ON THE MULTIPLICITY OF REES ALGEBRAS OF GOOD FILTRATIONS

Duong Quoc VIET and Le Van DINH

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Abstract. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A . Denote by $R(\mathcal{F}) = \bigoplus_{n \geq 0} I_n t^n$ the Rees algebra of $\mathcal{F}$. In this paper we investigate the mixed multiplicities of $\mathcal{F}$ and the relationship between the multiplicity of $R(\mathcal{F})$ and the mixed multiplicities of $\mathcal{F}$. As an application, we obtain several multiplicity formulas that express the multiplicity of $R(\mathcal{F})$ as a sum of Hilbert–Samuel multiplicities of some quotient rings of A .

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring of Krull dimension $d = \dim A > 0$ with infinite residue field $k = A/\mathfrak{m}$. Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A . Define

$$R(\mathcal{F}) = \bigoplus_{n \geq 0} I_n t^n$$

to be the *Rees algebra* of $\mathcal{F}$. In the case when $\mathcal{F} = \{I^n\}_{n \geq 0}$ is an I -adic filtration, $R(\mathcal{F})$ is called the *Rees algebra* of I and denoted by $R(I)$.

The multiplicity of the Rees algebra of an ideal has been studied in many works (see e.g. [3, 4, 11–13]. Verma in [11] expressed the Hilbert–Samuel multiplicity of $R(I)$ as the sum of all the mixed multiplicities of I . Viet in [12] described the multiplicity of $R(I)$ as a sum of Hilbert–Samuel multiplicities of some quotient rings of A .

The aim of this paper is to give generalizations of the above-mentioned results in [12] to the case of the Rees algebra of a good filtration. We first define mixed multiplicities of a good filtration (see Section 2). Next we interpret each mixed multiplicity of a good filtration as the Hilbert–Samuel multiplicity of a quotient ring of A (see Theorem 3.7). Then we express the multiplicity of $R(\mathcal{F})$ as the sum of all the mixed multiplicities of $\mathcal{F}$ (see Proposition 4.2). As a consequence of Theorem 3.7 and Proposition 4.2, we obtain an explicit formula for the Hilbert–Samuel multiplicity of the Rees algebra of a good filtration that expresses the multiplicity of $R(\mathcal{F})$ as a sum of Hilbert–Samuel multiplicities of some quotient rings of A (see Theorem 4.6). In the case when $\mathcal{F}$ is an equimultiple filtration, we obtain a multiplicity formula for $R(\mathcal{F})$ (see Corollary 4.7) which is a generalization of the multiplicity formula for the Rees algebra of an ideal in [13, Theorem 3.2]. Our main results not only cover several

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earlier results, but also yield interesting consequences (see, e.g., Remark 2.3, Lemma 3.3, Remark 3.8, Lemma 4.1, Remark 4.4, Corollary 4.5).

This paper is divided into four sections. Section 2 is devoted to the discussion of mixed multiplicities of a good filtration. In Section 3 we investigate the relationship between mixed multiplicities of a good filtration and Hilbert–Samuel multiplicities via (FC)-sequences (see Definition 3.1). Finally, in Section 4 we establish the multiplicity formula for $R(\mathcal{F})$ which expresses the multiplicity of $R(\mathcal{F})$ as a sum of Hilbert–Samuel multiplicities of some quotient rings of A .

2. Mixed multiplicities of a good filtration

In this section, we define mixed multiplicities of a good filtration. We first recall some notions that will be used in this paper.

A filtration $\mathcal{F} = \{I_n\}_{n \geq 0}$ of ideals in A is a chain of ideals I_n such that $I_0 = A$, I_1 is a proper ideal of A , $I_{n+1} \subseteq I_n$, and $I_m I_n \subseteq I_{m+n}$ for all $m, n \geq 0$. Let I be an ideal of A . $\mathcal{F}$ is called an I -good filtration if $I_n \subseteq I_{n+1}$ for all $n \geq 0$ and $I_{n+1} = II_n$ for all large n . In this case, $I \subseteq I_1$. $\mathcal{F}$ is called a good filtration if it is an I -good filtration for some ideal I of A . It is easily seen that $\mathcal{F}$ is a good filtration if and only if $\mathcal{F}$ is an I_1 -good filtration. There are numerous examples of good filtrations; for instance, for an ideal I of A , $\{\tilde{I}^n\}_{n \geq 0}$ is a good filtration, where $\tilde{I}^n$ is the Ratliff–Rush closure of I^n (see [6, Theorem 2.1]). Moreover, if A is analytically unramified containing a field, then $\{\overline{I^n}\}_{n \geq 0}$ and $\{(I^n)^*\}_{n \geq 0}$ are good filtrations; here $\overline{I^n}$ and $(I^n)^*$ denote the integer closure and tight closure of I^n , respectively (see [2, 7]).

From now on, let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A . We call $\mathcal{F}$ a nilpotent filtration if $I_n = 0$ for all large n . Observe that $\mathcal{F}$ is a nilpotent filtration if and only if I_1 is nilpotent. Let J be an $\mathfrak{m}$ -primary ideal of A . Denote by $F_J(\mathcal{F}) = \bigoplus_{n \geq 0} (I_n/JI_n)t^n$ the fiber cone of $\mathcal{F}$ with respect to J . If $\mathcal{F} = \{I^n\}_{n \geq 0}$ is an I -adic filtration, we denote $F_J(\mathcal{F})$ by $F_J(I)$ and call it the fiber cone of I with respect to J . The number $\ell(I) = \dim F_{\mathfrak{m}}(I)$ is called the analytic spread of I . Since J is an $\mathfrak{m}$ -primary ideal of A , in $R(\mathcal{F})$ we have $\sqrt{JR(\mathcal{F})} = \sqrt{\mathfrak{m}R(\mathcal{F})}$. Hence $\dim F_J(\mathcal{F}) = \dim F_{\mathfrak{m}}(\mathcal{F}) = \ell(I_1)$. It is well known that $\text{ht } I \leq \ell(I) \leq \dim A$, where $\text{ht } I$ is the height of I [5]. In the case when $\text{ht } I = \ell(I)$, I is called an equimultiple ideal. If I_1 is equimultiple, we call $\mathcal{F}$ an equimultiple filtration. Set $\mathcal{F}/I = \{I_n(A/I)\}_{n \geq 0}$ and

$$F_J(\mathcal{F}/I) = \bigoplus_{n \geq 0} [(I_n + I)/(JI_n + I)]t^n$$

for any ideal I of A .

Note 1. Put $\ell = \ell(I_1)$. Then $\ell_A(I_n/JI_n)$ is a polynomial of degree $\ell - 1$ for all large n by [16, Remark 2.1(vii)].

We now turn to the definition of mixed multiplicities of a good filtration. Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A and let J be an $\mathfrak{m}$ -primary ideal of A . Consider the following bigraded algebras:

$$T(J, \mathcal{F}) = \bigoplus_{m, n \geq 0} \frac{J^m I_n}{J^{m+1} I_n} t_1^m t_2^n \quad \text{and} \quad T(J, I_1) = \bigoplus_{m, n \geq 0} \frac{J^m I_1^n}{J^{m+1} I_1^n} t_1^m t_2^n,$$

where t_1 and t_2 are indeterminates. It is easily seen that $T(J, I_1)$ is a standard bigraded algebra. Set $T_j = \bigoplus_{m \geq 0, n \geq j} (J^m I_n / J^{m+1} I_n) t_1^m t_2^n$ for $j \geq 0$. Then T_j has a natural bigraded $T(J, I_1)$ -module structure given by

$$(a + J^{m+1} I_1^n)(x + J^{k+1} I_l) = ax + J^{m+k+1} I_{n+l}$$

for $a \in J^m I_1^n$ and $x \in J^k I_l$. Since $\mathcal{F}$ is a good filtration, we have

$$T_j = \bigoplus_{m \geq 0, n \geq j} \frac{J^m I_n}{J^{m+1} I_n} t_1^m t_2^n = T(J, I_1)(I_j / J I_j) t_2^j$$

for all large j . Hence T_j is a finitely generated $T(J, I_1)$ -module. Consequently, the *Bhattacharya function* $B(m, n) = \ell_A(J^m I_n / J^{m+1} I_n)$ is a polynomial for all large m, n . Denote by $P(m, n)$ the polynomial of the function $B(m, n)$. Then $P(m, n)$ is called the *Bhattacharya polynomial* of $B(m, n)$ [1].

Set $\mathfrak{a} : \mathfrak{b}^\infty = \bigcup_{n \geq 0} (\mathfrak{a} : \mathfrak{b}^n)$. We need the following lemma.

LEMMA 2.1. *Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Set $A^* = A/0 : I_1^\infty$ and $q = \dim A^*$. Then:*

- (i) $(0 : I_1^\infty) \cap I_n = 0$ for all large n ;
- (ii) $\text{grade } I_1 A^* \geq 1$ and $q \geq 1$.

Proof. Since $\mathcal{F}$ is a good filtration, there exists a non-negative integer u such that $I_n = I_1^{n-u} I_u$ for all large n . By the Artin–Rees lemma, there exists a non-negative integer v such that for all $n - u \geq v$,

$$(0 : I_1^\infty) \cap I_n \subseteq (0 : I_1^\infty) \cap I_1^{n-u} = ((0 : I_1^\infty) \cap I_1^v) I_1^{n-u-v} \subseteq (0 : I_1^\infty) I_1^{n-u-v}.$$

Hence, since $(0 : I_1^\infty) I_1^{n-u-v} = 0$ for n large enough, it follows that $(0 : I_1^\infty) \cap I_n = 0$ for all large n . We get (i).

To proof (ii), set $I_1^* = I_1 A^*$. Note that I_1 is non-nilpotent, $A^* \neq 0$. Since $(0 : I_1^\infty) : I_1 = 0 : I_1^\infty$, then $0_{A^*} : I_1^* = 0_{A^*}$. This implies that $\text{grade } I_1^* \geq 1$, and hence $q = \dim A^* \geq 1$. $\square$

PROPOSITION 2.2. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Set $q = \dim A/0 : I_1^\infty$. Then the Bhattacharya function $\ell_A(J^m I_n / J^{m+1} I_n)$ is a polynomial of degree $q - 1$ for all large m, n .*

Proof. Set $\mathcal{E} = \{J^n I_n\}_{n \geq 0}$. Since $\mathcal{F}$ is a good filtration, $\mathcal{E}$ is also a good filtration. Let $P(m, n)$ be the Bhattacharya polynomial of the function $\ell_A(J^m I_n / J^{m+1} I_n)$. By Note 1, $\deg P(n, n) = \dim F_J(\mathcal{E}) - 1 = \ell(JI_1) - 1$. On the other hand, $\deg P(m, n) = \deg P(n, n)$, since the highest coefficients of $P(m, n)$ are non-negative numbers, not all zero. Hence $\deg P(m, n) = \ell(JI_1) - 1$. Denote by $Q(m, n)$ the polynomial of the function $\ell_A(J^m I_1^n / J^{m+1} I_1^n)$. By [12, Proposition 3.1], $\deg Q(m, n) = q - 1$. By Note 1, $\deg Q(n, n) = \ell(JI_1) - 1$. It is easily seen that $\deg Q(m, n) = \deg Q(n, n)$. Hence we get $\deg P(m, n) = q - 1$. $\square$

Remark 2.3. From the proof of Proposition 2.2, it follows that if J is an $\mathfrak{m}$ -primary ideal and I is an arbitrary ideal then $\ell(JI) = \dim A/0 : I^\infty$. Note that in the case $\text{ht } I > 0$, $\ell(JI) = \dim A$. This fact was shown by Viet [15, Theorem 5.2(ii)].

Write the Bhattacharya polynomial of $B(m, n) = \ell_A(J^m I_n / J^{m+1} I_n)$ in the form

$$P(m, n) = \sum_{d_0+d_1=q-1} e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \binom{m+d_0}{d_0} \binom{n+d_1}{d_1} \\ + \sum_{i+j < q-1} e_A(J^{[i+1]}, \mathcal{F}^{[j]}) \binom{m+i}{i} \binom{n+j}{j};$$

then $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]})$ are non-negative integers not all zero. We call $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]})$ the *mixed multiplicity* of the good filtration $\mathcal{F}$ with respect to J of the type $(d_0 + 1, d_1)$.

3. Computing mixed multiplicities of a filtration

The relationship between mixed multiplicities of ideals and Hilbert–Samuel multiplicity was studied first for $\mathfrak{m}$ -primary ideals by Risler and Teissier (1973) [9], Rees (1984) [8] and then for arbitrary ideals by Viet (2000) [12]. In this section we develop the method of Viet in [12] to interpret mixed multiplicities of a good filtration as the Hilbert–Samuel multiplicity. We recall here the concept of (FC)-sequences of good filtrations that were introduced in [16]. This is a kind of variant of superficial sequences.

Definition 3.1. Let I be an ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . An element $x \in I_1$ is called an *(FC)-element* with respect to $(I, \mathcal{F})$ if the following conditions are satisfied:

- (FC1) $I^m I_n \cap (x) = x I^m I_{n-1}$ for all $m \geq 0$ and all large n ;
- (FC2) x is a filter-regular element with respect to I_1 , i.e., $0 : x \subseteq 0 : I_1^\infty$;
- (FC3) $\dim[A/(x) : I_1^\infty] = \dim[A/0 : I_1^\infty] - 1$.

We call x a *weak-(FC)-element* with respect to $(I, \mathcal{F})$ if x satisfies conditions (FC1) and (FC2).

Let $x_1, \dots, x_s$ be a sequence in I_1 . For each $i = 0, 1, \dots, s-1$, set $A_i = A/(x_1, \dots, x_i)$, $\bar{x}_{i+1}$ is the image of x_{i+1} in A_i , and $\mathcal{F}_i = \{I_n A_i\}_{n \geq 0}$. Then $x_1, \dots, x_s$ is called an *(FC)-sequence* (respectively, a *weak-(FC)-sequence*) with respect to $(I, \mathcal{F})$ if $\bar{x}_{i+1}$ is an (FC)-element (respectively, a weak-(FC)-element) with respect to $(I A_i, \mathcal{F}_i)$ for each $i = 0, 1, \dots, s-1$.

A weak-(FC)-sequence $x_1, \dots, x_s$ in I_1 with respect to $(I, \mathcal{F})$ is called a *maximal weak-(FC)-sequence* if $\mathcal{F}_{s-1}$ is a non-nilpotent filtration of A_{s-1} and $\mathcal{F}_s$ is a nilpotent filtration of A_s .

The existence of weak-(FC)-sequences of good filtrations was shown in [16].

LEMMA 3.2. [16, Proposition 2.3] *Let I be an ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Then there exists a weak-(FC)-element in I_1 with respect to $(I, \mathcal{F})$.*

Basic facts about weak-(FC)-sequences of good filtrations were pointed out in [16]. In the following we also give some properties of these sequences.

LEMMA 3.3. *Let I be an ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Suppose that x is a weak-(FC)-element in I_1 with respect to $(I, \mathcal{F})$, and that*

$\mathfrak{b} \subseteq \mathfrak{a}$ are ideals in A . Then we have the following isomorphisms of A -modules for all large n :

$$\mathfrak{a}I_n/\mathfrak{b}I_n \simeq x\mathfrak{a}I_n/x\mathfrak{b}I_n.$$

Proof. It is enough to show that $\mathfrak{a}I_n \simeq x\mathfrak{a}I_n$ for all large n . It will follow that

$$\mathfrak{a}I_n/\mathfrak{b}I_n \simeq x\mathfrak{a}I_n/x\mathfrak{b}I_n$$

for all large n . For each n , consider the map

$$\begin{aligned} \lambda_n : \mathfrak{a}I_n &\longrightarrow x\mathfrak{a}I_n \\ y &\longmapsto xy. \end{aligned}$$

Then λ_n is surjective and $\text{Ker } \lambda_n = \mathfrak{a}I_n \cap (0 : x)$. Since x is a weak-(FC)-element with respect to $(I, \mathcal{F})$, it follows from Lemma 2.1(i) that

$$\text{Ker } \lambda_n = \mathfrak{a}I_n \cap (0 : x) \subseteq I_n \cap (0 : I_1^\infty) = 0$$

for all large n . Hence λ_n are isomorphisms; that is, $\mathfrak{a}I_n \simeq x\mathfrak{a}I_n$ for all large n . $\square$

The length of maximal weak-(FC)-sequences of ideals was studied by Viet in [13] and [14]. In the case of good filtrations, we recall the following result from [16].

LEMMA 3.4. [16, Proposition 2.4(iii)] *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A . Then the length of maximal weak-(FC)-sequences in I_1 with respect to $(J, \mathcal{F})$ is $\ell(I_1)$.*

Note 2. Let $x \in I_1$ be a weak-(FC)-element with respect to $(J, \mathcal{F})$. Set $\bar{A} = A/(x)$, $\bar{J} = J\bar{A}$, and $\bar{I}_n = I_n\bar{A}$ for all $n \geq 0$. Then $\mathcal{F}/(x) = \{\bar{I}_n\}_{n \geq 0}$ and for all large n , we have

$$\begin{aligned} \ell_{\bar{A}}\left(\frac{\bar{I}_n}{\bar{J}\bar{I}_n}\right) &= \ell_A\left(\frac{I_n + (x)}{JI_n + (x)}\right) \\ &= \ell_A\left(\frac{I_n}{JI_n + (x) \cap I_n}\right) \\ &= \ell_A\left(\frac{I_n}{JI_n}\right) - \ell_A\left(\frac{JI_n + (x) \cap I_n}{JI_n}\right) \\ &= \ell_A\left(\frac{I_n}{JI_n}\right) - \ell_A\left(\frac{(x) \cap I_n}{(x) \cap JI_n}\right) \\ &= \ell_A\left(\frac{I_n}{JI_n}\right) - \ell_A\left(\frac{xI_{n-1}}{xJI_{n-1}}\right) \quad (\text{since } x \text{ satisfies condition (FC1)}) \\ &= \ell_A\left(\frac{I_n}{JI_n}\right) - \ell_A\left(\frac{I_{n-1}}{JI_{n-1}}\right) \quad (\text{by Lemma 3.3}). \end{aligned}$$

Hence

$$\ell_{\bar{A}}\left(\frac{\bar{I}_n}{\bar{J}\bar{I}_n}\right) = \ell_A\left(\frac{I_n}{JI_n}\right) - \ell_A\left(\frac{I_{n-1}}{JI_{n-1}}\right).$$

LEMMA 3.5. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Set $A^* = A/0 : I_1^\infty$ and $q = \dim A^*$. Then:*

- (i) $e(J, I_n) = e(J, A^*)$ for all large n ;
- (ii) $e_A(J^{[q]}, \mathcal{F}^{[0]}) = e(J, A^*)$. Moreover, if $\text{ht } I_1 > 0$ then $e_A(J^{[q]}, \mathcal{F}^{[0]}) = e(J, A)$.

Proof. We have $(0 : I_1^\infty) \cap I_n = 0$ for all large n by Lemma 2.1(i). It follows that the map

$$\begin{aligned} I_n &\longrightarrow A/0 : I_1^\infty \\ x &\longmapsto x + 0 : I_1^\infty \end{aligned}$$

is an injection for n large enough. Consequently, we get the following exact sequences

$$0 \longrightarrow I_n \longrightarrow A^* \longrightarrow A^*/I_n A^* \longrightarrow 0$$

for all large n . Note that $\text{grade } I_1 A^* \geq 1$ by Lemma 2.1(ii). Therefore, $\text{ht}(I_n A^*) = \text{ht}(I_1 A^*) > 0$ for all $n \geq 1$ since $I_1^n \subseteq I_n \subseteq I_1$. This implies that $\dim(A^*/I_n A^*) < \dim A^*$ for all $n \geq 1$. Now the above exact sequences give $e(J, I_n) = e(J, A^*)$ for all large n . We obtain (i). Let us prove (ii). It is easily seen that

$$\begin{aligned} e_A(J^{[q]}, \mathcal{F}^{[0]}) &= \lim_{m \rightarrow \infty} \frac{(q-1)!}{m^{q-1}} \ell_A(J^m I_n / J^{m+1} I_n) \quad \text{for all large } n \\ &= e(J, I_n) \quad \text{for all large } n. \end{aligned} \tag{1}$$

It now follows from (i) that $e_A(J^{[q]}, \mathcal{F}^{[0]}) = e(J, A^*)$. If $\text{ht } I_1 > 0$, then since $\text{ht } I_n = \text{ht } I_1 > 0$, it follows that $\dim(A/I_n) < \dim A$ for all $n \geq 1$. Hence the exact sequence

$$0 \longrightarrow I_n \longrightarrow A \longrightarrow A/I_n \longrightarrow 0$$

yields $e(J, I_n) = e(J, A)$, which together with (1) implies that $e_A(J^{[q]}, \mathcal{F}^{[0]}) = e(J, A)$. $\square$

PROPOSITION 3.6. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Set $A^* = A/0 : I_1^\infty$, $q = \dim A^*$, and $\ell = \ell(I_1)$. Let d_0, d_1 be non-negative integers such that $d_0 + d_1 = q - 1$. Then the following statements hold.*

- (i) *If $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0$ and $d_1 > 0$, then there exists an (FC)-element $x \in I_1$ with respect to $(J, \mathcal{F})$.*
- (ii) *If $x \in I_1$ is an (FC)-element with respect to $(J, \mathcal{F})$ and $d_1 > 0$, then*

$$e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e_{\bar{A}}(\bar{J}^{[d_0+1]}, \bar{\mathcal{F}}^{[d_1-1]}),$$

where $\bar{A} = A/(x)$, $\bar{J} = J\bar{A}$, and $\bar{\mathcal{F}} = \{I_n \bar{A}\}_{n \geq 0}$.

- (iii) *If $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0$ and $x_1, \dots, x_{d_1} \in I_1$ is a weak-(FC)-sequence with respect to $(J, \mathcal{F})$, then $x_1, \dots, x_{d_1}$ is an (FC)-sequence with respect to $(J, \mathcal{F})$.*
- (iv) *If $x_1, \dots, x_s \in I_1$ is a weak-(FC)-sequence with respect to $(J, \mathcal{F})$, then*

$$\dim A/[(x_1, \dots, x_s) : I_1^\infty] \leq q - s,$$

with equality if and only if $x_1, \dots, x_s$ is an (FC)-sequence with respect to $(J, \mathcal{F})$.

- (v) *Let $x_1, \dots, x_\ell$ be a maximal weak-(FC)-sequence in I_1 with respect to $(J, \mathcal{F})$. If $\text{ht } I_1 = h > 0$, then $x_1, \dots, x_{h-1}$ is an (FC)-sequence with respect to $(J, \mathcal{F})$.*

Proof. (i) By Lemma 3.2, there exists a weak-(FC)-element $x \in I_1$ with respect to $(J, \mathcal{F})$. Set $\bar{A} = A/(x)$, $\bar{J} = J\bar{A}$, and $\bar{I}_n = I_n\bar{A}$ for all $n \geq 0$. By Note 2, we have

$$\ell_{\bar{A}}\left(\frac{\bar{J}^m \bar{I}_n}{\bar{J}^{m+1} \bar{I}_n}\right) = \ell_A\left(\frac{J^m I_n}{J^{m+1} I_n}\right) - \ell_A\left(\frac{J^m I_{n-1}}{J^{m+1} I_{n-1}}\right) \quad (2)$$

for all large m, n . Since $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0$ and $d_1 > 0$, we obtain from (2) that $\ell_{\bar{A}}(\bar{J}^m \bar{I}_n / \bar{J}^{m+1} \bar{I}_n)$ is a polynomial of degree $q - 2$ for all large m, n . On the other hand, by Proposition 2.2, the degree of this polynomial is $\dim A/[(x) : I_1^\infty] - 1$. Therefore,

$$\dim[A/(x) : I_1^\infty] = q - 1 = \dim[A/0 : I_1^\infty] - 1.$$

Hence x is an (FC)-element with respect to $(J, \mathcal{F})$.

(ii) Denote by $P(m, n)$ and $\bar{P}(m, n)$ the Bhattacharya polynomials of $\mathcal{F}$ and $\bar{\mathcal{F}}$, respectively. By (2), we have

$$\bar{P}(m, n) = P(m, n) - P(m, n - 1) \quad (3)$$

for all large m, n . By Proposition 2.2, $P(m, n)$ has degree $q - 1$, while the degree of $\bar{P}(m, n)$ is $\dim \bar{A}/\bar{0} : \bar{I}_1^\infty - 1$. Since x is an (FC)-element with respect to $(J, \mathcal{F})$, we have

$$\dim \frac{\bar{A}}{\bar{0} : \bar{I}_1^\infty} = \dim \frac{A}{(x) : I_1^\infty} = \dim \frac{A}{0 : I_1^\infty} - 1 = q - 1.$$

Therefore, $\deg \bar{P}(m, n) = \deg P(m, n) - 1$. It now follows from (3) that

$$e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e_{\bar{A}}(\bar{J}^{[d_0+1]}, \bar{\mathcal{F}}^{[d_1-1]}).$$

(iii) We prove by induction on d_1 . When $d_1 = 1$, the result was proved in (i). Assume that the result holds for $d_1 - 1 \geq 1$. Then from the proof of (i) it follows that x_1 is an (FC)-element with respect to $(J, \mathcal{F})$. Hence by (ii), we have

$$e_{\bar{A}}(\bar{J}^{[d_0+1]}, \bar{\mathcal{F}}^{[d_1-1]}) = e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0,$$

where $\bar{A} = A/(x_1)$, $\bar{J} = J\bar{A}$, and $\bar{\mathcal{F}} = \{I_n \bar{A}\}_{n \geq 0}$. Denote by $\bar{x}_2, \dots, \bar{x}_{d_1}$ the images of $x_2, \dots, x_{d_1}$ in $\bar{A}$, respectively. Then $\bar{x}_2, \dots, \bar{x}_{d_1}$ is a weak-(FC)-sequence with respect to $(\bar{J}, \bar{\mathcal{F}})$. By the induction hypothesis, $\bar{x}_2, \dots, \bar{x}_{d_1}$ is an (FC)-sequence with respect to $(\bar{J}, \bar{\mathcal{F}})$. So $x_1, \dots, x_{d_1}$ is an (FC)-sequence with respect to $(J, \mathcal{F})$. We get (iii).

(iv) Assume that $x \in I_1$ is a weak-(FC)-element with respect to $(J, \mathcal{F})$. Set $\bar{A} = A/(x)$, $\bar{J} = J\bar{A}$, and $\bar{\mathcal{F}} = \{I_n \bar{A}\}_{n \geq 0}$. Since $\deg P(m, n) = q - 1$ and $\deg \bar{P}(m, n) = \dim A/[(x) : I_1^\infty] - 1$, by (3) we have $\dim A/[(x) : I_1^\infty] \leq q - 1$. By definition, the equality holds in the above inequality if and only if x is an (FC)-element with respect to $(J, \mathcal{F})$. The claim now follows by induction.

(v) By (iv), we have $\dim A/[(x_1, \dots, x_{h-1}) : I_1^\infty] \leq q - (h - 1)$. Set

$$\bar{A} = A/(x_1, \dots, x_{h-1}) \quad \text{and} \quad \bar{I}_1 = I_1 \bar{A}.$$

Since $\text{ht } I_1 = h$, it is easily seen that $\text{ht } \bar{I}_1 > 0$. This implies that $\bar{0} : \bar{I}_1^\infty$ is nilpotent in $\bar{A}$. Thus, $\dim \bar{A} = \dim \bar{A}/\bar{0} : \bar{I}_1^\infty = \dim A/[(x_1, \dots, x_{h-1}) : I_1^\infty] \leq q - (h - 1)$. But it is clear that $\dim \bar{A} = \dim A/(x_1, \dots, x_{h-1}) \geq \dim A - (h - 1) \geq q - (h - 1)$. Hence $\dim A/[(x_1, \dots, x_{h-1}) : I_1^\infty] = q - (h - 1)$. According to (iv), $x_1, \dots, x_{h-1}$ is an (FC)-sequence with respect to $(J, \mathcal{F})$. $\square$

Combining Lemma 3.5 and Proposition 3.6 we get a generalization of [12, Theorem 4.1] which is the main result of this section as follows.

THEOREM 3.7. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Set $q = \dim A/\bar{0} : I_1^\infty$, $h = \text{ht } I_1$, and $\ell = \ell(I_1)$. Let d_0, d_1 be non-negative integers such that $d_0 + d_1 = q - 1$. Then the following statements hold.*

- (i) $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0$ if and only if there exists an (FC)-sequence $x_1, \dots, x_{d_1}$ in I_1 with respect to $(J, \mathcal{F})$. In this case,

$$e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e(J, A/[(x_1, \dots, x_{d_1}) : I_1^\infty]).$$

- (ii) If $h > 0$, $d_1 < h$, and $x_1, \dots, x_{d_1} \in I_1$ is an (FC)-sequence with respect to $(J, \mathcal{F})$, then

$$e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e(J, A/(x_1, \dots, x_{d_1})).$$

- (iii) The length of maximal (FC)-sequences in I_1 with respect to $(J, \mathcal{F})$ is an invariant and this invariant is equal to

$$p = \max\{d_1 \mid e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0\}.$$

Moreover, $h - 1 \leq p \leq \ell - 1$.

Proof. We prove (i). If $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0$ then by induction on d_1 , using Proposition 3.6(i) and (ii), we can easily show the existence of an (FC)-sequence $x_1, \dots, x_{d_1}$ in I_1 with respect to $(J, \mathcal{F})$. Conversely, assume that $x_1, \dots, x_{d_1}$ is an (FC)-sequence in I_1 with respect to $(J, \mathcal{F})$. Set $\bar{A} = A/(x_1, \dots, x_{d_1})$, $\bar{J} = J\bar{A}$ and $\bar{\mathcal{F}} = \{I_n \bar{A}\}_{n \geq 0}$. Then by Proposition 3.6(ii) we get $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e_{\bar{A}}(\bar{J}^{[d_0+1]}, \bar{\mathcal{F}}^{[0]})$. Now by Lemma 3.5(ii),

$$e_{\bar{A}}(\bar{J}^{[d_0+1]}, \bar{\mathcal{F}}^{[0]}) = e(J, A/[(x_1, \dots, x_{d_1}) : I_1^\infty]).$$

Hence

$$e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e(J, A/[(x_1, \dots, x_{d_1}) : I_1^\infty]).$$

We now prove (ii). Set $\bar{A} = A/(x_1, \dots, x_{d_1})$, $\bar{J} = J\bar{A}$, and $\bar{\mathcal{F}} = \{I_n \bar{A}\}_{n \geq 0}$. By Proposition 3.6(ii) we easily get

$$e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e_{\bar{A}}(\bar{J}^{[d_0+1]}, \bar{\mathcal{F}}^{[0]}).$$

Since $\text{ht } I_1 = h > d_1$, $\text{ht}(I_1 \bar{A}) > 0$. Lemma 3.5(ii) now yields

$$e_{\bar{A}}(\bar{J}^{[d_0+1]}, \bar{\mathcal{F}}^{[0]}) = e(\bar{J}, \bar{A}) = e(J, \bar{A}).$$

Hence $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e(J, \bar{A})$.

We now prove (iii). (i) immediately gives the first assertion of (iii) that the length of maximal (FC)-sequences in I_1 with respect to $(J, \mathcal{F})$ is an invariant and this invariant is equal to $p = \max\{d_1 \mid e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0\}$. By Proposition 3.6(v), $h - 1 \leq p$. It remains to show that $p \leq \ell - 1$. First, note that $p \leq d_0 + d_1 = q - 1$. Now assume that $x_1, \dots, x_p$ is a maximal (FC)-sequence in I_1 with respect to $(J, \mathcal{F})$. Set $\bar{A} = A/(x_1, \dots, x_p)$ and $\bar{I}_1 = I_1 \bar{A}$. Then by Proposition 3.6(iv), we have

$$\dim \bar{A}/\bar{0} : \bar{I}_1^\infty = \dim A/[(x_1, \dots, x_p) : I_1^\infty] = q - p \geq 1.$$

From this it follows that $\bar{I}_1$ is non-nilpotent in $\bar{A}$. So by Lemma 3.2, there exists $x \in I_1$ such that $x_1, \dots, x_p, x$ is a weak-(FC)-sequence with respect to $(J, \mathcal{F})$. Since the length of maximal weak-(FC)-sequences in I_1 with respect to $(J, \mathcal{F})$ is ℓ (by Lemma 3.4), we get $p + 1 \leq \ell$, or $p \leq \ell - 1$. $\square$

Remark 3.8. If we call $\max\{d_1 \mid e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0\}$ the *index of positive definition* of mixed multiplicities of $(J, \mathcal{F})$, then Theorem 3.7(iii) shows that this index is equal to the length of maximal (FC)-sequences in I_1 with respect to $(J, \mathcal{F})$, i.e., $p = \max\{d_1 \mid e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0\}$. In the case of ideals, this fact was mentioned by Viet [12] in 2000, and Trung [10] in 2001 constructed an example where $p < \ell - 1$.

Recall that if $\mathcal{F}$ is an equimultiple filtration then $\text{ht } I_1 = \ell(I_1)$. Hence as an immediate consequence of Theorem 3.7 we obtain the following interesting corollary.

COROLLARY 3.9. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Suppose that $\mathcal{F}$ is an equimultiple filtration with $\text{ht } I_1 = \ell(I_1) = h > 0$. Then the following statements hold.*

- (i) *The length of maximal (FC)-sequences in I_1 with respect to $(J, \mathcal{F})$ is $(h - 1)$.*
- (ii) *$e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \neq 0$ if and only if $d_1 \leq h - 1$.*
- (iii) *If $x_1, \dots, x_{h-1}$ is an (FC)-sequence in I_1 with respect to $(J, \mathcal{F})$, then*

$$e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e(J, A/(x_1, \dots, x_{d_1}))$$

for all $d_1 = 0, 1, \dots, h - 1$.

In particular, if I_1 is an $\mathfrak{m}$ -primary ideal, then we get the following.

COROLLARY 3.10. *Let J, I_1 be $\mathfrak{m}$ -primary ideals of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A . Then there exists an (FC)-sequence $x_1, \dots, x_{d-1}$ in I_1 with respect to $(J, \mathcal{F})$ and we have $e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) = e(J, A/(x_1, \dots, x_{d_1}))$ for all $d_1 = 0, 1, \dots, d - 1$.*

Proof. Since I_1 is an $\mathfrak{m}$ -primary ideal, $\text{ht } I_1 = \ell(I_1) = d$, i.e., $\mathcal{F}$ is an equimultiple filtration. Hence by Corollary 3.9, we obtain the proof. $\square$

4. Some multiplicity formulas for Rees algebras

In this section, we first extend the multiplicity formula for the Rees algebra of an ideal obtained by Verma [11] to the Rees algebra of a good filtration. Together with the results just established in Section 3, we obtain some multiplicity formulas for Rees algebras of good filtrations.

Let S be a finitely-generated graded algebra over a Noetherian local ring A with the homogeneous maximal ideal $\mathfrak{M}$, and let M be a finitely generated graded S -module. We denote by $e(M)$ the Hilbert–Samuel multiplicity $e(M_{\mathfrak{M}})$ of the module $M_{\mathfrak{M}}$ over the local ring $S_{\mathfrak{M}}$, and by S_+ the set of elements of positive degrees of S . If I is a homogeneous $\mathfrak{M}$ -primary ideal of S , we shall use the notation $e(I, M)$ instead of $e(IS_{\mathfrak{M}}, M_{\mathfrak{M}})$.

Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Denote by $R(\mathcal{F})$ the Rees algebra of $\mathcal{F}$ and $R_j = \bigoplus_{k \geq j} I_k t^k$ for any $j \geq 0$.

LEMMA 4.1. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Set $K = J \oplus (\bigoplus_{k \geq 1} I_k t^k)$ and $q = \dim A/0 : I_1^\infty$. Then $\dim R_j = q + 1$ and*

$$e(K, R_j) = \sum_{d_0+d_1=q-1} e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]})$$

for all large j .

Proof. Note that $R_v \subseteq R_u$ and $J^m R_u + R_v = J^m I_u t^u \oplus \cdots \oplus J^m I_{v-1} t^{v-1} \oplus R_v$ for all $v > u$. Since $\mathcal{F}$ is a good filtration, $I_u t^u R_j = R_{u+j}$ and $R_u R_j = R_{u+j}$ for all $u \geq 0$ and all large j . Hence we can prove by induction on m that

$$K^m R_j = J^m I_j t^j \oplus J^{m-1} I_{j+1} t^{j+1} \oplus \cdots \oplus J I_{j+m-1} t^{j+m-1} \oplus R_{m+j}$$

for all $m \geq 0$ and all large j . It follows that

$$\ell_A\left(\frac{K^m R_j}{K^{m+1} R_j}\right) = \sum_{k=0}^m \ell_A\left(\frac{J^{m-k} I_{j+k}}{J^{m-k+1} I_{j+k}}\right) \quad (4)$$

for all $m \geq 0$ and all large j . By Proposition 2.2, $B(m, n) = \ell_A(J^m I_n / J^{m+1} I_n)$ is the polynomial $P(m, n)$ of degree $q - 1$ for all large m, n . We may assume that $B(m, n) = P(m, n)$ for all m, n as it does not affect the top degree terms of the polynomial $P(m, n)$. Thus, from (4) we can write

$$\ell_A\left(\frac{K^m R_j}{K^{m+1} R_j}\right) = \sum_{k=0}^m \sum_{d_0+d_1=q-1} e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) \frac{(m-k)^{d_0} (j+k)^{d_1}}{d_0! d_1!} + V(m)$$

($\deg V(m) < q - 1$) for all $m \geq 0$ and all large j . Now, by using the same argument as in the proof of [11, Theorem 3.1] we get

$$\ell_A\left(\frac{K^m R_j}{K^{m+1} R_j}\right) = \frac{m^q}{q!} \sum_{d_0+d_1=q-1} e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}) + W(m)$$

($\deg W(m) < q$) for all $m \geq 0$ and all large j . From this it follows that $\dim R_j = q + 1$ and $e(K, R_j) = \sum_{d_0+d_1=q-1} e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]})$ for all large j . $\square$

In the following, let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A with $\text{ht } I_1 > 0$. The following proposition is a generalization of [11, Theorem 3.1].

PROPOSITION 4.2. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A with $\text{ht } I_1 > 0$. Set $K = J \oplus (\bigoplus_{k \geq 1} I_k t^k)$. Then*

$$e(K, R(\mathcal{F})) = \sum_{d_0+d_1=d-1} e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}).$$

Proof. Since $\text{ht } I_1 > 0$, then $0 : I_1^\infty$ is nilpotent. Hence $q = \dim A/0 : I_1^\infty = \dim A = d$. So by Lemma 4.1 we can choose j large enough such that

$$e(K, R_j) = \sum_{d_0+d_1=d-1} e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]}).$$

Put $S = R(\mathcal{F})/R_j$. We first prove that $\dim S = d$. Since $[R(\mathcal{F})_+]^j \subseteq R_j$, then S_+ is nilpotent. Therefore, every prime ideal of S has the form $P \oplus S_+$, where P is a prime ideal of A .

From this it follows that $\dim S = \dim A = d$. Since $\dim S < \dim R_j = d + 1$, from the exact sequence

$$0 \longrightarrow R_j \longrightarrow R(\mathcal{F}) \longrightarrow S \longrightarrow 0$$

we get $e(K, R(\mathcal{F})) = e(K, R_j) = \sum_{d_0+d_1=d-1} e_A(J^{[d_0+1]}, \mathcal{F}^{[d_1]})$. $\square$

As an immediate consequence of this proposition we have the following formula for the multiplicity of Rees algebras of good filtrations.

COROLLARY 4.3. *Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A with $\text{ht } I_1 > 0$. Then*

$$e(R(\mathcal{F})) = \sum_{d_0+d_1=d-1} e_A(\mathfrak{m}^{[d_0+1]}, \mathcal{F}^{[d_1]}).$$

Remark 4.4. The condition $\text{ht } I_1 > 0$ in Proposition 4.2 and Corollary 4.3 can be replaced by a weaker condition: $\dim A/0 : I_1^\infty = \dim A$. Moreover, from the proof of Lemma 4.1 and Proposition 4.2 we obtain the following interesting corollary.

COROLLARY 4.5. *Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a non-nilpotent good filtration of ideals in A . Then*

$$\dim R(\mathcal{F}) = \max\{\dim A, \dim[A/0 : I_1^\infty] + 1\}.$$

In particular, $\dim R(\mathcal{F}) = \dim A + 1$ if and only if $\dim A/0 : I_1^\infty = \dim A$.

Proof. Fix j large enough such that $\dim R_j = \dim[A/0 : I_1^\infty] + 1$ (this is possible by Lemma 4.1). Put $S = R(\mathcal{F})/R_j$. Then from the proof of Proposition 4.2 we have $\dim S = \dim A$. So

$$\dim R(\mathcal{F}) = \max\{\dim S, \dim R_j\} = \max\{\dim A, \dim[A/0 : I_1^\infty] + 1\}. \quad \square$$

Now from Proposition 4.2, Corollary 4.3 and Theorem 3.7 we get the following extension of [12, Theorem 4.2].

THEOREM 4.6. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A with $\text{ht } I_1 = h > 0$. Assume that $x_1, \dots, x_p$ is a maximal (FC)-sequence in I_1 with respect to $(J, \mathcal{F})$, and that $y_1, \dots, y_p$ is a maximal (FC)-sequence in I_1 with respect to $(\mathfrak{m}, \mathcal{F})$. Set $K = J \oplus (\bigoplus_{k \geq 1} I_k t^k)$. Then:*

- (i) $e(K, R(\mathcal{F})) = \sum_{i=0}^{h-1} e(J, A/(x_1, \dots, x_i)) + \sum_{i=h}^p e(J, A/[(x_1, \dots, x_i) : I_1^\infty]);$
- (ii) $e(R(\mathcal{F})) = \sum_{i=0}^{h-1} e(A/(y_1, \dots, y_i)) + \sum_{i=h}^p e(A/[(y_1, \dots, y_i) : I_1^\infty]).$

In the case when $\mathcal{F}$ is an equimultiple good filtration, we obtain the following result by combining Proposition 4.2 and Corollary 4.3 with Corollary 3.9. This is a generalization of [13, Theorem 3.2].

COROLLARY 4.7. *Let J be an $\mathfrak{m}$ -primary ideal of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be an equimultiple good filtration of ideals in A with $\text{ht } I_1 = \ell(I_1) = h > 0$. Assume that $x_1, \dots, x_{h-1}$ is a maximal (FC)-sequence in I_1 with respect to $(J, \mathcal{F})$, and that $y_1, \dots, y_{h-1}$ is a maximal (FC)-sequence in I_1 with respect to $(\mathfrak{m}, \mathcal{F})$. Set $K = J \oplus (\bigoplus_{k \geq 1} I_k t^k)$. Then:*

- (i) $e(K, R(\mathcal{F})) = \sum_{i=0}^{h-1} e(J, A/(x_1, \dots, x_i));$
- (ii) $e(R(\mathcal{F})) = \sum_{i=0}^{h-1} e(A/(y_1, \dots, y_i)).$

Finally, if I_1 is an $\mathfrak{m}$ -primary ideal, then as an immediate consequence of Proposition 4.2, Corollary 4.3 and Corollary 3.10, we have the following generalization of [12, Theorem 4.3] to the case of good filtrations.

COROLLARY 4.8. *Let J, I_1 be $\mathfrak{m}$ -primary ideals of A and let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a good filtration of ideals in A . Assume that $x_1, \dots, x_{d-1}$ is a maximal (FC)-sequence in I_1 with respect to $(J, \mathcal{F})$, and that $y_1, \dots, y_{d-1}$ is a maximal (FC)-sequence in I_1 with respect to $(\mathfrak{m}, \mathcal{F})$. Set $K = J \oplus (\bigoplus_{k \geq 1} I_k t^k)$. Then:*

- (i) $e(K, R(\mathcal{F})) = \sum_{i=0}^{d-1} e(J, A/(x_1, \dots, x_i));$
- (ii) $e(R(\mathcal{F})) = \sum_{i=0}^{d-1} e(A/(y_1, \dots, y_i)).$

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Duong Quoc Viet
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy street
Hanoi, Vietnam
(E-mail: duongquocviet@fmail.vnn.vn)

Le Van Dinh
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy street
Hanoi, Vietnam
(E-mail: dinhlevands@gmail.com)