ON THE REGULARITY INDEX OF $n + 3$ ALMOST EQUIMULTIPLE FAT POINTS IN $\mathbb{P}^n$

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Abstract. The aim of this paper is to prove a well-known conjecture on the regularity index of $n + 3$ almost equimultiple fat points in $\mathbb{P}^n$.

1. Introduction

We denote by $\mathbb{P}^n$ the projective space over an algebraically closed field k . Let $P \in \mathbb{P}^n$; then we say that a form f of the polynomial ring $R := k[x_0, \dots, x_n]$ has multiplicity m at P if all partial derivatives of f of order $< m$ vanish at P .

Let $X := \{P_1, \dots, P_s\} \subset \mathbb{P}^n$, let $m_1, \dots, m_s$ be positive integers, and denote by $Z := m_1 P_1 + \dots + m_s P_s$ the zero-scheme corresponding to the ideal of all forms of R vanishing at P_i with multiplicity at least m_i , for $i = 1, \dots, s$. This zero-scheme is called a *set of fat points*.

Let $\wp_1, \dots, \wp_s$ denote the defining ideals of $P_1, \dots, P_s$, respectively. Put

$$I := R/\wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}.$$

Since the coordinate ring R/I of the zero-scheme Z is a one-dimensional Cohen–Macaulay graded k -algebra, the Hilbert function $H_{R/I}(t) := \dim_k(R/I)_t$ is strictly increasing until it reaches the multiplicity

$$e(R/I) = \sum_{i=1}^s \binom{m_i + n - 1}{n}$$

at which it stabilizes. Note that the Hilbert polynomial $P_{R/I}(t)$ of R/I is constant and equal to $e(R/I)$. The least integer t for which $H_{R/I}(t) = P_{R/I}(t) = e(R/I)$ is called the *regularity index* of Z and is denoted by $\text{reg}(Z)$. It is well known that $\text{reg}(Z)$ is equal to the Castelnuovo regularity index $\text{reg}(R/I)$ (see [9] for a definition of the Castelnuovo regularity index of any k -algebra).

It was proved in [5] that $\text{reg}(Z) = (\sum_{i=1}^s m_i) - 1$ when the points are all on a line. Except for special cases, computation of $\text{reg}(Z)$ is difficult. Then one tries to find (possibly sharp) upper bounds for it.

Such a (sharp) upper bound was given by B. Segre in 1961 for fat points in general position in $\mathbb{P}^2$ (see [2, 10]) and was extended to fat points in general position in $\mathbb{P}^n$ by

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Catalisano *et al* in 1993 (see [4]). The bound was given by

$$\max \left\{ \max_{i,j} (m_i + m_j - 1), \left\lceil \frac{1}{n} \sum_{i=1}^s m_i + n - 2 \right\rceil \right\}$$

and it is called the *Segre bound*.

The Segre bound was extended to any set of fat points in $\mathbb{P}^2$ by Fatabbi in 1994 (see [6]) and, with different methods, by Thien in 1999 (see [11]).

In 1996, N. V. Trung formulated the following conjecture (see [12]) which was independently given by G. Fatabbi and A. Lorenzini (see [7]).

CONJECTURE. Let $Z := m_1 P_1 + \cdots + m_s P_s$ be a set of fat points in $\mathbb{P}^n$. For $j = 1, \dots, n$, let

$$T_j := \max \left\{ \left\lceil \frac{1}{j} (m_{i_1} + m_{i_2} + \cdots + m_{i_q} + j - 2) \right\rceil \mid P_{i_1}, P_{i_2}, \dots, P_{i_q} \text{ lie on a } j\text{-plane} \right\},$$

then

$$\text{reg}(Z) \leq T_Z := \max \{T_j \mid j = 1, \dots, n\}.$$

The upper bound T_Z is an extended form of the Segre bound and therefore it is called the *general Segre bound*. The conjecture was proved for the case of fat points in $\mathbb{P}^3$ by Thien in 2000 (see [12]) and independently by Fatabbi and Lorenzini in 2001 (see [7]). Thien also proved in 2002 the case of double points in $\mathbb{P}^4$ (see [13]). Recently, Benedetti *et al* [1] have proved the case of non-degenerate $n + 2$ fat points in $\mathbb{P}^n$. Note that, for any set Z of $s < n + 2$ fat points not in general position in $\mathbb{P}^n$, there exists an $(s - 2)$ -plane containing Z . It means that the result for the case of $n + 2$ fat points in $\mathbb{P}^n$ implies that the conjecture holds for all cases of smaller numbers of fat points in $\mathbb{P}^n$.

A set of fat points is called *almost equimultiple* if the multiplicities of the points are equal to k or $k - 1$ for a given integer $k \geq 2$. In this paper, we use the algebraic method as indicated in [14] to prove the conjecture for any set of $n + 3$ almost equimultiple, non-degenerate fat points in $\mathbb{P}^n$.

In the next section, we show several results which will be used to prove the conjecture in Section 3, and explain why the case of $n + 3$ fat points is more complicated than the case of $n + 2$ fat points.

2. Auxiliary results

The algebraic method used to estimate $\text{reg}(Z) = \text{reg}(R/I)$ in this paper as well as in [1, 4, 11–13] is based on the following lemma.

LEMMA 2.1. [4, Lemma 1] Let $P_1, \dots, P_r, P$ be distinct points in $\mathbb{P}^n$ and let $\wp$ be the defining prime ideal of P . If $m_1, \dots, m_r, a$ are positive integers, $J := \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$, and $I := J \cap \wp^a$, then

$$\text{reg}(R/I) = \max \{a - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp^a))\}.$$

In the above lemma, the Castelnuovo regularity index $\text{reg}(R/(J + \wp^a))$ of k -algebra $R/(J + \wp^a)$ is equal to the least integer t such that $(R/(J + \wp^a))_t = 0$.

In order to use the lemma, we choose a relevant point P_{i_0} in the support of Z and let $J := \bigcap_{j \neq i_0} \wp_j^{m_j}$. Since $\text{reg}(R/J)$, in many cases, can be computed by induction on the number of fat points, one needs to estimate $\text{reg}(R/(J + \wp_{i_0}^{m_{i_0}}))$, for which the following lemma provides a useful tool.

LEMMA 2.2. [4, Lemma 3] *Let $P_1, \dots, P_r, P$ be distinct points in $\mathbb{P}^n$ and let $\wp$ be the defining prime ideal of P . If $m_1, \dots, m_r, a$ are positive integers, $J := \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r}$ and $\wp = (x_1, \dots, x_n)$, then $\text{reg}(R/(J + \wp^a)) \leq b$ if and only if $x_0^{b-i} M \in J + \wp^{i+1}$ for any monomial $M \in \wp^i$, where $i = 0, \dots, a - 1$.*

In practice, we choose a relevant number T , then for any monomial $M \in \wp^i$ where $0 \leq i \leq a - 1$, we show that there exist hyperplanes $H_1, \dots, H_{T-i}$ avoiding P and satisfying $H_1 \cdots H_{T-i} M \in J$. We can write $H_j = L_j + x_0$, where $L_j \in \wp$ for $j = 1, \dots, T - i$. Then

$$H_1 \cdots H_{T-i} M = (L_1 + x_0) \cdots (L_{T-i} + x_0) M \in J$$

implies $x_0^{T-i} M \in J + \wp^{i+1}$. This means that $\text{reg}(R/(J + \wp^a)) \leq T$.

In particular, we have the following corollary.

COROLLARY 2.3. *Let $Z := m_1 P_1 + \dots + m_s P_s$ be a set of fat points in $\mathbb{P}^n$. Let $J := \wp_1^{m_1} \cap \dots \cap \wp_{s-1}^{m_{s-1}}$ and suppose that $P_1, \dots, P_{s-1}$ are contained in a hyperplane avoiding P_s . Then $\text{reg}(R/(J + \wp_s^{m_s})) \leq T_Z$.*

The result of the above corollary is attained by noting that if H is a hyperplane containing $P_1, \dots, P_{s-1}$ and avoiding P_s , then $H^{T_Z-i} \in J$ since $T_Z - i \geq T_Z - m_s + 1 \geq m_j$ for all $0 \leq i \leq m_s - 1$ and $1 \leq j \leq s - 1$.

For the case $s = n + 2$ and supposing that the points are not in general position and non-degenerate, then there exists a hyperplane H containing $n + 1$ points and not containing the remain one, say P_{i_0} . Therefore in this case, the corollary shows that $\text{reg}(R/J + \wp_{i_0}^{m_{i_0}}) \leq T_Z$ immediately.

For the case $s = n + 3$ fat points and supposing that they are non-degenerate and not in general position, it could be possible that there is no hyperplane containing $s - 1 = n + 2$ points of the support of the given set of fat points. Therefore showing the existence of P_{i_0} or proving $\text{reg}(R/J + \wp_{i_0}^{m_{i_0}}) \leq T_Z$ is more complicated even for the case of equimultiple fat points.

Moreover, for the case that there exists a hyperplane H containing $s - 1 = n + 2$ points and avoiding the remain point P_{i_0} , then by Lemma 2.1, we still need to show that $\text{reg}(R/J) \leq T_Z$, where $J := \bigcap_{j \neq i_0} \wp_j^{m_j}$. Note that the ideal J corresponds to the set of fat points Y whose support lies on a hyperplane $H \cong \mathbb{P}^{n-1}$. Denote by Y' the set of fat points in $\mathbb{P}^{n-1}$ corresponding to Y , then by induction we have $\text{reg}(Y') \leq T_{Y'}$. The following lemma shows that, in fact, $T_{Y'}$ is still an upper bound of $\text{reg}(Y) = \text{reg}(R/J)$.

LEMMA 2.4. [1, Lemma 4.4] *Let $Y := m_1 P_1 + \dots + m_{s-1} P_{s-1}$ be a set of fat points in $\mathbb{P}^n$ whose support is contained in a hyperplane $H \cong \mathbb{P}^{n-1}$. Denote by P'_j the point of $\mathbb{P}^{n-1}$ corresponding to P_j , for $1 \leq j \leq s - 1$. Let $Y' := m_1 P'_1 + \dots + m_{s-1} P'_{s-1}$ be the set of fat points in $\mathbb{P}^{n-1}$ corresponding to Y . If there is a non-negative integer t such that $\text{reg}(Y') \leq t$ then $\text{reg}(Y) \leq t$.*

3. Main results

In this section, we show that the conjecture holds for any set of $n + 3$ almost equimultiple fat points in $\mathbb{P}^n$. We only consider the cases of $n + 3$ fat points which are non-degenerate in $\mathbb{P}^n$ to avoid the fact that all the points lie on a hyperplane. Due to the result in [4], we only need to consider the case that the support $X = \{P_1, \dots, P_{n+3}\}$ of a set of fat points Z is not in general position. Set

$$d := \min\{t \mid \exists t\text{-plane containing } t + 2 \text{ points of } X\}.$$

Then it is easy to see that $1 \leq d \leq n - 1$.

As mentioned in the previous section, our method is based on Lemma 2.1. Therefore, in each case, we firstly choose P_{i_0} in the support X of Z , put $J := \bigcap_{j \neq i_0} \wp_j^{m_j}$ and show that $\text{reg}(R/J + \wp_{i_0}^{m_{i_0}}) \leq T_Z$.

We divide the section into several parts. In the first subsection, we show a definition of coverings of a set of fat points and elementary properties in order to have a convenient method for studying $\text{reg}(R/J + \wp_{i_0}^{m_{i_0}})$. In the second subsection, we show covering properties of a set of $n + 2$ almost equimultiple fat points. These properties give upper bounds of $\text{reg}(R/J + \wp_{i_0}^{m_{i_0}})$ according to sets of $n + 3$ almost equimultiple fat points in $\mathbb{P}^n$. The last section is intended to check the conjecture for this case.

3.1. Coverings of a set of fat points

Definition. Let s, a be non-negative integers and let $a \geq 1$. A fat point $Z := m_1 P_1 + \dots + m_r P_r$ is called (s, a) -covered if for every collection $\mathcal{C}$ of i hyperplanes of the form $x_j = 0$, $j = 1, \dots, n$ (not necessary distinct), $i = 0, \dots, a - 1$, there exists a collection $\mathcal{D}$ of $s - i$ hyperplanes avoiding $P = e_0 := (1, 0, \dots, 0)$ such that there are at least m_t hyperplanes in $\mathcal{C} \cup \mathcal{D}$ containing P_t for any $t = 1, \dots, r$.

A trivial verification shows the following properties.

LEMMA 3.1. *If $m_1 P_1 + \dots + m_r P_r$ is (s, a) -covered, then $n_1 P_1 + \dots + n_r P_r$ is (q, b) -covered for any sequence of non-negative integers $n_t \leq m_t$, $t = 1, \dots, r$, and for any $q \geq s$, $1 \leq b \leq a$.*

LEMMA 3.2. *If $m_1 P_1 + \dots + m_r P_r$ is (s, a) -covered and $n_1 P_1 + \dots + n_r P_r$ is (t, b) -covered, then $(m_1 + n_1) P_1 + \dots + (m_r + n_r) P_r$ is $(s + t, a + b - 1)$ -covered.*

In particular, with $n_1 = \dots = n_r = 0$, $b = t + 1$, we have the following lemma.

LEMMA 3.3. *If $m_1 P_1 + \dots + m_r P_r$ is (s, a) -covered, then it is $(s + t, a + t)$ -covered for any non-negative integer t .*

The meaning of the definition is shown in the following lemma.

LEMMA 3.4. *Let $\wp_1, \dots, \wp_r, \wp$ be the ideals of $P_1, \dots, P_r, P$, respectively, let $J := \bigcap_{i=1}^r \wp_i^{m_i}$ and $P = e_0$. If $m_1 P_1 + \dots + m_r P_r$ is (s, a) -covered, then $\text{reg}(R/(J + \wp^a)) \leq s$.*

In the next subsection, we will study covering properties of a set of $n + 2$ almost equimultiple fat points in $\mathbb{P}^n$.

3.2. Covering properties of a set of $n + 2$ almost equimultiple fat points

PROPOSITION 3.5. *Let $X := \{P_1, \dots, P_{n+3}\}$ be a non-degenerate set of distinct points, not in general position in $\mathbb{P}^n$, $n \geq 4$, and there are no $n + 2$ points of X lying on a hyperplane. Let d be the smallest number such that there exists a d -plane containing $d + 2$ points of X . Assume that $P_1, \dots, P_d, P_{n+2}, P_{n+3}$ lie on a d -plane and $P_{n+3} = e_0$. Then:*

- (1) $P_1 + 2P_2 + \dots + 2P_{n-1} + P_n + P_{n+1} + P_{n+2}$ is $(2, 1)$ -covered. Consequently, $P_1 + P_2 + \dots + P_{n+2}$ is $(2, 1)$ -covered and hence $2P_1 + 2P_2 + \dots + 2P_{n+2}$ is $(4, 1)$ -covered;
- (2) $P_1 + P_2 + \dots + P_{n+2}$ is $(2, 2)$ -covered if $d \geq 2$;
- (3) $2P_1 + \dots + 2P_{n+2}$ is $(3, 1)$ -covered if $d \geq 2$;
- (4) $2P_1 + \dots + 2P_{n+2}$ is $(3, 2)$ -covered if $d \geq 3, n \geq 5$.

Proof. We can assume that $P_i = e_i$, where $e_1 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$.

(1) Let $H_1 = V(x_0)$ and H_2 be the hyperplane determined by $P_2, \dots, P_{n-1}, P_{n+1}, P_{n+2}$ then $P_i \in H_1, i = 1, \dots, n$, and $P_{n+3} \notin H_1 \cup H_2$. It is clear that

$$P_1 + 2P_2 + \dots + 2P_{n-1} + P_n + P_{n+1} + P_{n+2}$$

is $(2, 1)$ -covered.

(2) According to the definition and from case (1), we now need only to check for $i = 1$. Consider $K = V(x_j)$.

If $1 \leq j \leq d$, then the plane containing P_j, P_{n+1}, P_{n+2} does not contain P_{n+3} since $d \geq 2$. Choose H to be a hyperplane containing P_j, P_{n+1}, P_{n+2} and avoiding P_{n+3} then $P_i \in K \cup H, i = 1, \dots, n + 2$. If $K = V(x_j)$ and $d < j \leq n$, then $P_{n+2}, P_i \in K, 1 \leq i \neq j \leq n$. Since there are no 3 points of X collinear, let H be a hyperplane containing P_j, P_{n+1} and avoiding P_{n+3} , then we have $P_i \in K \cup H, i = 1, \dots, n + 2$. This means that $P_1 + P_2 + \dots + P_{n+1}$ is $(2, 2)$ -covered if $d \geq 2$.

(3) Since $d \geq 2$, we see that the 3-plane determined by $P_n, P_{n+1}, P_{n+2}, P_{n+3}$ does not contain at least one point in $\{P_1, \dots, P_d\}$, which we can assume to be P_1 . This means that $P_1 + P_n + P_{n+1} + P_{n+2}$ is $(1, 1)$ -covered. Moreover $P_1 + 2P_2 + \dots + 2P_{n-1} + P_n + P_{n+1} + P_{n+2}$ is $(2, 1)$ -covered, by Lemma 3.2, we have $2P_1 + \dots + 2P_{n+2}$ is $(3, 1)$ -covered if $d \geq 2$.

(4) Since $2P_1 + \dots + 2P_{n+2}$ is $(3, 1)$ -covered, according to the covering definition, we only need to check for $i = 1$. Consider $K = V(x_j)$, then K contains $P_i, 1 \leq i \neq j \leq n$. The proof is divided into 2 cases.

Case 1: $1 \leq j \leq d$. Since $d \geq 3$, there exist distinct integers t, u such that $1 \leq t \neq i \neq u \leq d$. One of the points in $\{P_t, P_u\}$, say P_t , is not contained in the 4-plane determined by $P_j, P_n, P_{n+1}, P_{n+2}, P_{n+3}$. Let H_1 be a hyperplane containing the 4-plane determined by $P_t, P_j, P_n, P_{n+1}, P_{n+2}$ and avoiding P_{n+3} . Let H_2 be the hyperplane determined by $P_{n+1}, P_{n+2}, P_i, 1 \leq i \neq t \leq n - 1$, then $P_{n+3} \notin H_2$. It is clear that there exist two of $\{K, H_1, H_2\}$ containing P_i for any $1 \leq i \leq n + 2$.

Case 2: $d < j \leq n$. In this case the hyperplane $K = V(x_j)$ contains $P_{n+2}, P_i, 1 \leq i \neq j \leq n$.

If $d \leq n - 2$, then there exists an integer s such that $d < j \neq s \leq n$. There exists $1 \leq t \leq d$ such that P_t is not contained in the 3-plane determined by $P_j, P_s, P_{n+1}, P_{n+3}$. Let H_1

be a hyperplane containing P_t, P_j, P_s, P_{n+1} and avoiding P_{n+3} . Let H_2 be the hyperplane determined by all $P_i, i \notin \{t, s\}, i \leq n+2$, then $P_{n+3} \notin H_2$. It is clear that there exist two of $\{K, H_1, H_2\}$ containing P_i for any $1 \leq i \leq n+2$.

If $d = n - 1$ then $j = n$ and the hyperplane $K = V(x_n)$ contains $P_{n+2}, P_i, 1 \leq i \leq n - 1$. One of P_1, P_2 , say P_2 , is not contained in the hyperplane determined by $P_3, \dots, P_{n+1}, P_{n+3}$. Let H_1 be the hyperplane determined by $P_2, \dots, P_{n+1}$, then $P_{n+3} \notin H_1$. Since $d = n - 1 \geq 4$, the 3-plane determined by $P_1, P_n, P_{n+1}, P_{n+2}$ does not contain P_{n+3} . Let H_2 be a hyperplane containing $P_1, P_n, P_{n+1}, P_{n+2}$ and avoiding P_{n+3} . It is clear that there exist two of $\{K, H_1, H_2\}$ containing P_i for any $1 \leq i \leq n+2$.

This means that $2P_1 + \dots + 2P_{n+2}$ is (3, 2)-covered and the proof is complete. $\square$

COROLLARY 3.6. *With the hypothesis as in Proposition 3.5, we have the following:*

- (1) $m_1P_1 + 2P_2 + \dots + 2P_{n+1} + m_{n+2}P_{n+2}$ is (3, 1)-covered, where $m_1, m_{n+2} \geq 1, m_1 + m_{n+2} = 3$;
- (2) $2P_1 + 3P_2 + \dots + 3P_{n+1} + 2P_{n+2}$ is (5, 2)-covered;
- (3) for $n = 4, d = 3$, $2P_1 + 2P_2 + 2P_3 + P_4 + 2P_5 + 2P_6$ is (3, 2)-covered and $3P_1 + \dots + 3P_6$ is (5, 3)-covered.

Proof. Choose coordinates such that $P_i = e_i, i = 1, \dots, n$.

(1) By Proposition 3.5, we have $P_1 + 2P_2 + \dots + 2P_{n-1} + P_n + P_{n+1} + P_{n+2}$ is (2, 1)-covered. On the other hand, both $P_1 + P_n + P_{n+1}$ and $P_n + P_{n+1} + P_{n+2}$ are (1, 1)-covered, the result is attained by Lemma 3.2.

(2) If $d \geq 2$, we know from Proposition 3.5 that $P_1 + \dots + P_{n+2}$ is (2, 2)-covered and $P_1 + 2P_2 + \dots + 2P_{n+1} + P_{n+2}$ is (3, 1)-covered as in the above. Then the assertion follows by Lemma 3.2. If $d = 1$, then $P_1 + 2P_2 + \dots + 2P_{n-1} + P_n + P_{n+1} + P_{n+2}$ is (2, 1)-covered. Similarly, it is easily seen that $P_1 + P_2 + P_3 + \dots + 2P_n + 2P_{n+1} + P_{n+2}$ is (2, 1)-covered. This implies that $2P_1 + 3P_2 + \dots + 3P_{n+1} + 2P_{n+2}$ is (4, 1)-covered and hence (5, 2)-covered.

(3) We see that $P_1 + 2P_2 + 2P_3 + P_4 + P_5 + P_6$ is (2, 1)-covered by Proposition 3.5, $P_1 + P_5 + P_6$ is (1, 1)-covered. Thus $2P_1 + 2P_2 + 2P_3 + P_4 + 2P_5 + 2P_6$ is (3, 1)-covered.

Consider $K = V(x_1)$, we see that $P_2, P_3, P_4 \in K$. Consider hyperplane H_1 determined by P_1, P_2, P_5, P_6 and hyperplane H_2 determined by P_1, P_3, P_5, P_6 , then $P_7 \notin H_1 \cup H_2$. Note that $P_4 \in K$ and there exist two of $\{K, H_1, H_2\}$ containing P_i for any $1 \leq i \leq 6, i \neq 4$. A similar consideration applies to the case $K = V(x_j), j = 2, 3$.

If $K = V(x_4)$, then $P_i \in K, i = 1, 2, 3$. Any of two planes determined by $\{P_1, P_2, P_5\}$ and $\{P_1, P_3, P_5\}$, respectively, does not contain P_3 . Let H_1, H_2 be two hyperplanes, respectively, containing these planes. Note that $P_4 \in K$ and there exist two of $\{K, H_1, H_2\}$ containing P_i for any $1 \leq i \leq 6, i \neq 4$. By the definition of covering, we have $2P_1 + 2P_2 + 2P_3 + P_4 + 2P_5 + 2P_6$ is (3, 2)-covered.

Since $P_1 + \dots + P_6$ is (2, 2)-covered and $2P_1 + \dots + 2P_6$ is (3, 1)-covered, we have $3P_1 + \dots + 3P_3$ is (5, 2)-covered. Thus to prove $3P_1 + \dots + 3P_3$ is (5, 3)-covered, we need only to check for $i = 2$, which is straightforward. $\square$

From Proposition 3.5 and Corollary 3.6, we now consider covering properties of some particular sets of $n+2$ almost equimultiple fat points in $\mathbb{P}^n$ with an assumption that the number of points having multiplicities $k-1$ is not more than two.

PROPOSITION 3.7. *With the hypothesis as in Proposition 3.5, for any integer $k \geq 2$, we have:*

- (1) $m_1 P_1 + k P_2 + \cdots + k P_{n+1} + m_{n+2} P_{n+2}$, for $m_1, m_{n+2} \geq 1$, $m_1 + m_{n+2} = 2k - 1$, is $(s, k - 1)$ -covered where $s = 3k - 3$ if $d = 1$ and $s = 2k - 1$ if $d \geq 2$;
- (2) $k P_1 + k P_2 + \cdots + k P_{n+2}$ is $(s, k - 1)$ -covered, where $s = 3k - 2$ if $d = 1$ and $s = 2k - 1$ if $d \geq 2$;
- (3) $k P_1 + k P_2 + \cdots + k P_{n+2}$ is (s, k) -covered, where $s = 3k - 1$ if $d = 1$, $s = 2k - 1$ if $d \geq 3$, $n + k \geq 7$ and $s = 2k$ otherwise;
- (4) $(k - 1) P_1 + k P_2 + \cdots + k P_{n+1} + (k - 1) P_{n+2}$ is $(s, k - 1)$ -covered, where $s = 3k - 4$ if $d = 1$, $k \geq 3$ and $s = 2k - 1$ if $d \geq 2$ or $k = 2$.

Proof. Assume that $P_i = e_i$, $i = 1, \dots, n$. We prove mainly by induction on k .

(1) For $k = 2$, the assertion is attained by Corollary 3.6. Suppose that the conclusion holds for $k \geq 2$. We know from Proposition 3.5 that $P_1 + \cdots + P_{n+2}$ is $(3, 2)$ -covered if $d = 1$ and it is $(2, 2)$ -covered if $d \geq 2$. This implies that $(m_1 + 1)P_1 + (k + 1)P_2 + \cdots + (k + 1)P_{n+1} + (m_{n+2} + 1)P_{n+2}$ is $(3k, k)$ -covered if $d = 1$ and $(2k + 1, k)$ -covered if $d \geq 2$. Thus the conclusion holds for $k + 1$ and hence the proof is complete.

(2) For $k = 2$, since $2P_1 + \cdots + 2P_{n+2}$ is $(4, 1)$ -covered if $d = 1$ and it is $(3, 1)$ -covered if $d \geq 2$ by Proposition 3.5, the conclusion follows. The same argument as in case (1) works for this case.

(3) From the above, we have $k P_1 + k P_2 + \cdots + k P_{n+2}$ is $(3k - 2, k - 1)$ -covered if $d = 1$ and $(2k - 1, k - 1)$ -covered if $d = 2$. This implies that $k P_1 + k P_2 + \cdots + k P_{n+2}$ is $(3k - 1, k)$ -covered if $d = 1$ and $(2k, k)$ -covered if $d = 2$.

For $d \geq 3$, from Proposition 3.5, we see that $2P_1 + \cdots + 2P_{n+2}$ is $(3, 2)$ -covered if $n \geq 5$ and $3P_1 + \cdots + 3P_6$ is $(5, 3)$ -covered for $n = 4$ by Corollary 3.6. Suppose that $k P_1 + k P_2 + \cdots + k P_{n+2}$ is $(2k - 1, k)$ -covered if $n \geq 5$, $k \geq 2$ or $n = 4$, $k \geq 3$. Since $P_1 + \cdots + P_{n+2}$ is $(2, 2)$ -covered, we have $(k + 1)P_1 + \cdots + (k + 1)P_{n+2}$ is $(2k + 1, k + 1)$ -covered for $n \geq 5$, $k \geq 2$ or for $n = 4$, $k \geq 3$. By the induction principle, we have $k P_1 + k P_2 + \cdots + k P_{n+2}$ is $(2k - 1, k)$ -covered for $d \geq 3$ and any $n + k \geq 7$. Finally, for the case $n = 4$, $k = 2$, we have $2P_1 + \cdots + 2P_6$ is $(3, 1)$ -covered, hence it is $(4, 2)$ -covered. This completes the proof of (3).

(4) From case (1), we see that $(k - 1)P_1 + k P_2 + \cdots + k P_{n+1} + (k - 1)P_{n+2}$ is $(2k - 1, k - 1)$ -covered if $d \geq 2$. For $k = 2$, we have $P_1 + 2P_2 + \cdots + 2P_{n+1} + P_{n+2}$ is $(3, 1)$ -covered by Corollary 3.6.

Now, consider the case $d = 1$, $k \geq 3$. If $k = 3$, we know from Corollary 3.6 that $2P_1 + 3P_2 + \cdots + 3P_{n+1} + 2P_{n+2}$ is $(5, 2)$ -covered. Suppose that $(k - 1)P_1 + k P_2 + \cdots + k P_{n+1} + (k - 1)P_{n+2}$ is $(3k - 4, k - 1)$ -covered. We see from Proposition 3.5 that $P_1 + \cdots + P_{n+2}$ is $(3, 2)$ covered. Therefore, $k P_1 + (k + 1)P_2 + \cdots + (k + 1)P_{n+1} + k P_{n+2}$ is $(3k - 1, k)$ -covered. This means that the conclusion holds for $k + 1$, hence the proof is complete. $\square$

The above results yield the following covering property of a general set of $n + 2$ almost equimultiple fat points in a relationship to the general Segre bound T_Z and with an assumption that the point $P = e_0$ has multiplicity $k - 1$.

COROLLARY 3.8. *Let $X := \{P_1, \dots, P_{n+3}\}$ be a non-degenerate set of distinct points, not in general position in $\mathbb{P}^n$, $n \geq 4$, such that there are no $n + 2$ points of X lying on*

a hyperplane. Let d be the smallest number such that there exists a d -plane containing $d + 2$ points of X . Assume that $P_{n+3} = e_0$ and $P_1, \dots, P_d, P_{n+2}, P_{n+3}$ lie on a d -plane. Let $k \geq 2$ be an integer, and let $Z := m_1 P_1 + \dots + m_{n+3} P_{n+3}$, where $k - 1 \leq m_i \leq k$, $i = 1, \dots, n + 2$ and $m_{n+3} = k - 1$. Set

$$T_j := \max \left\{ \left\lceil \frac{1}{j} (m_{i_1} + m_{i_2} + \dots + m_{i_q} + j - 2) \right\rceil \mid P_{i_1}, P_{i_2}, \dots, P_{i_q} \text{ lie on a } j\text{-plane} \right\}$$

and

$$T_Z := \max \{T_j \mid j = 1, \dots, n\}.$$

Then $m_1 P_1 + \dots + m_{n+2} P_{n+2}$ is $(T_Z, k - 1)$ -covered.

Proof. As usual, we assume that $P_i = e_i, i = 1, \dots, n$. We divide the proof into three cases.

Case 1: there exist two points in X with multiplicities k . If $m_1 = m_{n+2} = k$, then $T_Z = 3k - 2$ if $d = 1$ and $T_Z \geq T_1 = 2k - 1$ if $d \geq 2$. From Proposition 3.7, we have $k P_1 + \dots + k P_{n+2}$ is $(T_Z, k - 1)$ -covered, hence $m_1 P_1 + \dots + m_{n+2} P_{n+2}$ is $(T_Z, k - 1)$ -covered.

If $m_1 + m_{n+2} = 2k - 1$, then $T_Z \geq 3k - 3$ if $d = 1$ and $T_Z \geq T_1 = 2k - 1$ if $d \geq 2$. From Proposition 3.7, we have $m_1 P_1 + k P_2 + \dots + k P_{n+1} + m_{n+2} P_{n+2}$ is $(T_Z, k - 1)$ -covered, hence $m_1 P_1 + \dots + m_{n+2} P_{n+2}$ is $(T_Z, k - 1)$ -covered.

If $m_1 = m_{n+2} = k - 1$, then $T_Z \geq 3k - 4$ if $d = 1, k \geq 3$ and $T_Z \geq T_1 = 2k - 1$ if $d \geq 2$ or $k = 2$. From Proposition 3.7, we have $(k - 1) P_1 + k P_2 + \dots + k P_{n+1} + (k - 1) P_{n+2}$ is $(T_Z, k - 1)$ -covered, hence $m_1 P_1 + \dots + m_{n+2} P_{n+2}$ is $(T_Z, k - 1)$ -covered.

Case 2: there exists only one point in X with multiplicity k . Let $1 \leq t \leq n + 2$ such that $m_t = k$.

If $d = 1$ and $t \in \{1, n + 2\}$, it means that $m_1 + m_{n+2} = 2k - 1$, then $T_Z = 3k - 3$. By Proposition 3.7, $m_1 P_1 + k P_2 + \dots + k P_{n+1} + m_{n+2} P_{n+2}$ is $(T_Z, k - 1)$ -covered, hence $m_1 P_1 + \dots + m_{n+2} P_{n+2}$ is $(T_Z, k - 1)$ -covered.

If $d = 1$ and $1 < t < n + 2$, then $T_Z \geq 3k - 4$. By Proposition 3.7, $(k - 1) P_1 + k P_2 + \dots + k P_{n+1} + (k - 1) P_{n+2}$ is $(3k - 4, k - 1)$ -covered for $k \geq 3$. For $k = 2$, we see from Proposition 3.5 that $P_1 + \dots + P_{t-1} + 2 P_t + P_{t+1} + \dots + P_{n+2}$ is $(2, 1)$ -covered. Thus $m_1 P_1 + \dots + m_{n+2} P_{n+2}$ is $(T_Z, k - 1)$ -covered.

If $d \geq 2$, then $T_Z = T_1 = 2k - 2$. We will prove that

$$(k - 1) P_1 + \dots + (k - 1) P_{t-1} + k P_t + (k - 1) P_{t+1} + \dots + (k - 1) P_{n+2}$$

is $(2k - 2, k - 1)$ -covered by induction on k . If $k = 2$, it follows from Proposition 3.5 that $P_1 + \dots + P_{t-1} + 2 P_t + P_{t+1} + \dots + P_{n+2}$ is $(2, 1)$ -covered. The conclusion holds for $k = 2$. Suppose that the conclusion holds for k . We know that $P_1 + \dots + P_{n+2}$ is $(2, 2)$ -covered for $d \geq 2$. This means that $k P_1 + \dots + k P_{t-1} + (k + 1) P_t + k P_{t+1} + \dots + k P_{n+2}$ is $(2k, k)$ -covered for $d \geq 2$. The conclusion holds for $k + 1$.

Case 3: all points in X have multiplicities $k - 1$. For $k = 2$, we have $T_Z = T_n = 2$ and $P_1 + \dots + P_{n+1}$ is $(2, 1)$ -covered. The conclusion holds for $k = 2$. If $k \geq 3$, then $T_Z = 3(k - 1) - 1$ if $d = 1$, $T_Z = 2(k - 1) - 1$ for $d \geq 3$, $n + k - 1 \geq 7$ and $T_Z = 2(k - 1)$ otherwise. Proposition 3.7 now shows that $(k - 1) P_1 + \dots + (k - 1) P_{n-1}$ is $(T_Z, k - 1)$ -covered. $\square$

3.3. Regularity index of a set of $n + 3$ almost equimultiple fat points

We now check the conjecture for any non-degenerate set of $n + 3$ almost equimultiple fat points in $\mathbb{P}^n$.

PROPOSITION 3.9. *Let $X := \{P_1, \dots, P_{n+3}\}$ be a non-degenerate set of distinct points, not in general position in $\mathbb{P}^n$, $n \geq 4$, such that there are no $n + 2$ points of X lying on a hyperplane. Let $k \geq 2$ be an integer and let $Z := m_1 P_1 + \dots + m_{n+3} P_{n+3}$, where $k - 1 \leq m_i \leq k$, $i = 1, \dots, n + 3$. Set*

$$T_j := \max \left\{ \left\lceil \frac{1}{j} (m_{i_1} + m_{i_2} + \dots + m_{i_q} + j - 2) \right\rceil \mid P_{i_1}, P_{i_2}, \dots, P_{i_q} \text{ lie on a } j\text{-plane} \right\}$$

and

$$T_Z := \max\{T_j \mid j = 1, \dots, n\}.$$

Then there exists a point $P_{i_0} \in X$ such that if $P_{i_0} = e_0$, we have $\sum_{j \neq i_0} m_j P_j$ is (T_Z, m_{i_0}) -covered. Consequently,

$$\text{reg}(R/(J + \wp_{i_0}^{m_{i_0}})) \leq T_Z,$$

where $\wp_j$ is the ideal corresponding to P_j and $J := \bigcap_{j \neq i_0} \wp_j^{m_j}$.

Proof. Let d be the smallest number such that there exists a d -plane containing $d + 2$ points of X . The proof will be divided into two cases.

Case 1: there exists a d -plane K containing $d + 2$ points of X , one of which has multiplicity $k - 1$. Choose $P_{i_0} \in K$, a point of X with multiplicity $k - 1$ and assume that $P_{i_0} = e_0$. Then Proposition 3.7 shows that $\sum_{j \neq i_0} m_j P_j$ is (T_Z, m_{i_0}) -covered.

Case 2: all points in any d -plane containing $d + 2$ points of X have multiplicities k . Consider the case $d \leq 2$ or $n + k \geq 7$, then $T_Z = 3k - 1$ if $d = 1$, $T_Z = 2k - 1$ if $d \geq 3$ and $T_Z = 2k$ otherwise. Choose P_{i_0} to be a point in a d -plane containing $d + 2$ points of X and assume that $P_{i_0} = e_0$. Proposition 3.7 shows that $k P_1 + \dots + k P_{n+2}$ is (T_Z, k) -covered. This implies that $\sum_{j \neq i_0} m_j P_j$ is (T_Z, m_{i_0}) -covered.

Consider the case $n = 4$, $k = 2$ and $d = 3$. Choose $P_{i_0} = P_7$ to be a point on a 3-plane containing P_1, P_2, P_3, P_6, P_7 with multiplicities 2.

If $m_4 = m_5 = 2$ then $T_Z = T_4 = 4$. Since $2P_1 + \dots + 2P_6$ is $(3, 1)$ -covered by Proposition 3.5, it is $(4, 2)$ -covered.

If $2 \leq m_4 + m_5 \leq 3$ then $T_Z = 3$. We can assume that $m_4 = 1$. We know from Corollary 3.6 that $2P_1 + 2P_2 + 2P_3 + P_4 + 2P_5 + 2P_6$ is $(3, 2)$ -covered. This implies that $2P_1 + 2P_2 + 2P_3 + m_4 P_4 + m_5 P_5 + 2P_6$ is $(3, 2)$ -covered.

The proof is complete. $\square$

THEOREM 3.10. *Let n, k be positive integers and let $k \geq 2$. Let $X := \{P_1, \dots, P_{n+3}\}$ be a set of non-degenerate distinct points in $\mathbb{P}^n$ and let $Z := m_1 P_1 + \dots + m_{n+3} P_{n+3}$, where $k - 1 \leq m_i \leq k$, $i = 1, \dots, n + 3$. Let $\wp_j$ be the ideal corresponding to P_j and let $I := \wp_1^{m_1} \cap \dots \cap \wp_{n+3}^{m_{n+3}}$. Then*

$$\text{reg}(R/I) \leq T_Z,$$

where

$$T_j := \max \left\{ \left\lceil \frac{1}{j} (m_{i_1} + m_{i_2} + \dots + m_{i_q} + j - 2) \right\rceil \mid P_{i_1}, P_{i_2}, \dots, P_{i_q} \text{ lie on a } j\text{-plane} \right\}$$

and

$$T_Z := \max\{T_j \mid j = 1, \dots, n\}.$$

Proof. We only need to consider the case where X is non-general position and $n \geq 4$. Other cases follow from [4, 7, 11].

Suppose that there exists a hyperplane containing $n + 2$ points, say $P_1, \dots, P_{n+2}$ of X . Assume that $P_{n+3} = e_0$ and set $Y := \sum_{j=1}^{n+2} m_j P_j$, $J := \bigcap_{j=1}^{n+2} \wp_j^{m_j}$. Then we can consider that $P_1, \dots, P_{n+2}$ are contained in a projective space $\mathbb{P}^{n-1}$, and let Y' be the set of fat points in $\mathbb{P}^{n-1}$ corresponding to Y . By induction, we have $\text{reg}(Y') \leq T_{Y'}$. By Lemma 2.4, we have $\text{reg}(R/J) = \text{reg}(Y) \leq T_{Y'} \leq T_Y \leq T_Z$. Moreover, Corollary 2.3 implies

$$\text{reg}(R/(J + \wp_{n+3}^{m_{n+3}})) \leq T_Z.$$

By Lemma 2.1, we have

$$\text{reg}(R/I) = \max\{m_{n+3} - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp_{n+3}^{m_{n+3}}))\} \leq T_Z.$$

We now consider the case that there is no hyperplane containing $n + 2$ points of X . By Proposition 3.9, there exists $P_{i_0} \in X$ such that

$$\text{reg}(R/J + \wp_{i_0}^{m_{i_0}}) \leq T_Z,$$

where $J := \bigcap_{j \neq i_0} \wp_j^{m_j}$.

Moreover, the ideal J corresponds to $n + 2$ non-degenerate points in $\mathbb{P}^n$ and we have, by [1], that

$$\text{reg}(R/J) \leq T_Y \leq T_Z,$$

where $Y := \sum_{j \neq i_0} m_j P_j$.

By Lemma 2.1, we have

$$\text{reg}(R/I) = \max\{m_{i_0} - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp_{i_0}^{m_{i_0}}))\} \leq T_Z.$$

The proof is complete. $\square$

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