THE ANTI-DERIVATIVE METHOD IN THE HALF SPACE AND APPLICATION TO DAMPED WAVE EQUATIONS WITH NON-CONVEX CONVECTION

Itsuko HASHIMOTO and Yoshihiro UEDA
(Received 17 July 2011)

Abstract. We study the asymptotic stability of nonlinear waves for damped wave equations with a non-convex convection term on the half line. In the case where the convection term satisfies the convexity and sub-characteristic conditions at the origin, it is shown by our previous work [Osaka J. Math. To appear] that the solution tends to the stationary wave. In this paper, we deal with the damped wave equations with convection term which is not necessarily the convexity at the origin, and we show not only the asymptotic stability of the non-degenerate stationary wave but also for the degenerate stationary wave. Furthermore, for the non-degenerate case, we also show that the time convergence rate is polynomially (respectively exponentially) fast if the initial perturbation decays polynomially (respectively exponentially) as $x$ goes up to infinity. Our proofs are based on a combination of the anti-derivative method and the $L^2$ weighted energy method.

1. Introduction

We consider the initial-boundary value problem on the half line for damped wave equations with a nonlinear convection term:

\[
\begin{aligned}
&u_{tt} - u_{xx} + u_t + f(u) x = 0, & x > 0, t > 0, \\
&u(0, t) = u_-, & t > 0, \\
&\lim_{x \to \infty} u(x, t) = 0, & t > 0, \\
&u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x > 0,
\end{aligned}
\]

(1.1)

where the function $f = f(u)$ is a given smooth function satisfying $f(0) = 0$ and $u_-$ is a given constant with $u_- < 0$. In this problem, we assume that the initial data $u_0(x)$ satisfies $u_0(0) = u_-$ and $\lim_{x \to \infty} u_0(x) = 0$ as the compatibility conditions. Throughout this paper, we impose that the convection term satisfies the following condition:

\[
|f''(0)| < 1, \quad f(u) > f(0)(= 0) \quad \text{for } u \in [u_-, 0). \tag{1.2}
\]

We note that the first condition in (1.2) is the so-called sub-characteristic condition. Ueda and Kawashima [12] and Ueda [10, 11] dealt with the damped wave equation (1.1) as a

2010 Mathematics Subject Classification: Primary 35B40; Secondary 35L70.
Keywords: asymptotic stability; stationary solution; anti-derivative method; energy method; damped wave equation.
derivation of a relaxation system and by applying the Chapman–Enskog expansion to the relaxation system, they suggested that the dissipative structure of (1.1) is similar to one of the viscous conservation laws. Indeed, Ueda \[10\] considered the case where \( f(u) \) of (1.1) satisfies
\[
f''(u) > 0, \quad |f'(u)| < 1 \quad \text{for} \quad u \in [u_-, 0],
\]
which is the case when the solution of the corresponding viscous conservation laws tends toward a stationary solution. Then Ueda \[10\] actually showed that the solution of (1.1) tends toward the stationary solution \( \phi \), provided that the initial perturbation is suitably small. Here, the stationary solution \( \phi = \phi(x) \) is defined by the solution of the stationary problem corresponding to (1.1):
\[
\begin{align*}
f(\phi) &= \phi_x, & x > 0, \\
\phi(0) &= u_-, & \lim_{x \to \infty} \phi(x) = 0.
\end{align*}
\]
(1.4)

Additionally, Ueda \[10\] derived the polynomial and exponential convergence rate to the non-degenerate stationary solution, and Ueda et al \[13\] obtained the polynomial convergence rate to the degenerate stationary solution under condition (1.3). As a direct expansion, Hashimoto and Ueda \[2\] considered the case when the convection term satisfies
\[
f''(0) > 0, \quad |f'(0)| < 1, \quad f(u) > 0 \quad \text{for} \quad u \in [u_-, 0].
\]
(1.5)

In our paper \[2\], we also consider the case \( u_- < 0 < u_+ \) for some positive constant \( u_+ \) with \( \lim_{t \to \infty} u(x, t) = u_+ \), which is the case when the solution of corresponding viscous conservation laws tends toward a superposition of the stationary solution and rarefaction wave. Then Hashimoto and Ueda \[2\] actually showed that the superposition of the stationary solution and rarefaction wave is asymptotically stable for problem (1.1) by using a technical \( L^2 \) weighted energy method.

On the other hand, Liu and Nishihara in \[6\] investigated the asymptotic stability of travelling waves for viscous conservation laws with non-convex condition in the one-dimensional whole space. In this paper, they employed the anti-derivative method and got the desired result. Inspired by this argument, we try to relax condition (1.5), and achieve it. Namely, by using the anti-derivative method in the half space, we can derive the asymptotic stability of the stationary wave to (1.1) under condition (1.2).

In the present paper, we show the asymptotic stability of both the non-degenerate stationary wave and degenerate stationary wave. For the degenerate case, we need the assumption that the initial data belong to the corresponding weighted Sobolev space. Moreover, we focus on the non-degenerate case and obtain the polynomial and exponential convergence rates of the stationary waves.

This paper is organized as follows. The main theorems are given in Section 2. In Section 3, we reformulate our initial-boundary value problem (1.1). Moreover, we show the existence of the corresponding stationary waves, and prepare the weight function and some preliminaries. In Section 4, we prove the asymptotic stability result under the non-convex condition (1.2) by using the anti-derivative method and the weighted energy method. By using the space-time weighted energy method, we obtain the polynomial and exponential convergence rate in Section 5. Finally, we give some remarks in the appendix.
Notation

Let \( L^2 = L^2(\mathbb{R}_+) \) and \( H^s = H^s(\mathbb{R}_+) \) denote the usual Lebesgue space of square integrable functions and the \( s \)th order Sobolev space on the half line \((0, \infty)\) with norms \( \| \cdot \|_{L^2} \) and \( \| \cdot \|_{H^s} \), respectively.

For \( \alpha > 0 \), \( L^2_{\alpha} = L^2_{\alpha}(\mathbb{R}_+) \) denotes the polynomially weighted \( L^2 \) space with the norm
\[
\|u\|_{L^2_{\alpha}} = \left( \int_0^\infty (1 + x)^\alpha |u(x)|^2 \, dx \right)^{1/2},
\]
while \( L^2_{\alpha,\exp} = L^2_{\alpha,\exp}(\mathbb{R}_+) \) denotes the exponentially weighted \( L^2 \) space with the norm
\[
\|u\|_{L^2_{\alpha,\exp}} = \left( \int_0^\infty e^{\alpha x} |u(x)|^2 \, dx \right)^{1/2}.
\]

Similarly, we define the corresponding weighted Sobolev spaces \( H^s_{\alpha} = H^s_{\alpha}(\mathbb{R}_+) \) and \( H^s_{\alpha,\exp} = H^s_{\alpha,\exp}(\mathbb{R}_+) \) with a positive integer \( s \). Especially, we note that \( H^0_{\alpha} = H^0 \).

For an interval \( I \) and a Banach space \( X, C^s(I; X) \) denotes the space of \( s \)-times continuously differentiable functions on the interval \( I \) with values in \( X \). Finally, \( C \) and \( c \) in this paper are defined as positive generic constants unless they need to be distinguished.

2. Main theorems

In this section, we state our main results. The first theorem is the asymptotic stability of the stationary solution. To investigate the stability of the stationary wave \( \phi \), we assume that \( u_0 - \phi \) and \( u_1 \) are integrable. Then we can define the following functions:
\[
z_0(x) = -\int_x^\infty (u_0(y) - \phi(y)) \, dy \in L^2, \quad z_1(x) = -\int_x^\infty u_1(y) \, dy \in L^2.
\]

By using these functions, we obtain the stability results as follows.

**Theorem 2.1.** Suppose that \( f(u) \) satisfies (1.2). Let \( \phi(x) \) be the stationary solution satisfying problem (1.4). Then the following results hold.

(i) For the non-degenerate case \( f'(0) < 0 \), assume that \( z_0 \in H^2 \) and \( z_1 \in H^1 \). Then there is a positive constant \( \varepsilon_0 \) such that if \( \|z_0\|_{H^2} + \|z_1\|_{H^1} \leq \varepsilon_0 \), then the initial-boundary value problem (1.1) has a unique global solution \( u(x, t) \) satisfying \( u - \phi \in C^0([0, \infty); H^1) \cap C^1([0, \infty); L^2) \) and asymptotic behavior
\[
\lim_{t \to \infty} \sup_{x > 0} |u(x, t) - \phi(x)| = 0. \tag{2.1}
\]

(ii) For the degenerate case \( f'(0) = 0 \), assume that \( z_0 \in H^2_1 \) and \( z_1 \in H^1_1 \). Then there is a positive constant \( \varepsilon_1 \) such that if \( \|z_0\|_{H^2} + \|z_1\|_{H^1} \leq \varepsilon_1 \), then problem (1.1) has a unique global solution in time \( u(x, t) \) satisfying \( u - \phi \in C^0([0, \infty); H^1) \cap C^1([0, \infty); L^2) \) and asymptotic behavior (2.1).

The second purpose of this paper is to obtain the convergence rates of the solution \( u \) toward the stationary wave \( \phi \). Theorems 2.2 and 2.3 give the polynomial and the exponential stability result, respectively.
THEOREM 2.2. Suppose the same assumptions as in Theorem 2.1(i) hold true. Assume that $z_0 \in H^2_\alpha$ and $z_1 \in H^1_\alpha$ for $\alpha > 0$. Let $u(x, t)$ be the global solution to problem (1.1), which is constructed in Theorem 2.1(i). Then it holds that

$$
\|u(t) - \phi\|_{H^1} \leq CE(1 + t)^{-\alpha/2},
$$

for $t \geq 0$, where $C$ is a positive constant and $E = \|z_0\|_{H^2_\alpha} + \|z_1\|_{H^1_\alpha}$.

THEOREM 2.3. Suppose the same assumptions as in Theorem 2.1(i) hold true. Assume that $z_0 \in H^2_{\alpha, \exp}$ and $z_1 \in H^1_{\alpha, \exp}$ for $\alpha > 0$. Let $u(x, t)$ be the global solution to problem (1.1), which is constructed in Theorem 2.1(i). Then it holds that

$$
\|u(t) - \phi\|_{H^1} \leq CE_{\alpha, \exp} e^{-\beta t},
$$

for $t \geq 0$, where $\beta$ is a positive constant depending on $\alpha$, $C$ is a positive constant and $E_{\alpha, \exp} = \|z_0\|_{H^2_{\alpha, \exp}} + \|z_1\|_{H^1_{\alpha, \exp}}$.

3. Preliminaries and reformulation of the problem

The purpose of this section is to prepare for the proof of the main theorems. First of all, we review the fundamental properties of the stationary solution $\phi(x)$ which is defined as the solution of the stationary problem (1.4). To this end, we apply Taylor expansion for the nonlinear term $f(u)$. Since the nonlinear term $f(u)$ is smooth, it is described as

$$
f(u) = a_{q+1}(0)(-u)^{q+1} + a_{q+2}(\theta u)(-u)^{q+2},
$$

for some constant $\theta$ with $0 < \theta < 1$, where $q$ is a non-negative integer (called the degeneracy exponent) and $a_q(u)$ defined by

$$
a_q(u) = \frac{(-1)^q}{q!} \frac{d^q}{du^q} f(u)
$$

is not equal to zero. Under assumption (1.2), we see that $a_{q+1}(0)$ is a positive constant.

LEMMA 3.1. Suppose that $f(u)$ satisfies (1.2). Then the stationary problem (1.4) has a unique smooth solution $\phi(x)$ satisfying $u_- < \phi(x) < 0$ and $\phi(x) > 0$ for $x > 0$. Moreover, the following properties hold:

(i) for the non-degenerate case $f'(0) < 0$, it holds that

$$
ce^{-c \alpha x} \leq -\phi(x) \leq Ce^{-c \alpha x};
$$

(ii) for the degenerate case $f'(0) = 0$, it holds that

$$
c(1 + x)^{-1/q} \leq -\phi(x) \leq C(1 + x)^{-1/q}.
$$

Here, non-negative integer $q$ denotes the degeneracy exponent defined by (3.1), and $C$ and $c$ are positive constants.

For its proof, we refer the reader to [5, 6, 10].
Now we rewrite our original problem (1.1). Let \( \phi(x) \) be the stationary solution satisfying (1.4). We introduce a new unknown function \( v(x, t) \) and its integral \( z(x, t) \) by

\[
v(x, t) = u(x, t) - \phi(x),
\]

\[
z(x, t) = -\int_{x}^{\infty} v(y, t) \, dy.
\]

Here, we assume the integrability of \( z(x, t) \) over \( \mathbb{R}_+ \). This transformation is motivated by the argument in Liu and Nishihara [6]. By using (1.1) and (1.4), we reformulate (1.1) in terms of \( v(x, t) \) as

\[
\begin{aligned}
v_{tt} - v_{xx} + v_t + \{f(\phi + v) - f(\phi)\}_x &= 0, & x > 0, \ t > 0, \\
v(0, t) &= 0, \ & t > 0, \\
v(x, 0) &= v_0(x), \ v_t(x, 0) = v_1(x), \ & x > 0,
\end{aligned}
\]

(3.2)

where we put \( v_0(x) = u_0(x) - \phi(x) \) and \( v_1(x) = u_1(x) \). We also reformulate (1.1) in terms of \( z(x, t) \) as

\[
\begin{aligned}
z_{tt} - z_{xx} + z_t + \{f(\phi + z_x) - f(\phi)\} &= 0, & x > 0, \ t > 0, \\
z_x(0, t) &= 0, \ & t > 0, \\
z(x, 0) &= z_0(x), \ z_t(x, 0) = z_1(x), \ & x > 0.
\end{aligned}
\]

(3.3)

where we put

\[
z_0(x) = -\int_{x}^{\infty} (u_0(y) - \phi(y)) \, dy \quad \text{and} \quad z_1(x) = -\int_{x}^{\infty} u_1(y) \, dy.
\]

In order to derive our main theorems, we will discuss the reformulated problem (3.3) and use the weighted energy method in Sections 4 and 5.

The application of weight functions to construct the first energy has been used in several works (cf. [1, 2, 4, 7, 8, 9]). The stability of shock profiles for non-convex systems of viscoelasticity was investigated by Nishihara [9], Kawashima and Matsumura [4] and Matsumura and Mei [7]. Matsumura and Nishihara [8] and Hashimoto and Matsumura [1] considered the asymptotic behavior of solutions for scalar viscous conservation laws with non-convex nonlinearity. On the other hand, the asymptotic behavior of solutions for damped wave equations with non-convex convection term was considered by Hashimoto and Ueda [2] as we described in Section 1.

In the present paper, we refer to [8] to construct the weight function. We define a weight function as

\[
w(u) = \frac{(-e^{Au} + 1)}{f(u)} \quad \text{for} \ u \in [u_-, 0],
\]

(3.4)

where \( A \) is a positive constant determined later. We use this weight function when we derive \textit{a priori} estimate in the latter section. For this weight function, we obtain the following lemma.

**Lemma 3.2.** Suppose that \( f(u) \) satisfies (1.2). Let \( w(u) \) be the weight function defined in (3.4). Then there exists a positive constant \( \delta \) such that if \( A \geq \delta \), then \( w(u) \) satisfies the following conditions:

(i) \( c(-u)^{-q} < w(u) < C(-u)^{-q} \),

(ii) \( ((fw)'(u))^2 < w^2(u) \),

(iii) \( (fw)''(u) < 0 \),

(iv) \( (fw)'(u_-) < 0 \),
for \( u \in [u_-, 0] \), where non-negative integer \( q \) is the degeneracy exponent defined by (3.1), and \( C \) and \( c \) are some positive constants.

By using Lemmas 3.1 and 3.2, we can obtain the following corollary immediately. We omit the proof in detail.

**Corollary 3.3.** Let \( \phi \) be the stationary solution of (1.4). Then the weight function \( w(\phi) \) defined in (3.4) satisfies the following properties:

(i) for the non-degenerate case \( f'(0) < 0 \), the weight function \( w(\phi) \) satisfies the condition

\[
c < w(\phi) < C \quad \text{for } \phi \in [u_-, 0];\]

(ii) for the degenerate case \( f'(0) = 0 \), the weight function \( w(\phi) \) satisfies the condition

\[
c(1 + x) < w(\phi) < C(1 + x) \quad \text{for } \phi \in [u_-, 0].\]

Here, \( C \) and \( c \) are some positive constants which are independent of \( x \).

**Proof of Lemma 3.2.** Since (iii) and (iv) are trivial, we prove only (i) and (ii).

We first prove (i). In order to derive the desired estimate, we divide the interval \([u_-, 0]\) into \([u_-, -r]\) and \([-r, 0]\) for some positive constant \( r \) determined later. First, we consider the case in the interval \([-r, 0]\). By using (3.1) and \( e^{Au} = 1 + Ae^{\tilde{\theta}u} \) for some real number \( \tilde{\theta} \) with \( 0 < \tilde{\theta} < 1 \), we obtain

\[
w(u) = (-u)^{-q} \frac{Ae^{\tilde{\theta}u}}{a_{q+1}(0) + a_{q+2}(\theta u)(-u)}, \quad (3.5)
\]

where \( \theta \) is introduced in (3.1). Now we estimate that \( Ae^{-Ar} \leq Ae^{\tilde{\theta}u} \leq A \) for \( u \in [-r, 0] \). Moreover, letting \( r \) be sufficiently small and using the positivity of \( a_{q+1}(0) \), we obtain

\[
\frac{1}{2} a_{q+1}(0) \leq a_{q+1}(0) + a_{q+2}(\theta u)(-u) \leq 2a_{q+1}(0),
\]

for \( u \in [-r, 0] \). Therefore, applying the above estimates to (3.5), we get

\[
\frac{2Ae^{-Ar}}{3a_{q+1}(0)} (-u)^{-q} \leq w(u) \leq \frac{2A}{a_{q+1}(0)} (-u)^{-q}.
\]

We next consider the case in the interval \([u_-, -r]\). By a simple calculation, we have

\[
\frac{1 - e^{-Ar}}{\max_{u \in [u_-, -r]} |f(u)|} \leq w(u) \leq \frac{1 - e^{Au_+}}{\min_{u \in [u_-, -r]} |f(u)|}.
\]

Thus, taking \( A \) suitably large, we obtain \( c \leq w(u) \leq C \) for \( u \in [u_-, -r] \). Finally, combining the above two estimates, we complete the proof of (i).

We next derive (ii). Similar to the previous argument, we also divide the interval \([u_-, 0]\) into \([u_-, -r]\) and \([-r, 0]\). We prove \(|(fw)'(u)|/w(u) < 1\) which is enough to derive the inequality (iii). We first consider the case in the interval \([-r, 0]\). It follows from the representation (3.1) that

\[
\left| \frac{(fw)'(u)}{w(u)} \right| = \left| f(u)Ae^{Au} \frac{1}{1 - e^{Au}} \right|
\]

\[
= |a_{q+1}(0) + a_{q+2}(\theta u)(-u)|e^{A(1-\tilde{\theta}u)}(-u)^q
\]

\[
\leq (a_{q+1}(0) + |a_{q+2}(\theta u)r|r)q,
\]
for \( u \in [-r, 0] \). Therefore, by using the sub-characteristic condition \(|f'(0)| < 1\) for \( q = 0 \), and choosing \( r \) suitably small, we obtain \(|((fw)')/w(u)| < 1\) for any \( q \geq 0 \) and \( u \in [-r, 0] \). In the case \( u \in [u_-, -r] \), we have

\[
\left| \frac{(fw)'}{w}(u) \right| = \left| \frac{f(u)Ae^{Au}}{1 - e^{Au}} \right| \leq \frac{MA}{|e^{Au} - 1|},
\]

where \( M = \max_{u \in [u_-, -r]} |f(u)| \). Hence, taking \( A \) sufficiently large, we have \( MA/(e^{Ar} - 1) < 1 \) and obtain the desired estimate for \( u \in [u_-, -r] \). Thus the proof of (ii) and hence Lemma 3.2 is completed.

In order to derive the existence of the global solution in time, which is described in Theorem 2.1, we need to construct the local existence theorem. For this purpose, we define the solution space for an arbitrary \( T > 0, M > 0 \) and \( k = 0, 1 \) as

\[
X_{k,M}(T) := \{ z \in C^0([0, T]; H^2_k); z_t \in C^0([0, T]; H^1_k), N(T) \leq M \},
\]

where we define

\[
N(T) = \sup_{0 \leq t \leq T} (\|z(t)\|_{H^2} + \|z_t(t)\|_{H^1}).
\]

Then the local existence theorem of the solution \( z \) for (3.3) in the solution space \( X_{k,M}(T) \) is stated as follows.

**Proposition 3.4.** (Local existence) Let \( k = 0, 1 \). For any positive constant \( M \), there exists a positive constant \( t_0 = t_0(M) \) such that if \( z_0 \in H^2_k, z_1 \in H^1_k \) and \( \|z_0\|_{H^2} + \|z_1\|_{H^1} \leq M \), then the initial boundary value problem (3.3) has a unique solution \( z \in X_{k,2M}(t_0) \).

We prove Proposition 3.4 by using a standard iterative method and omit the proof.

**4. A priori estimate**

The aim of this section is to prove Theorem 2.1. For this purpose, it is important to derive the following a priori estimate of solutions \( z \) for (3.3) in the Sobolev space \( H^2 \).

**Proposition 4.1.** (A priori estimate) Suppose that the same assumptions as in Theorem 2.1 hold true. Let \( k = 0, 1 \). Then there exists a positive constant \( \varepsilon_2 \) such that if \( z \in X_{k,\varepsilon_2}([0, T]) \) is the solution of the problem (3.3) for some \( T > 0 \), then the following a priori estimates hold:

(i) for the non-degenerate case \( f'(0) < 0 \), it holds that

\[
\|z(t)\|_{H^2_k}^2 + \|z_t(t)\|_{H^1_k}^2 + \int_0^t (\|z_{tt}(\tau)\|_{H^2_k}^2 + \|\sqrt{\phi_\lambda}z(\tau)\|_{L^2_k}^2) d\tau \leq C(\|z_0\|_{H^2_k}^2 + \|z_1\|_{H^1_k}^2),
\]

with \( k = 0 \), where \( t \in [0, T] \) and \( C \) is a positive constant which is independent of \( T \);

(ii) for the degenerate case \( f'(0) = 0 \), the a priori estimate (4.1) with \( k = 1 \) holds true.
Proof. Throughout this proof, we use the weighted $L^2$ norm:
\[ \| f \|_{L^2_w} := \left( \int_0^\infty w(\phi(x)) |f(x)|^2 \, dx \right)^{1/2}, \]
where $w$ is the weight function defined by (3.4). Moreover, we define the corresponding weighted Sobolev norm $\| \cdot \|_{H^s_w}$ with a positive integer $s$. For this weighted norm, by using Corollary 3.3, we see the following properties:
\[ c \| \cdot \|_{L^2} \leq \| \cdot \|_{L^2_w} \leq C \| \cdot \|_{L^2} \]
for the non-degenerate case $f'(0) < 0$,
\[ c \| \cdot \|_{L^2} \leq \| \cdot \|_{L^2_w} \leq C \| \cdot \|_{L^2} \]
for the degenerate case $f'(0) = 0$.

We prove (i) and (ii) in Proposition 4.1 in parallel. Under the smallness of $N(t)$, we derive the a priori estimate (4.1). The first equality of (3.3) is rewritten as
\[ z_{tt} - z_{xx} + z_t + f'(\phi)z_x = g, \]
where
\[ g = - \left\{ f(\phi + z_x) - f(\phi) - f'(\phi)z_x \right\} = O(|z_x|^2). \]

Multiplying (4.3) by $w(\phi)z$, we then get
\[ \left\{ \frac{1}{2} w(\phi)z_t^2 + w(\phi)zz_t \right\}_t + \left\{ \frac{1}{2} f'(\phi)w(\phi)z_x^2 - w(\phi)zz_x + \frac{1}{2} w'(\phi)\phi_x z^2 \right\}_x \]
\[ - w(\phi)z_t^2 + w(\phi)z_x^2 - \frac{1}{2} (fw)'(\phi)\phi_x z^2 = w(\phi) zg. \]

We next multiply (4.3) by $w(\phi)z_t$. Then this yields
\[ \frac{1}{2} \{ w(\phi)z_t^2 + w(\phi)z_x^2 \}_t - \{ w(\phi)z_t z_x \}_x + w(\phi)z_x^2 + (fw)'(\phi)z_t z_x = w(\phi) z_t g. \]

Therefore, making a combination (4.5) $+ 2 \times (4.6)$, we obtain the energy equality
\[ E_t + D + F_x = R, \]
where we define
\[ E = w(\phi)(\frac{1}{2} z_t^2 + z_x^2 + z_t^2 + z_x^2), \quad R = w(\phi)(z + 2z_t)g, \]
\[ D = w(\phi)z_t^2 + w(\phi)z_x^2 + 2(fw)'(\phi)z_t z_x - \frac{1}{4} (fw)''(\phi)\phi_x z^2, \]
\[ F = \frac{1}{2} (fw)'(\phi)z_x^2 - w(\phi)zz_x - 2w(\phi)z_t z_x. \]

Since $E$ is a quadratic form in terms of $z$, $z_x$ and $z_t$, then calculating the discriminant of $E$, we have the condition
\[ E \sim w(\phi)(z_t^2 + z_x^2 + z_t^2). \]

In a similar way, since $D$ is a quadratic form in terms of $z$, $z_x$, $z_t$ and $\sqrt{\phi_x z}$, then by using Lemma 3.2(ii), we have
\[ w(\phi)z_x^2 + w(\phi)z_t^2 + 2(fw)'(\phi)z_t z_x = w(\phi) \left\{ z_x^2 + z_t^2 + 2 \frac{(fw)'(\phi)z_t z_x}{w(\phi)z_t z_x} \right\} \]
\[ \geq cw(\phi)(z_x^2 + z_t^2). \]
Thus, by using this inequality and Lemma 3.2(iii), this yields
\[ D \geq c\omega(t)(z_x^2 + z_t^2) + c\phi_xz^2. \] (4.10)

Now, integrating (4.7) with respect to \( x \) and \( t \), we have
\[
\int_0^\infty E(x, t) \, dx + \int_0^t \int_0^\infty D(x, \tau) \, dx \, d\tau - \frac{1}{2} \left( (f\omega)'(u_-) \right) \int_0^t z(0, \tau)^2 \, d\tau \\
\leq \int_0^\infty E(x, 0) \, dx + \int_0^t \int_0^\infty R(x, \tau) \, dx \, d\tau.
\]
Therefore, applying (4.4), (4.9), (4.10) and Lemma 3.2(iv), we get
\[
\|(z, z_t, z_x)(t)\|_{L^2_w}^2 + \int_0^t (\|(z_t, z_x)(\tau)\|_{L^2_w}^2 + \|\phi_xz(\tau)\|_{L^2_z}^2) \, d\tau + \int_0^t z(0, \tau)^2 \, d\tau \\
\leq C\|(z_0, z_1, z_{0,x})\|_{L^2_w}^2 + C\|(z, z_t)(t)\|_{L^\infty} \int_0^t \|z_x(\tau)\|_{L^2_w}^2 \, d\tau.
\]
Finally, using the relation \( \|(z, z_t)(t)\|_{L^\infty} \leq CN(t) \) and letting \( N(t) \) be suitably small, we arrive at the energy estimate for the lower-order derivatives as
\[
\|(z, z_t, z_x)(t)\|_{L^2_w}^2 + \int_0^t (\|(z_t, z_x)(\tau)\|_{L^2_w}^2 + \|\phi_xz(\tau)\|_{L^2_z}^2) \, d\tau \leq C\|(z_0, z_1, z_{0,x})\|_{L^2_w}^2.
\] (4.11)

Next, we proceed to the estimates of the higher-order derivatives \( z_x \). We note that the differentiation of the solution \( z \) satisfies (3.2) with \( v = z_x \). Thus, by employing (3.2), we calculate the \( L^2 \)-estimate of \( v \) and derive the \textit{a priori} estimates of \( z_x \).

Multiplying (3.2) by \( w(\phi)v \), we obtain
\[
\left\{ \frac{1}{2}w(\phi)v^2 + w(\phi)v v_t \right\}_t + \mathcal{F}_x - w(\phi)v_t^2 + w(\phi)v_x^2 + \frac{1}{2}(fw)''(\phi)\phi_xv^2 + \mathcal{R} = 0,
\] (4.12)where
\[
\mathcal{F} = \frac{1}{2}w'(\phi)\phi_xv^2 - w(\phi)v v_x + \frac{1}{2}f'(\phi + v)w(\phi)v^2,
\]
\[
\mathcal{R} = -\frac{1}{2}f''(\phi + v)w(\phi)v_xv^2 - \frac{1}{2}(f''(\phi + v) - f''(\phi))w(\phi)\phi_xv^2
\]
\[
- \frac{1}{2}(f'(\phi + v) - f'(\phi))w'(\phi)\phi_xv^2.
\]

We next multiply (3.2) by \( w(\phi)v_t \). Then we have
\[
\left\{ \frac{1}{2}w(\phi)v_t^2 + \frac{1}{2}w(\phi)v_x^2 + w(\phi)\phi_x \int_0^\eta f'(\phi + \eta) - f'(\phi) \, d\eta \right\}_t - \{w(\phi)v_t v_x\}_x
\]
\[
+ w(\phi)v_t^2 + (fw)'(\phi)v_t v_x + \{f'(\phi + v) - f'(\phi)\}w(\phi)v_t v_x = 0.
\] (4.13)
By making a combination (4.12) + 2× (4.13), we then have
\[
\{\tilde{E} + \tilde{R}_1\}_t + \tilde{D} + \tilde{F}_x = \tilde{R}_2,
\] (4.14)where \( \tilde{E}, \tilde{D}, \tilde{F}, \tilde{R}_1 \) and \( \tilde{R}_2 \) are defined by
\[
\tilde{E} = w(\phi)(\frac{1}{2}v^2 + v_t^2 + v_x^2 + vv_t), \quad \tilde{F} = \mathcal{F} - 2w(\phi)v_t v_x,
\]
\[
\tilde{D} = w(\phi)v_t^2 + w(\phi)v_x^2 + (fw)'(\phi)v_t v_x + \frac{1}{2}(fw)''(\phi)\phi_xv^2,
\]
\[
\tilde{R}_1 = w(\phi)\phi_x \int_0^\eta f'(\phi + \eta) - f'(\phi) \, d\eta, \quad \tilde{R}_2 = \mathcal{R} + 2\{f'(\phi + v) - f'(\phi)\}w(\phi)v_t v_x.
\]
For the terms $\tilde{E}$ and $\tilde{D}$, we employ Lemma 3.2 and the same argument as used for the derivation of (4.9) and (4.10). Then we can obtain

$$\tilde{E} \sim w(\phi)(v^2 + v_t^2 + v_x^2), \quad \tilde{D} \geq cw(\phi)(v_t^2 + v_x^2) + c\phi_x v^2.$$  \hspace{1cm} (4.15)

We estimate the remainder terms $\tilde{R}_1$ and $\tilde{R}_2$. By a straightforward calculation, we have

$$|\tilde{R}_1| \leq Cw(\phi)v^2.$$  \hspace{1cm} (4.16)

On the other hand, noting that $w(\phi)\phi_x$ and $w'(\phi)\phi_x$ are bounded for $\phi \in [u_-, 0]$, we can estimate that $|\tilde{R}_2| \leq Cw(\phi)|v_x||v|^2 + C|v|^3$. Thus this yields that

$$|\tilde{R}_2| \leq Cw(\phi)(v^2 + v_t^2 + v_x^2).$$  \hspace{1cm} (4.17)

Therefore, integrating (4.14) with respect to $x$ and $t$, and applying (4.15), (4.16) and (4.17) to the resultant equation, we obtain

$$\| (v, v_t, v_x)(t) \|^2_{L^2_w} + \int_0^t (\| (v, v_x)(\tau) \|^2_{L^2_w} + \| \sqrt{\phi_x} v(\tau) \|^2_{L^2}) \, d\tau$$

$$\leq C\| (v_0, v_1, v_0, x) \|^2_{L^2_w} + C\| v(t) \|^2_{L^2} + C \sup_{0 \leq \tau \leq t} \| v(\tau) \|_{L^\infty} \int_0^t \| (v, v_t, v_x)(\tau) \|^2_{L^2_w} \, d\tau.$$

Then, we make use of $v = z_x$ and apply the smallness of $N(t)$ to the above inequality. Furthermore, applying (4.11) to the resultant inequality, we arrive at

$$\| (z_x, z_{tx}, z_{xx})(t) \|^2_{L^2_w} + \int_0^t (\| (z_{tx}, z_{xx})(\tau) \|^2_{L^2_w} + \| \sqrt{\phi_x} z_x(\tau) \|^2_{L^2}) \, d\tau$$

$$\leq C\| (z_0, z_1, z_0, x) \|^2_{H^1_w}.$$  \hspace{1cm} (4.18)

Therefore, combining (4.11) and (4.18), we get

$$\| z(t) \|^2_{H^k_w} + \| z_t(t) \|^2_{H^1_w}$$

$$+ \int_0^t (\| z_t(\tau) \|^2_{H^k_w} + \| z_x(\tau) \|^2_{H^1_w} + \| \sqrt{\phi_x} z(\tau) \|^2_{L^2} + \| \sqrt{\phi_x} z_x(\tau) \|^2_{L^2}) \, d\tau$$

$$\leq C(\| z_0 \|^2_{H^k_w} + \| z_1 \|^2_{H^1_w}).$$

Finally, by applying (4.2) to the above estimate, we derive the desired estimate (4.1) with $k = 0, 1$. Hence the proof of Proposition 4.1 is completed.

**Proof of Theorem 2.1.** The global existence of solutions to the initial-boundary value problem (1.1) can be proved by the continuation argument based on a local existence result in Proposition 3.4 combined with the corresponding a priori estimate in Proposition 4.1. Hence, we omit the detail of the proof.

**5. Convergence rates of stationary solutions**

In this section, we prove Theorems 2.2 and 2.3. The main ideas of the proofs are due to Ueda [10]. We use the space-time weighted energy method introduced in Kawashima and Matsumura [3].
Proof of Theorem 2.2. We start with the energy equality (4.7). By the definition of weight \( w \), we calculate that 
\[
(fw)'(u) = -AE^{Au} - AE^{Au-}
\] for \( u \in [u_-, 0] \). Thus, by applying this inequality and Corollary 3.3(i) to \( F \) in (4.8), we have the estimate
\[
-F = -\frac{1}{2}(fw)'(\phi)z^2 + w(\phi)(zz_{x} + 2z_{t}z_{x})
\]
\[
\geq \frac{1}{2}AE^{Au-}z^2 + w(\phi)(zz_{x} + 2z_{t}z_{x}) \geq cz^2 - C(z^2 + \gamma^2), \tag{5.1}
\]
where \( c \) and \( C \) are positive constants.

Let \( \gamma \) and \( \beta \) be any positive constants satisfying \( 0 \leq \gamma, \beta \leq \alpha \). We multiply the equality (4.7) by \( (1 + t)^\gamma (1 + x)^\beta \), obtaining
\[
\{(1 + t)^\gamma (1 + x)^\beta E_t - \gamma(1 + t)^{\gamma-1}(1 + x)^\beta E + (1 + t)^\gamma (1 + x)^\beta D
\]
\[
+ \{(1 + t)^\gamma (1 + x)^\beta F\} \leq \beta(1 + t)^\gamma (1 + x)^{\beta-1}F = (1 + t)^\gamma (1 + x)^\beta R. \tag{5.2}
\]

Integrating (5.2) over \( (0, \infty) \times (0, t) \), substituting (4.9), (4.10) and (5.1) into the resultant inequality and letting \( \sup_{0 \leq t < \infty} \|z(t)\|_{H^1} \) be suitably small, we have
\[
(1 + t)^\gamma \|(z,t,z)\|_{L^2_{\beta}}^2 + \int_0^t (1 + \tau)^{\gamma-1} \|(z,t,z)\|_{L^2_{\beta-1}}^2 d\tau \leq CE_{\beta}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma-1} \|(z,t,z)\|_{L^2_{\beta-1}}^2 d\tau + \beta C \int_0^t (1 + \tau)^{\gamma-1} \|(z,t,z)\|_{L^2_{\beta-1}}^2 d\tau, \tag{5.3}
\]
for an arbitrary \( \gamma \) and \( \beta \) with \( 0 \leq \gamma, \beta \leq \alpha \), where \( C \) is a constant independent of \( \gamma \) and \( \beta \).

Next, we proceed to the higher-order inequality. Let \( \gamma \) and \( \beta \) be any positive constants satisfying \( 0 \leq \gamma, \beta \leq \alpha \). We multiply the equality (4.14) by \( (1 + t)^\gamma (1 + x)^\beta \), integrate the resultant equality over \( (0, \infty) \times (0, t) \) and take \( \sup_{0 \leq t < \infty} \|v(t)\|_{H^1} \) suitably small. Then we obtain
\[
(1 + t)^\gamma \|(v,t,v)\|_{L^2_{\beta}}^2 + \int_0^t (1 + \tau)^{\gamma-1} \|(v,t,v)\|_{L^2_{\beta-1}}^2 d\tau \leq CE_{\beta}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma-1} \|(v,t,v)\|_{L^2_{\beta-1}}^2 d\tau
\]
\[
+ \beta C \int_0^t (1 + \tau)^{\gamma-1} \|(v,t,v)\|_{L^2_{\beta-1}}^2 d\tau + C \int_0^t (1 + \tau)^{\gamma-1} \|(v,t,v)\|_{L^2_{\beta-1}}^2 d\tau. \tag{5.4}
\]

Therefore, combining (5.3) and (5.4), we obtain
\[
(1 + t)^\gamma \|(z,t,z)\|_{H^1_{\beta}}^2
\]
\[
+ \int_0^t (1 + \tau)^{\gamma-1} \|(z,t,z)\|_{H^1_{\beta}}^2 + \|\sqrt{\phi_x z(t)}\|_{L^2_{\beta}}^2 + \|\sqrt{\phi_x z(t)}\|_{L^2_{\beta-1}}^2 d\tau \leq CE_{\beta}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma-1} \|(z,t,z)\|_{H^1_{\beta}}^2 d\tau + \beta C \int_0^t (1 + \tau)^{\gamma-1} \|(z,t,z)\|_{H^1_{\beta-1}}^2 d\tau. \tag{5.5}
\]

Finally, applying the induction argument with respect to \( \gamma \) and \( \beta \) to (5.5), we arrive at the desired estimate in Theorem 2.2. This completes the proof of Theorem 2.2. \( \square \)
In the rest of this paper, we prove Theorem 2.3 by using the space-time weighted energy method.

Proof of Theorem 2.3. Let $\alpha, \beta > 0$. Multiplying (4.7) by $e^{\beta t} e^{\alpha x}$, we obtain

$$\{e^{\beta t} e^{\alpha x} E\} - \beta e^{\beta t} e^{\alpha x} E + e^{\beta t} e^{\alpha x} D + \{e^{\beta t} e^{\alpha x} F\}_x - \alpha e^{\beta t} e^{\alpha x} F = e^{\beta t} e^{\alpha x} R. \quad (5.6)$$

Then, integrating (5.6) over $(0, \infty) \times (0, t)$, substituting (4.9), (4.10) and (5.1) into the resultant inequality, and letting $\sup_{0 \leq t < \infty} \|z(t)\|_{H^1}$ be suitably small, we obtain

$$e^{\beta t} \|z(t)\|_{L^2_{a, exp}}^2 + \int_0^t e^{\beta \tau} \|z(\tau)\|_{L^2_{a, exp}}^2 d\tau + \alpha \int_0^t e^{\beta \tau} \|z(\tau)\|_{L^2_{a, exp}}^2 d\tau \leq CE_2^2 + \beta C \int_0^t e^{\beta \tau} \|z(\tau)\|_{L^2_{a, exp}}^2 d\tau,$$

where $C$ is a positive constant independent of $\alpha$ and $\beta$.

On the other hand, we multiply the equality (4.14) by $e^{\beta t} e^{\alpha x}$, integrate the resultant equality over $(0, \infty) \times (0, t)$ and take $\sup_{0 \leq t < \infty} \|v(t)\|_{H^1}$ suitably small. Then we obtain

$$e^{\beta t} \|v(t)\|_{L^2_{a, exp}}^2 + \int_0^t e^{\beta \tau} \|v(\tau)\|_{L^2_{a, exp}}^2 d\tau + \alpha \int_0^t e^{\beta \tau} \|v(\tau)\|_{L^2_{a, exp}}^2 d\tau \leq CE_2^2 + \beta C \int_0^t e^{\beta \tau} \|v(\tau)\|_{L^2_{a, exp}}^2 d\tau,$$

where $C_0$ and $C_1$ are positive constants independent of $\alpha$ and $\beta$. Finally, letting $\alpha > 0$ and $\beta > 0$ suitably small such that $\beta C_0 \leq \alpha$ and $(\alpha + \beta) C_1 \leq 1$, we arrive at the desired estimate in Theorem 2.3, and complete the proof.

A. Appendix

Until the previous section, we considered the asymptotic stability of stationary solutions under the assumption $u_- < 0$ and (1.2). In this section, we remark that even if $u_+ > 0$ and $|f'(0)| < 1, \quad f(u) < f(0)(= 0) \quad \text{for } u \in (0, u_-)$, \quad (A.1)

instead of (1.2), we can derive the same result as Theorems 2.1–2.3. Actually, under the assumption (A.1), we can expect that the solution also tends to the stationary solution through
the characteristic curve method. For this purpose, we introduce a new weight function instead of (3.4) as

\[ \tilde{w}(u) = \frac{e^{-Au} - 1}{f(u)} \quad \text{for } u \in [0, u_-]. \] (A.2)

Then we have the following key lemma for the weight function (A.2).

**Lemma A.1.** Suppose that \( u_- > 0 \) and \( f(u) \) satisfies (A.1). Let \( \tilde{w}(u) \) be the weight function defined in (A.2). Then there exists a positive constant \( \tilde{\delta} \) such that if \( A \geq \tilde{\delta} \), then \( \tilde{w}(u) \) satisfies the following conditions:

(i) \( cu^{-q} < \tilde{w}(u) < Cu^{-q} \),

(ii) \( (f \tilde{w})' (u) \leq 0 \),

(iii) \( (f \tilde{w})''(u) > 0 \),

(iv) \( (f \tilde{w})'(u_-) < 0 \),

for \( u \in [0, u_-] \), where \( q \) is the degeneracy exponent of \( f(u) \), and \( C \) and \( c \) are some positive constants.

Here, the degeneracy exponent \( q \) which is a non-negative integer is defined by Taylor expansion of \( f(u) \). Namely, it holds that

\[ f(u) = a_{q+1}(0)u^{q+1} + a_{q+2}(\theta u)u^{q+2}, \]

for some constant \( \theta \) with \( 0 < \theta < 1 \), where \( a_q(u) \) is defined by

\[ a_q(u) = \frac{1}{q!} \frac{d^q}{du^q} f(u). \]

By using Lemma A.1 and using the same strategy as in the previous section, we can obtain the same result as in Theorems 2.1–2.3 in the case \( u_- > 0 \) and (A.1). Hence the proof is omitted here.

**Acknowledgements.** The research of the second author is partially supported by a Grant-in-Aid for Young Scientists (B) No. 21740111 from Japan Society for the Promotion of Science.

**References**


Itsuko Hashimoto
Division of Mathematics and Physics
Kanazawa University
Kanazawa
920-1192 Japan
(E-mail: itsuko@staff.kanazawa-u.ac.jp)

Yoshihiro Ueda
Graduate School of Maritime Sciences
Kobe University
Kobe
658-0022 Japan
(E-mail: ueda@maritime.kobe-u.ac.jp)