MONODROMY REPRESENTATIONS ASSOCIATED WITH APPELL'S HYPERGEOMETRIC FUNCTION F_1 USING INTEGRALS OF A MULTIVALUED FUNCTION

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Abstract. Monodromy representations on the solution space of Appell's system of differential equations E_1 are studied by using integrals of a multivalued function. In particular, we realize the representations in the reducible cases and give a complete list of finite reducible representations.

0. Introduction

This paper is a continuation of [MS1, MS2] and is devoted to studying the monodromy representation associated with Appell's hypergeometric function F_1 , especially in the reducible cases.

Appell's hypergeometric function F_1 is the analytic continuation of the series defined by

$$F_1(\alpha, \beta_1, \beta_2, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}m!n!} x^m y^n, \quad |x|, |y| < 1, \quad (0.1)$$

where $(a)_m = a(a+1) \cdots (a+m-1)$. It satisfies the system E_1 of differential equations:

$$(E_1) : \begin{cases} [\theta_1(\theta + \gamma - 1) - x(\theta + \alpha)(\theta_1 + \beta_1)]F = 0, \\ [\theta_2(\theta + \gamma - 1) - y(\theta + \alpha)(\theta_2 + \beta_2)]F = 0, \\ [x(\theta_1 + \beta_1)\theta_2 - y(\theta_2 + \beta_2)\theta_1]F = 0, \end{cases}$$

where $\theta_1 = x\partial/\partial x$, $\theta_2 = y\partial/\partial y$ and $\theta = \theta_1 + \theta_2$, on the space

$$\mathbb{C}^2 \setminus \{\{x=0\} \cup \{y=0\} \cup \{x=1\} \cup \{y=1\} \cup \{x=y\}\} \subset (\mathbb{P}^1)^2.$$

The rank of E_1 is 3. The function F_1 has an integral representation in the form

$$\begin{aligned} F_1(\alpha, \beta_1, \beta_2, \gamma; x, y) \\ = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_1^\infty t^{\beta_1+\beta_2-\gamma} (t-x)^{-\beta_1} (t-y)^{-\beta_2} (t-1)^{\gamma-\alpha-1} dt, \end{aligned}$$

where $\operatorname{Re}(\gamma - \alpha) > 0$ and $\operatorname{Re}(\alpha) > 0$.

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In this paper, we consider the integrals in the form

$$\int_C t^{\lambda_0} (t-x)^{\lambda_1} (t-y)^{\lambda_2} (t-y)^{\lambda_3} dt, \quad (0.2)$$

where C is a suitable chain which is not necessarily a cycle. We call (0.2) a *hypergeometric integral associated with Appell's hypergeometric function* F_1 .

In [MS2], the following is proved: (i) the integral (0.2) with $\lambda_0 = \beta_1 + \beta_2 - \gamma$, $\lambda_1 = -\beta_1$, $\lambda_2 = -\beta_2$, $\lambda_3 = \gamma - \alpha - 1$ (and $\lambda_\infty = -\lambda_{0123} - 2 = -\alpha - 3$) satisfies the system E_1 ; (ii) a necessary and sufficient condition for the irreducibility of the system E_1 is that none of α , $\gamma - \alpha$, $\gamma - \beta_1 - \beta_2$, β_1 and β_2 are integers. Here the irreducibility of the differential equation is defined to be that of the monodromy representation on the solution space of the equation.

In the present paper, we realize representations in reducible cases. These realizations show the structure of the representations, in particular, how a subrepresentation is included in each of the reducible representations. They also give a complete list of the finite reducible representations.

In this paper, the symbols

$$\begin{aligned} e(A) &= \exp(\pi\sqrt{-1}A), \quad \lambda_{ij\dots k} = \lambda_i + \lambda_j + \dots + \lambda_k, \\ \mu_i &= e(2\lambda_i), \quad \mu_{ij\dots k} = \mu_i \mu_j \dots \mu_k \end{aligned}$$

and

ϵ : a sufficiently small positive number

are frequently used.

1. Preliminaries

Let z_0, z_1, z_2, z_3 be distinct points of $\mathbb{C}$, and $U(t)$ a function

$$U(t) = \prod_{0 \leq j \leq 3} (t - z_j)^{\lambda_j},$$

defined on $T_z = \mathbb{C} \setminus \{z_0, z_1, z_2, z_3\}$, with exponents $\lambda_0, \dots, \lambda_3 \in \mathbb{C}$. Let λ_∞ , which corresponds to the exponent at ∞ , be

$$\lambda_\infty = -\sum_{j=0}^3 \lambda_j - 2.$$

When all the points $z_0, \dots, z_3$ are real points, for an interval D of $T_{\mathbb{R}} = \mathbb{R} \setminus \{z_0, \dots, z_3\}$ (the real locus of T_z), we let $U_D(t)$ denote the function

$$U_D(t) = \prod_{j=0}^3 (\epsilon_j (t - z_j))^{\lambda_j},$$

where $\epsilon_j = \pm 1$ is so determined that each $\epsilon_j (t - z_j)$ is positive and the argument of each factor is zero on D . For instance, when $z_0 < z_1 < z_2 < z_3$, $U_{(z_0, z_1)}(t) = (t - z_0)^{\lambda_0} \prod_{j=1}^3 (z_j - t)^{\lambda_j}$

with each argument zero on the interval (z_0, z_1) , and $U_{(z_3, \infty)}(t) = \prod_{j=0}^3 (t - z_j)^{\lambda_j}$ with each argument zero on (z_3, ∞) .

The function $U_p(t)$ for $p \in T_{\mathbb{R}}$ is defined similarly. For example, when $z_0 < z_1 < z_2 < z_3$, $U_{z_0+\epsilon}(t) = U_{z_1-\epsilon}(t) = U_{(z_0, z_1)}(t)$ and $U_{z_3+\epsilon}(t) = U_{\infty-\epsilon}(t) = U_{(z_3, \infty)}(t)$. Recall that ϵ denotes a small positive number.

In what follows in this section, we fix $z_0 < z_1 < z_2 < z_3$ and denote ∞ by $z_4 = z_{-1}$ and λ_{∞} by $\lambda_4 = \lambda_{-1}$.

In the case when $\lambda_{k-1} \notin \mathbb{Z}_{<0}$ and $\lambda_k \notin \mathbb{Z}_{<0}$, a regularization of the path $\overrightarrow{(z_{k-1}, z_k)}$, which has a natural orientation, associated with the function $U(t)$ is defined to be

$$\text{reg}(z_{k-1}, z_k) = \begin{cases} \frac{1}{d_{\lambda_{k-1}}} S(z_{k-1} + \epsilon) + \overrightarrow{[z_{k-1} + \epsilon, z_k - \epsilon]} - \frac{1}{d_{\lambda_k}} S(z_k - \epsilon) & \text{if } \lambda_{k-1}, \lambda_k \notin \mathbb{Z}, \\ \frac{1}{d_{\lambda_{k-1}}} S(z_{k-1} + \epsilon) + \overrightarrow{[z_{k-1} + \epsilon, z_k]} & \text{if } \lambda_{k-1} \notin \mathbb{Z}, \lambda_k \in \mathbb{Z}_{\geq 0}, \\ \overrightarrow{[z_{k-1}, z_k - \epsilon]} - \frac{1}{d_{\lambda_k}} S(z_k - \epsilon) & \text{if } \lambda_{k-1} \in \mathbb{Z}_{\geq 0}, \lambda_k \notin \mathbb{Z}, \\ \overrightarrow{[z_{k-1}, z_k]} & \text{if } \lambda_{k-1}, \lambda_k \in \mathbb{Z}_{\geq 0}. \end{cases}$$

Here d_{λ} denotes the value $e(2\lambda) - 1$, and here and in what follows $S(z_k + \epsilon)$ or $S(z_k - \epsilon)$ stands for the positively oriented circle that is centered at the point z_k and that starts and ends at the point $z_k + \epsilon$ or $z_k - \epsilon$, as is depicted in Figure 1. In particular, $S(\infty - \epsilon)$ or $S(\infty + \epsilon)$ means a negatively oriented circle centered at the origin with sufficiently large radius ϵ^{-1} , which might be written by the negatively oriented circle $S^{-1}(0 + \epsilon^{-1})$ or $S^{-1}(0 - \epsilon^{-1})$, as is depicted in Figure 2.

FIGURE 1. $S(z_k + \epsilon)$ and $S(z_k - \epsilon)$.

FIGURE 2. $S(\infty - \epsilon)$ and $S^{-1}(0 + \epsilon^{-1})$.

In the case when $\lambda_{k-1} \notin \mathbb{Z}_{<0}$ and $\lambda_k \notin \mathbb{Z}_{<0}$, we consider the integral

$$\int_{\text{reg}(z_{k-1}, z_k)} U_{(z_{k-1}, z_k)}(t) dt, \quad (1.1)$$

where the arguments of $t - z_{k-1}$ and $z_k - t$ take values from 0 to 2π on $S(z_{k-1} + \epsilon)$ and on $S(z_k - \epsilon)$, respectively.

In the case when $\lambda_k \in \mathbb{Z}_{<0}$, we consider the integral

$$\int_{S(z_k + \epsilon)} U_{z_k + \epsilon}(t) dt \quad \text{or} \quad \int_{S(z_k - \epsilon)} U_{z_k - \epsilon}(t) dt. \quad (1.2)$$

To obtain solutions of E_1 , we set $(z_0, z_1, z_2, z_3) = (0, x, y, 1)$ and

$$\lambda_0 = \beta_1 + \beta_2 - \gamma, \quad \lambda_1 = -\beta_1, \quad \lambda_2 = -\beta_2, \quad \lambda_3 = \gamma - \alpha - 1$$

in $U(t)$:

$$U(t) = t^{\lambda_0} (t - x)^{\lambda_1} (t - y)^{\lambda_2} (t - 1)^{\lambda_3}.$$

With this notation, Proposition 1.4 in [MS2] implies the following.

PROPOSITION 1.1.

(1) If $\lambda_k, \lambda_{k+1} \notin \mathbb{Z}_{<0}$, then $\int_{\text{reg}(z_k, z_{k+1})} U(t) dt$ satisfies the system of differential equations E_1 .

(2) If $\lambda_k \in \mathbb{Z}_{<0}$, then $\int_{S(z_k + \epsilon)} U(t) dt$ satisfies the system of differential equations E_1 .

The hypergeometric integrals with variables $z = (x, y)$ in the form (1.1) and (1.2) are defined on $\mathbb{C}^2 \setminus S \subset (\mathbb{P}^1)^2$, where

$$S = \{x = 0\} \cup \{y = 0\} \cup \{x = 1\} \cup \{y = 1\} \cup \{x = y\}.$$

In the following, we study a monodromy representation

$$\rho : \pi_1(\mathbb{C}^2 \setminus S) \longrightarrow GL(4, \mathbb{C})$$

given by

$$\gamma^*(f_1(z), f_2(z), \dots, f_4(z)) = (f_1(z), f_2(z), \dots, f_4(z))\rho(\gamma) \quad (1.3)$$

for $\gamma \in \pi_1(\mathbb{C}^2 \setminus S)$ and a set of linearly independent functions $f_1(z), \dots, f_4(z)$ in the form (1.1) or (1.2). The matrix $\rho(\gamma)$ in (1.3) will be denoted by M_γ , and called *the matrix of the representation of γ in the basis $(f_1(z), \dots, f_4(z))$* . Here each $\gamma^* f_j(z)$ is the consequence of the analytic continuation of $f_j(z)$, and $\gamma^*(f_1(z), \dots, f_4(z)) = (\gamma^* f_1(z), \dots, \gamma^* f_4(z))$. Note that Proposition 1.1 guarantees that the monodromy representations on the solution space of E_1 can be identified with these representations.

For a fixed real base point $z^0 = (x^0, y^0) \in \mathbb{C}^2 \setminus S$ satisfying $0 < x^0 < y^0 < 1$, the symbols $\gamma_{x, \{0\}}$, $\gamma_{x, \{y\}}$ and $\gamma_{x, \{y, 1\}}$ designate the paths that express the moving of the coordinate x as in Figure 3, and the symbols $\gamma_{y, \{0, x\}}$, $\gamma_{y, \{x\}}$ and $\gamma_{y, \{1\}}$ designate the paths that express the moving of the coordinate y as in Figure 4. It is known that these paths give a set of generators of the fundamental group $\pi_1(z^0, \mathbb{C}^2 \setminus S)$ with a base point $z^0 = (x^0, y^0)$.

Here and in what follows, we denote the function $\int_C U(t) dt$ by C for brevity. For instance, $\text{reg}(z_{k-1}, z_k)$ expresses $\int_{\text{reg}(z_{k-1}, z_k)} U_{(z_{k-1}, z_k)}(t) dt$.

Proposition 2.1 in [MS2] implies the following.

FIGURE 3. Paths for the moves of x . FIGURE 4. Paths for the moves of y .

PROPOSITION 1.2. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \notin \mathbb{Z}_{<0}$ and $\lambda_\infty \notin \mathbb{Z}_{\geq 0}$, the matrices of the representation are*

$$\begin{aligned}
 M_{x,\{0\}} &= \begin{pmatrix} \mu_{01} & 1 - \mu_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_{x,\{y,1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ \mu_1(1 - \mu_{23}) & \mu_{123} & 0 \\ \mu_{12}(1 - \mu_3) & \mu_{12}(\mu_3 - 1) & \mu_1 \end{pmatrix}, \\
 M_{x,\{y\}} &= \begin{pmatrix} 1 & 0 & 0 \\ \mu_1(1 - \mu_2) & \mu_{12} & 1 - \mu_1 \\ 0 & 0 & 1 \end{pmatrix}, & & (1.4) \\
 M_{y,\{1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mu_2(1 - \mu_3) & \mu_{23} \end{pmatrix}, & M_{y,\{0,x\}} &= \begin{pmatrix} \mu_2 & \mu_2(\mu_0 - 1) & 1 - \mu_0 \\ 0 & \mu_{012} & 1 - \mu_{01} \\ 0 & 0 & 1 \end{pmatrix},
 \end{aligned}$$

in the basis $(e(\lambda_{12})\text{reg}(0, x), e(\lambda_2)\text{reg}(x, y), \text{reg}(y, 1))$.

The next proposition is a special case of (2) of Theorem 3.1 in [MS2].

PROPOSITION 1.3. *A necessary and sufficient condition for the irreducibility of E_1 is that none of λ_i ($0 \leq i \leq 3$) and λ_∞ are integers.*

The purpose of the present paper is to study the monodromy representations in reducible cases, especially to realize a representation in each of the reducible cases, and to make a complete list of the finite reducible representations. For such a purpose, the following lemmas will play important roles.

LEMMA 1.4. *Fix a base point to be $0 < x < y < 1$.*

(1) *If $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \notin \mathbb{Z}_{<0}$ and $\lambda_\infty \notin \mathbb{Z}_{\geq 0}$, the integrals $\text{reg}(0, x)$, $\text{reg}(x, y)$, and $\text{reg}(y, 1)$ are linearly independent.*

(2) *If $\lambda_\infty, \lambda_0, \lambda_1, \lambda_2 \notin \mathbb{Z}_{<0}$ and $\lambda_3 \notin \mathbb{Z}_{\geq 0}$, the integrals $\text{reg}(\infty, 0)$, $\text{reg}(0, x)$, and $\text{reg}(x, y)$ are linearly independent.*

(3) If $\lambda_1, \lambda_2, \lambda_3, \lambda_\infty \notin \mathbb{Z}_{<0}$ and $\lambda_0 \notin \mathbb{Z}_{\geq 0}$, the integrals $\text{reg}(x, y)$, $\text{reg}(y, 1)$, and $\text{reg}(1, \infty)$ are linearly independent.

Proof. (1) is a special case of Lemma 2.3 of [MS2]. (2) follows from (1) by considering the symmetry (2.3) explained later. Similarly, (3) is obtained. $\square$

The next two lemmas give conditions on non-triviality of the integrals.

LEMMA 1.5. Fix a base point to be $0 < x < y < 1$.

(1) For $\lambda_0, \lambda_1 \notin \mathbb{Z}_{<0}$, $\text{reg}(0, x) = 0$ if $\lambda_{01} \in \mathbb{Z}_{\leq -2}$ and $\text{reg}(0, x) \neq 0$ otherwise.

(2) For $\lambda_1, \lambda_2 \notin \mathbb{Z}_{<0}$, $\text{reg}(x, y) = 0$ if $\lambda_{12} \in \mathbb{Z}_{\leq -2}$ and $\text{reg}(x, y) \neq 0$ otherwise.

(3) For $\lambda_2, \lambda_3 \notin \mathbb{Z}_{<0}$, $\text{reg}(y, 1) = 0$ if $\lambda_{23} \in \mathbb{Z}_{\leq -2}$ and $\text{reg}(y, 1) \neq 0$ otherwise.

Proof. If $\text{Re}(\lambda_0) > -1$ and $\text{Re}(\lambda_1) > -1$, the change of the integration variable $t \rightarrow xt$ makes

$$\begin{aligned} \int_0^x U_{(0,x)}(t) dt &= x^{\lambda_{01}+1} y^{\lambda_2} \int_0^1 t^{\lambda_0} (1-t)^{\lambda_1} \left(1 - \frac{x}{y}t\right)^{\lambda_2} (1-xt)^{\lambda_3} dt \\ &= x^{\lambda_{01}+1} y^{\lambda_2} \sum_{i,j \geq 0} \frac{(-\lambda_2)_i}{i!} \frac{(-\lambda_3)_j}{j!} \left(\frac{x}{y}\right)^i y^j \int_0^1 t^{\lambda_0+i+j} (1-t)^{\lambda_1} dt \\ &= B(\lambda_0 + 1, \lambda_1 + 1) x^{\lambda_{01}+1} y^{\lambda_2} \sum_{i,j \geq 0} \frac{(-\lambda_2)_i (-\lambda_3)_j (\lambda_0 + 1)_{i+j}}{i! j! (\lambda_{01} + 2)_{i+j}} \left(\frac{x}{y}\right)^i y^j. \end{aligned}$$

The analytic continuation implies

$$\text{reg}(0, x) = x^{\lambda_{01}+1} y^{\lambda_2} \frac{\Gamma(\lambda_0 + 1) \Gamma(\lambda_1 + 1)}{\Gamma(\lambda_{01} + 2)} F_1 \left(\lambda_0 + 1, -\lambda_2, -\lambda_3, \lambda_{01} + 2; \frac{x}{y}, x \right),$$

which results in the assertion (1). Note that $\Gamma(\lambda)^{-1} = 0$ for $\lambda \in \mathbb{Z}_{\leq 0}$. Similarly, the equalities

$$\begin{aligned} \text{reg}(x, y) &= x^{\lambda_0} (y-x)^{\lambda_{12}+1} (1-x)^{\lambda_3} \\ &\times \frac{\Gamma(\lambda_1 + 1) \Gamma(\lambda_2 + 1)}{\Gamma(\lambda_{12} + 2)} F_1 \left(\lambda_1 + 1, -\lambda_0, -\lambda_3, \lambda_{12} + 2; \frac{x-y}{x}, \frac{y-x}{1-x} \right) \end{aligned}$$

and

$$\begin{aligned} \text{reg}(y, 1) &= (1-x)^{\lambda_1} (1-y)^{\lambda_{23}+1} \\ &\times \frac{\Gamma(\lambda_2 + 1) \Gamma(\lambda_3 + 1)}{\Gamma(\lambda_{23} + 2)} F_1 \left(\lambda_3 + 1, -\lambda_0, -\lambda_1, \lambda_{23} + 2; 1-y, \frac{1-y}{1-x} \right) \end{aligned}$$

imply (2) and (3), respectively. $\square$

LEMMA 1.6.

(1) In the case $\lambda_j \in \mathbb{Z}_{<0}$ for some j ($0 \leq j \leq 3$), the integral $S(z_j + \epsilon)$ is not zero, unless $\lambda_0, \dots, \widehat{\lambda_j}, \dots, \lambda_3, \lambda_\infty \in \mathbb{Z}_{\geq 0}$.

(2) In the case $\lambda_\infty \in \mathbb{Z}_{<0}$, the integral $S(\infty - \epsilon)$ is not zero, unless $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_{\geq 0}$.

This lemma follows from Lemma 3.2 in [MS2].

2. Symmetry of the monodromy representations

In a study of the monodromy representation, a certain symmetry of the integrals that comes from a simple transformation of the coordinates is useful.

For a fixed pair of complex numbers (x, y) , we consider a fractional linear transformation

$$t = \psi(s) := \frac{as + b}{cs + d}$$

satisfying

$$\psi(\{0, 1, \infty\}) \subset \{0, x, y, 1, \infty\}.$$

We denote the inverse of ψ by ϕ and define the points ξ, η to satisfy

$$\phi(\{0, x, y, 1, \infty\} \setminus \psi(\{0, 1, \infty\})) = \{\xi, \eta\}.$$

Then ϕ carries the points $t = 0, x, y, 1, \infty$ to the points $s = 0, \xi, \eta, 1, \infty$, and ψ carries the points $s = 0, \xi, \eta, 1, \infty$ to the points $t = 0, x, y, 1, \infty$. Moreover, we have

$$\begin{aligned} & \int_C t^{v_0} (t-x)^{v_x} (t-y)^{v_y} (t-1)^{v_1} dt \\ &= \sigma(x, y) \int_{C'} s^{v_{\psi(0)}} (s-\xi)^{v_{\psi(\xi)}} (s-\eta)^{v_{\psi(\eta)}} (s-1)^{v_{\psi(1)}} ds, \end{aligned}$$

where $C = \psi(C')$ and $\sigma(x, y)$ designates a multivalued function with singularities (at most) around $x = 0, x = 1, y = 0, y = 1$, and $x = y$. This means that if each element of generators of $\pi_1(\mathbb{C}^2 \setminus S)$ with base point (ξ, η) is mapped from $\pi_1(\mathbb{C}^2 \setminus S)$ with base point (x, y) by ϕ , then the monodromy representation in the basis

$$\int_{C'_j} s^{v_{\psi(0)}} (s-\xi)^{v_{\psi(\xi)}} (s-\eta)^{v_{\psi(\eta)}} (s-1)^{v_{\psi(1)}} ds \quad (j = 1, 2, 3)$$

is isomorphic to that in the basis

$$\sigma(x, y)^{-1} \int_{C_j} t^{v_0} (t-x)^{v_x} (t-y)^{v_y} (t-1)^{v_1} dt \quad (j = 1, 2, 3)$$

and hence is projectively isomorphic to that in the basis

$$\int_{C_j} t^{v_0} (t-x)^{v_x} (t-y)^{v_y} (t-1)^{v_1} dt \quad (j = 1, 2, 3),$$

where $\psi(C'_j) = C_j$. Denoting the isomorphic class of the projective representation given by

$$\int_{C'_j} s^{v_{\psi(0)}} (s-\xi)^{v_{\psi(\xi)}} (s-\eta)^{v_{\psi(\eta)}} (s-1)^{v_{\psi(1)}} ds \quad (j = 1, 2, 3)$$

by $P(v_{\psi(0)}, v_{\psi(\xi)}, v_{\psi(\eta)}, v_{\psi(1)}, v_{\psi(\infty)})$, we will show that

$$P(v_0, v_x, v_y, v_1, v_\infty) \cong P(v_x, v_y, v_1, v_\infty, v_0), \quad (2.1)$$

$$P(v_0, v_x, v_y, v_1, v_\infty) \cong P(v_x, v_0, v_y, v_1, v_\infty). \quad (2.2)$$

Then, as a corollary, we have the following.

PROPOSITION 2.1. *We have the isomorphisms*

$$P(v_0, v_x, v_y, v_1, v_\infty) \cong P(v_\sigma(0), v_\sigma(x), v_\sigma(y), v_\sigma(1), v_\sigma(\infty))$$

for any $\sigma \in \text{Sym}\{0, x, y, 1, \infty\}$.

Proof of (2.1). When

$$s = \phi(t) := 1 - \frac{x}{t} \quad \left(t = \psi(s) := \frac{x}{1-s} \right),$$

we have

$$\phi((0, x, y, 1, \infty)) = (\infty, 0, \xi, \eta, 1),$$

where

$$\xi := \phi(y) = 1 - \frac{x}{y}, \quad \eta := \phi(1) = 1 - x,$$

and we have

$$\begin{aligned} & \int_{C_j} t^{v_0} (t-x)^{v_x} (t-y)^{v_y} (t-1)^{v_1} dt \\ &= x^{v_0+1} y^{v_y} \int_{C'_j} s^{v_x} (s-\xi)^{v_y} (s-\eta)^{v_1} (1-s)^{v_\infty} ds, \end{aligned} \quad (2.3)$$

where $v_\infty = -2 - v_0 - v_x - v_y - v_1$. Moreover, if $0 < x < y < 1$, then we have $0 < \xi < \eta < 1$,

$$\begin{aligned} \psi(\gamma_{\xi, \{0\}}) &= \gamma_{y, \{x\}}, & \psi(\gamma_{\xi, \{\eta\}}) &= \gamma_{y, \{1\}}, & \psi(\gamma_{\xi, \{\eta, 1\}}) &= \gamma_{y, \{0, x\}}, \\ \psi(\gamma_{\eta, \{1\}}) &= \gamma_{y, \{0, x\}} \gamma_{x, \{0\}}, & \psi(\gamma_{\eta, \{\xi\}}) &= \gamma_{y, \{1\}}, & \psi(\gamma_{\eta, \{0, \xi\}}) &= \gamma_{y, \{1\}} \gamma_{x, \{y, 1\}} \gamma_{x, \{y\}}^{-1}, \end{aligned}$$

and

$$\begin{aligned} \phi(\gamma_{x, \{0\}}) &= \gamma_{\xi, \{\eta, 1\}} \gamma_{\eta, \{1\}}, & \phi(\gamma_{x, \{y\}}) &= \gamma_{\xi, \{0\}}, & \phi(\gamma_{x, \{y, 1\}}) &= \gamma_{\xi, \{0\}} \gamma_{\eta, \{0, \xi\}} \gamma_{\eta, \{\xi\}}^{-1}, \\ \phi(\gamma_{y, \{0, x\}}) &= \gamma_{\xi, \{\eta, 1\}}, & \phi(\gamma_{y, \{x\}}) &= \gamma_{\xi, \{0\}}, & \phi(\gamma_{y, \{1\}}) &= \gamma_{\eta, \{\xi\}}. \end{aligned}$$

Note that

$$\begin{aligned} \gamma_{y, \{1\}} \gamma_{x, \{y, 1\}} \gamma_{x, \{y\}}^{-1} &= \gamma_{x, \{y\}}^{-1} \gamma_{x, \{y, 1\}} \gamma_{y, \{1\}}, \\ \gamma_{\xi, \{0\}} \gamma_{\eta, \{0, \xi\}} \gamma_{\eta, \{\xi\}}^{-1} &= \gamma_{\eta, \{\xi\}}^{-1} \gamma_{\eta, \{0, \xi\}} \gamma_{\xi, \{0\}}, \\ \gamma_{x, \{y\}} &= \gamma_{y, \{x\}}, \\ \gamma_{\xi, \{\eta\}} &= \gamma_{\eta, \{\xi\}}. \end{aligned}$$

Therefore, we obtain the isomorphism (2.1). □

Proof of (2.2). When

$$s = \phi(t) := \frac{t-x}{1-x} \quad (t = \psi(s) := x + (1-x)s),$$

we have

$$\phi((0, x, y, 1, \infty)) = (\xi, 0, \eta, 1, \infty),$$

where

$$\xi := \phi(0) = \frac{-x}{1-x}, \quad \eta := \phi(y) = \frac{y-x}{1-x},$$

and we have

$$\begin{aligned} & \int_{C_j} t^{v_0} (t-x)^{v_x} (t-y)^{v_y} (t-1)^{v_1} dt \\ &= (1-x)^{v_{0xy}+1} \int_{C'_j} s^{v_x} (s-\xi)^{v_0} (s-\eta)^{v_y} (s-1)^{v_1} ds. \end{aligned} \quad (2.4)$$

Moreover, if $0 < x < y < 1$, then we have $\xi < 0 < \eta < 1$,

$$\begin{aligned} \psi(\gamma_{\xi,\{0\}}) &= \gamma_{x,\{0\}}, & \psi(\gamma_{\xi,\{0,\eta\}}) &= \gamma_{x,\{0\}} \gamma_{y,\{0,x\}} \gamma_{y,\{x\}}^{-1}, & \psi(\gamma_{\xi,\{0,\eta,1\}}) &= \gamma_{x,\{y,1\}}^{-1} \gamma_{y,\{1\}}^{-1}, \\ \psi(\gamma_{\eta,\{\xi,0\}}) &= \gamma_{y,\{0,x\}}, & \psi(\gamma_{\eta,\{0\}}) &= \gamma_{y,\{x\}}, & \psi(\gamma_{\eta,\{1\}}) &= \gamma_{y,\{1\}}, \end{aligned}$$

and

$$\begin{aligned} \phi(\gamma_{x,\{0\}}) &= \gamma_{\xi,\{0\}}, & \phi(\gamma_{x,\{y\}}) &= \gamma_{\eta,\{0\}}, & \phi(\gamma_{x,\{y,1\}}) &= \gamma_{\xi,\{0,\eta,1\}}^{-1} \gamma_{\eta,\{1\}}^{-1}, \\ \phi(\gamma_{y,\{0,x\}}) &= \gamma_{\eta,\{\xi,0\}}, & \phi(\gamma_{y,\{x\}}) &= \gamma_{\eta,\{0\}}, & \phi(\gamma_{y,\{1\}}) &= \gamma_{\eta,\{1\}}. \end{aligned}$$

Here the paths $\gamma_{\xi,\{0\}}$, $\gamma_{\xi,\{0,\eta\}}$ and $\gamma_{\xi,\{0,\eta,1\}}$ are defined as in Figure 5, the paths $\gamma_{\eta,\{\xi,0\}}$, $\gamma_{\eta,\{0\}}$ and $\gamma_{\eta,\{1\}}$ are defined as in Figure 6.

FIGURE 5. Paths for the moves of ξ .

FIGURE 6. Paths for the moves of η .

Note that

$$\gamma_{\xi,\{0,\eta\}} \gamma_{\xi,\{0\}}^{-1} = \gamma_{\eta,\{0\}} \gamma_{\eta,\{\xi,0\}}, \quad \gamma_{\eta,\{\xi,0\}} \gamma_{\eta,\{0\}}^{-1} = \gamma_{\xi,\{0\}} \gamma_{\xi,\{0,\eta\}}.$$

Therefore, we obtain the isomorphism (2.2). $\square$

3. Monodromy representations in reducible cases

In this section we realize the monodromy representation in the cases when at least one of the exponents λ_j is an integer. These cases are divided into four classes: (1) one and only one of the exponents is an integer; (2) two and only two of the exponents are integers; (3) three and only three of the exponents are integers; and (4) all the exponents are integers.

For our convenience, we introduce the notation $(+i_1, -i_2, *i_3)$, which means that i_1 of the λ_j are non-negative integers, i_2 of the λ_j are negative integers, and i_3 of the λ_j are not integers; any term with 0 such as $0i_j$ is not written; for example, $(+1, *4)$ stands for $(+1, -0, *4)$. Furthermore, we introduce another set of notation such as $(+, *, *, *, *)$ to mean $\lambda_0 \in \mathbb{Z}_{\geq 0}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_\infty \notin \mathbb{Z}$ and $(-, *, *, *, +)$ to mean $\lambda_0 \in \mathbb{Z}_{< 0}$, $\lambda_\infty \in \mathbb{Z}_{\geq 0}$, and $\lambda_1, \lambda_2, \lambda_3 \notin \mathbb{Z}$. This notation is used to assign the case of a representation.

This section is devoted to realizing the representations of the type

(1)	$(+1, *4),$	$(-1, *4),$		
(2)	$(+2, *3),$	$(+1, -1, *3),$	$(-2, *3),$	
(3)	$(+3, *2),$	$(+2, -1, *2),$	$(+1, -2, *2),$	$(-3, 2*),$
(4)	$(+4, -1),$	$(+3, -2),$	$(+2, -3),$	$(+1, -4).$

3.1. The class where just one of the λ_i is an integer

The reducible representations where just one of the λ_i are integers fall into two types, $(+1, *4)$ and $(-1, *4)$.

We compute the monodromy representation of the case $(+, *, *, *, *)$ as a representative of type $(+1, *4)$ and the case $(-, *, *, *, *)$ as a representative of type $(-1, *4)$.

PROPOSITION 3.1. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_0 \in \mathbb{Z}_{\geq 0}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_\infty \notin \mathbb{Z}$, then the matrices of the representation are*

$$\begin{aligned}
 M_{x,\{0\}} &= \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_{x,\{y,1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ \mu_1(1-\mu_{23}) & \mu_{123} & 0 \\ \mu_{12}(1-\mu_3) & \mu_{12}(\mu_3-1) & \mu_1 \end{pmatrix}, \\
 M_{x,\{y\}} &= \begin{pmatrix} 1 & 0 & 0 \\ \mu_1(1-\mu_2) & \mu_{12} & 1-\mu_1 \\ 0 & 0 & 1 \end{pmatrix}, \\
 M_{y,\{1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mu_2(1-\mu_3) & \mu_{23} \end{pmatrix}, & M_{y,\{0,x\}} &= \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_{12} & 1-\mu_1 \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

in the basis $(e(\lambda_{12})\text{reg}(0, x), e(\lambda_2)\text{reg}(x, y), \text{reg}(y, 1))$.

Proof. Proposition 1.2 under the condition $\lambda_0 \in \mathbb{Z}_{\geq 0}$ implies the result. $\square$

The expression of the matrices shows that the space generated by $\text{reg}(x, y)$ and $\text{reg}(y, 1)$ is an invariant subspace.

PROPOSITION 3.2. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \notin \mathbb{Z}$, then the matrices of the representation are*

$$\begin{aligned} M_{x,\{0\}} &= \begin{pmatrix} \mu_{01} & 1 - \mu_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_{x,\{y,1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ \mu_1(1 - \mu_{23}) & \mu_{123} & 0 \\ \mu_{12}(1 - \mu_3) & \mu_{12}(\mu_3 - 1) & \mu_1 \end{pmatrix}, \\ M_{x,\{y\}} &= \begin{pmatrix} 1 & 0 & 0 \\ \mu_1(1 - \mu_2) & \mu_{12} & 1 - \mu_1 \\ 0 & 0 & 1 \end{pmatrix}, \\ M_{y,\{1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mu_2(1 - \mu_3) & \mu_{23} \end{pmatrix}, & M_{y,\{0,x\}} &= \begin{pmatrix} \mu_2 & \mu_2(\mu_0 - 1) & 1 - \mu_0 \\ 0 & \mu_{012} & 1 - \mu_{01} \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

in the basis $(e(\lambda_{12})\text{reg}(0, x), e(\lambda_2)\text{reg}(x, y), \text{reg}(y, 1))$. The invariant subspace is one-dimensional and generated by

$$(1 - \mu_0)e(\lambda_{12})\text{reg}(0, x) + (1 - \mu_{01})e(\lambda_2)\text{reg}(x, y) + (1 - \mu_{012})\text{reg}(y, 1),$$

which equals to $e(-\lambda_3)S(\infty - \epsilon)$.

Proof. The expression of the matrix in Proposition 1.2 is valid even in the case $\lambda_\infty \in \mathbb{Z}_{<0}$ and the linear independence of the basis follows from (1) of Lemma 1.4. This completes the proof. $\square$

3.2. The class where two of the λ_i are integers

We consider the cases $(+, +, *, *, *)$, $(+, *, *, *, -)$ and $(-, -, *, *, *)$ as representatives of the types $(+2, *3)$, $(+1, -1, *3)$ and $(-2, *3)$.

PROPOSITION 3.3. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_0, \lambda_1 \in \mathbb{Z}_{\geq 0}$ and $\lambda_2, \lambda_3, \lambda_\infty \notin \mathbb{Z}$, the matrices of the representation are*

$$\begin{aligned} M_{x,\{0\}} &= \text{Id}_3, & M_{x,\{y,1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 - \mu_{23} & \mu_{23} & 0 \\ \mu_2(1 - \mu_3) & \mu_2(\mu_3 - 1) & 1 \end{pmatrix}, \\ M_{x,\{y\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 - \mu_2 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ M_{y,\{1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mu_2(1 - \mu_3) & \mu_{23} \end{pmatrix}, & M_{y,\{0,x\}} &= \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

in the basis $(e(\lambda_{12})\text{reg}(0, x), e(\lambda_2)\text{reg}(x, y), \text{reg}(y, 1))$.

Proof. Proposition 1.2 under the condition $\lambda_0, \lambda_1 \in \mathbb{Z}_{\geq 0}$ implies the result. $\square$

PROPOSITION 3.4. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_\infty \in \mathbb{Z}_{<0}$, $\lambda_0 \in \mathbb{Z}_{\geq 0}$ and $\lambda_1, \lambda_2, \lambda_3 \notin \mathbb{Z}$, then the matrices of the representation are*

$$M_{x,\{0\}} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 - 1 & 1 & 0 \\ \mu_{12} - 1 & 1 - \mu_{12} & \mu_1 \end{pmatrix},$$

$$M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1(1 - \mu_2) & \mu_{12} & 1 - \mu_1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mu_2(1 - \mu_3) & \mu_{23} \end{pmatrix}, \quad M_{y,\{0,x\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_{12} & 1 - \mu_1 \\ 0 & 0 & 1 \end{pmatrix},$$

in the basis $(e(\lambda_{12})\text{reg}(0, x), e(\lambda_2)\text{reg}(x, y), \text{reg}(y, 1))$. The invariant subspace is generated by $\text{reg}(x, y)$ and $\text{reg}(y, 1)$.

Proof. The expression of the matrix and linear independence of the basis in Proposition 1.2 is valid even in the case when $\lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_0 \in \mathbb{Z}_{\geq 0}$, which implies the result. Note that $\mu_{123} = 1$ since $\mu_{0\infty} = 1$. $\square$

PROPOSITION 3.5. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_0, \lambda_1 \in \mathbb{Z}_{<0}$ and $\lambda_2, \lambda_3, \lambda_\infty \notin \mathbb{Z}$, the matrices of the representation are*

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_{23} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_{23} \end{pmatrix}, \quad M_{y,\{0,x\}} = \begin{pmatrix} \mu_2 & 0 & -1 \\ 0 & \mu_2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $(e(\lambda_{12})S(0 + \epsilon), e(\lambda_2)S(x + \epsilon), \text{reg}(y, 1))$.

Proof. We have

$$\gamma_{x,\{y\}}(\text{reg}(y, 1)) = \left\{ \begin{array}{c} \circ \\ 0 \end{array} \begin{array}{c} \circ \\ x \end{array} \begin{array}{c} \circ \\ y \end{array} \begin{array}{c} \circ \\ 1 \end{array} \right\}$$

$$= \text{reg}(y, 1) - e(\lambda_2)S(x + \epsilon),$$

$$\gamma_{y,\{0,x\}}(\text{reg}(y, 1)) = \left\{ \begin{array}{c} \circ \\ 0 \end{array} \begin{array}{c} \circ \\ x \end{array} \begin{array}{c} \circ \\ y \end{array} \begin{array}{c} \circ \\ 1 \end{array} \right\}$$

$$= \text{reg}(y, 1) - e(\lambda_2)S(x + \epsilon) - e(\lambda_{12})S(0 + \epsilon).$$

The other actions are trivially derived. Note that $\mu_{23} \neq 1$ from $\lambda_{23\infty} \in \mathbb{Z}$ with $\lambda_\infty \notin \mathbb{Z}$. The non-triviality of $S(0 + \epsilon)$ and $S(x + \epsilon)$ follows from Lemma 1.6 and the non-triviality of

$\text{reg}(y, 1)$ follows from (3) of Lemma 1.5. Linear independence of the basis is seen by applying the following Lemma 3.6. $\square$

LEMMA 3.6. *Let f_1 and f_2 be linear transformations acting on the space spanned by three non-trivial vectors v_1, v_2 , and v_3 . If*

$$f_1(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f_2(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

with $\lambda \neq 1$ and $\mu \neq 1$, then v_1, v_2 , and v_3 are linearly independent.

Proof. Let $w := c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ be a linear relation. Then $(f_1 - 1)(w) = c_2(\lambda - 1)v_2 = 0$ implies $c_2 = 0$, and $(f_2 - 1)(w) = c_3(\mu - 1)v_3 = 0$ implies $c_3 = 0$. Hence, $c_1 = 0$. This completes the proof. $\square$

3.3. The class where three of the λ_i are integers

We study the cases $(+3, *2)$, $(+2, -1, *2)$, $(+1, -2, *2)$ and $(-3, *2)$ in this ordering. The cases $(+, +, *, *, -)$ and $(-, -, +, *, *)$ are considered in Propositions 3.8 and 3.9 as representatives of the types $(+2, -1, *2)$ and $(+1, -2, *2)$, respectively. On the other hand, all possible cases are considered for the types $(+3, *2)$ and $(-3, *2)$ in Propositions 3.7 and 3.10, because the representations in these cases might be finite.

PROPOSITION 3.7. *For a fixed base point such that $0 < x < y < 1$, assign*

$$K_0 = e(\lambda_{012}) \text{reg}(\infty, 0), \quad K_1 = e(\lambda_{12}) \text{reg}(0, x), \quad K_2 = e(\lambda_2) \text{reg}(x, y), \\ K_3 = \text{reg}(y, 1), \quad K_4 = e(-\lambda_3) \text{reg}(1, \infty).$$

Then the matrices of the representation are expressed as follows.

(1) $(+, +, +, *, *, *)$: *If $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 0}$ and $\lambda_3, \lambda_\infty \notin \mathbb{Z}$, then*

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ M_{y,\{0,x\}} = \text{Id}_3, \quad M_{y,\{x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

in the basis $(K_1 + K_2 + K_3, K_2 + K_3, K_3)$.

(2) $(+, +, *, +, *)$: *If $\lambda_0, \lambda_1, \lambda_3 \in \mathbb{Z}_{\geq 0}$ and $\lambda_2, \lambda_\infty \notin \mathbb{Z}$, then*

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ M_{y,\{0,x\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}$$

in the basis $(K_1 + K_2, K_2, K_3)$.

(3) $(+, +, *, *, +)$: If $\lambda_0, \lambda_1, \lambda_\infty \in \mathbb{Z}_{\geq 0}$ and $\lambda_2, \lambda_3 \notin \mathbb{Z}$, then

$$\begin{aligned} M_{x,\{0\}} &= \text{Id}_3, & M_{x,\{y\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}, & M_{x,\{y,1\}} &= \text{Id}_3, \\ M_{y,\{0,x\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}, & M_{y,\{x\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}, & M_{y,\{1\}} &= \text{Id}_3 \end{aligned}$$

in the basis $(K_0 + K_1 + K_2, K_1 + K_2, K_2)$.

(4) $(+, *, +, +, *)$: If $\lambda_0, \lambda_2, \lambda_3 \in \mathbb{Z}_{\geq 0}$ and $\lambda_1, \lambda_\infty \notin \mathbb{Z}$, then

$$\begin{aligned} M_{x,\{0\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, & M_{x,\{y\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_{x,\{y,1\}} &= \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ M_{y,\{0,x\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_{y,\{x\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_{y,\{1\}} &= \text{Id}_3 \end{aligned}$$

in the basis $(K_2 + K_3, K_2, K_1)$.

(5) $(+, *, +, *, +)$: If $\lambda_0, \lambda_2, \lambda_\infty \in \mathbb{Z}_{\geq 0}$ and $\lambda_1, \lambda_3 \notin \mathbb{Z}$, then

$$\begin{aligned} M_{x,\{0\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & M_{x,\{y\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, & M_{x,\{y,1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, \\ M_{y,\{0,x\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, & M_{y,\{x\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, & M_{y,\{1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1^{-1} \end{pmatrix} \end{aligned}$$

in the basis $(K_0 + K_1, K_1, K_2)$.

(6) $(+, *, *, +, +)$: If $\lambda_0, \lambda_3, \lambda_\infty \in \mathbb{Z}_{\geq 0}$ and $\lambda_1, \lambda_2 \notin \mathbb{Z}$, then

$$\begin{aligned} M_{x,\{0\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, & M_{x,\{y\}} &= \text{Id}_3, & M_{x,\{y,1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ M_{y,\{0,x\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}, & M_{y,\{x\}} &= \text{Id}_3, & M_{y,\{1\}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

in the basis $(K_3 + K_4, K_3, K_1)$.

(7) $(*, +, +, +, *)$: If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_{\geq 0}$ and $\lambda_0, \lambda_\infty \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_0 \end{pmatrix}, \quad M_{x,\{y\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \text{Id}_3,$$

$$M_{y,\{0,x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \text{Id}_3$$

in the basis $(K_1 + K_2 + K_3, K_1 + K_2, K_1)$.

(8) $(*, +, +, *, +)$: If $\lambda_1, \lambda_2, \lambda_\infty \in \mathbb{Z}_{\geq 0}$ and $\lambda_0, \lambda_3 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_0 \end{pmatrix}, \quad M_{x,\{y\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_0^{-1} \end{pmatrix},$$

$$M_{y,\{0,x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_0^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $(K_0, K_1 + K_2, K_1)$.

(9) $(*, +, *, +, +)$: If $\lambda_1, \lambda_3, \lambda_\infty \in \mathbb{Z}_{\geq 0}$ and $\lambda_0, \lambda_2 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2^{-1} \end{pmatrix}, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}, \quad M_{y,\{x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $(K_3 + K_4, K_3, K_2)$.

(10) $(*, *, +, +, +)$: If $\lambda_2, \lambda_3, \lambda_\infty \in \mathbb{Z}_{\geq 0}$ and $\lambda_0, \lambda_1 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \text{Id}_3, \quad M_{y,\{x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, \quad M_{y,\{1\}} = \text{Id}_3$$

in the basis $(K_2 + K_3 + K_4, K_2 + K_3, K_2)$.

Proof. The linear independence of each of the integrals is guaranteed by Lemma 1.4. The monodromy representations of (1) and (2) follow from Proposition 1.2.

(3) The action of $\gamma_{x,\{y,1\}}$ on $\text{reg}(x, y)$ is described as

$$\begin{aligned}
 \gamma_{x,\{y,1\}}(\text{reg}(x, y)) &= \left\{ \begin{array}{c} \circ \\ 0 \end{array} \quad \begin{array}{c} \circ \\ x \end{array} \quad \begin{array}{c} \circ \\ y \end{array} \quad \begin{array}{c} \circ \\ 1 \end{array} \right\} \\
 &= \left\{ \begin{array}{c} \circ \\ 0 \end{array} \quad \begin{array}{c} \circ \\ x \end{array} \quad \begin{array}{c} \circ \\ y \end{array} \quad \begin{array}{c} \circ \\ 1 \end{array} \right\} \\
 &+ \left\{ \begin{array}{c} \circ \\ 0 \end{array} \quad \begin{array}{c} \circ \\ x \end{array} \quad \begin{array}{c} \circ \\ y \end{array} \quad \begin{array}{c} \circ \\ 1 \end{array} \right\} = \text{reg}(x, y).
 \end{aligned}$$

The action of $\gamma_{y,\{1\}}$ on $\text{reg}(x, y)$ is, similarly,

$$\begin{aligned}
 \gamma_{y,\{1\}}(\text{reg}(x, y)) &= \left\{ \begin{array}{c} 0 \\ \circ \end{array} \quad \begin{array}{c} x \\ \circ \end{array} \quad \begin{array}{c} y \\ \circ \end{array} \quad \begin{array}{c} 1 \\ \circ \end{array} \right\} \\
 &= \left\{ \begin{array}{c} 0 \\ \circ \end{array} \quad \begin{array}{c} x \\ \circ \end{array} \quad \begin{array}{c} y \\ \circ \end{array} \quad \begin{array}{c} 1 \\ \circ \end{array} \right\} = \text{reg}(x, y).
 \end{aligned}$$

The other relations are easily derived.

(4) is obtained from (2) by changing the role of $(0, x, y, 1)$ with that of $(1, y, x, 0)$.

(5) The action of $\gamma_{x,\{y,1\}}$ on $\text{reg}(0, x)$ and $\text{reg}(x, y)$ is

$$\begin{aligned}
 \gamma_{x,\{y,1\}}(\text{reg}(0, x)) &= \left\{ \begin{array}{c} 0 \\ \circ \end{array} \quad \begin{array}{c} x \\ \circ \end{array} \quad \begin{array}{c} y \\ \circ \end{array} \quad \begin{array}{c} 1 \\ \circ \end{array} \right\} = \text{reg}(0, x), \\
 \gamma_{x,\{y,1\}}(\text{reg}(x, y)) &= \left\{ \begin{array}{c} \circ \\ 0 \end{array} \quad \begin{array}{c} \circ \\ x \end{array} \quad \begin{array}{c} \circ \\ y \end{array} \quad \begin{array}{c} \circ \\ 1 \end{array} \right\} \mu_1
 \end{aligned}$$

$$= \left\{ \begin{array}{c} \circ \\ 0 \end{array} \quad \begin{array}{c} \circ \\ x \end{array} \quad \begin{array}{c} \circ \\ y \end{array} \quad \begin{array}{c} \circ \\ 1 \end{array} \right\} \mu_1 = \text{reg}(x, y) \mu_1.$$

Similarly, we have

$$\gamma_{y,\{1\}}(\text{reg}(x, y)) = \left\{ \begin{array}{c} 0 \\ \circ \end{array} \quad \begin{array}{c} x \\ \circ \end{array} \quad \begin{array}{c} y \\ \circ \end{array} \quad \begin{array}{c} 1 \\ \circ \end{array} \right\} \mu_3$$

$$= \left\{ \begin{array}{c} \circ \\ 0 \end{array} \quad \begin{array}{c} \circ \\ x \end{array} \quad \begin{array}{c} \circ \\ y \end{array} \quad \begin{array}{c} \circ \\ 1 \end{array} \right\} \mu_3 = \text{reg}(x, y) \mu_1^{-1}.$$

(6) The action of $\gamma_{x,\{y\}}$ on $\text{reg}(0, x)$ or $\text{reg}(y, 1)$ is

$$\gamma_{x,\{y\}}(\text{reg}(0, x)) = \left\{ \begin{array}{c} 0 \\ \circ \end{array} \quad \begin{array}{c} x \\ \circ \end{array} \quad \begin{array}{c} y \\ \circ \end{array} \right\} = \text{reg}(0, x),$$

$$\gamma_{x,\{y\}}(\text{reg}(y, 1)) = \left\{ \begin{array}{c} x \\ \circ \end{array} \quad \begin{array}{c} y \\ \circ \end{array} \quad \begin{array}{c} 1 \\ \circ \end{array} \right\} = \text{reg}(y, 1).$$

Similarly, the action of $\gamma_{y,\{0,x\}}$ on $\text{reg}(y, 1)$ is

$$\gamma_{y,\{0,x\}}(\text{reg}(y, 1)) = \left\{ \begin{array}{c} 0 \\ \circ \end{array} \quad \begin{array}{c} x \\ \circ \end{array} \quad \begin{array}{c} y \\ \circ \end{array} \quad \begin{array}{c} 1 \\ \circ \end{array} \right\} = \text{reg}(y, 1).$$

The other relations are easily derived.

(7) is obtained from (1) by changing the role of $(0, x, y, 1)$ with that of $(1, y, x, 0)$.

(8) The action of $\gamma_{x,\{y,1\}}$ on $\text{reg}(0, x)$ and $\text{reg}(x, y)$ is

$$\begin{aligned}
 & \gamma_{x,\{y,1\}}(\text{reg}(0, x)) \\
 &= \left\{ \begin{array}{c} \text{Diagram 1: A horizontal line with points 0, x, y, 1. An arrow points up at 0. A loop connects x and y, passing through 1.} \end{array} \right\} \\
 &= \left\{ \begin{array}{c} \text{Diagram 2: A horizontal line with points 0, x, y, 1. An arrow points up at 0. A large circle encloses the points x, y, 1, with an arrow pointing up at x.} \end{array} \right\} \mu_3 \\
 &= \text{reg}(0, x) \mu_0^{-1}, \\
 & \gamma_{y,\{1\}}(\text{reg}(0, x) + e(-\lambda_1)\text{reg}(x, y)) \\
 &= \gamma_{y,\{1\}}(\text{reg}(0, x) + e(\lambda_1)\text{reg}(x, y)) \\
 &= \left\{ \begin{array}{c} \text{Diagram 3: A horizontal line with points 0, x, y, 1. An arrow points up at 0. A loop connects 0 and x, passing through y and 1.} \end{array} \right\} = \left\{ \begin{array}{c} \text{Diagram 4: A horizontal line with points 0, x, y, 1. An arrow points up at 0. A large circle encloses the points 0, x, y, 1, with an arrow pointing up at 0.} \end{array} \right\} \mu_3 \\
 &= (\text{reg}(0, x) + e(-\lambda_1)\text{reg}(x, y))\mu_0^{-1}.
 \end{aligned}$$

The other relations are similarly derived.

(9) and (10) are obtained from (5) and (3), respectively, by changing the role of $(0, x, y, 1)$ with that of $(1, y, x, 0)$. This completes the proof. $\square$

Remark. In each case, the basis above is chosen in order to obtain a simple expression of the representation matrices and in order to avoid using a trivial integral. For instance, in (1), we have $K_4 = 0$ for $\lambda_{3\infty} \in \mathbb{Z}_{\leq -2}$; in (3), $K_3 = 0$ for $\lambda_{23} \in \mathbb{Z}_{\leq -2}$; in (6), $K_2 = 0$ for $\lambda_{12} \in \mathbb{Z}_{\leq -2}$; in (7), $K_0 = 0$ for $\lambda_{\infty 0} \in \mathbb{Z}_{\leq -2}$; and in (10), $K_1 = 0$ for $\lambda_{01} \in \mathbb{Z}_{\leq -2}$. We refer the reader to Lemma 1.5.

PROPOSITION 3.8. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_\infty \in \mathbb{Z}_{<0}$, $\lambda_0, \lambda_1 \in \mathbb{Z}_{\geq 0}$ and $\lambda_2, \lambda_3 \notin \mathbb{Z}$, then the matrices of the representation are*

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu_2 - 1 & 1 - \mu_2 & 1 \end{pmatrix},$$

$$M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 - \mu_2 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mu_2 - 1 & 1 \end{pmatrix}, \quad M_{y,\{0,x\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in the basis $(e(\lambda_{12})\text{reg}(0, x), e(\lambda_2)\text{reg}(x, y), \text{reg}(y, 1))$. The space generated by $\text{reg}(x, y)$ and $\text{reg}(y, 1)$ and the space generated only by $\text{reg}(y, 1)$ are respectively invariant.

Proof. Proposition 1.2 and Lemma 1.4 imply the result. Note that $\mu_{23} = 1$ from $\mu_{\infty 01} = 1$. $\square$

PROPOSITION 3.9. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_0, \lambda_1 \in \mathbb{Z}_{<0}$, $\lambda_2 \in \mathbb{Z}_{\geq 0}$ and $\lambda_3, \lambda_{\infty} \notin \mathbb{Z}$, then the matrices of the representation are*

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad M_{y,\{0,x\}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $(e(\lambda_{12})S(0 + \epsilon), e(\lambda_2)S(x + \epsilon), \text{reg}(y, 1))$.

Proof. The argument in the proof of Proposition 3.5 is valid even in the case when $\lambda_2 \in \mathbb{Z}_{\geq 0}$, which implies the result. $\square$

PROPOSITION 3.10. *For a fixed base point such that $0 < x < y < 1$, the matrices of the representation are expressed as follows.*

(1) $(-, -, -, *, *)$: If $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}_{<0}$ and $\lambda_3, \lambda_{\infty} \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \text{Id}_3, \quad M_{y,\{x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

in the basis $(S(0 + \epsilon), S(x + \epsilon), S(y + \epsilon))$.

(2) $(-, -, *, -, *)$: If $\lambda_0, \lambda_1, \lambda_3 \in \mathbb{Z}_{<0}$ and $\lambda_2, \lambda_{\infty} \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}$$

in the basis $(S(0 + \epsilon), S(x + \epsilon), S(1 - \epsilon))$.

(3) $(-, -, *, *, -)$: If $\lambda_0, \lambda_1, \lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_2, \lambda_3 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \text{Id}_3,$$

$$M_{y,\{0,x\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{1\}} = \text{Id}_3$$

in the basis $(S(0 + \epsilon), S(x + \epsilon), S(\infty - \epsilon))$.

(4) $(-, *, -, -, *)$: If $\lambda_0, \lambda_2, \lambda_3 \in \mathbb{Z}_{<0}$ and $\lambda_1, \lambda_\infty \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{1\}} = \text{Id}_3$$

in the basis $(S(0 + \epsilon), S(y + \epsilon), S(1 - \epsilon))$.

(5) $(-, *, -, *, -)$: If $\lambda_0, \lambda_2, \lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_1, \lambda_3 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $(S(0 + \epsilon), S(y + \epsilon), S(\infty - \epsilon))$.

(6) $(-, *, *, -, -)$: If $\lambda_0, \lambda_3, \lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_1, \lambda_2 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $(S(0 + \epsilon), S(1 - \epsilon), S(\infty - \epsilon))$.

(7) $(*, -, -, -, *)$: If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_{<0}$ and $\lambda_0, \lambda_\infty \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \text{Id}_3,$$

$$M_{y,\{0,x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \text{Id}_3$$

in the basis $(S(x + \epsilon), S(y + \epsilon), S(1 - \epsilon))$.

(8) $(*, -, -, *, -)$: If $\lambda_1, \lambda_2, \lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_0, \lambda_3 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \begin{pmatrix} \mu_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $(S(x + \epsilon), S(y + \epsilon), S(\infty - \epsilon))$.

(9) $(*, -, *, -, -)$: If $\lambda_1, \lambda_3, \lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_0, \lambda_2 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $(S(x + \epsilon), S(1 - \epsilon), S(\infty - \epsilon))$.

(10) $(*, *, -, -, -)$: If $\lambda_2, \lambda_3, \lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_0, \lambda_1 \notin \mathbb{Z}$, then

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y\}} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \text{Id}_3, \quad M_{y,\{x\}} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{1\}} = \text{Id}_3$$

in the basis $(S(y + \epsilon), S(1 - \epsilon), S(\infty - \epsilon))$.

Proof. In each case, the matrix elements are easily derived. Linear independence of the basis follows from Lemma 1.6 together with an application of Lemma 3.6. $\square$

3.4. The class where all of the λ_i are integers

The types when all of the λ_i are integers fall into four cases, $(+4, -1)$, $(+3, -2)$, $(+2, -3)$, and $(+1, -4)$, because $\lambda_{0123\infty} = -2$. Since all the exponents are integers, the integrals are single-valued.

We consider the cases $(+, +, +, +, -)$, $(-, +, +, +, -)$, $(+, +, -, -, -)$, and $(-, -, -, -, +)$ as representatives of the types $(+4, -1)$, $(+3, -2)$, $(+2, -3)$, and $(+1, -4)$.

PROPOSITION 3.11. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_{\geq 0}$, then all the matrices of the representation are the identity in the basis $(\text{reg}(0, x), \text{reg}(x, y), \text{reg}(y, 1))$.*

Proof. Proposition 1.2 implies the result. $\square$

PROPOSITION 3.12. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_0, \lambda_\infty \in \mathbb{Z}_{<0}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_{\geq 0}$, then the matrices of the monodromy representation are*

$$M_{x,\{0\}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y\}} = \text{Id}_3, \quad M_{x,\{y,1\}} = \text{Id}_3,$$

$$M_{y,\{0,x\}} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{y,\{x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \text{Id}_3$$

in the basis $(e(\lambda_{12})S(0 + \epsilon), e(\lambda_2)\text{reg}(x, y), \text{reg}(y, 1))$.

Proof. The matrices are easily derived. Non-triviality of the elements of the basis follows from Lemmas 1.5 and 1.6. Linear independence of the basis is shown as follows. Let $w := c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ be a linear relation, where

$$v_1 = e(\lambda_{12}) S(0 + \epsilon), \quad v_2 = e(\lambda_2) \text{reg}(x, y), \quad v_3 = \text{reg}(y, 1).$$

Then $(M_{x,\{0\}} - 1)(w) = c_2(-v_2) = 0$ and $(M_{y,\{0,x\}} - 1)(w) = (c_2 - c_3)v_1 = 0$ imply $c_2 = c_3 = 0$. Hence, $c_1 = 0$. This completes the proof. $\square$

PROPOSITION 3.13. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_0, \lambda_1 \in \mathbb{Z}_{\geq 0}$ and $\lambda_2, \lambda_3, \lambda_\infty \in \mathbb{Z}_{<0}$, then the matrices of the monodromy representation are*

$$M_{x,\{0\}} = \text{Id}_3, \quad M_{x,\{y\}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{x,\{y,1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$M_{y,\{0,x\}} = \text{Id}_3, \quad M_{y,\{1\}} = \text{Id}_3$$

in the basis $(\text{reg}(0, x), e(\lambda_1)S(y - \epsilon), e(\lambda_{12})S(1 - \epsilon))$.

Proof. The matrices are easily derived. Non-triviality of the elements of the basis follows from Lemmas 1.5 and 1.6. Linear independence of the basis follows from the following. Let $w := c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ be a linear relation, where

$$v_1 = \text{reg}(0, x), \quad v_2 = e(\lambda_1)S(y - \epsilon), \quad v_3 = e(\lambda_{12})S(1 - \epsilon).$$

Then $(M_{x,\{y\}} - 1)(w) = c_1 v_2 = 0$, which implies $c_1 = 0$. On the other hand, by the expressions (3.2) and (3.3), $c_2 v_2 + c_3 v_3 = 0$ implies $c_2 = c_3 = 0$. This completes the proof. $\square$

PROPOSITION 3.14. *For a fixed base point such that $0 < x < y < 1$, if $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_{<0}$ and $\lambda_\infty \in \mathbb{Z}_{\geq 0}$, then all the matrices of the representation are identity in the basis $(S(x - \epsilon), S(y - \epsilon), S(1 - \epsilon))$.*

Proof. It is enough only to prove the linear independence of $S(x - \epsilon)$, $S(y - \epsilon)$, and $S(1 - \epsilon)$. First, the change of the variable $t \rightarrow xt$ implies that

$$\begin{aligned} \int_{S(x-\epsilon)} U_{x-\epsilon}(t) dt &= x^{\lambda_{01}+1} y^{\lambda_2} \int_{|1-t|=\epsilon} t^{\lambda_0} (1-t)^{\lambda_1} \left(1 - \frac{xt}{y}\right)^{\lambda_2} (1-xt)^{\lambda_3} dt \\ &= x^{\lambda_{01}+1} y^{\lambda_2} \int_{|t-1|=\epsilon} t^{\lambda_0} (1-t)^{\lambda_1} dt (1 + O(r)) \quad (r \rightarrow 0) \end{aligned} \quad (3.1)$$

where $r = \max\{|x|, |x/y|\}$. Second, the change of the variable $t \rightarrow yt$ implies that

$$\begin{aligned} \int_{S(y-\epsilon)} U_{y-\epsilon}(t) dt &= y^{\lambda_{012}+1} \int_{|t-1|=\epsilon} t^{\lambda_{01}} \left(1 - \frac{x}{yt}\right)^{\lambda_1} (1-t)^{\lambda_2} (1-yt)^{\lambda_3} dt \\ &= y^{\lambda_{012}+1} \int_{|t-1|=\epsilon} t^{\lambda_{01}} (1-t)^{\lambda_2} dt (1 + O(r)) \quad (r \rightarrow 0) \end{aligned} \quad (3.2)$$

where $r = \max\{|y|, |x/y|\}$. Third,

$$\begin{aligned} \int_{S(1-\epsilon)} U_{1-\epsilon}(t) dt &= \int_{|t-1|=\epsilon} t^{\lambda_{012}} \left(1 - \frac{x}{t}\right)^{\lambda_1} \left(1 - \frac{y}{t}\right)^{\lambda_2} (1-t)^{\lambda_3} dt \\ &= \int_{|t-1|=\epsilon} t^{\lambda_{012}} (1-t)^{\lambda_3} dt (1 + O(r)) \quad (r \rightarrow 0) \end{aligned} \quad (3.3)$$

where $r = \max\{|x|, |y|\}$.

Finally, note that

$$\begin{aligned} \int_{|t|=\epsilon} t^\alpha (1-t)^\beta dt &= \sum_{j \geq 0} \frac{(-\beta)_j}{j!} \int_{|t|=\epsilon} t^{\alpha+j} dt \\ &= 2\pi \sqrt{-1} \frac{(-\beta)_{-1-\alpha}}{(-1-\alpha)!} \end{aligned}$$

for $\alpha \in \mathbb{Z}_{<0}$. Hence the linear independence follows. This completes the proof. $\square$

4. Finite reducible monodromy groups

In the previous section, we determined the monodromy representations for each possible types of parameters. We determine here when the monodromy group is finite or infinite by assuming that the group is reducible. A summary of the results below is given in the next theorem.

THEOREM 4.1. *Assume that the monodromy group is reducible. Then it is finite only when (1) $(\pm 3, *2)$ and when (2) $(+4, -1)$ and $(+1, -4)$. In case (1), the group is isomorphic to $\mathbb{Z}_k^{\times 2}$ or $\mathbb{Z}_k^{\times 3}$ for an integer k , and in case (2), the group is trivial.*

We remark here that the irreducible finite monodromy groups for Lauricella's system E_D are classified in Theorem 2 of [S]. Such a classification is given in [CW] by an arithmetic study of the system; however, the Hesse group of order 648 is missing in the list.

PROPOSITION 4.2. *When one and only one of the parameters λ_i is an integer, the projective monodromy group is of infinite order. In particular, the monodromy group itself is also of infinite order.*

Proof. We write the element $g_1 g_2 g_1^{-1} g_2^{-1}$ by $[g_1, g_2]$.

In the case $(+1, *4)$, we have

$$[M_{x,\{0\}}, M_{x,\{y\}}] = \begin{pmatrix} 1 & 0 & 0 \\ (1 - \mu_1)(1 - \mu_2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $M_{x,\{0\}}$ and $M_{x,\{y\}}$ in Proposition 3.1, which implies the infinity, since $\mu_1 \neq 1$ and $\mu_2 \neq 1$.

In the case $(-1, *4)$, for $M_{x,\{0\}}$, $M_{x,\{y\}}$, and $M_{x,\{y,1\}}$ in Proposition 3.2, define $M_{x,\{\infty\}}$ to be $M_{x,\{0\}}^{-1} M_{x,\{y,1\}}^{-1}$, $N_{x,\{\infty\}}$ to be $P^{-1} M_{x,\{\infty\}} P$, and $N_{x,\{y\}}$ to be $P^{-1} M_{x,\{y\}} P$, where

$$P = \begin{pmatrix} 1 - \mu_0 & 0 & 0 \\ 1 - \mu_{01} & 1 & 0 \\ 1 - \mu_{012} & 0 & 1 \end{pmatrix}.$$

Then we have

$$[N_{x,\{\infty\}}, N_{x,\{y\}}] = \begin{pmatrix} 1 & (1 - \mu_{12})/\mu_{12} & -(1 - \mu_1)/\mu_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which implies the infinity, since $\mu_1 \neq 1$.

In the case $(+2, *3)$, we have

$$[M_{x,\{y\}}, M_{y,\{1\}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(1 - \mu_2)(1 - \mu_3) & (1 - \mu_2)(1 - \mu_3) & 1 \end{pmatrix}$$

for $M_{x,\{y\}}$ and $M_{y,\{1\}}$ in Proposition 3.3, which implies the infinity.

In the case $(+1, -1, *3)$,

$$[M_{x,\{0\}}, M_{x,\{y\}}] = \begin{pmatrix} 1 & 0 & 0 \\ (1 - \mu_1)(1 - \mu_2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $M_{x,\{0\}}$ and $M_{x,\{y\}}$ in Proposition 3.4 shows that the monodromy group is of infinite order.

In the case $(-2, *3)$,

$$[M_{x,\{y\}}, M_{y,\{1\}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & (1 - \mu_{23})/\mu_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

for $M_{x,\{y\}}$ and $M_{y,\{1\}}$ in Proposition 3.5 shows that the monodromy group is of infinite order, since $\mu_{23} \neq 1$.

In the case $(+2, -1, *2)$, $M_{y,\{1\}}$ in Proposition 3.8 for the case $(+, +, *, *, -)$ is of infinite order because $\mu_2 \neq 1$. Hence the monodromy group is of infinite order.

Similarly, $M_{x,\{y\}}$ in Proposition 3.9, $M_{x,\{0\}}$ in Proposition 3.12, and $M_{x,\{y\}}$ in Proposition 3.13 show that the monodromy group is of infinite order in the case $(+1, -2, *2)$, in the case $(+3, -2)$, and in the case $(+2, -3)$, respectively. $\square$

Summarizing the results in Propositions 3.11 and 3.14, we obtain the following.

PROPOSITION 4.3.

- (1) *The monodromy group for the type $(+3, *2)$ is isomorphic to either $G(\mu) \times G(\mu) \times G(\mu)$ or $G(\mu) \times G(\mu)$. The former occurs for the cases $(+, +, *, +, *)$ and $(+, *, +, +, *)$, where $\mu = \mu_2$ and $\mu = \mu_1$, respectively. The latter occurs for the remaining cases, where $\mu = \mu_3, \mu_2, \mu_1, \mu_1, \mu_0, \mu_0, \mu_2$, and μ_1 according to the cases $(+, +, +, *, *)$, $(+, +, *, *, +)$, $(+, *, +, *, +)$, $(+, *, *, +, +)$, $(*, +, +, +, *)$, $(*, +, +, *, +)$, $(*, +, *, +, +)$, and $(*, *, +, +, +)$, respectively.*
- (2) *The monodromy group is of finite order when μ above is a rational number.*

PROPOSITION 4.4.

- (1) *The monodromy group for the type $(-3, *2)$ is isomorphic to either $G(\mu) \times G(\mu) \times G(\mu)$ or $G(\mu) \times G(\mu)$. The former occurs for the cases $(-, -, *, -, *)$ and $(-, *, -, -, *)$, where $\mu = \mu_2$ and $\mu = \mu_1$, respectively. The latter occurs for the remaining cases, where $\mu = \mu_3, \mu_2, \mu_1, \mu_1, \mu_0, \mu_0, \mu_2$, and μ_1 according to the cases $(-, -, -, *, *)$, $(-, -, *, *, -)$, $(-, *, -, *, -)$, $(-, *, *, -, -)$, $(*, -, -, -, *)$, $(*, -, -, *, -)$, $(*, -, *, -, -)$, and $(*, *, -, -, -)$, respectively.*
- (2) *The monodromy group is of finite order when μ above is a rational number.*

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