ON MODULI SUBSETS OF CENTRAL EXTENSIONS OF RATIONAL H-SPACES

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Abstract. We investigate the moduli sets of central extensions of H-spaces enjoying inversivity, power associativity and Moufang properties. By considering rational H-extensions, it turns out that there is no relationship between the first and the second properties in general.

1. Introduction

We assume that a space has the homotopy type of a path connected CW-complex with a nondegenerate base point $*$ and that all maps are based maps.

An *H-space* is a space X endowed with a map $\mu : X \times X \rightarrow X$, called a multiplication, such that both the restrictions $\mu|_{X \times *}$ and $\mu|_{* \times X}$ are homotopic to the identity map of X . The multiplication naturally induces a binary operation on the homotopy set $[Y, X]$ for any space Y . In [4], James proved that $[Y, X]$ is an algebraic loop; that is, it has a two-sided unit element and, for any elements x and $y \in [Y, X]$, the equations $xa = y$ and $bx = y$ have unique solutions $a, b \in [Y, X]$.

Loop theoretic properties of H-spaces have been considered by several authors; for example, Curjel [3] and Norman [7]. In [1], Arkowitz and Lupton considered H-space structures with inversive, power associative and Moufang properties. They consider whether there exists an H-space structure which does not satisfy the properties. Thanks to the general theory of algebraic loops, we see that the Moufang property implies inversivity and power associativity. However, it is expected that there is no relationship between the latter two properties in general.

In [5], Kachi introduced central extensions of H-spaces which are called central H-extensions; see Definition 2.7. Roughly speaking, for a given homotopy associative and homotopy commutative H-space X_1 and an H-space X_2 , a central H-extension of X_1 by X_2 is defined to be the product $X_1 \times X_2$ with a twisted multiplication. He also gave a classification theorem for the extensions. In fact, a quotient set of an appropriate homotopy set classifies the equivalence classes of central H-extensions; see Theorem 2.14. Such a quotient set is called *the moduli set* of H-extensions. Moreover, we refer to the subset of the moduli set corresponding to the set of the equivalence classes of H-extensions enjoying a property P via the bijection in the classifying theorem as *the moduli subset* of H-extensions associated with the property P .

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The objective of this paper is to investigate the moduli subsets of central extensions of H-spaces associated with inversive, power associative or Moufang properties. If a given H-space is $\mathbb{Q}$ -local, then the moduli set of its central H-extensions is endowed with a vector space structure over $\mathbb{Q}$. It turns out that the moduli subset mentioned above inherits the vector space structure. This fact enables us to compare such moduli subsets as a vector space, and thus our main theorem (Theorem 4.7) deduces the following result.

ASSERTION 1.1. *Let S_{inv} , $S_{\text{p.a}}$ and S_{Mo} be the moduli subsets of central H-extensions of the Eilenberg–MacLane space $K(\mathbb{Q}, n)$ by $K(\mathbb{Q}, m)$ associated with inversive, power associative and Moufang properties, respectively. Here, the Eilenberg–MacLane space may be considered as a based loop space; that is, a homotopy associative and commutative H-space.*

- (1) *If m is even, $n = km$ and $k \geq 4$ is even, then S_{Mo} is the proper subset of $S_{\text{p.a}}$ and $S_{\text{p.a}}$ is the proper subset of S_{inv} :*

$$S_{\text{Mo}} \subsetneq S_{\text{p.a}} \subsetneq S_{\text{inv}}.$$

- (2) *If m is even, $n = km$ and $k \geq 5$ is odd, then S_{Mo} is the proper subset of $S_{\text{p.a}} \cap S_{\text{inv}}$. Moreover, $S_{\text{p.a}} \cap S_{\text{inv}}$ is a proper subset of $S_{\text{p.a}}$ and S_{inv} :*

$$S_{\text{Mo}} \subsetneq S_{\text{p.a}} \cap S_{\text{inv}} \subsetneq S_{\text{p.a}} \quad \text{and} \quad S_{\text{p.a}} \cap S_{\text{inv}} \subsetneq S_{\text{inv}}.$$

- (3) *Otherwise, $S_{\text{inv}} = S_{\text{p.a}} = S_{\text{Mo}}$.*

Kachi [5] proved that the space $K(\mathbb{Z}, 2n) \times S^n$ ($n = 1, 3, 7$) admits infinitely many essentially different multiplications. As a corollary of Assertion 1.1, we have the following result.

COROLLARY 1.2. *The product space $K(\mathbb{Z}, 2n) \times S^n$ ($n = 1, 3, 7$) has infinitely many essentially different multiplications which satisfy the Moufang property.*

The organization of this paper is as follows. In Section 2, we will recall several fundamental definitions and facts on H-spaces and algebraic loops. The classification theorem of central H-extensions are also described. In Section 3, we will present necessary and sufficient conditions for central H-extensions to be inversive, power associative and Moufang. The conditions allows us to describe the moduli set of extensions enjoying each of the properties in terms of a homotopy set of maps. In Section 4, we will deal with rational H-spaces and the dimensions of the moduli sets mentioned above. Moreover, several examples are presented. Assertion 1.1 and Corollary 1.2 are proved at the end of Section 4.

2. Preliminaries

We begin by recalling the definition of an algebraic loop.

Definition 2.1. An algebraic loop $(Q, \cdot)$ is a set Q with a map

$$\cdot : Q \times Q \rightarrow Q; (x, y) \mapsto xy$$

such that the equations $ax = b$ and $ya = b$ have unique solutions $x, y \in Q$ for all $a, b \in Q$. Moreover, there exists an element $e \in Q$ such that $xe = x = ex$ for all $x \in Q$. The element e is called a *unit*.

Algebraic loops with particular multiplications are considered.

Definition 2.2. An algebraic loop $(Q, \cdot)$ is called

- (1) *inversive* if, for any element x of Q , there exists a unique element x^{-1} of Q such that the equalities $x^{-1}x = e = xx^{-1}$ hold;
- (2) *power associative* if $x(xx) = (xx)x$ for any $x \in Q$;
- (3) *Moufang* if $(x(yz))x = (xy)(zx)$ for any $x, y, z \in Q$; and
- (4) *flexible* if $(xy)x = x(yx)$ for any $x, y \in Q$.

In Section 4, we will focus on the flexible property to examine the Moufang property.

LEMMA 2.3. [2, Lemma 2A] *If an algebraic loop Q is Moufang, then it is inversive.*

Thus we have the following implication between properties for multiplication described in Definition 2.2:

$$\boxed{\text{power associative}} \Longleftarrow \boxed{\text{flexible}} \Longleftarrow \boxed{\text{Moufang}} \Longrightarrow \boxed{\text{inversive}} \quad (2.1)$$

We bring the loop theoretic notion to the realm of H-spaces.

Definition 2.4. Let (X, μ) be an H-space. A *left inverse* $l : X \rightarrow X$ and a *right inverse* $r : X \rightarrow X$ of an H-space (X, μ) are maps such that $\mu(l \times id)\Delta \simeq * \simeq \mu(id \times r)\Delta$.

We see that an H-space has a left inverse and a right inverse unique up to homotopy [4].

Definition 2.5. An H-space (X, μ) is called

- (1) *inversive* if a left inverse and a right inverse of (X, μ) are homotopic;
- (2) *power associative* if $\mu(\mu \times id)(\Delta \times id)\Delta \simeq \mu(id \times \mu)(\Delta \times id)\Delta$;
- (3) *Moufang* if $\mu(\mu \times \mu)\theta \simeq \mu(\mu \times id)(id \times \mu \times id)\theta$, where $\theta : X^3 \rightarrow X^4$ is defined by $\theta(x, y, z) = (x, y, z, x)$; and
- (4) *flexible* if $\mu(\mu \times id)(id \times t)(\Delta \times id) \simeq \mu(id \times \mu)(id \times t)(\Delta \times id)$, where $t : X^2 \rightarrow X^2$ is defined by $t(x, y) = (y, x)$.

The following proposition characterizes the above conditions on H-spaces by homotopy sets.

PROPOSITION 2.6. [1] *Let (X, μ) be an H-space. Then (X, μ) is inversive (respectively, power associative, Moufang and flexible) if and only if the homotopy set $[Y, X]$ is inversive (respectively, power associative, Moufang and flexible) for any space Y .*

Here we recall the definition of the central extension of an H-space.

Definition 2.7. [5] Let (X_1, μ_1) be a homotopy associative and homotopy commutative H-space and let (X_2, μ_2) be an H-space. An H-space (X, μ) is a *central H-extension* of (X_1, μ_1) by (X_2, μ_2) if there exists a sequence of H-maps

$$(X_1, \mu_1) \xrightarrow{f_1} (X, \mu) \xrightarrow{f_2} (X_2, \mu_2)$$

such that the sequence

$$e \longrightarrow [Y, X_1] \xrightarrow{f_{1*}} [Y, X] \xrightarrow{f_{2*}} [Y, X_2] \longrightarrow e$$

is exact as algebraic loops for any space Y and the image of f_{1*} is contained in the center of $[Y, X]$. Here f_{i*} is the algebraic loop homomorphism induced by f_i and the center of an algebraic loop Q is the following subset of Q [9, p. 83]:

$$\{x \in Q \mid rx = xr, (rs)x = r(sx), (xr)s = x(rs) \text{ for any } r, s \in Q\}.$$

In order to describe the classification theorem for central H-extensions presented by Kachi, we recall the definition of H-deviations. Let (X, μ) be an H-space and let Y be a space. For any $[f], [g] \in [Y, X]$, let $D(f, g)$ be a unique element of $[Y, X]$ such that $D(f, g)[g] = [f]$.

Definition 2.8. [11, Definition 1.4.1] Let (X, μ) and (Y, μ') be H-spaces, and let $f : X \rightarrow Y$ be a map. An *H-deviation of the map f* is an element

$$HD(f) \in [X \wedge X, Y]$$

such that $q^*(HD(f)) = D(f\mu, \mu'(f \times f))$, where $q : X \times X \rightarrow X \wedge X$ is the quotient map.

The existence of H-deviations is ensured by the following lemma.

LEMMA 2.9. [11, Lemma 1.3.5] *Let (X, μ) be an H-space. Then, for any spaces Y and Z ,*

$$e \longrightarrow [Y \wedge Z, X] \xrightarrow{q^*} [Y \times Z, X] \xrightarrow{i^*} [Y \vee Z, X] \longrightarrow e$$

is the short exact sequence as algebraic loops, where $i : Y \vee Z \rightarrow Y \times Z$ is the inclusion and $q : Y \times Z \rightarrow Y \wedge Z$ is the quotient map.

Since $i^*D(f\mu, \mu'(f \times f)) = D(f\mu i, \mu'(f \times f)i) = e$, there exists an element $HD(f) \in [X \wedge X, Y]$ such that $q^*(HD(f)) = D(f\mu, \mu'(f \times f))$. If $f \simeq f' : X \rightarrow Y$, then $HD(f) = HD(f')$. Therefore, the H-deviation map

$$HD : [X, Y] \rightarrow [X \wedge X, Y]$$

is defined by sending a class $[f]$ to the class $HD(f)$.

LEMMA 2.10. *Let (X, μ) and (Y, μ') be H-spaces and let $f : X \rightarrow Y$ be a map such that $[f\mu], [fp_1]$ and $[fp_2]$ are in the center of $[X \times X, Y]$, where p_1 and p_2 are the projections of $X \times X$ on the first and second factors respectively. Then, for any map $g : X \rightarrow Y$, $HD(f \cdot g) = HD(f)HD(g)$.*

Remark 2.11. In [5, Lemma 2.3], Kachi claimed Lemma 2.10 with different hypotheses. However, it is necessary to modify the hypotheses and the proof of [5, Lemma 2.3], and hence we modify the assertion of Lemma 2.3 in [5] as in Lemma 2.10. We note that there is no effect on the arguments in paper [5], especially, on the argument of [5, p. 255], even if we replace Lemma 2.3 in [5] with Lemma 2.10.

Proof of Lemma 2.10. By the definition of the H-deviation, the equality $HD(f \cdot g) = H(f)H(g)$ holds if and only if

$$D((f \cdot g)\mu, \mu'(f \cdot g \times f \cdot g)) = D(f\mu, \mu'(f \times f))D(g\mu, \mu'(g \times g))$$

for any $f, g : X \rightarrow Y$. Since $[fp_1]$ and $[fp_2]$ are in the center, the equations $[\mu'(f \times f)] = [fp_1] \cdot [fp_2]$ and $[\mu'(g \times g)] = [gp_1] \cdot [gp_2]$ show that

$$\begin{aligned} [\mu'(f \times f) \cdot \mu'(g \times g)] &= ([fp_1][fp_2])([gp_1][gp_2]) \\ &= ([fp_1][gp_1])([fp_2][gp_2]) = [\mu'(f \cdot g \times f \cdot g)] \end{aligned}$$

in $[X \times X, Y]$. It means that

$$D((f \cdot g)\mu, \mu'(f \cdot g \times f \cdot g)) = D((f\mu) \cdot (g\mu), \mu'(f \times f) \cdot \mu'(g \times g)).$$

On the other hand, since $[\mu'(f \times f)] = [fp_1][fp_2]$ and $[f\mu]$ are in the center, $D(f\mu, \mu'(f \times f))$ is also in the center. Then, the following equations hold:

$$\begin{aligned} [(f\mu) \cdot (g\mu)] &= [f\mu][g\mu] \\ &= (D(f\mu, \mu'(f \times f))[\mu'(f \times f)])(D(g\mu, \mu'(g \times g))[\mu'(g \times g)]) \\ &= (D(f\mu, \mu'(f \times f))D(g\mu, \mu'(g \times g)))[\mu'(f \times f) \cdot \mu'(g \times g)]. \end{aligned}$$

Therefore, $D((f \cdot g)\mu, \mu'(f \cdot g \times f \cdot g)) = D(f\mu, \mu'(f \times f))D(g\mu, \mu'(g \times g))$. $\square$

COROLLARY 2.12. [5, Corollary 2.4] *If Y is a homotopy associative and homotopy commutative H-space, then the H-deviation map is an algebraic loop homomorphism.*

Proof. Since any maps from X to Y satisfy the condition of Lemma 2.10, it follows the assertion. $\square$

Definition 2.13. [5] Two central H-extensions

$$(X, \mu_X) \xrightarrow{f_i} (Z_i, \mu_i) \xrightarrow{g_i} (Y, \mu_Y) \quad (i = 1, 2)$$

are said to be *equivalent* if there exists an H-map $h : (Z_1, \mu_1) \rightarrow (Z_2, \mu_2)$ such that the following diagram is homotopy commutative.

$$\begin{array}{ccccc} X & \xrightarrow{f_1} & Z_1 & \xrightarrow{g_1} & Y \\ \downarrow & & \downarrow h & & \downarrow \\ X & \xrightarrow{f_2} & Z_2 & \xrightarrow{g_2} & Y \end{array}$$

It is readily seen that this relation is an equivalence relation. We denote by $CH(X_1, \mu_1; X_2, \mu_2)$ the set of equivalence classes of central H-extensions of (X_1, μ_1) by (X_2, μ_2) , and denote by $[(X, \mu), f_1, f_2]$ the equivalence class of the central H-extension

$$(X_1, \mu_1) \xrightarrow{f_1} (X, \mu) \xrightarrow{f_2} (X_2, \mu_2).$$

THEOREM 2.14. [5, Theorem 4.3] *Let (X_1, μ_1) be a homotopy associative and homotopy commutative H-space and let (X_2, μ_2) be an H-space. Define*

$$\Phi : [X_2 \wedge X_2, X_1]/\text{Im } HD \longrightarrow CH(X_1, \mu_1; X_2, \mu_2)$$

by $\Phi([\omega]) = [(X_1 \times X_2, \mu_\omega), i_1, p_2]$, where the multiplication μ_ω of $X_1 \times X_2$ is defined by

$$\mu_\omega((x_1, x_2), (y_1, y_2)) = (\mu_1(\mu_1(x_1, y_1), \omega q(x_2, y_2)), \mu_2(x_2, y_2)),$$

where $q : X_2 \times X_2 \rightarrow X_2 \wedge X_2$ is the quotient map. Then Φ is bijective.

We now check that $(X_1 \times X_2, \mu_\omega)$ is a central H-extension of X_1 by X_2 . It is easily seen that, for any space Y , the sequence of algebraic loops

$$e \longrightarrow [Y, X_1] \xrightarrow{i_{1*}} [Y, X_1 \times X_2] \xrightarrow{p_{2*}} [Y, X_2] \longrightarrow e$$

is exact. For any $[f] \in [Y, X_1]$, $[g], [h] \in [Y, X_1 \times X_2]$ and $y \in Y$, we put $f_1 := f(y)$, $g_i := p_i g(y)$ and $h_i := p_i h(y)$ for convenience. Then, we have

$$\begin{aligned} \mu_\omega(g \times i_1 f) \Delta(y) &= (\mu_1(\mu_1(g_1, f_1), \omega q(g_2, *)), \mu_2(g_2, *)) \\ &\simeq (\mu_1(\mu_1(f_1, g_1), \omega q(*, g_2)), \mu_2(*, g_2)) = \mu_\omega(i_1 f \times g) \Delta(y) \end{aligned}$$

and

$$\begin{aligned} \mu_\omega(\mu_\omega(g \times h) \Delta \times i_1 f) \Delta(y) &= (\mu_1(\mu_1(\mu_1(\mu_1(g_1, h_1), \omega q(g_2, h_2)), f_1), \omega q(\mu_2(g_2, h_2), *)), \mu_2(\mu_2(g_2, h_2), *)) \\ &\simeq (\mu_1(\mu_1(g_1, \mu_1(\mu_1(h_1, f_1), \omega q(h_2, *)), \omega q(g_2, h_2))), \mu_2(g_2, \mu_2(h_2, *))) \\ &= \mu_\omega(g \times \mu_\omega(h \times i_1 f) \Delta) \Delta(y). \end{aligned}$$

Thus, the equations $[i_1 f][g] = [g][i_1 f]$ and $([g][h])[i_1 f] = [g]([h][i_1 f])$ hold. A similar argument shows that $[i_1 f]([g][h]) = ([i_1 f][g])[h]$. Therefore, $(X_1 \times X_2, \mu_\omega)$ is a central H-extension.

Remark 2.15. We recall the inverse map of Φ, Φ' , defined in [5, p. 256]. For any central H-extension $[(X, \mu), f_1, f_2]$ of X_1 by X_2 , the exact sequence

$$e \longrightarrow [X_2, X_1] \xrightarrow{f_{1*}} [X_2, X] \xrightarrow{f_{2*}} [X_2, X_2] \longrightarrow e$$

shows that there is a map $g : X_2 \rightarrow X$ such that $f_2 g \simeq id_{X_2}$. We consider the exact sequence

$$e \longrightarrow [X_2 \wedge X_2, X_1] \xrightarrow{f_{1*}} [X_2 \wedge X_2, X] \xrightarrow{f_{2*}} [X_2 \wedge X_2, X_2] \longrightarrow e.$$

Since $f_{2*}(HD(g)) = HD(f_2 g) = 0$ in $[X_2 \wedge X_2, X_2]$, there exists $\theta \in [X_2 \wedge X_2, X_1]$ such that $f_{1*}(\theta) = HD(g)$. Then, we define a map

$$\Phi' : CH(X_1, \mu_1; X_2, \mu_2) \longrightarrow [X_2 \wedge X_2, X_1] / \text{Im } HD$$

by $\Phi'([(X, \mu), f_1, f_2]) = [-\theta]$.

We note that, in [5, Theorem 4.3], Kachi defined the map $\overline{\Phi}$ from $[X_2 \wedge X_2, X_1] / \text{Im } HD$ to $CH(X_1, \mu_1; X_2, \mu_2)$ by sending $[\omega]$ to $[(X_1 \times X_2, \mu_{-\omega}), i_1, p_2]$ and claimed that this map is bijective and Φ' is an inverse map of $\overline{\Phi}$. However, this map is not bijective since $\Phi' \overline{\Phi}([\omega]) = -[\omega]$. We hence need to modify the definition of $\overline{\Phi}$ a little and we see that Φ is bijective and Φ' is its inverse map using an argument similar to [5, Theorem 4.3]. Then combining the reason mentioned above and Remark 2.11, Theorem 4.3 in [5] is true without changing the proof.

By Theorem 2.14, for an equivalence class $\mathcal{E}$ in $CH(X_1, \mu_1; X_2, \mu_2)$, there exists a map $\omega : X_2 \wedge X_2 \rightarrow X_1$ such that $\Phi([\omega]) = [\mathcal{E}]$, the map ω is called *the classifying map* of the central H-extensions $\mathcal{E}$. Moreover, we refer to the set $[X_2 \wedge X_2, X_1] / \text{Im } HD$ as *the moduli set* of H-extensions of (X_1, μ_1) by (X_2, μ_2) .

3. Moduli subsets of central H-extensions

We retain the notation and terminology described in the previous section. Let (X_1, μ_1) be a homotopy associative and homotopy commutative H-space, and let (X_2, μ_2) be an H-space. Then, by Theorem 2.14, a central H-extension of (X_1, μ_1) by (X_2, μ_2) is of the form $(X_1 \times X_2, \mu_\omega)$. Let $i_j : X_j \rightarrow X_1 \times X_2$ and $p_j : X_1 \times X_2 \rightarrow X_j$ denote the inclusions and the projections, respectively. Let $\Delta_j : X_j \rightarrow X_j \times X_j$ and $\Delta : X_1 \times X_2 \rightarrow (X_1 \times X_2)^2$ be diagonal maps. We denote by id_j and id the identity maps of X_j and $X_1 \times X_2$, respectively. For convenience, we put $xx' := \mu_1(x, x')$ and $yy' := \mu_2(y, y')$ for any $x, x' \in X_1$ and $y, y' \in X_2$.

PROPOSITION 3.1. *The central H-extension $(X_1 \times X_2, \mu_\omega)$ is inversive if and only if (X_2, μ_2) is inversive and $\omega q(l_2 \times id_2)\Delta_2 \simeq \omega q(id_2 \times r_2)\Delta_2$, where l_2 and r_2 are the left inverse and the right inverse of (X_2, μ_2) , respectively.*

Before proving Proposition 3.1, we give the following lemmas.

LEMMA 3.2. *Let (X, μ) , (Y, μ') be H-spaces and let $f : X \rightarrow Y$ be a map. If f is an H-map, then $fl \simeq l'f$, $fr \simeq r'f$, where l, l' are left inverses of (X, μ) , (Y, μ') and r, r' are right inverses of (X, μ) , (Y, μ') , respectively.*

Proof. We see that $[fl][f] = [*] = [l'f][f]$ in $[X, Y]$ since f is an H-map. Hence, we get $[fl] = [l'f]$. Similarly, one obtains the equality $[fr] = [r'f]$. $\square$

LEMMA 3.3. *Let Y, Z be spaces and let $i : Y \rightarrow Y \times Z$, $p : Y \times Z \rightarrow Y$ be the inclusion and the projection. Then one has*

$$\text{Ker}\{i^* : [Y \times Z, X] \rightarrow [Y, X]\} \cap \text{Im}\{p^* : [Y, X] \rightarrow [Y \times Z, X]\} = \{e\}$$

for any H-space X .

Proof. For any $[f] \in \text{Ker } i^* \cap \text{Im } p^*$, there is a map $g : Y \rightarrow X$ such that $f \simeq gq$. Since $[f]$ is in $\text{Ker } i^*$, it follows that $* \simeq fi \simeq g$ and hence $f \simeq *$. $\square$

Proof of Proposition 3.1. Let l and r be a left inverse and a right inverse of $(X_1 \times X_2, \mu_\omega)$, respectively. Note that $l \simeq r$ if and only if $p_j l \simeq p_j r$ for $j = 1, 2$. In order to prove Proposition 3.1, it is enough to show

- (1) $p_1 l \simeq p_1 r$ if and only if $\omega q(l_2 \times id_2)\Delta_2 \simeq \omega q(id_2 \times r_2)\Delta_2$ and
- (2) $p_2 l \simeq p_2 r$ if and only if (X_2, μ_2) is inversive.

Since

$$* \simeq p_1 \mu_\omega(l \times id)\Delta = \mu_1(\mu_1(p_1 l \times p_1)\Delta \times \omega q(p_2 l \times p_2)\Delta)\Delta,$$

it follows that

$$[*] = ([p_1 l][p_1])[\omega q(p_2 l \times p_2)\Delta]$$

in $[X_1 \times X_2, X_1]$. Similarly, we have

$$[*] = ([p_1][p_1 r])[\omega q(p_2 \times p_2 r)\Delta]$$

in $[X_1 \times X_2, X_1]$. Since $[X_1 \times X_2, X_1]$ is an abelian group, we see that $[p_1 l] = [p_1 r]$ if and only if $[\omega q(p_2 l \times p_2)\Delta] = [\omega q(p_2 \times p_2 r)\Delta]$. Lemma 3.2 allows us to obtain that

$$[\omega q(p_2 l \times p_2)\Delta] = [\omega q(l_2 \times id_2)\Delta_2 p_2]$$

and

$$[\omega q(p_2 \times p_2 r) \Delta] = [\omega q(id_2 \times r_2) \Delta_2 p_2].$$

If $[\omega q(l_2 \times id_2) \Delta_2 p_2] = [\omega q(id_2 \times r_2) \Delta_2 p_2]$, then

$$[\omega q(l_2 \times id_2) \Delta_2] = [\omega q(l_2 \times id_2) \Delta_2 p_2 i_2] = [\omega q(id_2 \times r_2) \Delta_2 p_2 i_2] = [\omega q(id_2 \times r_2) \Delta_2].$$

Therefore we have assertion (1).

Suppose that $p_2 l \simeq p_2 r$. Then $l_2 \simeq p_2 l i_2 \simeq p_2 r i_2 \simeq r_2$; that is, (X_2, μ_2) is inversive. Conversely, suppose that (X_2, μ_2) is inversive. Then it follows that $p_2 l i_2 \simeq p_2 r i_2$ and hence

$$[*] = [p_2 l i_2][p_2 r i_2]^{-1} = [p_2 l i_2][r_2 p_2 r i_2] = i_2^*([p_2 l][r_2 p_2 r]).$$

Therefore we see that $[p_2 l][r_2 p_2 r] \in \text{Ker } i_2^*$. On the other hand,

$$[p_2 l][r_2 p_2 r] = [l_2 p_2][r_2 r_2 p_2] = p_2^*([l_2][r_2 r_2]) \in \text{Im } p_2^*.$$

In view of Lemma 3.3, we have

$$[*] = [p_2 l][r_2 p_2 r] = [p_2 l][p_2 r]^{-1}.$$

Hence, it follows that $[p_2 l] = [p_2 r]$. We have the second assertion. $\square$

PROPOSITION 3.4. *The central H -extension $(X_1 \times X_2, \mu_\omega)$ is power associative if and only if (X_2, μ_2) is power associative and $\omega q(\mu_2 \times id_2) \overline{\Delta}_2 \simeq \omega q(id_2 \times \mu_2) \overline{\Delta}_2$, where $\overline{\Delta}_2 = (\Delta_2 \times id_2) \Delta_2$.*

Proof. As in the proof of Proposition 3.1, it is enough to show the following:

- (1) $p_1 \mu_\omega(\mu_\omega \times id) \overline{\Delta} \simeq p_1 \mu_\omega(id \times \mu_\omega) \overline{\Delta}$ if and only if $\omega q(\mu_2 \times id_2) \overline{\Delta}_2 \simeq \omega q(id_2 \times \mu_2) \overline{\Delta}_2$, where $\overline{\Delta} = (\Delta \times id) \Delta$;
- (2) $p_2 \mu_\omega(\mu_\omega \times id) \overline{\Delta} \simeq p_2 \mu_\omega(id \times \mu_\omega) \overline{\Delta}$ if and only if (X_2, μ_2) is power associative.

Given $(x, y) \in X_1 \times X_2$, then, we have

$$\begin{aligned} \mu_\omega(\mu_\omega \times id) \overline{\Delta}(x, y) &= \mu_\omega(((xx)\omega q(y, y), yy), (x, y)) \\ &= (((((xx)\omega q(y, y))x)\omega q(yy, y), (yy)y) \end{aligned}$$

and

$$\begin{aligned} \mu_\omega(id \times \mu_\omega) \overline{\Delta}(x, y) &= \mu_\omega((x, y), ((xx)\omega q(y, y), yy)) \\ &= ((x((xx)\omega q(y, y)))\omega q(y, yy), y(yy)). \end{aligned}$$

Hence, assertions (1) and (2) follow from the equations. $\square$

The same argument as in Proposition 3.4 does work to prove the following propositions.

PROPOSITION 3.5. *The central H -extension $(X_1 \times X_2, \mu_\omega)$ is Moufang if and only if (X_2, μ_2) is Moufang and*

$$\begin{aligned} &[\mu_1(\omega q \times \omega q) \theta_2][\omega q(\mu_2 \times \mu_2) \theta_2] \\ &= [\mu_1(\omega q \times \omega q)(id_2 \times (\mu_2 \times id_2 \times id_2) \Delta'_2)][\omega q(\mu_2(id_2 \times \mu_2) \times id_2) \theta_2] \end{aligned}$$

in $[X_2^3, X_1]$, where Δ'_2 is the diagonal map of $X_2 \times X_2$ and $\theta_2 : X_2^3 \longrightarrow X_2^4$ is defined by $\theta_2(x, y, z) = (x, y, z, x)$.

Proof. Let $(x_i, y_i) \in X_1 \times X_2$ for $i = 1, 2, 3$. The following equations hold:

$$\begin{aligned} & \mu_\omega(\mu_\omega \times \mu_\omega)\theta((x_1, y_1), (x_2, y_2), (x_3, y_3)) \\ &= \mu_\omega(((x_1x_2)\omega q(y_1, y_2), y_1y_2), ((x_3x_1)\omega q(y_3, y_1), y_3y_1)) \\ &= (((x_1x_2)\omega q(y_1, y_2))((x_3x_1)\omega q(y_3, y_1)))\omega q(y_1y_2, y_3y_1), (y_1y_2)(y_3y_1)) \end{aligned}$$

and

$$\begin{aligned} & \mu_\omega(\mu_\omega \times id)(id \times \mu_\omega \times id)\theta((x_1, y_1), (x_2, y_2), (x_3, y_3)) \\ &= \mu_\omega(\mu_\omega \times id)((x_1, y_1), ((x_2x_3)\omega q(y_2, y_3), y_2y_3), (x_1, y_1)) \\ &= \mu_\omega(((x_1((x_2x_3)\omega q(y_2, y_3)))\omega q(y_1, y_2y_3), y_1(y_2y_3)), (x_1, y_1)) \\ &= (((x_1((x_2x_3)\omega q(y_2, y_3)))\omega q(y_1, y_2y_3))x_1)\omega q(y_1(y_2y_3), y_1), (y_1(y_2y_3))y_1). \end{aligned}$$

These imply the assertion. $\square$

PROPOSITION 3.6. *The central H-extension $(X_1 \times X_2, \mu_\omega)$ is flexible if and only if (X_2, μ_2) is flexible and*

$$\begin{aligned} & \mu_1(\omega q \times \omega q)(t \times (id_2 \times \mu_2 t)(\Delta_2 \times id_2))\Delta'_2 \\ & \simeq \mu_1(\omega q \times \omega q)(id_2 \times id_2 \times (\mu_2 \times id_2)(id_2 \times t)(\Delta_2 \times id_2))\Delta'_2. \end{aligned}$$

Proof. For any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$, we have

$$\begin{aligned} & \mu_\omega(\mu_\omega \times id)(id \times t)(\Delta \times id)((x_1, y_1), (x_2, y_2)) \\ &= \mu_\omega(((x_1x_2)\omega q(y_1, y_2), y_1y_2), (x_1, y_1)) \\ &= (((x_1x_2)\omega q(y_1, y_2))x_1)\omega q(y_1y_2, y_1), (y_1y_2)y_1 \end{aligned}$$

and

$$\begin{aligned} & \mu_\omega(id \times \mu_\omega)(id \times t)(\Delta \times id)((x_1, y_1), (x_2, y_2)) \\ &= \mu_\omega((x_1, y_1), ((x_2x_1)\omega q(y_2, y_1), y_2y_1)) \\ &= ((x_1((x_2x_1)\omega q(y_2, y_1)))\omega q(y_1, y_2y_1), y_1(y_2y_1)). \end{aligned}$$

Therefore, we obtain the result. $\square$

Next, we consider the following subsets of the homotopy set $[X_2 \wedge X_2, X_1]$:

$$\begin{aligned} G_{\text{inv}} &= \{[\omega] \in [X_2 \wedge X_2, X_1] \mid \omega q(l_2 \times id_2)\Delta_2 \simeq \omega q(id_2 \times r_2)\Delta_2\}, \\ G_{\text{p.a}} &= \{[\omega] \in [X_2 \wedge X_2, X_1] \mid \omega q(\mu_2 \times id_2)\overline{\Delta}_2 \simeq \omega q(id_2 \times \mu_2)\overline{\Delta}_2\}, \\ G_{\text{Mo}} &= \{[\omega] \in [X_2 \wedge X_2, X_1] \mid \Gamma_{\text{Mo}}(\omega) \simeq \Gamma'_{\text{Mo}}(\omega)\}, \\ G_{\text{fle}} &= \{[\omega] \in [X_2 \wedge X_2, X_1] \mid \mu_1(\omega \times \omega)\Gamma_{\text{fle}} \simeq \mu_1(\omega \times \omega)\Gamma'_{\text{fle}}\}, \end{aligned}$$

where

$$\begin{aligned} \Gamma_{\text{Mo}}(\omega) &= \mu_1(\mu_1(\omega q \times \omega q)\theta_2 \times \omega q(\mu_2 \times \mu_2)\theta_2)\Delta_{X_2^3}, \\ \Gamma'_{\text{Mo}}(\omega) &= \mu_1(\mu_1(\omega q \times \omega q)(id_2 \times (\mu_2 \times id_{X_2^2})\Delta'_2) \times \omega q(\mu_2(id_2 \times \mu_2) \times id_2)\theta_2)\Delta_{X_2^3}, \\ \Gamma_{\text{fle}} &= (q \times q)(t \times (id_2 \times \mu_2 t)(\Delta_2 \times id_2))\Delta'_2, \\ \Gamma'_{\text{fle}} &= (q \times q)(id_2 \times id_2 \times (\mu_2 \times id_2)(id_2 \times t)(\Delta_2 \times id_2))\Delta'_2. \end{aligned}$$

LEMMA 3.7. *The sets G_{inv} , $G_{\text{p.a.}}$, G_{Mo} and G_{fle} are subgroups of $[X_2 \wedge X_2, X_1]$.*

Proof. For any $[\omega_1], [\omega_2] \in G_{\text{inv}}$, we have

$$\begin{aligned} [\mu_1(\omega_1 \times \omega_2)\Delta_2''q(l_2 \times id_2)\Delta_2] &= [\omega_1q(l_2 \times id_2)\Delta_2][\omega_2q(l_2 \times id_2)\Delta_2] \\ &= [\omega_1q(id_2 \times r_2)\Delta_2][\omega_2q(id_2 \times r_2)\Delta_2] \\ &= [\mu_1(\omega_1 \times \omega_2)\Delta_2''q(id_2 \times r_2)\Delta_2] \end{aligned}$$

where Δ_2'' is the diagonal map of $X_2 \wedge X_2$. Hence $[\omega_1][\omega_2] \in G_{\text{inv}}$. Let l_1 be a left inverse of (X_1, μ_1) . Then we have

$$[l_1\omega_1q(l_2 \times id_2)\Delta_2] = [l_1\omega_1q(id_2 \times r_2)\Delta_2].$$

Hence we obtain $[\omega_1]^{-1} = [l_1\omega_1] \in G_{\text{inv}}$. Similarly, we can show that $G_{\text{p.a.}}$, G_{Mo} and G_{fle} are subgroups of $[X_2 \wedge X_2, X_1]$. $\square$

Let $\Phi : [X_2 \wedge X_2, X_1]/\text{Im } HD \rightarrow CH(X_1, \mu_1; X_2, \mu_2)$ be the bijection mentioned in Theorem 2.14.

LEMMA 3.8. *If (X_2, μ_2) is inversive (respectively, power associative, Moufang and flexible), then $\text{Im } HD \subset G_{\text{inv}}$ (respectively, $\text{Im } HD \subset G_{\text{p.a.}}$, $\text{Im } HD \subset G_{\text{Mo}}$ and $\text{Im } HD \subset G_{\text{fle}}$).*

Proof. For any $[\omega] \in \text{Im } HD$, $\Phi([\omega]) = [(X_1 \times X_2, \mu_1 \times \mu_2), i_1, p_2]$. Since (X_2, μ_2) is inversive, $(X_1 \times X_2, \mu_1 \times \mu_2)$ is also inversive, and hence $\omega q(l_2 \times id_2)\Delta_2 \simeq \omega q(id_2 \times r_2)\Delta_2$ by Proposition 3.1. We thus see that $[\omega] \in G_{\text{inv}}$. A similar argument shows the other cases. $\square$

THEOREM 3.9. *The following statements hold.*

(1) *If (X_2, μ_2) is inversive, then*

$$\Phi_{\text{inv}} : G_{\text{inv}}/\text{Im } HD \rightarrow \{[(X, \mu), f_1, f_2] \in CH(X_1, \mu_1; X_2, \mu_2) \mid (X, \mu) \text{ is inversive}\},$$

which is the restricted homomorphism of Φ to $G_{\text{inv}}/\text{Im } HD$, is bijective.

(2) *If (X_2, μ_2) is power associative, then*

$$\begin{aligned} \Phi_{\text{p.a.}} : G_{\text{p.a.}}/\text{Im } HD &\rightarrow \{[(X, \mu), f_1, f_2] \in CH(X_1, \mu_1; X_2, \mu_2) \mid \\ &\quad (X, \mu) \text{ is power associative}\}, \end{aligned}$$

which is the restricted homomorphism of Φ to $G_{\text{p.a.}}/\text{Im } HD$, is bijective.

(3) *If (X_2, μ_2) is Moufang, then*

$$\Phi_{\text{Mo}} : G_{\text{Mo}}/\text{Im } HD \rightarrow \{[(X, \mu), f_1, f_2] \in CH(X_1, \mu_1; X_2, \mu_2) \mid (X, \mu) \text{ is Moufang}\},$$

which is the restricted homomorphism of Φ to $G_{\text{Mo}}/\text{Im } HD$, is bijective.

(4) *If (X_2, μ_2) is flexible, then*

$$\Phi_{\text{fle}} : G_{\text{fle}}/\text{Im } HD \rightarrow \{[(X, \mu), f_1, f_2] \in CH(X_1, \mu_1; X_2, \mu_2) \mid (X, \mu) \text{ is flexible}\},$$

which is the restricted homomorphism of Φ to $G_{\text{fle}}/\text{Im } HD$, is bijective.

Proof. By Proposition 3.1, we see that the map Φ_{inv} is a well-defined map and hence it is injective. Let (X, μ) be an inversive central H-extension of (X_1, μ_1) by (X_2, μ_2) . Theorem 2.14 yields that there exists a map $\omega : X_2 \wedge X_2 \rightarrow X_1$ such that $[(X, \mu), f_1, f_2] = [(X_1 \times X_2, \mu_\omega), i_1, p_2]$. Since (X, μ) is inversive, $(X_1 \times X_2, \mu_\omega)$ is also inversive, and hence $\omega q(l_2 \times id_2)\Delta_2 \simeq \omega q(id_2 \times r_2)\Delta_2$ by Proposition 3.1. Therefore $\Phi_{\text{inv}}([\omega]) = [(X, \mu), f_1, f_2]$ and Φ_{inv} is surjective. In the same manner, we can show that $\Phi_{\text{p.a.}}$, Φ_{Mo} and Φ_{fle} are bijective. $\square$

We denote $G_{\text{inv}}/\text{Im } HD$, $G_{\text{p.a.}}/\text{Im } HD$, $G_{\text{Mo}}/\text{Im } HD$ and $G_{\text{fle}}/\text{Im } HD$ by S_{inv} , $S_{\text{p.a.}}$, S_{Mo} and S_{fle} , respectively, and call them *the moduli subset* of the H-extensions associated with the corresponding properties.

4. The moduli sets of central H-extensions of rational H-spaces

In this section, we will investigate the moduli set $G/\text{Im } HD$ and its subset S_{inv} , $S_{\text{p.a.}}$, S_{Mo} and S_{fle} in the rational cases. If (X, μ) is a $\mathbb{Q}$ -local, simply-connected H-space and the homotopy group $\pi_*(X)$ are of finite type, then, by [10, §8, Theorem 12], $H^*(X; \mathbb{Q}) \cong \bigotimes_i \mathbb{Q}[x_i] \otimes \bigotimes_j \Lambda(y_j)$ as an algebra, where $\mathbb{Q}[x_i]$ is a polynomial algebra and $\Lambda(y_j)$ is an exterior algebra.

THEOREM 4.1. [8, Proposition 1] *Let (X, μ) be a $\mathbb{Q}$ -local, simply-connected H-space where $\pi_*(X)$ are of finite type, and let Y be a space. Then the canonical map*

$$[Y, X] \longrightarrow \text{Hom}_{\text{Alg}}(H^*(X; \mathbb{Q}), H^*(Y; \mathbb{Q})), f \longmapsto H^*(f)$$

is bijective.

According to Arkowitz and Lupton [1], we give $\text{Hom}_{\text{Alg}}(H^*(X; \mathbb{Q}), H^*(Y; \mathbb{Q}))$ an algebraic loop structure so that the above canonical map is an algebraic loop homomorphism.

Definition 4.2. [1, Definition 2.1] Let M be a free commutative graded algebra over $\mathbb{Q}$ of the form $\bigotimes_i \mathbb{Q}[x_i] \otimes \bigotimes_j \Lambda(y_j)$. A homomorphism $v : M \rightarrow M \otimes M$ is called a *diagonal* if the following diagram is commutative, where $\varepsilon : M \rightarrow \mathbb{Q}$ is the augmentation.

$$\begin{array}{ccccc} & & M & & \\ & \swarrow \cong & \downarrow v & \searrow \cong & \\ \mathbb{Q} \otimes M & \xleftarrow{\varepsilon \otimes id} & M \otimes M & \xrightarrow{id \otimes \varepsilon} & M \otimes \mathbb{Q} \end{array}$$

We note that, for any H-space (X, μ) , $H^*(\mu) : H^*(X; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q}) \otimes H^*(X; \mathbb{Q})$ is a diagonal.

THEOREM 4.3. [1, Lemma 3.1] *Let $M = \bigotimes_i \mathbb{Q}[x_i] \otimes \bigotimes_j \Lambda(y_j)$ with the diagonal map $v : M \rightarrow M \otimes M$, and let A be a graded algebra. We define the product of $\text{Hom}_{\text{Alg}}(M, A)$ by*

$$\alpha \cdot \beta = m(\alpha \otimes \beta)v,$$

where m is the product of A . Then $\text{Hom}_{\text{Alg}}(M, A)$ is an algebraic loop endowed with the product.

Let γ_1 and γ_2 be the elements in $\text{Hom}_{\text{Alg}}(M, A)$ such that $\gamma_1 \cdot \alpha = \beta$ and $\alpha \cdot \gamma_2 = \beta$. Then, by the proof of Theorem 4.3, the following equations hold:

$$\gamma_1(z) = \beta(z) - \alpha(z) - m(\gamma_1 \otimes \alpha)P(z),$$

$$\gamma_2(z) = \beta(z) - \alpha(z) - m(\alpha \otimes \gamma_2)P(z),$$

where z is a generator of M and $P(z) = v(z) - z \otimes 1 - 1 \otimes z$.

Definition 4.4. A left inverse λ (respectively, a right inverse ρ) of an algebraic loop $\text{Hom}_{\text{Alg}}(M, M)$ is an element of $\text{Hom}_{\text{Alg}}(M, M)$ such that $\lambda \cdot id = e$ (respectively, $id \cdot \rho = e$), where e is the unit of $\text{Hom}_{\text{Alg}}(M, M)$.

Let A be a graded algebra and let M be a free commutative graded algebra of the form $\bigotimes_i \mathbb{Q}[x_i] \otimes \bigotimes_j \Lambda(y_j)$ for which each x_i and y_j are primitive; that is, $v(x_i) = x_i \otimes 1 + 1 \otimes x_i$ and $v(y_j) = y_j \otimes 1 + 1 \otimes y_j$. We consider the bijection of sets

$$\Psi : \text{Hom}_{\text{Alg}}(M, A) \xrightarrow{\cong} \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, A)$$

which is defined by

$$\Psi(\alpha)(x_i) = \alpha(x_i), \quad \Psi(\alpha)(y_j) = \alpha(y_j) \quad \text{for } \alpha \in \text{Hom}_{\text{Alg}}(M, A),$$

where $\mathbb{Q}\langle x_i, y_j \rangle$ denotes the graded $\mathbb{Q}$ -vector space with basis x_i and y_j for any i and j . Let $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, A)$ denote the set of $\mathbb{Q}$ -linear maps from $\mathbb{Q}\langle x_i, y_j \rangle$ to A . Since $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, A)$ is a $\mathbb{Q}$ -vector space with respect to the canonical sum and the canonical scalar multiple, we can give the $\mathbb{Q}$ -vector space structure to $\text{Hom}_{\text{Alg}}(M, A)$ via Ψ . We regard $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, A)$ as an algebraic loop with the canonical sum. Then Ψ is a morphism of algebraic loops. In fact, for any basis z of $\mathbb{Q}\langle x_i, y_j \rangle$,

$$(\alpha \cdot \beta)(z) = m(\alpha \otimes \beta)v(z) = \alpha(z) + \beta(z) = (\alpha + \beta)(z).$$

Let (X_1, μ_1) be a $\mathbb{Q}$ -local, simply-connected, homotopy associative and homotopy commutative H-space and the homotopy group of X_1 are of finite type, let (X_2, μ_2) be an H-space, and let $H^*(X_1; \mathbb{Q}) = \bigotimes_i \mathbb{Q}[x_i] \otimes \bigotimes_j \Lambda(y_j)$. Since (X_1, μ_1) is a homotopy associative and homotopy commutative H-space, we see that each x_i and y_j are primitive; see [6, Corollary 4.18].

LEMMA 4.5. *Let X and Y be spaces. Then, for any $n > 0$, there is an isomorphism of vector spaces*

$$H^n(X \wedge Y; \mathbb{Q}) \cong (H^{>0}(X; \mathbb{Q}) \otimes H^{>0}(Y; \mathbb{Q}))^n.$$

Proof. For any $n > 0$, we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & [X \wedge Y, K(\mathbb{Q}, n)] & \xrightarrow{q^*} & [X \times Y, K(\mathbb{Q}, n)] & \xrightarrow{i^*} & [X \vee Y, K(\mathbb{Q}, n)] \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & H^n(X \wedge Y; \mathbb{Q}) & \xrightarrow{H^n(q)} & H^n(X \times Y; \mathbb{Q}) & \xrightarrow{H^n(i)} & H^n(X \vee Y; \mathbb{Q}) \longrightarrow 0 \end{array}$$

Then it follows that

$$H^n(X \wedge Y; \mathbb{Q}) \cong \text{Ker } H^n(i) = (H^{>0}(X; \mathbb{Q}) \otimes H^{>0}(Y; \mathbb{Q}))^n. \quad \square$$

Define the map

$$\overline{HD} : \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, H^*(X_2; \mathbb{Q})) \rightarrow \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, H^*(X_2 \wedge X_2; \mathbb{Q}))$$

by

$$\overline{HD}(\alpha)(z) = P_2(\alpha(z)) \quad \text{for } \alpha \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, H^*(X_2; \mathbb{Q})),$$

where $P_2(z) = H^*(\mu_2)(z) - z \otimes 1 - 1 \otimes z$ for any $z \in H^*(X_2; \mathbb{Q})$. Then we see that $\overline{HD}$ is a $\mathbb{Q}$ -linear map.

PROPOSITION 4.6. *The following diagram of algebraic loops is commutative.*

$$\begin{array}{ccc} \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, H^*(X_2; \mathbb{Q})) & \xrightarrow{\overline{HD}} & \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, H^*(X_2 \wedge X_2; \mathbb{Q})) \\ \Psi H^* \uparrow \cong & & \cong \uparrow \Psi H^* \\ [X_2, X_1] & \xrightarrow{HD} & [X_2 \wedge X_2, X_1] \end{array}$$

Proof. Using the isomorphism Ψ , we see that for any $f \in [X_2, X_1]$ there exists a unique element $\overline{D}(f) \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, H^*(X_2 \times X_2; \mathbb{Q}))$ such that

$$\overline{D}(f) + (H^*(f) \otimes H^*(f))\Psi H^*(\mu_1) = H^*(\mu_2)\Psi H^*(f).$$

Therefore, it follows that $\overline{D}(f) = \Psi H^*(D(f\mu_1, \mu_2(f \times f)))$. Observe that each x_i and y_j are primitive. In view of the proof of Theorem 4.3, we have

$$\overline{D}(f)(z) = P_2(\Psi H^*(f)(z))$$

for any generators z of $\mathbb{Q}\langle x_i, y_j \rangle$. Lemma 4.5 implies that

$$\overline{D}(f)(z) = H^*(q)\overline{HD}(\Psi H^*(f))(z).$$

Hence, by the definition of an H-deviation, we obtain

$$H^*(q)\Psi H^*(HD(f)) = \Psi H^*(D(f\mu_1, \mu_2(f \times f))) = H^*(q)\overline{HD}(\Psi H^*(f)).$$

Since $H^*(q)$ is injective, it follows from Lemma 2.9 and Theorem 4.1 that $\Psi H^*(HD(f)) = \overline{HD}(\Psi H^*(f))$. $\square$

Put $V = \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x_i, y_j \rangle, H^*(X_2 \wedge X_2; \mathbb{Q}))$ and set

$$V_{\text{inv}} = \{\alpha \in V \mid m_2(\lambda_2 \otimes id)H^*(q)\alpha = m_2(id \otimes \rho_2)H^*(q)\alpha\},$$

$$V_{\text{p.a}} = \{\alpha \in V \mid \overline{m}_2(H^*(\mu_2) \otimes id)H^*(q)\alpha = \overline{m}_2(id \otimes H^*(\mu_2))H^*(q)\alpha\},$$

$$V_{\text{Mo}} = \{\alpha \in V \mid \overline{\Gamma}_{\text{Mo}}(\alpha) = \overline{\Gamma}'_{\text{Mo}}(\alpha)\},$$

$$V_{\text{fle}} = \{\alpha \in V \mid H^*(\Gamma_{\text{fle}})(\tilde{\alpha} \otimes \tilde{\alpha})\Psi(H^*(\mu_1)) = H^*(\Gamma'_{\text{fle}})(\tilde{\alpha} \otimes \tilde{\alpha})\Psi(H^*(\mu_1))\},$$

where m_2 is the product of $H^*(X_2; \mathbb{Q})$, $\overline{m}_2 = m_2(m_2 \otimes id)$, $\tilde{\alpha} = \Psi^{-1}(\alpha)$ and λ_2, ρ_2 are a left inverse and a right inverse of $\text{Hom}_{\text{Alg}}(H^*(X_2; \mathbb{Q}), H^*(X_2; \mathbb{Q}))$, respectively. Let

$$\overline{\Gamma}_{\text{Mo}}(\alpha) = H^*(\Gamma_{\text{Mo}}(\omega)), \quad \overline{\Gamma}'_{\text{Mo}}(\alpha) = H^*(\Gamma'_{\text{Mo}}(\omega))$$

where $\omega : X_2 \wedge X_2 \rightarrow X_1$ satisfies $H^*(\omega) = \alpha$. Then, V_{inv} , $V_{\text{p.a}}$, V_{Mo} and V_{fle} are subspaces of V .

THEOREM 4.7. *The following statements hold.*

- (1) *If (X_2, μ_2) is inversive, then $\text{Im } \overline{HD} \subset V_{\text{inv}}$ and the canonical map $S_{\text{inv}} \rightarrow V_{\text{inv}}/\text{Im } \overline{HD}$ is an isomorphism of algebraic loops.*
- (2) *If (X_2, μ_2) is power associative, then $\text{Im } \overline{HD} \subset V_{\text{p.a}}$ and the canonical map $S_{\text{p.a}} \rightarrow V_{\text{p.a}}/\text{Im } \overline{HD}$ is an isomorphism of algebraic loops.*
- (3) *If (X_2, μ_2) is Moufang, then $\text{Im } \overline{HD} \subset V_{\text{Mo}}$ and the canonical map $S_{\text{Mo}} \rightarrow V_{\text{Mo}}/\text{Im } \overline{HD}$ is an isomorphism of algebraic loops.*
- (4) *If (X_2, μ_2) is flexible, then $\text{Im } \overline{HD} \subset V_{\text{fle}}$ and the canonical map $S_{\text{fle}} \rightarrow V_{\text{fle}}/\text{Im } \overline{HD}$ is an isomorphism of algebraic loops.*

Proof. The assertions follow from Lemma 3.8 and Proposition 4.6. $\square$

In the rest of this section, we give some examples of V , V_{inv} , $V_{\text{p.a}}$, V_{Mo} , V_{fle} and $\text{Im } \overline{HD}$. We suppose that X_1 is a rational H-space with $H^*(X_1; \mathbb{Q}) = \mathbb{Q}[x]$ or $\Lambda(x)$ in the following two examples.

Example 4.8. Let X_2 be a rational H-space such that $H^*(X_2; \mathbb{Q}) = \Lambda(y)$. Then we have $V = V_{\text{inv}} = V_{\text{p.a}} = V_{\text{Mo}} = V_{\text{fle}}$ and $\text{Im } \overline{HD} = \{0\}$.

Proof. Since $y^2 = 0$, it follows that $V = V_{\text{inv}} = V_{\text{p.a}} = V_{\text{Mo}} = V_{\text{fle}}$. For any α in $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x \rangle, \Lambda(y))$, we may write $\alpha(x) = ry$ for some $r \in \mathbb{Q}$ if $\deg x = \deg y$ and $\alpha(x) = 0$ if $\deg x \neq \deg y$ by degree reasons. Then, we have $P_2(\alpha(x)) = 0$; that is, $\text{Im } \overline{HD} = \{0\}$. $\square$

Example 4.9. Let X_2 be a rational H-space such that $H^*(X_2; \mathbb{Q}) = \mathbb{Q}[y]$. Then one has the following.

1. If $\deg x = 2 \deg y$, then

$$V = V_{\text{inv}} = V_{\text{p.a}} = V_{\text{Mo}} = V_{\text{fle}} = \text{Im } \overline{HD} \cong \mathbb{Q}.$$

2. If $\deg x = 3 \deg y$, then

$$V \cong \mathbb{Q}^2 \quad \text{and} \quad V_{\text{inv}} = V_{\text{p.a}} = V_{\text{Mo}} = V_{\text{fle}} = \text{Im } \overline{HD} \cong \{(r_1, r_2) \in \mathbb{Q}^2 \mid r_1 = r_2\}.$$

3. If $\deg x = m \deg y$ and $m \geq 4$, then $V \cong \mathbb{Q}^{m-1}$,

$$(i) \quad V_{\text{inv}} \cong \begin{cases} V(m : \text{even}), \\ \left\{ (r_1, \dots, r_{m-1}) \in \mathbb{Q}^{m-1} \mid \sum_{j=1}^{m-1} (-1)^j r_j = 0 \right\} (m : \text{odd}), \end{cases}$$

$$(ii) \quad V_{\text{p.a}} \cong \left\{ (r_1, \dots, r_{m-1}) \in \mathbb{Q}^{m-1} \mid \sum_{j=1}^{m-1} (2^j - 2^{m-j}) r_j = 0 \right\},$$

$$(iii) \quad V_{\text{fle}} \cong \{(r_1, \dots, r_{m-1}) \in \mathbb{Q}^{m-1} \mid r_l = r_{m-l}, 1 \leq l \leq m-1\},$$

$$(iv) \quad \text{Im } \overline{HD} \cong \{(r_1, \dots, r_{m-1}) \in \mathbb{Q}^{m-1} \mid (m-1)!r_1 = \dots = i!(m-i)!r_i = \dots = (m-1)!r_{m-1}\}.$$

4. Other cases, $V = V_{\text{inv}} = V_{\text{p.a}} = V_{\text{Mo}} = V_{\text{fle}} = \text{Im } \overline{HD} = \{0\}$.

Proof. 1. By degree reasons, V is a one-dimensional vector space with the base σ defined by $\sigma(x) = y \otimes y$. Suppose that, for any $r \in \mathbb{Q}$, $r\sigma = \overline{HD}(\beta)$ for some β . We may write $\beta(x) = r_1 y^2$ for some $r_1 \in \mathbb{Q}$ and then we have $r = 2r_1$. This implies that $V = \text{Im } \overline{HD} \cong \mathbb{Q}$. Since V_{inv} , $V_{\text{p.a}}$, V_{Mo} and V_{fle} include $\text{Im } \overline{HD}$ by Theorem 4.7, statement 1 is shown.

2. We choose the basis σ_1 and σ_2 of V defined by

$$\sigma_1(x) = y \otimes y^2 \quad \text{and} \quad \sigma_2(x) = y^2 \otimes y.$$

Then, for any element $\alpha = r_1\sigma_1 + r_2\sigma_2$ ($r_i \in \mathbb{Q}$) of V , we have

$$m_2(\lambda_2 \otimes id)H^*(q)\alpha(x) - m_2(id \otimes \rho_2)H^*(q)\alpha(x) = (-2r_1 + 2r_2)y^3,$$

$$\overline{m}_2(H^*(\mu_2) \otimes id)H^*(q)\alpha(x) - \overline{m}_2(id \otimes H^*(\mu_2))H^*(q)\alpha(x) = (-2r_1 + 2r_2)y^3.$$

The equalities show that $\alpha \in V_{\text{inv}}$ ($\alpha \in V_{\text{p.a}}$) if and only if $r_1 = r_2$. The element α is in $\text{Im } \overline{HD}$ if and only if there exists β in $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x \rangle, \mathbb{Q}\langle y \rangle)$ such that $\alpha = \overline{HD}(\beta)$. Put $\beta(x) = ry^3$ ($r \in \mathbb{Q}$), then we have $r_1 = 3r = r_2$. Hence,

$$V_{\text{p.a}} = V_{\text{inv}} = \text{Im } \overline{HD} \cong \{(r_1, r_2) \in \mathbb{Q}^2 \mid r_1 = r_2\}.$$

For this reason, we obtain $V_{\text{inv}} = V_{\text{p.a}} = V_{\text{Mo}} = V_{\text{fle}} = \text{Im } \overline{HD}$ by the implication (2.1) and Theorem 4.7. Therefore statement 2 is shown.

3. We can choose the basis σ_i of V defined by

$$\sigma_i(x) = y^i \otimes y^{m-i} \quad (1 \leq i \leq m-1)$$

and, for any $\alpha = \sum_i r_i \sigma_i$ ($r_i \in \mathbb{Q}$) in V , we have the following equations:

$$m_2(\lambda_2 \otimes id)H^*(q)\alpha(x) - m_2(id \otimes \rho_2)H^*(q)\alpha(x) = \sum_{i=1}^{m-1} ((-1)^i - (-1)^{m-i})r_i y^m,$$

$$\overline{m}_2(H^*(\mu_2) \otimes id)H^*(q)\alpha(x) - \overline{m}_2(id \otimes H^*(\mu_2))H^*(q)\alpha(x) = \sum_{i=1}^{m-1} (2^i - 2^{m-i})r_i y^m.$$

The equalities show (i) and (ii). A straightforward calculation shows that

$$\begin{aligned} & H^*(\Gamma_{\text{fle}})(\tilde{\alpha} \otimes \tilde{\alpha})\Psi(H^*(\mu_1))(x) \\ &= H^*((q \times q)(t \times (id_2 \times \mu_2 t)(\Delta_2 \times id_2))\Delta'_2)(\tilde{\alpha}(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \tilde{\alpha}(x)) \\ &= H^*(t) \left(\sum_{i=1}^{m-1} r_i y^i \otimes y^{m-i} \right) + H^*(\Delta_2 \times id_2) \left(\sum_{i=1}^{m-1} \sum_{j=0}^{m-i} \binom{m-i}{j} r_i y^i \otimes y^{m-i-j} \otimes y^j \right) \\ &= \sum_{i=1}^{m-1} r_i y^{m-i} \otimes y^i + \sum_{i=1}^{m-1} \sum_{j=0}^{m-i} \binom{m-i}{j} r_i y^{m-j} \otimes y^j. \end{aligned}$$

We see here that

$$\begin{aligned} & \sum_{i=1}^{m-1} \sum_{j=0}^{m-i} \binom{m-i}{j} r_i y^{m-j} \otimes y^j \\ &= \sum_{j=1}^{m-1} \binom{m-j}{0} r_j y^m \otimes 1 + \sum_{j=1}^{m-1} \binom{m-j}{1} r_j y^{m-1} \otimes y + \cdots \\ & \quad + \sum_{j=1}^{m-i} \binom{m-j}{i} r_j y^{m-i} \otimes y^i + \cdots + r_1 y \otimes y^{m-1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{m-1} r_j y^m \otimes 1 + \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} \binom{m-j}{i} r_j y^{m-i} \otimes y^i \\
&= \sum_{j=1}^{m-1} r_j y^m \otimes 1 + \sum_{i=1}^{m-1} \sum_{j=1}^i \binom{m-j}{m-i} r_j y^i \otimes y^{m-i}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&H^*(\Gamma'_{\text{fle}})(\tilde{\alpha} \otimes \tilde{\alpha})\Psi(H^*(\mu_1))(x) \\
&= H^*((id_2 \times id_2 \times (\mu_2 \times id_2)(id_2 \times t)(\Delta_2 \times id_2))\Delta'_2)(\tilde{\alpha}(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \tilde{\alpha}(x)) \\
&= \sum_{i=1}^{m-1} r_i y^i \otimes y^{m-i} + H^*(\Delta_2 \times id_2) \left(\sum_{i=1}^{m-1} \sum_{j=0}^i \binom{i}{j} y^j \otimes y^{m-i} \otimes y^{i-j} \right) \\
&= \sum_{i=1}^{m-1} r_i y^i \otimes y^{m-i} + \sum_{i=1}^{m-1} \sum_{j=0}^i \binom{i}{j} r_i y^{m-i+j} \otimes y^{i-j}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{m-1} \sum_{j=0}^i \binom{i}{j} r_i y^{m-i+j} \otimes y^{i-j} \\
&= \sum_{j=1}^{m-1} \binom{j}{j} r_j y^m \otimes 1 + \sum_{j=1}^{m-1} \binom{j}{j-1} r_j y^{m-1} \otimes y + \cdots \\
&\quad + \sum_{j=i}^{m-1} \binom{j}{j-i} r_j y^{m-i} \otimes y^i + \cdots + r_{m-1} y \otimes y^{m-1} \\
&= \sum_{j=1}^{m-1} r_j y^m \otimes 1 + \sum_{i=1}^{m-1} \sum_{j=i}^{m-1} \binom{j}{j-i} r_j y^{m-i} \otimes y^i \\
&= \sum_{j=1}^{m-1} r_j y^m \otimes 1 + \sum_{i=1}^{m-1} \sum_{j=m-i}^{m-1} \binom{j}{j-m+i} r_j y^i \otimes y^{m-i}.
\end{aligned}$$

Combining these results, we have

$$\begin{aligned}
&H^*(\Gamma_{\text{fle}})(\tilde{\alpha} \otimes \tilde{\alpha})\Psi(H^*(\mu_1))(x) - H^*(\Gamma'_{\text{fle}})(\tilde{\alpha} \otimes \tilde{\alpha})\Psi(H^*(\mu_1))(x) \\
&= \sum_{i=1}^{m-1} \left(r_{m-i} + \sum_{j=1}^i \binom{m-j}{m-i} r_j - r_i - \sum_{j=m-i}^{m-1} \binom{j}{j-m+i} r_j \right) y^i \otimes y^{m-i}.
\end{aligned}$$

Thus α is in V_{fle} if and only if the following equation holds for $i = 1, 2, \dots, m-1$:

$$r_{m-i} + \sum_{j=1}^i \binom{m-j}{m-i} r_j = r_i + \sum_{j=m-i}^{m-1} \binom{j}{j-m+i} r_j. \quad (4.1)$$

In the case where $i = 2$, we have

$$r_{m-2} + \binom{m-1}{m-2} r_1 + \binom{m-2}{m-2} r_2 = r_2 + \binom{m-2}{0} r_{m-2} + \binom{m-1}{1} r_{m-1}$$

by equation (4.1). It follows that $r_1 = r_{m-1}$. Suppose that the equations $r_j = r_{m-j}$ hold for some $1 \leq l \leq m-3$ and any $1 \leq j \leq l$. Then, equation (4.1) in the case where $i = l+2$ shows that

$$\begin{aligned} r_{m-(l+2)} + \sum_{j=1}^l \binom{m-j}{m-(l+2)} r_j + \binom{m-(l+1)}{m-(l+2)} r_{l+1} + r_{l+2} \\ = r_{l+2} + \sum_{j=m-l}^{m-1} \binom{j}{j-m+(l+2)} r_j + r_{m-(l+2)} + \binom{m-l-1}{1} r_{m-(l+1)}. \end{aligned}$$

The assumption implies that

$$\sum_{j=1}^l \binom{m-j}{m-(l+2)} r_j = \sum_{j=1}^l \binom{m-j}{m-(l+2)} r_{m-j} = \sum_{j=m-l}^{m-1} \binom{j}{j-m+(l+2)} r_j$$

and it follows that $r_{l+1} = r_{m-(l+1)}$. Conversely, we see that the equalities $r_l = r_{m-l}$ for $1 \leq l \leq m-1$ imply the equations

$$\sum_{j=1}^i \binom{m-j}{m-i} r_j = \sum_{j=1}^i \binom{m-j}{m-i} r_{m-j} = \sum_{j=m-i}^{m-1} \binom{j}{j-m+i} r_j;$$

that is, equation (4.1) holds. Hence (iii) is shown. If α is in $\text{Im } \overline{HD}$, that is, $\alpha = \overline{HD}(\beta)$ for some β in $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle x \rangle, \mathbb{Q}\langle y \rangle)$, then we may write $\beta(x) = ry^m$ for some $r \in \mathbb{Q}$ and the equation

$$\alpha(x) = \overline{HD}(\beta)(x) = \sum_{j=1}^{m-1} \binom{m}{j} r y^j \otimes y^{m-j}$$

shows that $r_i = \binom{m}{i} r$ for any i . Therefore (iv) is shown.

4. We readily see that $V = \{0\}$ by degree reasons. It implies the statement 4. □

Proof of Assertion 1.1. If $n = km$ and $k \geq 4$ is even, by Example 4.9, we have

$$V_{\text{Mo}}/\text{Im } HD \subseteq V_{\text{fe}}/\text{Im } HD \subsetneq V_{\text{p.a}}/\text{Im } HD \subsetneq V/\text{Im } HD = V_{\text{inv}}/\text{Im } HD.$$

Thus statement (1) holds. A similar argument shows the other cases. □

Proof of Corollary 1.2. The map

$$l : CH(K(\mathbb{Z}, 2n); S^n) \longrightarrow CH(K(\mathbb{Q}, 2n); S_{\mathbb{Q}}^n)$$

induced by the localization maps $K(\mathbb{Z}, 2n) \rightarrow K(\mathbb{Q}, 2n)$ and $S^n \rightarrow S_{\mathbb{Q}}^n$ is injective since $CH(K(\mathbb{Z}, 2n); S^n) \cong \mathbb{Z}$ and $CH(K(\mathbb{Q}, 2n); S^n) \cong \mathbb{Q}$. We see that the map l is indeed the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. On the other hand, by Example 4.8, any central H-extensions of $K(\mathbb{Q}, 2n)$ by $S_{\mathbb{Q}}^n$ are Moufang. Therefore, any central H-extensions of $K(\mathbb{Z}; 2n)$ by S^n are also Moufang. Since $CH(K(\mathbb{Z}, 2n); S^n)$ is a subset of the set of essentially different multiplications of $K(\mathbb{Z}, 2n) \times S^n$, this completes the proof. □

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