation (1.1) with $\mathcal{G}_{s_h, m_h, 0}$ in place of $\mathcal{G}_{s,m,k}$, which is given by (1.6). By means of (1.6) and (1.7), we suppose $p_h^*(t, x, y)$ is represented as

$$p_h^*(t, x, y) = \int_{[\beta, \infty)} e^{-\lambda t} \psi_h^*(x, \lambda) \psi_h^*(y, \lambda) d\sigma_h^*(\lambda), \quad t > 0, x, y \in I, \quad (1.8)$$

and $\psi_h^*(x, \lambda)$ and $d\sigma_h^*(\lambda)$ are represented as

$$\psi_h^*(x, \lambda) = C_o \psi(x, \lambda - \beta) / h(x), \quad d\sigma_h^*(\lambda) = C_o^{-2} d_\lambda \sigma(\lambda - \beta), \quad (1.9)$$

for $\lambda \geq \beta$, where C_o is a positive constant function. In Section 5 we show that $\psi_h^*(x, \lambda)$ satisfies (5.1) or (5.2) with (s, m, k) replaced by $(s_h, m_h, 0)$. Further, we will find there that C_o only depends on the behavior of s, m, k and h near the boundary l_1 , which leads us to an interesting behavior of Lévy measure density of inverse local time corresponding to a diffusion process on I with the scale function s_h and the speed measure m_h . We will discuss this in another paper.

The organization of this paper is as follows. In Section 2 we give the precise definitions of ODGDO $\mathcal{G}_{s,m,k}$, the domain $D(\mathcal{G}_{s,m,k})$ and the corresponding items. In Section 3 we give the precise definitions of $\mathcal{M}_{s,m}$ and $\mathcal{H}_{s,m,k,\beta}$, and prove Proposition 1.1 and Theorems 1.2 and 1.4. In Section 4 we prove Theorem 1.5. In Section 5 we show that (1.8) holds true for $\psi_h^*(x, \lambda)$ and $d\sigma_h^*(\lambda)$ given by (1.9) with a suitable constant C_o . It is easy to see that if $H_{h_1}^o \mathcal{G}_{s,m,0} = H_{h_2}^o \mathcal{G}_{s,m,0}$ (respectively $H_{h_1}^* \mathcal{G}_{s,m,k+\beta m} = H_{h_2}^* \mathcal{G}_{s,m,k+\beta m}$) for some $h_1, h_2 \in \mathcal{M}_{s,m}$ (respectively $h_1, h_2 \in \mathcal{H}_{s,m,k,\beta}$), then there is a positive constant K such that $h_1 = Kh_2$. However, it is not necessarily true that if $H_{h_1}^o \mathcal{G}_{s_1,m_1,0} = H_{h_2}^o \mathcal{G}_{s_2,m_2,0}$ (respectively $H_{h_1}^* \mathcal{G}_{s_1,m_1,k_1+\beta_1 m_1} = H_{h_2}^* \mathcal{G}_{s_2,m_2,k_2+\beta_2 m_2}$) for some $h_i \in \mathcal{M}_{s_i,m_i}$ (respectively $h_i \in \mathcal{H}_{s_i,m_i,k_i,\beta_i}$) ($i = 1, 2$), then $h_1 = Kh_2$ for some positive constant K and $\mathcal{G}_{s_1,m_1,0} = \mathcal{G}_{s_2,m_2,0}$ (respectively $\mathcal{G}_{s_1,m_1,k_1+\beta_1 m_1} = \mathcal{G}_{s_2,m_2,k_2+\beta_2 m_2}$). We give such typical examples in Section 6.

2. Preliminaries

In this section we give the precise definitions of ODGDO $\mathcal{G}_{s,m,k}$, the domain $D(\mathcal{G}_{s,m,k})$ and the corresponding items.

Let s be a continuous increasing function on an open interval $I = (l_1, l_2)$, where $-\infty \leq l_1 < l_2 \leq \infty$, let m be a right-continuous nondecreasing function on I and let k be a right-continuous nondecreasing function on I . We sometimes use the same symbols s, m and k for the induced measures $ds(x)$, $dm(x)$ and $dk(x)$, respectively. For a function u on I , we set $u(l_i) = \lim_{x \rightarrow l_i, x \in I} u(x)$ if the limit exists, for $i = 1, 2$. We set

$$I_*(\mu) = \{x \in I; \mu(x_1) < \mu(x_2) \text{ for } \ell_1 < \forall x_1 < x < \forall x_2 < \ell_2\}, \quad (2.1)$$

for a nondecreasing right-continuous function μ on I . $I_*(\mu)$ is the same as the support of the measure induced by μ . We assume $I_*(m) \neq \emptyset$ and $I_*(k) \subset I_*(m)$ throughout this paper. Further, we set

$$I_{**}(m) = I_*(m) \cup \{x; x = \ell_i \text{ with } |m(\ell_i)| + |s(\ell_i)| + |k(\ell_i)| < \infty, i = 1, 2\}.$$

Let us fix a point $c_o \in I_*(m)$ arbitrarily and set

$$J_{\mu,v}(x) = \int_{(c_o,x]} d\mu(y) \int_{(c_o,y]} dv(z),$$

for $x \in I$, where μ and ν are Borel measures on I , and the integral $\int_{(a,b]}$ is read as $-\int_{(b,a]}$ if $a > b$. Following [1], we say the boundary l_i is

$$\begin{array}{ll} (s, m, k)\text{-regular} & \text{if } J_{s,m+k}(l_i) < \infty \text{ and } J_{m+k,s}(l_i) < \infty, \\ (s, m, k)\text{-exit} & \text{if } J_{s,m+k}(l_i) < \infty \text{ and } J_{m+k,s}(l_i) = \infty, \\ (s, m, k)\text{-entrance} & \text{if } J_{s,m+k}(l_i) = \infty \text{ and } J_{m+k,s}(l_i) < \infty, \\ (s, m, k)\text{-natural} & \text{if } J_{s,m+k}(l_i) = \infty \text{ and } J_{m+k,s}(l_i) = \infty. \end{array}$$

Let $D(\mathcal{G}_{s,m,k})$ be the space of all functions $u \in L^2(I, m)$ which have a continuous version u (we use the same symbol) satisfying the following conditions:

- (i) there exist two constants A, B and a function $f_u \in L^2(I, m)$ such that

$$\begin{aligned} u(x) = A + Bs(x) + \int_{(c_o,x]} \{s(x) - s(y)\} f_u(y) dm(y) \\ + \int_{(c_o,x]} \{s(x) - s(y)\} u(y) dk(y), \quad x \in I; \end{aligned} \quad (2.2)$$

- (ii) if l_i is (s, m, k) -regular, then $u(l_i) = 0$ for each $i = 1, 2$.

By virtue of (2.2), f_u is uniquely determined as a function of $L^2(I, m)$ if it exists. The operator $\mathcal{G}_{s,m,k}$ from $D(\mathcal{G}_{s,m,k})$ into $L^2(I, m)$ is defined by $\mathcal{G}_{s,m,k}u = f_u$, and it is called the one-dimensional generalized diffusion operator with the scale function s , the speed measure m , and the killing measure k (ODGDO with (s, m, k)). The above condition (ii) implies that the absorbing boundary condition is posed at the regular boundary.

It is easy to see that $\mathcal{G}_{s_1,m_1,k_1}$ coincides with $\mathcal{G}_{s_2,m_2,k_2}$ if and only if there are a positive constant K and constants $K_i, i = 1, 2, 3$, such that $s_1 = Ks_2 + K_1, m_1 = K^{-1}m_2 + K_2$, and $k_1 = K^{-1}k_2 + K_3$ (see [3]).

We note that l_i is $(s, m, k + \beta m)$ -regular, -exit, -entrance, or -natural according to if l_i is (s, m, k) -regular, exit, entrance, or natural, for $i = 1, 2$ and $\beta \geq 0$. Combining this fact with (2.2), we immediately obtain the following:

$$D(\mathcal{G}_{s,m,k}) = D(\mathcal{G}_{s,m,k+\beta m}), \quad (2.3)$$

$$\mathcal{G}_{s,m,k}u - \beta u = \mathcal{G}_{s,m,k+\beta m}u \quad \text{for } u \in D(\mathcal{G}_{s,m,k}), \quad (2.4)$$

where $\beta \geq 0$.

In the following, for a measurable function u on I , $D_s u(x)$ stands for the right derivative with respect to $s(x)$, that is, $D_s u(x) = \lim_{\varepsilon \downarrow 0} \{u(x + \varepsilon) - u(x)\} / \{s(x + \varepsilon) - s(x)\}$, provided it exists. It is obvious that $u \in D(\mathcal{G}_{s,m,k})$ has the right derivative $D_s u$ and it satisfies

$$D_s u(y) - D_s u(x) = \int_{(x,y]} \mathcal{G}_{s,m,k}u(z) dm(z) + \int_{(x,y]} u(z) dk(z), \quad x, y \in I.$$

So we sometimes use the symbol $\mathcal{G}_{s,m,k}u = (dD_s u - u dk)/dm$.

Following McKean [5] (see also Section 4.11 of [3]), we can define the elementary solution $p(t, x, y)$ of equation (1.1). It is known that $p(t, x, y)$ satisfies the following

properties:

$$p(t, x, y) = p(t, y, x) > 0, \quad t > 0, x, y \in I;$$

$$p(t, x, y) \text{ is continuous on } (0, \infty) \times I \times I;$$

$$p(s + t, x, y) = \int_I p(s, x, z) p(t, z, y) dm(z), \quad s, t > 0, x, y \in I;$$

$$p(t, l_i, y) = 0, \quad t > 0, y \in I, \quad \text{if } l_i \text{ is not entrance};$$

$$D_s p(t, l_i, y) = 0, \quad t > 0, y \in I, \quad \text{if } l_i \text{ is entrance},$$

where $D_s p(t, x, y) = \lim_{\varepsilon \downarrow 0} \{p(t, x + \varepsilon, y) - p(t, x, y)\} / \{s(x + \varepsilon) - s(x)\}$. It is also known that there exists a ODGDP $\mathbb{D}_{s,m,k} = [X(t) : t \geq 0, P_x : x \in I_{**}(m)]$ such that

$$P_x(X(t) \in E) = \int_E p(t, x, y) dm(y), \quad t > 0, x \in I_{**}(m), E \in \mathcal{B}(I),$$

where $\mathcal{B}(I)$ stands for the set of all Borel sets of I . For this reason, $p(t, x, y)$ is sometimes called the transition probability density with respect to m .

For $\alpha \geq 0$ and $i = 1, 2$, let $g_i(\cdot, \alpha)$ be a function on I satisfying the following properties (2.5)–(2.9):

$$g_i(x, \alpha) \text{ is positive and continuous in } x; \quad (2.5)$$

$$g_1(x, \alpha) \text{ is nondecreasing in } x; \quad (2.6)$$

$$g_2(x, \alpha) \text{ is nonincreasing in } x; \quad (2.7)$$

$$g_i(l_i, \alpha) = 0 \text{ if } |s(l_i)| < \infty; \quad (2.8)$$

$$\begin{aligned} g_i(x, \alpha) &= g_i(c_o, \alpha) + D_s g_i(c_o, \alpha) \{s(x) - s(c_o)\} \\ &\quad + \int_{(c_o, x]} \{s(x) - s(y)\} g_i(y, \alpha) \{\alpha dm(y) + dk(y)\}, \quad x \in I. \end{aligned} \quad (2.9)$$

Here, $D_s g_i(x, \alpha) = \lim_{\varepsilon \downarrow 0} \{g_i(x + \varepsilon, \alpha) - g_i(x, \alpha)\} / \{s(x + \varepsilon) - s(x)\}$, $i = 1, 2$. It is known that there exist functions $g_i(\cdot, \alpha)$, $i = 1, 2$, satisfying the properties (2.5)–(2.9) (see Section 4.6 of [3]). We summarize some properties of $g_i(\cdot, \alpha)$, $i = 1, 2$, which we need later.

PROPOSITION 2.1.

(i) [3] Assume that k is not a null measure or $\alpha > 0$. Then it holds true that:

$$\begin{aligned} g_i(l_i, \alpha) &\begin{cases} \in (0, \infty) & \text{if } l_i \text{ is } (s, m, k)\text{-entrance,} \\ = 0 & \text{if } l_i \text{ is not } (s, m, k)\text{-entrance;} \end{cases} \\ g_j(l_i, \alpha) &\begin{cases} \in (0, \infty) & \text{if } l_i \text{ is } (s, m, k)\text{-regular or -exit,} \\ = \infty & \text{if } l_i \text{ is } (s, m, k)\text{-entrance or -natural;} \end{cases} \end{aligned}$$

$$|D_s g_i(l_i, \alpha)| \begin{cases} \in (0, \infty) & \text{if } l_i \text{ is } (s, m, k)\text{-regular or -exit,} \\ = 0 & \text{if } l_i \text{ is } (s, m, k)\text{-entrance or -natural;} \end{cases}$$

$$|D_s g_j(l_i, \alpha)| \begin{cases} \in (0, \infty) & \text{if } l_i \text{ is } (s, m, k)\text{-regular or -entrance,} \\ = \infty & \text{if } l_i \text{ is } (s, m, k)\text{-exit or -natural;} \end{cases}$$

$$\lim_{x \rightarrow l_i} g_i(x, \alpha) D_s g_j(x, \alpha) = 0 \quad \text{if } l_i \text{ is } (s, m, k)\text{-exit,}$$

$$\lim_{x \rightarrow l_i} g_j(x, \alpha) D_s g_i(x, \alpha) = 0 \quad \text{if } l_i \text{ is } (s, m, k)\text{-entrance;}$$

where $i, j = 1, 2$ and $i \neq j$.

- (ii) Assume that k is the null measure and $\alpha = 0$. Then $g_i(x, 0)$, $i = 1, 2$ are represented as follows:

$$g_1(x, 0) = \begin{cases} C_1 & \text{if } s(l_1) = -\infty, \\ C_1\{s(x) - s(l_1)\} & \text{if } s(l_1) > -\infty; \end{cases}$$

$$g_2(x, 0) = \begin{cases} C_2 & \text{if } s(l_2) = \infty, \\ C_2\{s(l_2) - s(x)\} & \text{if } s(l_2) < \infty; \end{cases}$$

where C_1 and C_2 are positive constants.

Statement (i) is shown in Section 4.6 of [3] and statement (ii) follows from (2.5)–(2.8) hence we omit the proof.

We set $W(\alpha) = D_s g_1(x, \alpha) g_2(x, \alpha) - g_1(x, \alpha) D_s g_2(x, \alpha)$. Note that $W(\alpha)$ is a positive number independent of $x \in I$. We put

$$G(\alpha, x, y) = G(\alpha, y, x) = W(\alpha)^{-1} g_1(x, \alpha) g_2(y, \alpha), \quad (2.10)$$

for $\alpha > 0$, $x, y \in I$, $x \leq y$. We call $G(\alpha, x, y)$ the α -Green function corresponding to the ODGDO $\mathcal{G}_{s,m,k}$. It is also known that

$$G(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y) dt, \quad \alpha > 0, x, y \in I_{**}(m). \quad (2.11)$$

It is easy to see that, if $k \neq 0$, then there exists $G(0, x, y)$ which is given by

$$G(0, x, y) = G(0, y, x) = W^{-1} g_1(x) g_2(y), \quad x, y \in I, x \leq y, \quad (2.12)$$

where $g_i(x) = g_i(x, 0)$, $i = 1, 2$, and $W = D_s g_1(x) g_2(x) - g_1(x) D_s g_2(x)$, which is a positive constant independent of $x \in I$. It follows from Proposition 2.1 that, in the case $k = 0$, there exists $G(0, x, y) \in (0, \infty)$ if and only if $|s(l_i)| < \infty$ for $i = 1$ or 2 .

We denote by G_α ($\alpha > 0$) the Green operator corresponding to $\mathcal{G}_{s,m,k}$:

$$G_\alpha f(x) = \int_I G(\alpha, x, y) f(y) dm(y), \quad f \in L^2(I, m). \quad (2.13)$$

It is well known that

$$G_\alpha(L^2(I, m)) = D(\mathcal{G}_{s,m,k}), \quad (2.14)$$

$$G_\alpha(\alpha - \mathcal{G}_{s,m,k})u = u, \quad u \in D(\mathcal{G}_{s,m,k}), \quad (2.15)$$

$$(\alpha - \mathcal{G}_{s,m,k})G_\alpha f = f, \quad f \in L^2(I, m) \quad (2.16)$$

(see [2] and [3]).

3. h transform of ODGDOs

In this section, we give the precise definitions of $\mathcal{M}_{s,m}$ and $\mathcal{H}_{s,m,k,\beta}$, and prove Proposition 1.1 and Theorems 1.2 and 1.4. We use the same notations as in the preceding sections.

3.1. h transform of $\mathcal{G}_{s,m,0}$ with $h \in \mathcal{M}_{s,m}$

Let $\mathcal{M}_{s,m}$ be the set of all positive continuous functions h on I such that h has the right derivative $D_s h$, $D_s h$ is right continuous and nonincreasing, and the set $\{x \in I; D_s h(x_1) > D_s h(x_2) \text{ for } l_1 < \forall x_1 < x < \forall x_2 < l_2\}$ is included in $I_*(m)$. For $h \in \mathcal{M}_{s,m}$, we consider s_h , m_h and k_h given by (1.3), (1.4) and (1.5), respectively. Note that $I_*(m) = I_*(m_h)$ and $I_*(k_h) \subset I_*(m_h)$ for $h \in \mathcal{M}_{s,m}$ (see [4]). Let $p^o(t, x, y)$ be the elementary solution of (1.1) with $\mathcal{G}_{s,m,k}$ replaced by $\mathcal{G}_{s,m,0}$. Let $G^o(\alpha, x, y)$ be the α -Green function corresponding to the ODGDO $\mathcal{G}_{s,m,0}$. For $h \in \mathcal{M}_{s,m}$, we consider $p_h^o(t, x, y)$ given by (1.2) and set

$$G_h^o(\alpha, x, y) = G^o(\alpha, x, y) / h(x)h(y), \quad x, y \in I. \quad (3.1)$$

The following result is obtained by Maeno (see Theorem 2.2 of [4]).

PROPOSITION 3.1. [4] $p_h^o(t, x, y)$ is the elementary solution of (1.1) with $\mathcal{G}_{s,m,k}$ replaced by $\mathcal{G}_{s_h,m_h,k_h}$, and $G_h^o(\alpha, x, y)$ is the α -Green function corresponding to the ODGDO $\mathcal{G}_{s_h,m_h,k_h}$.

3.2. h transform of $\mathcal{G}_{s,m,k}$ with $h \in \mathcal{H}_{s,m,k,\beta}$

For $\beta \geq 0$, let $h_\beta(\cdot)$ be a positive continuous function on I satisfying

$$\begin{aligned} h_\beta(x) &= h_\beta(c_o) + D_s h_\beta(c_o)\{s(x) - s(c_o)\} \\ &\quad + \int_{(c_o,x]} \{s(x) - s(y)\} h_\beta(y) \{\beta dm(y) + dk(y)\}, \quad x \in I. \end{aligned} \quad (3.2)$$

There exists such a function $h_\beta(\cdot)$. Indeed, it is represented as a linear combination of $g_i(\cdot, \beta)$, $i = 1, 2$, given in the preceding section. Let $\mathcal{H}_{s,m,k,\beta}$ be the set of all positive functions h_β satisfying (3.2). It immediately follows from (3.2) that

$$\mathcal{H}_{s,m,k,\beta} = \mathcal{H}_{s,m,k+\beta m,0}, \quad \beta \geq 0.$$

For $h \in \mathcal{H}_{s,m,k,\beta}$ and $g_i(x, \alpha)$, $i = 1, 2$, satisfying (2.5)–(2.9), we set

$$g_{h,i}(x, \alpha) = g_i(x, \alpha) / h(x), \quad i = 1, 2, \alpha \geq 0. \quad (3.3)$$

Let $G_h^*(\gamma, x, y)$ be the γ -Green function corresponding to $\mathcal{G}_{s_h,m_h,0}$, where s_h and m_h are given by (1.3) and (1.4), respectively. Now we obtain the following proposition. Under the assumption $I_*(m) = I$, the following result is obtained as Proposition 2.2 and Lemma 3.3 of [8]. It is not difficult to see that their proofs are available without the assumption $I_*(m) = I$. Therefore we omit the proof of the following proposition.

PROPOSITION 3.2. Let $\alpha \geq \beta \geq 0$ and $h \in \mathcal{H}_{s,m,k,\beta}$.

(i) Let $i = 1, 2$. If $\alpha = \beta$ and $|s_h(l_i)| = \infty$, then $g_{h,i}(x, \alpha)$ is a positive constant function on I . If $\alpha > \beta$ or $|s_h(l_i)| < \infty$, then $g_{h,i}(x, \alpha)$ satisfies the following properties:

- (i-1) $g_{h,i}(x, \alpha)$ is positive and continuous on I ;
- (i-2) $g_{h,1}(x, \alpha)$ is nondecreasing on I and $g_{h,2}(x, \alpha)$ is nonincreasing on I ;
- (i-3) if $|s_h(l_i)| < \infty$ or $|m_h(l_i)| = \infty$, then $g_{h,i}(l_i, \alpha) = 0$;
- (i-4) if $|s_h(l_i)| = \infty$, then $D_{s_h} g_{h,i}(l_i, \alpha) = 0$;
- (i-5) $g_{h,i}(x, \alpha)$ satisfies

$$\begin{aligned} g_{h,i}(x, \alpha) &= g_{h,i}(c_o, \alpha) + D_{s_h} g_{h,i}(c_o, \alpha) \{s_h(x) - s_h(c_o)\} \\ &\quad + (\alpha - \beta) \int_{(c_o, x]} \{s_h(x) - s_h(y)\} g_{h,i}(y, \alpha) dm_h(y), \end{aligned} \quad (3.4)$$

for $x \in I$.

(ii) The following equation (3.5) holds true:

$$\begin{aligned} G_h^*(\alpha - \beta, x, y) &= G_h^*(\alpha - \beta, y, x) \\ &= W(\alpha)^{-1} g_{h,1}(x, \alpha) g_{h,2}(y, \alpha) \\ &= G(\alpha, x, y) / h(x)h(y), \end{aligned} \quad (3.5)$$

for $l_1 < x \leq y < l_2$.

Let $p(t, x, y)$ be the elementary solution of equation (1.1) and consider $p_h^*(t, x, y)$ defined by (1.6). By virtue of (2.11) and (3.5), we get the following.

PROPOSITION 3.3. $p_h^*(t, x, y)$ is the elementary solution of equation (1.1) with $\mathcal{G}_{s,m,k}$ replaced by $\mathcal{G}_{s_h, m_h, 0}$, and $G_h^*(\alpha, x, y)$ is the α -Green function corresponding to $\mathcal{G}_{s_h, m_h, 0}$.

3.3. Proof of Proposition 1.1

(i) We set $\Lambda = \{h \in \mathcal{M}_{s,m}; D_s h(x) \text{ is a constant function on } I\}$. It follows from (3.2) that

$$D_s h(y) - D_s h(x) = \int_{(x, y]} h(z) \{\beta dm(z) + dk(z)\}, \quad (3.6)$$

for $h \in \mathcal{H}_{s,m,k,\beta}$. Therefore, h belongs to $\mathcal{H}_{s,m,0,0}$ if and only if $D_s h$ is a constant function, from which the set $\{x \in I; D_s h(x_1) > D_s h(x_2) \text{ for } l_1 < \forall x_1 < x < \forall x_2 < l_2\}$ is empty and hence it is included in $I_*(m)$. Thus $\mathcal{H}_{s,m,0,0}$ is included in the set Λ . Conversely, $h \in \Lambda$ if and only if (3.6) holds true for the null killing measure and $\beta = 0$, which implies (3.2) with $h = h_\beta$, $k = 0$ and $\beta = 0$. Therefore the set Λ is included in $\mathcal{H}_{s,m,0,0}$.

(ii) Assume that k is not a null measure or $\beta > 0$. Let $h \in \mathcal{H}_{s,m,k,\beta}$. By means of (3.6) we see that $D_s h(x) < D_s h(y)$ whenever $x < y$ and $(x, y] \cap I_*(m) \neq \emptyset$. Therefore $D_s h$ is increasing on $I_*(m)$ and it is nondecreasing on I . Thus h does not belong to $\mathcal{M}_{s,m}$, which implies the second statement. $\square$

3.4. Proofs of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. Let u be a measurable function on I . We show the first statement (i). Let $G^o(\alpha, x, y)$ be the α -Green function corresponding to $\mathcal{G}_{s,m,0}$. Let $h \in \mathcal{M}_{s,m}$ and consider $G_h^o(\alpha, x, y)$ given by (3.1). By means of (2.14) and Proposition 3.1, u belongs to $D(\mathcal{G}_{s_h, m_h, k_h})$

if and only if there is a function $f \in L^2(I, m_h)$ such that

$$u(x) = \int_I G_h^o(\alpha, x, y) f(y) dm_h(y), \quad (3.7)$$

or equivalently,

$$u(x) = \frac{1}{h(x)} \int_I G^o(\alpha, x, y) f(y) h(y) dm(y). \quad (3.8)$$

We note that fh belongs to $L^2(I, m)$. Therefore (3.8) holds true if and only if hu belongs to $D(\mathcal{G}_{s,m,0})$. Proposition 3.1 combined with (3.7) and (3.8) implies

$$u(x) = G_{h,\alpha}^o f(x) = \frac{1}{h(x)} G_\alpha^o(fh)(x),$$

where $G_{h,\alpha}^o$ and G_α^o are the α -Green operators corresponding to $\mathcal{G}_{s_h, m_h, k_h}$ and $\mathcal{G}_{s,m,0}$, respectively. By means of (2.16),

$$(\alpha - \mathcal{G}_{s_h, m_h, k_h}) G_{h,\alpha}^o f = f, \quad (\alpha - \mathcal{G}_{s,m,0}) G_\alpha^o(fh) = fh,$$

and hence we obtain

$$(\alpha - \mathcal{G}_{s_h, m_h, k_h}) u = \frac{1}{h} (\alpha - \mathcal{G}_{s,m,0})(hu);$$

that is,

$$\mathcal{G}_{s_h, m_h, k_h} u = \frac{1}{h} \mathcal{G}_{s,m,0}(hu).$$

Thus we get statement (i).

Statement (ii) is obtained in the same way as above and so we omit the proof. $\square$

Proof of Corollary 1.3. Statement (i) immediately follows from Proposition 1.1 and Theorem 1.2.

(ii) Let $h \in \mathcal{M}_{s,m}$. By means of Theorem 1.2, $H_h^o \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0}$ if and only if k_h is the null measure, and $s_h(x) = As(x) + A_1$, $m_h(x) = A^{-1}m(x) + A_2$ for some positive constant A and $A_i \in \mathbb{R}$, $i = 1, 2$, or equivalently h is a positive constant function.

(iii) Let $h \in \mathcal{H}_{s,m,k,\beta}$. By means of Theorem 1.2, $H_h^* \mathcal{G}_{s,m,k+\beta m} = \mathcal{G}_{s,m,k+\beta m}$ if and only if $k + \beta m$ is the null measure, and $s_h(x) = Bs(x) + B_1$, $m_h(x) = B^{-1}m(x) + B_2$ for some positive constant B and $B_i \in \mathbb{R}$, $i = 1, 2$. This is equivalent to k being the null measure, $\beta = 0$, and h being a positive constant function. $\square$

3.5. Proof of Theorem 1.4

(i) Let $h \in \mathcal{M}_{s,m}$. Then $D_{s_h} h^{-1} = -D_s h$ and hence

$$\begin{aligned} D_{s_h} h^{-1}(y) - D_{s_h} h^{-1}(x) &= -D_s h(y) + D_s h(x) \\ &= - \int_{(x,y]} dD_s h(z) = \int_{(x,y]} h^{-1}(z) dk_h(z). \end{aligned}$$

This shows that h^{-1} belongs to $\mathcal{H}_{s_h, m_h, k_h, 0}$. By means of Theorem 1.2(ii), $H_{h^{-1}}^* \mathcal{G}_{s_h, m_h, k_h} = \mathcal{G}_{s,m,0}$, from which $H_{h^{-1}}^* H_h^o \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0}$.

(ii) Let $h \in \mathcal{H}_{s,m,k,\beta}$. Then $D_{s_h} h^{-1} = -D_s h$, and by means of (3.6),

$$\begin{aligned} D_{s_h} h^{-1}(y) - D_{s_h} h^{-1}(x) &= -D_s h(y) + D_s h(x) \\ &= - \int_{(x,y]} h(z) \{ \beta \, dm(z) + dk(z) \}. \end{aligned} \quad (3.9)$$

Therefore $D_{s_h} h^{-1}$ is nonincreasing on I and the set $\Lambda = \{x \in I : D_{s_h} h^{-1}(x_1) > D_{s_h} h^{-1}(x_2) \text{ for } l_1 < \forall x_1 < x < \forall x_2 < l_2\}$ is included in $I_*(m) \cup I_*(k)$. Since $I_*(m) \cup I_*(k) = I_*(m) = I_*(m_h)$, we get $h^{-1} \in \mathcal{M}_{s_h, m_h}$. Calculating the right-hand side of (1.5) with s and h replaced by s_h and h^{-1} , respectively, we see by virtue of (3.9) that

$$- \int_{(c_o, x]} h^{-1}(y) \, dD_{s_h} h^{-1}(y) = \int_{(c_o, x]} \{ \beta \, dm(y) + dk(y) \}.$$

Combining this with Theorem 1.2(i), we get $H_{h^{-1}}^o \mathcal{G}_{s_h, m_h, 0} = \mathcal{G}_{s, m, k+\beta m}$, from which $H_{h^{-1}}^o H_h^* \mathcal{G}_{s, m, k+\beta m} = \mathcal{G}_{s, m, k+\beta m}$. $\square$

4. Global properties of ODGDOs and their h transform

In this section we show Theorem 1.5. First we prepare two lemmas on properties of $h \in \mathcal{M}_{s,m}$ and $h \in \mathcal{H}_{s,m,k,\beta}$.

LEMMA 4.1. *Assume that $s(l_1) = -\infty$ and $s(l_2) = \infty$. If $h \in \mathcal{M}_{s,m}$ and k_h is the null measure, then h is a positive constant function. If $h \in \mathcal{H}_{s,m,0,\beta}$, $s_h(l_1) = -\infty$ and $s_h(l_2) = \infty$, then h is a positive constant function and $\beta = 0$.*

Proof. Assume that $s(l_1) = -\infty$ and $s(l_2) = \infty$.

(i) Suppose that $h \in \mathcal{M}_{s,m}$ and k_h is the null measure. By means of (1.5), $D_s h$ is a constant function. Therefore $h(x)$ is represented as $h(x) = C_1 s(x) + C_2$ for some $C_i \in \mathbb{R}$, $i = 1, 2$. Since $s(l_1) = -\infty$, $s(l_2) = \infty$ and $h(x)$ is a positive function, we get $C_1 = 0$ and $C_2 > 0$.

(ii) Suppose that $h \in \mathcal{H}_{s,m,0,\beta}$, $s_h(l_1) = -\infty$ and $s_h(l_2) = \infty$. By using Lemma 3.2(i) of [8], we see that $h(l_i) < \infty$, $i = 1, 2$. Note that $h(x)$ is represented as $h(x) = C_3 g_1(x, \beta) + C_4 g_2(x, \beta)$, where $g_i(x, \beta)$, $i = 1, 2$, are functions satisfying (2.5)–(2.9) with $\alpha = \beta$ and $k = 0$. If $\beta > 0$, by means of Proposition 2.1(i), $g_i(l_i, \beta) < \infty$ for $i = 1, 2$, and $g_i(l_j, \beta) = \infty$ for $i \neq j$. Combining these with $h(l_i) < \infty$ for $i = 1, 2$, we see $C_3 = C_4 = 0$, and hence $h(x) = 0$. This contradicts $h(x) > 0$ on I . Thus $\beta = 0$. Noting $s(l_1) = -\infty$, $s(l_2) = \infty$ and Proposition 2.1(ii), we obtain that h is a positive constant function. $\square$

LEMMA 4.2.

- (i) *Let $h \in \mathcal{M}_{s,m}$. If $s(l_1) > -\infty$ or $s(l_2) < \infty$, then $s_h(l_1) > -\infty$, or $s_h(l_2) < \infty$, or k_h is not a null measure.*
- (ii) *Let $h \in \mathcal{H}_{s,m,k,\beta}$. If $s(l_1) > -\infty$, or $s(l_2) < \infty$, or k is not a null measure, then $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$.*

Proof. (i) Let $h \in \mathcal{M}_{s,m}$, and $s(l_1) > -\infty$ or $s(l_2) < \infty$. It is enough to show that $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$ in the case that k_h is the null measure. Since k_h is the null measure, $D_s h$ is a constant function by virtue of (1.5).

Suppose $s(l_1) > -\infty$. By means of Lemma 2.1(ii) of [4], we have $h(l_1) \in [0, \infty)$. If $h(l_1) \in (0, \infty)$, then $s_h(l_1) > -\infty$. Let $h(l_1) = 0$. Noting that $D_s h$ is a constant function, we see that $h(x) = C_1\{s(x) - s(l_1)\}$ for some $C_1 > 0$. Equation (1.3) coupled with this implies $s_h(l_2) < \infty$.

In the same way as above, we get $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$ when $s(l_2) < \infty$.

(ii) Let $h \in \mathcal{H}_{s,m,k,\beta}$ and assume that $s(l_1) > -\infty$, or $s(l_2) < \infty$, or k is not a null measure. Note $h(x)$ is represented as $h(x) = C_2 g_1(x, \beta) + C_3 g_2(x, \beta)$, where $g_i(x, \beta)$, $i = 1, 2$, are functions satisfying (2.5)–(2.9) with $\alpha = \beta$. We divide the proof into four cases.

Case 1: $h(l_1) = \infty$. Then $s_h(l_1) > -\infty$ by virtue of Lemma 3.2(i) of [8].

Case 2: $h(l_1) \in (0, \infty)$ and $s(l_1) > -\infty$. Then $s_h(l_1) > -\infty$ by means of (1.3).

Case 3: $h(l_1) \in (0, \infty)$ with $s(l_1) = -\infty$. Since $g_2(l_1, \beta) = \infty$ for $\beta \geq 0$ by virtue of Proposition 2.1, we get $C_3 = 0$ and $h(x) = C_2 g_1(x, \beta)$.

(3-1) Let $s(l_2) < \infty$. Then

$$s_h(l_2) = \int_{(c_o, l_2)} h(y)^{-2} ds(y) \leq C_2^{-2} g_1(c_o, \beta)^{-2} \{s(l_2) - s(c_o)\} < \infty.$$

(3-2) Let $s(l_2) = \infty$. Since k is not a null measure, we have, by virtue of Proposition 2.1(i), $h(l_2) = C_2 g_1(l_2, \beta) = \infty$ for $\beta \geq 0$. Therefore $s_h(l_2) < \infty$ by virtue of Lemma 3.2(i) of [8].

Case 4: $h(l_1) = 0$. Then $h(x)$ is represented as $h(x) = C_2 g_1(x, \beta)$, because of $g_2(l_1, \beta) > 0$. Thus we have $s_h(l_2) < \infty$ in the same way as in (3-1) and (3-2). $\square$

Proof of Theorem 1.5. (i) Let $\mathcal{G}_{s,m,0} \in \Phi^R$. Hence $s(l_1) = -\infty$ and $s(l_2) = \infty$.

Suppose that $H_h^o \mathcal{G}_{s,m,0} = \mathcal{G}_{s_h,m_h,k_h} \in \Phi^R$ for some $h \in \mathcal{M}_{s,m}$. Then k_h is the null measure. By virtue of Lemma 4.1, h is a positive constant function.

Suppose that $H_h^* \mathcal{G}_{s,m,\beta m} = \mathcal{G}_{s_h,m_h,0} \in \Phi^R$ for some $h \in \mathcal{H}_{s,m,0,\beta}$. Then $s_h(l_1) = -\infty$ and $s_h(l_2) = \infty$. By virtue of Lemma 4.1, h is a positive constant function and $\beta = 0$.

(ii) The former statement follows from Corollary 1.3. The later follows from Lemma 4.1.

(iii) (1) Let $\mathcal{G}_{s,m,0} \in \Phi^T$ and $h \in \mathcal{M}_{s,m}$. Since $s(l_1) > -\infty$ or $s(l_2) < \infty$, by means of Lemma 4.2(i), $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$ or k_h is not a null measure. Thus $H_h^o \mathcal{G}_{s,m,0} = \mathcal{G}_{s_h,m_h,k_h} \in \Phi^T$.

(2) Let $\mathcal{G}_{s,m,k} \in \Phi^T$ and $h \in \mathcal{H}_{s,m,k,\beta}$. Noting that $s(l_1) > -\infty$, or $s(l_2) < \infty$, or k is not a null measure, and using Lemma 4.2(ii), we see $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$. This implies $H_h^* \mathcal{G}_{s,m,k+\beta m} = \mathcal{G}_{s_h,m_h,0} \in \Phi^T$. $\square$

5. Spectral representation of the elementary solution

Let $\mathcal{G}_{s,m,k}$ be an ODGDO and let $p(t, x, y)$ be the elementary solution of equation (1.1). We assume that $I_*(m) = I$ and l_1 is not (s, m, k) -natural. Further, we assume that $p(t, x, y)$ is represented as (1.7). Namely,

$$p(t, x, y) = \int_{[0,\infty)} e^{-\lambda t} \psi(x, \lambda) \psi(y, \lambda) d\sigma(\lambda), \quad t > 0, \quad x, y \in I,$$

where $d\sigma(\lambda)$ is a Borel measure on $[0, \infty)$; $\sigma(\{0\}) = 0$ if l_1 is (s, m, k) -regular or -exit. Further, $\psi(x, \lambda)$, $x \in I$, $\lambda \geq 0$, is the solution of the integral equation

$$\psi(x, \lambda) = \begin{cases} s(x) - s(l_1) + \int_{(l_1, x]} \{s(x) - s(y)\} \psi(y, \lambda) \{-\lambda dm(y) + dk(y)\}, \\ \text{if } l_1 \text{ is } (s, m, k)\text{-regular or -exit,} \\ 1 + \int_{(l_1, x]} \{s(x) - s(y)\} \psi(y, \lambda) \{-\lambda dm(y) + dk(y)\}, \\ \text{if } l_1 \text{ is } (s, m, k)\text{-entrance.} \end{cases} \quad (5.1)$$

In the following we fix an $h \in \mathcal{H}_{s,m,k,\beta}$ arbitrarily. Then $\mathcal{G}_{s_h, m_h, 0}$ is an ODGDO and $p_h^*(t, x, y)$ given by (1.6) is the elementary solution of equation (1.1) with $\mathcal{G}_{s_h, m_h, 0}$ in place of $\mathcal{G}_{s,m,k}$. We set $\psi_h(x, \lambda) = \psi(x, \lambda)/h(x)$. By virtue of (1.6) and (1.7), we obtain

$$\begin{aligned} p_h^*(t, x, y) &= e^{-\beta t} \frac{p(t, x, y)}{h(x)h(y)} \\ &= \int_{[\beta, \infty)} e^{-\lambda t} \psi_h(x, \lambda - \beta) \psi_h(y, \lambda - \beta) d\sigma_\lambda(\lambda - \beta). \end{aligned} \quad (5.3)$$

In the same way as in the proof of Lemma 3.3 of [8], we see that $\psi_h(x, \lambda)$ satisfies

$$D_{s_h} \psi_h(y, \lambda) - D_{s_h} \psi_h(x, \lambda) = -(\lambda + \beta) \int_{(x, y]} \psi_h(z, \lambda) dm_h(z), \quad (5.4)$$

for $l_1 < x \leq y < l_2$. We note the following lemma.

LEMMA 5.1.

(i) Let l_1 be (s, m, k) -regular or -exit. Then $h(l_1) \in [0, \infty)$ and

$$\psi_h(l_1, \lambda) = D_s h(l_1)^{-1} \in (0, \infty), \quad D_{s_h} \psi_h(l_1, \lambda) = 0, \quad \text{if } h(l_1) = 0; \quad (5.5)$$

$$\psi_h(l_1, \lambda) = 0, \quad D_{s_h} \psi_h(l_1, \lambda) = h(l_1) \in (0, \infty), \quad \text{if } h(l_1) \in (0, \infty). \quad (5.6)$$

(ii) Let l_1 be (s, m, k) -entrance. Then $h(l_1) \in (0, \infty]$ and

$$\psi_h(l_1, \lambda) = h(l_1)^{-1} \in (0, \infty), \quad D_{s_h} \psi_h(l_1, \lambda) = 0, \quad \text{if } h(l_1) \in (0, \infty); \quad (5.7)$$

$$\psi_h(l_1, \lambda) = 0, \quad D_{s_h} \psi_h(l_1, \lambda) = -D_s h(l_1) \in (0, \infty), \quad \text{if } h(l_1) = \infty. \quad (5.8)$$

Proof. Note that h is represented as $h(x) = C_1 g_1(x, \beta) + C_2 g_2(x, \beta)$, where $g_i(x, \beta)$, $i = 1, 2$, are functions satisfying (2.5)–(2.9) with $\alpha = \beta$.

(i) Let l_1 be (s, m, k) -regular or -exit. By means of Theorem 1.1 of [7], $0 \leq h(l_1) < \infty$.

First we consider the case $h(l_1) = 0$, and hence $C_1 > 0$ and $C_2 = 0$, by virtue of Proposition 2.1. Since $D_s h(l_1) = C_1 D_s g_1(l_1, \beta) \in (0, \infty)$, we get

$$\psi_h(l_1, \lambda) = \lim_{x \rightarrow l_1} \frac{\psi(x, \lambda)}{h(x)} = \lim_{x \rightarrow l_1} \frac{D_s \psi(x, \lambda)}{D_s h(x)} = \frac{1}{D_s h(l_1)} \in (0, \infty),$$

$$D_{s_h} \psi_h(l_1, \lambda) = \lim_{x \rightarrow l_1} \{h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x)\} = 0.$$

Thus we obtain (5.5).

Next we consider the case $0 < h(l_1) < \infty$. Then $C_2 > 0$ and $\psi_h(l_1, \lambda) = \psi(l_1, \lambda)/h(l_1) = 0$. If l_1 is (s, m, k) -regular, then by means of Proposition 2.1, $D_s h(l_1) = C_1 D_s g_1(l_1, \beta) + C_2 D_s g_2(l_1, \beta) \in \mathbb{R}$ and hence

$$D_{s_h} \psi_h(l_1, \lambda) = \lim_{x \rightarrow l_1} \{h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x)\} = h(l_1) \in (0, \infty).$$

If l_1 is (s, m, k) -exit, then by means of Proposition 2.1

$$\lim_{x \rightarrow l_1} \frac{\psi(x, \lambda)}{g_1(x, \beta)} = \lim_{x \rightarrow l_1} \frac{D_s \psi(x, \lambda)}{D_s g_1(x, \beta)} = \frac{1}{D_s g_1(l_1, \beta)} \in (0, \infty),$$

$$\lim_{x \rightarrow l_1} \psi(x, \lambda) D_s h(x) = \frac{1}{D_s g_1(l_1, \beta)} \lim_{x \rightarrow l_1} g_1(x, \beta) \{C_1 D_s g_1(x, \beta) + C_2 D_s g_2(x, \beta)\} = 0.$$

Therefore we arrive at

$$\begin{aligned} D_{s_h} \psi_h(l_1, \lambda) &= \lim_{x \rightarrow l_1} \{h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x)\} \\ &= h(l_1) \in (0, \infty). \end{aligned}$$

Thus we have (5.6).

(ii) Let l_1 be (s, m, k) -entrance. By means of Theorem 1.1 of [7], $0 < h(l_1) \leq \infty$.

First we consider the case $0 < h(l_1) < \infty$, and hence $C_1 > 0$ and $C_2 = 0$ by virtue of Proposition 2.1. Then

$$\psi_h(l_1, \lambda) = \frac{1}{h(l_1)} = \frac{1}{C_1 g_1(l_1, \beta)} \in (0, \infty),$$

$$D_{s_h} \psi_h(l_1, \lambda) = h(l_1) D_s \psi(l_1, \lambda) - \psi(l_1, \lambda) D_s h(l_1) = -C_1 D_s g_1(l_1, \beta) = 0.$$

These show (5.7).

Next we consider the case $h(l_1) = \infty$. Then $C_2 > 0$ and $\psi_h(l_1, \lambda) = \psi(l_1, \lambda)/h(l_1) = 0$. We show that

$$D_{s_h} \psi_h(l_1, \lambda) = -D_s h(l_1) = -C_2 D_s g_2(l_1, \beta) \in (0, \infty). \quad (5.9)$$

Here is the proof of (5.9). Since l_1 is (s, m, k) -entrance, for any positive ε there exists an $r \in I$ such that

$$\begin{aligned} \{s(r) - s(x)\} \int_{(l_1, x]} (\lambda dm(z) + dk(z)) &\leq \int_{(l_1, x]} \{s(r) - s(z)\} (\lambda dm(z) + dk(z)) \\ &\leq \int_{(l_1, r]} \{s(r) - s(z)\} (\lambda dm(z) + dk(z)) < \varepsilon, \end{aligned} \quad (5.10)$$

for $l_1 < x < r$. It is easy to derive the following estimate from (5.2):

$$|D_s \psi(x, \lambda)| \leq \int_{(l_1, x]} (\lambda dm(z) + dk(z)) \exp \left\{ \int_{(l_1, x]} (\lambda dm(z) + dk(z)) \int_{(z, x]} ds(y) \right\}.$$

Combining this with (5.10), we find

$$\limsup_{x \rightarrow l_1} \{s(r) - s(x)\} |D_s \psi(x, \lambda)| < \varepsilon.$$

Therefore

$$\limsup_{x \rightarrow l_1} h(x) |D_s \psi(x, \lambda)| \leq \varepsilon \lim_{x \rightarrow l_1} \frac{h(x)}{s(r) - s(x)} = -\varepsilon D_s h(l_1) = -\varepsilon C_2 D_s g_2(l_1, \beta).$$

Since $D_s g_2(l_1, \beta) \in (-\infty, 0)$ by virtue of Proposition 2.1, letting $\varepsilon \downarrow 0$ leads us to $\lim_{x \rightarrow l_1} h(x) D_s \psi(x, \lambda) = 0$, and

$$\begin{aligned} D_{s_h} \psi_h(l_1, \lambda) &= \lim_{x \rightarrow l_1} \{h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x)\} \\ &= -D_s h(l_1) = -C_2 D_s g_2(l_1, \beta) \in (0, \infty), \end{aligned}$$

which shows (5.9). Thus we obtain (5.8). $\square$

Now we define $\psi_h^*(x, \lambda)$, $x \in I$, $\lambda \geq \beta$, and $d\sigma_h^*(\lambda)$, $\lambda \geq \beta$, as follows.

Case 1: l_1 is (s, m, k) -regular or -exit. If $h(l_1) = 0$, then

$$\psi_h^*(x, \lambda) = D_s h(l_1) h(x)^{-1} \psi(x, \lambda - \beta), \quad (5.11)$$

$$d\sigma_h^*(\lambda) = \{D_s h(l_1)\}^{-2} d_\lambda \sigma(\lambda - \beta). \quad (5.12)$$

If $h(l_1) \in (0, \infty)$, then

$$\psi_h^*(x, \lambda) = \{h(l_1) h(x)\}^{-1} \psi(x, \lambda - \beta), \quad (5.13)$$

$$d\sigma_h^*(\lambda) = h(l_1)^2 d_\lambda \sigma(\lambda - \beta). \quad (5.14)$$

Case 2: l_1 is (s, m, k) -entrance. If $h(l_1) \in (0, \infty)$, then

$$\psi_h^*(x, \lambda) = h(l_1) h(x)^{-1} \psi(x, \lambda - \beta), \quad (5.15)$$

$$d\sigma_h^*(\lambda) = h(l_1)^{-2} d_\lambda \sigma(\lambda - \beta). \quad (5.16)$$

If $h(l_1) = \infty$, then

$$\psi_h^*(x, \lambda) = \{-D_s h(l_1) h(x)\}^{-1} \psi(x, \lambda - \beta), \quad (5.17)$$

$$d\sigma_h^*(\lambda) = \{D_s h(l_1)\}^2 d_\lambda \sigma(\lambda - \beta). \quad (5.18)$$

We note that $\sigma_h^*(\{\beta\}) = 0$ in Case 1, but $\sigma_h^*(\{\beta\}) \geq 0$ in Case 2.

By means of Theorem 1.1 of [7], l_1 is not $(s_h, m_h, 0)$ -natural. More precisely, in the case l_1 is (s, m, k) -regular (respectively -exit),

$$\begin{aligned} l_1 \text{ is } (s_h, m_h, 0)\text{-regular (respectively -exit)} & \quad \text{if } h(l_1) \in (0, \infty), \\ l_1 \text{ is } (s_h, m_h, 0)\text{-entrance} & \quad \text{if } h(l_1) = 0; \end{aligned}$$

in the case l_1 is (s, m, k) -entrance,

$$\begin{aligned} l_1 \text{ is } (s_h, m_h, 0)\text{-entrance} & \quad \text{if } h(l_1) \in (0, \infty), \\ l_1 \text{ is } (s_h, m_h, 0)\text{-regular} & \quad \text{if } h(l_1) = \infty \text{ and } |m_h(l_1)| < \infty, \\ l_1 \text{ is } (s_h, m_h, 0)\text{-exit} & \quad \text{if } h(l_1) = \infty \text{ and } |m_h(l_1)| = \infty. \end{aligned}$$

Therefore (5.3), (5.4) and Lemma 5.1 lead us to the following result.

PROPOSITION 5.2. $\psi_h^*(x, \lambda)$ satisfies (5.1) or (5.2) with s, m and k replaced by s_h, m_h and 0, respectively. $d\sigma_h^*(\lambda)$ is a Borel measure on $[\beta, \infty)$. $p_h^*(t, x, y)$ is represented as

$$p_h^*(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \psi_h^*(x, \lambda) \psi_h^*(y, \lambda) d\sigma_h^*(\lambda), \quad t > 0, x, y \in I. \quad (5.19)$$

6. Examples

Example 6.1. First we consider the following ODGDO $\mathcal{G}_{s,m,k}$ on $I = (0, \infty)$ with constant coefficients:

$$\mathcal{G}_{s,m,k} = a \frac{d^2}{dx^2} + b \frac{d}{dx} - c, \quad (6.1)$$

where $a > 0$, $b \in \mathbb{R}$ and $c \geq 0$. We may set

$$ds(x) = e^{-(b/a)x} dx, \quad dm(x) = a^{-1} e^{(b/a)x} dx, \quad dk(x) = (c/a) e^{(b/a)x} dx. \quad (6.2)$$

The end point 0 is (s, m, k) -regular and the end point ∞ is (s, m, k) -natural. For $\alpha \geq 0$, we set

$$\lambda_i(\alpha) = \frac{1}{2a} \{(-1)^i b + \sqrt{b^2 + 4a(c + \alpha)}\}, \quad i = 1, 2.$$

The α -Green function $G(\alpha, x, y)$ corresponding to $\mathcal{G}_{s,m,k}$ is given by

$$G(\alpha, x, y) = G(\alpha, y, x) = \begin{cases} (\lambda_1(\alpha) + \lambda_2(\alpha))^{-1} g_1(x, \alpha) g_2(y, \alpha), & \text{if } b^2 + c + \alpha > 0, \\ g_1(x, \alpha) g_2(y, \alpha), & \text{if } b^2 + c + \alpha = 0, \end{cases}$$

for $0 \leq x \leq y < \infty$, where

$$g_1(x, \alpha) = \begin{cases} e^{\lambda_1(\alpha)x} - e^{-\lambda_2(\alpha)x}, & \text{if } b^2 + c + \alpha > 0, \\ x, & \text{if } b^2 + c + \alpha = 0, \end{cases}$$

$$g_2(x, \alpha) = \begin{cases} e^{-\lambda_2(\alpha)x}, & \text{if } b^2 + c + \alpha > 0, \\ 1, & \text{if } b^2 + c + \alpha = 0. \end{cases}$$

The elementary solution $p(t, x, y)$ is given by

$$p(t, x, y) = \frac{1}{2} \sqrt{\frac{a}{\pi t}} \exp \left\{ -\frac{b}{2a}(x+y) - At \right\} \{e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at}\}$$

$$= \int_A^\infty e^{-\lambda t} \psi(x, \lambda) \psi(y, \lambda) d\sigma(\lambda),$$

where $A = b^2/4a + c$ and

$$\psi(x, \lambda) = e^{-(b/2a)x} \sqrt{\frac{a}{\lambda - A}} \sin \left(\sqrt{\frac{\lambda - A}{a}} x \right),$$

$$d\sigma(\lambda) = \frac{1}{\pi} \sqrt{\frac{\lambda - A}{a}} d\lambda,$$

for $x \in I$ and $\lambda > A$.

Let $h \in \mathcal{H}_{s,m,k,\beta}$. Then the h transform of $\mathcal{G}_{s,m,k}$ is reduced to

$$\mathcal{G}_{s_h, m_h, 0} = a \frac{d^2}{dx^2} + \left(b + 2a \frac{h'(x)}{h(x)} \right) \frac{d}{dx}. \quad (6.3)$$

(1) Assume $h(0) \in (0, \infty)$. Then h is represented as $h(x) = C_1 g_1(x, \beta) + C_2 g_2(x, \beta)$ for some constants $C_1 \geq 0$ and $C_2 > 0$. The end point 0 is $(s_h, m_h, 0)$ -regular and we

get (5.19), where by means of (5.13) and (5.14),

$$\psi_h^*(x, \lambda) = \{C_2 h(x)\}^{-1} \psi(x, \lambda - \beta), \quad d\sigma_h^*(\lambda) = C_2^2 d_\lambda \sigma(\lambda - \beta). \quad (6.4)$$

(2) Assume $h(0) = 0$. Then $h(x) = C_1 g_1(x, \beta)$ with $C_1 > 0$. The end point 0 is $(s_h, m_h, 0)$ -entrance and we get (5.19), where by means of (5.11) and (5.12),

$$\psi_h^*(x, \lambda) = \tilde{C} h(x)^{-1} \psi(x, \lambda - \beta), \quad d\sigma_h^*(\lambda) = \tilde{C}^{-2} d_\lambda \sigma(\lambda - \beta).$$

Here

$$\tilde{C} = D_s h(0) = \begin{cases} C_1(\lambda_1(\beta) + \lambda_2(\beta)), & \text{if } b^2 + c + \beta > 0, \\ C_1, & \text{if } b^2 + c + \beta = 0. \end{cases}$$

Especially, if $b = c = 0$ and $C_1 = 1$, then $\lambda_i(\beta) = \sqrt{\beta/a}$, $i = 1, 2$, and

$$h(x) = \begin{cases} 2 \sinh(\sqrt{\beta/a} x), & \text{if } \beta > 0, \\ x, & \text{if } \beta = 0, \end{cases} \quad \tilde{C} = \begin{cases} 2\sqrt{\beta/a}, & \text{if } \beta > 0, \\ 1, & \text{if } \beta = 0. \end{cases}$$

Therefore (6.3) and (5.19) are reduced to the following. If $\beta > 0$, then

$$\begin{aligned} \mathcal{G}_{s_h, m_h, 0} &= a \frac{d^2}{dx^2} + 2\sqrt{a\beta} \frac{\cosh(\sqrt{\beta/a} x)}{\sinh(\sqrt{\beta/a} x)} \frac{d}{dx}, \\ p_h^*(t, x, y) &= \frac{1}{8} \sqrt{\frac{a}{\pi t}} e^{-\beta t} \{e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at}\} / \sinh(\sqrt{\beta/a} x) \sinh(\sqrt{\beta/a} y) \\ &= \int_{\beta}^{\infty} e^{-\lambda t} \psi_h^*(x, \lambda) \psi_h^*(y, \lambda) d\sigma_h^*(\lambda), \end{aligned}$$

where

$$\psi_h^*(x, \lambda) = \sqrt{\frac{\beta}{\lambda - \beta}} \frac{\sin(\sqrt{(\lambda - \beta)/a} x)}{\sinh(\sqrt{\beta/a} x)}, \quad d\sigma_h^*(\lambda) = \frac{1}{4\pi} \frac{\sqrt{(\lambda - \beta)/\beta}}{\sqrt{\beta/a}} d\lambda.$$

If $\beta = 0$, then

$$\mathcal{G}_{s_h, m_h, 0} = a \frac{d^2}{dx^2} + \frac{2a}{x} \frac{d}{dx}, \quad (6.5)$$

$$\begin{aligned} p_h^*(t, x, y) &= \frac{1}{2} \sqrt{\frac{a}{\pi t}} \frac{1}{xy} \{e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at}\} \\ &= \int_0^{\infty} e^{-\lambda t} \psi_h^*(x, \lambda) \psi_h^*(y, \lambda) d\sigma_h^*(\lambda), \end{aligned} \quad (6.6)$$

where

$$\psi_h^*(x, \lambda) = \frac{\sin(\sqrt{\lambda/a} x)}{\sqrt{\lambda/a} x}, \quad d\sigma_h^*(\lambda) = \frac{1}{\pi} \sqrt{\frac{\lambda}{a}} d\lambda.$$

When $a = 1/2$, (6.5) is the generator of a three-dimensional Bessel process which we denote by $\tilde{\mathcal{G}}$; that is,

$$\tilde{\mathcal{G}} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx}.$$

Since $\tilde{h}(x) = 1/x$ satisfies $\tilde{\mathcal{G}}\tilde{h} = 0$, we can consider the h transform of $\tilde{\mathcal{G}}$ with respect to $\tilde{h}(x) = 1/x$, which we denote by $\mathcal{G}$. Then we get

$$\mathcal{G} = \frac{1}{2} \frac{d^2}{dx^2}.$$

This is the generator of Brownian motion on $(0, \infty)$. The above observation shows that $\mathcal{G}$ is h transformed to $\tilde{\mathcal{G}}$ in terms of $h(x) = x$, and $\tilde{\mathcal{G}}$ is h transformed to $\mathcal{G}$ in terms of $\tilde{h}(x) = 1/x$. This fact is also obtained by means of Theorem 1.4.

(3) We denote $\mathcal{G}_{s,m,k}$ with $b^2 + c > 0$ (respectively $b^2 + c = 0$) by $\mathcal{G}^{(1)}$ (respectively $\mathcal{G}^{(2)}$). Fix a $\beta \geq 0$ arbitrarily, and set $h^{(1)}(x) = e^{\lambda_1(\beta)x}$. Since $\mathcal{G}^{(1)}h^{(1)} = \beta h^{(1)}$, we can consider an h transform of $\mathcal{G}^{(1)}$ with respect to $h^{(1)}$ which is denoted by $\tilde{\mathcal{G}}^{(1)}$; that is,

$$\begin{aligned} \tilde{\mathcal{G}}^{(1)} &= a \frac{d^2}{dx^2} + \left(b + 2a \frac{h^{(1)'}(x)}{h^{(1)}(x)} \right) \frac{d}{dx} \\ &= a \frac{d^2}{dx^2} + \sqrt{b^2 + 4a(c + \beta)} \frac{d}{dx}. \end{aligned} \quad (6.7)$$

We put $\kappa = b^2/4a + c + \beta$ and $h^{(2)}(x) = e^{\sqrt{\kappa/a}x}$. Since $\mathcal{G}^{(2)}h^{(2)} = \kappa h^{(2)}$, we can consider an h transform of $\mathcal{G}^{(2)}$ with respect to $h^{(2)}$ which is denoted by $\tilde{\mathcal{G}}^{(2)}$; that is,

$$\begin{aligned} \tilde{\mathcal{G}}^{(2)} &= a \frac{d^2}{dx^2} + 2a \frac{h^{(2)'}(x)}{h^{(2)}(x)} \frac{d}{dx} \\ &= a \frac{d^2}{dx^2} + 2a\sqrt{\kappa/a} \frac{d}{dx} \\ &= a \frac{d^2}{dx^2} + \sqrt{b^2 + 4a(c + \beta)} \frac{d}{dx}. \end{aligned} \quad (6.8)$$

We find that $\mathcal{G}^{(1)} \neq \mathcal{G}^{(2)}$, but $\tilde{\mathcal{G}}^{(1)} = \tilde{\mathcal{G}}^{(2)}$. This shows that an h transform of ODGDO is not a one-to-one correspondence.

Example 6.2. Let us consider the following ODGDO $\mathcal{G}_{(\kappa)}$ on $I = (0, \infty)$:

$$\mathcal{G}_{(\kappa)} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{\kappa}{2x} \frac{d}{dx},$$

where κ is a real number. $\mathcal{G}_{(0)}$ is the generator of Brownian motion on $(0, \infty)$, and $\mathcal{G}_{(\kappa)}$ is the generator of a $(\kappa + 1)$ -dimensional Bessel process for $\kappa \in \mathbb{N}$. We may set the scale function $s_{(\kappa)}$ and the speed measure $m_{(\kappa)}$ as

$$ds_{(\kappa)}(x) = x^{-\kappa} dx, \quad dm_{(\kappa)}(x) = 2x^\kappa dx.$$

The end point 0 is $(s_{(\kappa)}, m_{(\kappa)}, 0)$ -exit, or -regular, or -entrance according to if $\kappa \leq -1$, or $-1 < \kappa < 1$, or $1 \leq \kappa$. The end point ∞ is always natural. The α -Green function $G(\alpha, x, y)$ corresponding to $\mathcal{G}_{(\kappa)}$ is given by the following. We put $\mu = (1 - \kappa)/2$, and hence $\mu > 0$ or $\mu \leq 0$ according to if $\kappa < 1$ or $\kappa \geq 1$:

$$G_{(\kappa)}(\alpha, x, y) = G_{(\kappa)}(\alpha, y, x) = \begin{cases} (xy)^\mu I_{|\mu|}(\sqrt{2\alpha}x) K_{|\mu|}(\sqrt{2\alpha}y), & \text{if } \alpha > 0, \\ (2\mu)^{-1} x^{2\mu}, & \text{if } \kappa < 1, \alpha = 0, \end{cases}$$

for $0 < x \leq y < \infty$, where I_ν and K_ν are the modified Bessel functions defined by

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n! \Gamma(\nu + n + 1)}, \quad \nu > -1,$$

$$K_\nu(x)(x) = K_{-\nu}(x) = \begin{cases} \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi}, & \nu \in \mathbb{Z}, \\ \frac{(-1)^\nu}{2} \lim_{\xi \rightarrow \nu} \left(\frac{\partial I_{-\xi}(x)}{\partial \xi} - \frac{\partial I_\xi(x)}{\partial \xi} \right), & \nu \notin \mathbb{Z}. \end{cases}$$

The elementary solution $p_{(\kappa)}(t, x, y)$ is given as

$$p_{(\kappa)}(t, x, y) = \frac{1}{2t} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} (xy)^\mu I_{|\mu|} \left(\frac{xy}{t} \right), \quad t, x, y > 0. \quad (6.9)$$

We note the formula (see [6, p. 200])

$$\int_0^\infty e^{-\lambda t} J_\nu(\sqrt{\lambda} x) J_\nu(\sqrt{\lambda} y) d\lambda = \frac{1}{t} \exp \left\{ -\frac{x^2 + y^2}{4t} \right\} I_\nu \left(\frac{xy}{2t} \right), \quad (6.10)$$

for $t, x, y > 0$ and $\nu > -1$, where J_ν is the Bessel function defined by

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(\nu + n + 1)}.$$

By means of (6.9) and (6.10), we get the representation

$$p_{(\kappa)}(t, x, y) = \int_0^\infty e^{-\lambda t} \psi_{(\kappa)}(x, \lambda) \psi_{(\kappa)}(y, \lambda) \sigma_{(\kappa)}(\lambda) d\lambda,$$

where

$$\psi_{(\kappa)}(x, \lambda) = C_{(\kappa)}(\lambda) x^\mu J_{|\mu|}(\sqrt{2\lambda} x), \quad \sigma_{(\kappa)}(\lambda) = C_{(\kappa)}(\lambda)^{-2},$$

and $C_{(\kappa)}(\lambda)$ is a positive number satisfying

$$\lim_{x \rightarrow 0} \psi_{(\kappa)}(x, \lambda) = 1 \quad \text{if } \kappa \geq 1,$$

$$\lim_{x \rightarrow 0} D_{s_{(\kappa)}} \psi_{(\kappa)}(x, \lambda) = 1 \quad \text{if } \kappa < 1,$$

so that,

$$C_{(\kappa)}(\lambda) = \begin{cases} 2^{-1} (\lambda/2)^{-\mu/2} \Gamma(\mu), & \text{if } \kappa < 1, \\ (\lambda/2)^{-|\mu|/2} \Gamma(1 + |\mu|), & \text{if } \kappa \geq 1. \end{cases}$$

For $\beta > 0$, we set

$$h_{(\kappa)}(x) = (\sqrt{2\beta} x)^\mu K_{|\mu|}(\sqrt{2\beta} x). \quad (6.11)$$

The function $h_{(\kappa)}$ is positive and satisfies $\mathcal{G}^{(\kappa)} h_{(\kappa)} = \beta h_{(\kappa)}$. We denote by $\tilde{\mathcal{G}}_{(\kappa)}$ the h transform of $\mathcal{G}_{(\kappa)}$ with respect to $h_{(\kappa)}$; that is,

$$\begin{aligned} \tilde{\mathcal{G}}_{(\kappa)} &= \frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{\kappa}{2x} + \frac{h'_{(\kappa)}(x)}{h_{(\kappa)}(x)} \right) \frac{d}{dx} \\ &= \frac{1}{2} \frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \sqrt{2\beta} \frac{K'_{|\mu|}(\sqrt{2\beta} x)}{K_{|\mu|}(\sqrt{2\beta} x)} \right\} \frac{d}{dx}. \end{aligned} \quad (6.12)$$

Since $h_{(\kappa)}(0) \in (0, \infty)$ (respectively $= \infty$) if $\kappa \in (-\infty, 1)$ (respectively $\in [1, \infty)$), and $|\tilde{m}_{(\kappa)}(0)| \in [0, \infty)$ (respectively $= \infty$) if $\kappa \in [1, 3)$ (respectively $\in [3, \infty)$), the end point 0 is $(\tilde{s}_{(\kappa)}, \tilde{m}_{(\kappa)}, 0)$ -regular or -exit according to if $\kappa < 3$ or $\kappa \geq 3$, where

$$d\tilde{s}_{(\kappa)}(x) = h_{(\kappa)}(x)^{-2} x^{-\kappa} dx, \quad d\tilde{m}_{(\kappa)}(x) = 2h_{(\kappa)}(x)^2 x^{\kappa} dx. \quad (6.13)$$

We denote by $\tilde{p}_{(\kappa)}(t, x, y)$ the elementary solution of equation (1.1) with $\mathcal{G}_{s,m,k}$ replaced by $\tilde{\mathcal{G}}_{(\kappa)}$. Then we get

$$\begin{aligned} \tilde{p}_{(\kappa)}(t, x, y) &= e^{-\beta t} p_{(\kappa)}(t, x, y) / h_{(\kappa)}(x) h_{(\kappa)}(y) \\ &= \frac{1}{2(2\beta)^{\mu t}} \exp \left\{ -\beta t - \frac{x^2 + y^2}{2t} \right\} I_{|\mu|} \left(\frac{xy}{t} \right) / K_{|\mu|}(\sqrt{2\beta} x) K_{|\mu|}(\sqrt{2\beta} y) \\ &= \int_{\beta}^{\infty} e^{-\lambda t} \tilde{\psi}_{(\kappa)}(x, \lambda) \tilde{\psi}_{(\kappa)}(y, \lambda) \tilde{\sigma}_{(\kappa)}(\lambda) d\lambda, \end{aligned} \quad (6.14)$$

where $\tilde{\psi}_{(\kappa)}(x, \lambda)$ and $\tilde{\sigma}_{(\kappa)}(\lambda)$ are obtained by means of (5.13), (5.14), (5.17) and (5.18).

We now take a $\kappa_1 \in (-\infty, 1)$ and put $\kappa_2 = 2 - \kappa_1 \in (1, \infty)$. Then $|(1 - \kappa_1)/2| = |(1 - \kappa_2)/2|$. We thus find that $\mathcal{G}_{(\kappa_1)} \neq \mathcal{G}_{(\kappa_2)}$, but $\tilde{\mathcal{G}}_{(\kappa_1)} = \tilde{\mathcal{G}}_{(\kappa_2)}$. This also shows that an h transform of ODGDO is not a one-to-one correspondence.

Example 6.3. Finally we consider the following ODGDO $\mathcal{G}_o$ on $(0, \infty)$:

$$\mathcal{G}_o = \frac{1}{2} \frac{d^2}{dx^2} - cx^{-2}, \quad (6.15)$$

where c is a positive number. We may set

$$ds_o(x) = dx, \quad dm_o(x) = 2 dx, \quad dk_o(x) = 2cx^{-2} dx.$$

We note that both of the end points 0 and ∞ are (s_o, m_o, k_o) -natural.

Let $p_o(t, x, y)$ be the elementary solution of equation (1.1) with $\mathcal{G}_{s,m,k}$ replaced by $\mathcal{G}_o$. By using Proposition 3.3, we show that $p_o(t, x, y)$ is given by (6.19) below.

For $\beta > 0$, we set

$$h_o(x) = \sqrt{x} K_{\nu}(\sqrt{2\beta} x),$$

where $\nu = \sqrt{2c + 1/4} (> 1/2)$. The function h_o is positive and satisfies $\mathcal{G}_o h_o = \beta h_o$. We denote by $\tilde{\mathcal{G}}_o$ the h transform of $\mathcal{G}$ with respect to h_o . Then we find

$$\begin{aligned} \tilde{\mathcal{G}}_o &= \frac{1}{2} \frac{d^2}{dx^2} + \frac{h'_o(x)}{h_o(x)} \frac{d}{dx} \\ &= \frac{1}{2} \frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \sqrt{2\beta} \frac{K'_\nu(\sqrt{2\beta} x)}{K_\nu(\sqrt{2\beta} x)} \right\} \frac{d}{dx}. \end{aligned}$$

This shows that $\tilde{\mathcal{G}}_o$ coincides with $\tilde{\mathcal{G}}_{(\kappa)}$ with $\nu = |\mu| = |(1 - \kappa)/2|$ in the preceding example. It should be noted that we take the scale $\tilde{s}_o$ (respectively $\tilde{s}_{(\kappa)}$) and the speed measure $\tilde{m}_o$ (respectively $\tilde{m}_{(\kappa)}$) corresponding to $\tilde{\mathcal{G}}_o$ (respectively $\tilde{\mathcal{G}}_{(\kappa)}$), where

$$d\tilde{s}_o(x) = h_o(x)^{-2} dx, \quad d\tilde{m}_o(x) = 2h_o(x)^2 dx. \quad (6.16)$$

Let $\tilde{p}_o(t, x, y)$ be the elementary solution of equation (1.1) with $\mathcal{G}_{s,m,k}$ replaced by $\tilde{\mathcal{G}}_o$. Then we have

$$\tilde{p}_o(t, x, y) = e^{-\beta t} p_o(t, x, y) / h_o(x) h_o(y). \quad (6.17)$$

Let $\nu = |(1 - \kappa)/2|$. Since $\tilde{\mathcal{G}}_o$ coincides with $\tilde{\mathcal{G}}_{(\kappa)}$, noting (6.16) and (6.13), we see that

$$\tilde{p}_o(t, x, y) h_o(y)^2 = \tilde{p}_{(\kappa)}(t, x, y) h_{(\kappa)}(y)^2 y^\kappa. \quad (6.18)$$

By using (6.9), (6.11), (6.14), (6.17), and (6.18), we obtain

$$\begin{aligned} p_o(t, x, y) &= p_{(\kappa)}(t, x, y) \frac{h_{(\kappa)}(y)}{h_{(\kappa)}(x)} \frac{h_o(x)}{h_o(y)} y^\kappa \\ &= \frac{1}{2t} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} (xy)^{1/2} I_\nu \left(\frac{xy}{t} \right). \end{aligned} \quad (6.19)$$

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