HELICOIDAL SURFACES IN THE THREE-DIMENSIONAL LORENTZ–MINKOWSKI SPACE SATISFYING $\Delta^{\text{II}}r = Ar$

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Abstract. In this paper we study the helicoidal surfaces in the three-dimensional Lorentz–Minkowski space under the condition $\Delta^{\text{II}}r = Ar$, where A is a real 3×3 matrix and Δ^{II} is the Laplace operator with respect to the second fundamental form.

0. Introduction

Let $\mathbb{E}_1^3$ denote the three-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$g_L = ds^2 = -dx^2 + dy^2 + dz^2,$$

where (x, y, z) are the canonical coordinates in $\mathbb{R}^3$.

Let $r : M \rightarrow \mathbb{E}_1^3$ be an isometric immersion of a pseudo-Riemannian surface in $\mathbb{E}_1^3$.

The notion of finite-type submanifolds in Euclidean space or pseudo-Euclidean space was introduced by Chen [5]. In terms of Chen’s theory, a surface M is said to be of finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ . Chen posed the problem of classifying the finite-type surfaces in the three-dimensional Euclidean space $\mathbb{E}^3$. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space.

In [7], Kaimakamis and Papantoniou classified the first three types of surfaces of revolution without parabolic points in the three-dimensional Lorentz–Minkowski space, which satisfy the condition

$$\Delta^{\text{II}}r = Ar, \quad A \in \text{Mat}(3, \mathbb{R}),$$

where $\text{Mat}(3, \mathbb{R})$ is the set of 3×3 real matrices. They proved that such surfaces are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius.

In [2], Bekkar and Zoubir classified the surfaces of revolution with non-zero Gaussian curvature K_G in the three-dimensional Lorentz–Minkowski space $\mathbb{E}_1^3$, whose component functions are eigenfunctions of their Laplace operator, i.e.

$$\Delta x^i = \lambda^i x^i, \quad \lambda^i \in \mathbb{R}.$$

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In [1], Baba-Hamed and Bekkar studied the helicoidal surfaces without parabolic points in $\mathbb{E}_1^3$, which satisfy the condition

$$\Delta'' r_i = \lambda_i r_i.$$

In [3], Beneki *et al* obtained a classification of surfaces of revolution with constant Gauss curvature in $\mathbb{E}_1^3$ and in [4] defined four kinds of helicoidal surfaces in $\mathbb{E}_1^3$.

Also interesting computer graphics results concerning helicoidal surfaces were obtained in [6].

In this work we study the helicoidal surfaces in the three-dimensional Lorentz–Minkowski space under the condition

$$\Delta'' r = Ar \tag{1}$$

where $A \in \text{Mat}(3, \mathbb{R})$.

1. Preliminaries

A vector v of $\mathbb{E}_1^3$ is said to be space-like if $g_L(v, v) > 0$ or $v = 0$, time-like if $g_L(v, v) < 0$ and light-like if $g_L(v, v) = 0$ and $v \neq 0$. For two vectors v and w in $\mathbb{E}_1^3$ the Lorentz cross product of v and w is defined by

$$v \wedge_L w = (v_3 w_2 - v_2 w_3, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

The pseudo-vector product operation $\wedge_L$ is related to the determinant function by

$$\det(v, w, u) = g_L(v \wedge_L w, u).$$

The matrices

$$\begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 \\ \sinh \alpha & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & \sinh \alpha \\ 0 & \sinh \alpha & \cosh \alpha \end{pmatrix}, \quad \begin{pmatrix} \cosh \alpha & 0 & \sinh \alpha \\ 0 & 1 & 0 \\ \sinh \alpha & 0 & \cosh \alpha \end{pmatrix}$$

are called the Lorentzian rotation matrix in $\mathbb{E}_1^3$, where $\alpha \in \mathbb{R}$.

For an open interval $I \subset \mathbb{R}$, let $\gamma : I \rightarrow P$ be a curve in a plane P in $\mathbb{E}_1^3$ and let L be a straight line in P which does not intersect the curve γ (axis). A helicoidal surface in Minkowski space $\mathbb{E}_1^3$ is a surface invariant by a uniparametric group $G_{L,c} = \{g_v/g_v : \mathbb{E}_1^3 \rightarrow \mathbb{E}_1^3; v \in \mathbb{R}\}$ of helicoidal motions. Each helicoidal surface is given by a group of helicoidal motions and a generating curve. A helicoidal surface parametrizes as $r(u, v) = g_v(\gamma(u))$, $v \in \mathbb{R}$, $u \in I$. Each group of helicoidal motions is characterized by an axis L and a pitch $c \neq 0$. When $c = 0$, they reduce to rotation surfaces. The helicoidal surface is a generalization of a rotation surface in a Minkowski space.

If the axis L is space-like (respectively time-like), then L is transformed to the y -axis or z -axis (respectively x -axis) by the Lorentz transformation. Therefore, we may consider z -axis (respectively x -axis) as the axis if L is space-like (respectively time-like). If the axis L is light-like, then we may suppose that the axis is the line spanned by the vector $(1, 1, 0)$. We distinguish helicoidal surfaces in $\mathbb{E}_1^3$ into the following four types.

Case 1. The axis L is space-like, i.e. $(L = \langle(0, 0, 1)\rangle)$. Without loss of generality we may assume that the profile curve γ lies in the xz -plane or yz -plane. Hence, the curve γ can be represented by

$$\gamma(u) = (f(u), 0, g(u)) \quad \text{or} \quad \gamma(u) = (0, f(u), g(u)),$$

where $f = f(u)$ is a smooth positive function and $g = g(u)$ is a smooth function on I .

The helicoidal surfaces M in $\mathbb{E}_1^3$ given by [4] are defined by

$$r(u, v) = (f(u) \sinh v, f(u) \cosh v, cv + g(u)), \quad c \in \mathbb{R}^+ \quad (2)$$

or

$$r(u, v) = (f(u) \cosh v, f(u) \sinh v, cv + g(u)), \quad c \in \mathbb{R}^+. \quad (3)$$

We call (2) and (3) a helicoidal surface of type **I** and type **II**, respectively.

Case 2. The axis L is time-like, i.e. $(L = \langle(1, 0, 0)\rangle)$. In this case, we may assume that the profile curve γ lies in the xy -plane. So the curve γ is given by

$$\gamma(u) = (g(u), f(u), 0)$$

for a positive function $f = f(u)$ on I . Hence, the helicoidal surface M is given by [4]

$$r(u, v) = (g(u) + hv, f(u) \cos v, f(u) \sin v), \quad f(u) > 0, \quad c \in \mathbb{R}^+. \quad (4)$$

We call (4) a helicoidal surface of type **III**.

Case 3. The axis L is light-like, i.e. $(L = \langle(1, 1, 0)\rangle)$. In this case, we may assume that the profile curve γ lies in the xy -plane. Then its parametrization is given by

$$\gamma(u) = (f(u), g(u), 0), \quad u \in I,$$

where f and g are functions on I , such that $f(u) \neq g(u)$, $\forall u \in I$.

Therefore the helicoidal surface M may be parametrized as [4]

$$r(u, v) = \left(f(u) + \frac{v^2}{2}h(u) + cv, g(u) + \frac{v^2}{2}h(u) + cv, vh(u) \right), \quad c \in \mathbb{R}, \quad (5)$$

where $h(u) = f(u) - g(u)$. We call (5) a helicoidal surface of type **IV**.

We say that a helicoidal surface M in $\mathbb{E}_1^3$ is of type **I**[−] or **I**⁺ (respectively **II**[−] or **II**⁺, **III**[−] or **III**⁺, **IV**[−] or **IV**⁺) if it is time-like or space-like.

The immersion (M, r) is said to be of finite Chen type k if the position vector r admits the following spectral decomposition

$$r = r_0 + \sum_{j=1}^k r_j,$$

where r_j are $\mathbb{E}_1^3$ -valued eigenfunctions of the Laplacian of (M, r) : $\Delta r_j = \lambda_j r_j$, $\lambda_j \in \mathbb{R}$, $j = 1, 2, \dots, k$ (see [5]). If λ_j are different, then M is said to be of k -type.

Let $\{x^i, x^j\}$ be a local coordinate system of M . For the components h_{ij} ($i, j = 1, 2$) of the second fundamental form II on M we denote by (h^{ij}) the inverse matrix of the matrix (h_{ij}) .

The Laplace operator Δ^H of the second fundamental form H on M is formally defined by

$$\Delta^H = -\frac{1}{\sqrt{|h|}} \sum_{1 \leq i, j \leq 2} \left(\frac{\partial}{\partial x^i} \left(\sqrt{|h|} h^{ij} \frac{\partial}{\partial x^j} \right) \right),$$

where $h = \det(h_{ij})$.

Denote E, F, G the coefficients of the first fundamental form and L, M, N the coefficients of the second fundamental form.

If $\varphi : M \rightarrow \mathbb{R}$, $(u, v) \rightarrow \varphi(u, v)$ is a smooth function and Δ^H the Laplace operator with respect the second fundamental form, then the following holds:

$$\Delta^H \varphi = \frac{-1}{\sqrt{|LN - M^2|}} \left[\frac{\partial}{\partial u} \left(\frac{N\varphi_u - M\varphi_v}{\sqrt{|LN - M^2|}} \right) - \frac{\partial}{\partial v} \left(\frac{M\varphi_u - L\varphi_v}{\sqrt{|LN - M^2|}} \right) \right], \quad (6)$$

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

The Gaussian curvature K_G and the mean curvature H of M are given by

$$K_G = \frac{g_L(\mathbf{N}, \mathbf{N})}{EG - F^2} (LN - M^2)$$

$$H = \frac{1}{2|EG - F^2|} (EN + GL - 2FM),$$

where $\mathbf{N}$ is the unit normal vector to M .

2. Helicoidal surfaces of type I, II

In this section we are concerned with non-degenerate helicoidal surfaces M without parabolic points satisfying the condition (1).

Suppose that M is given by (2) or, equivalently, by

$$r(u, v) = (u \sinh v, u \cosh v, cv + g(u)), \quad c \in \mathbb{R}^+. \quad (7)$$

The coefficients of the first and the second fundamental form are:

$$E = 1 + g'^2, \quad F = cg', \quad G = c^2 - u^2$$

$$L = \frac{-ug''}{W}, \quad M = \frac{c}{W}, \quad N = \frac{u^2g'}{W}, \quad (8)$$

where $g' = dg/du$.

We define smooth function W as

$$W = \sqrt{\varepsilon g_L(r_u \wedge_L r_v, r_u \wedge_L r_v)} = \sqrt{\varepsilon(u^2(1 + g'^2) - c^2)},$$

where ε denotes the sign of the vector $r_u \wedge_L r_v$ in $\mathbb{E}_1^3$.

From these we find that the curvature K_G and the mean curvature H of (7) are given by

$$K_G = \frac{-1}{W^4} (u^3 g' g'' + c^2)$$

and

$$\begin{aligned} H &= \frac{1}{2W^3}(u^2g'(1+g'^2) - 2c^2g' - ug''(c^2 - u^2)) \\ &= \frac{1}{2u} \left(\frac{u^2g'}{W} \right)'. \end{aligned}$$

If a surface M in $\mathbb{E}_1^3$ has no parabolic points, then we have

$$u^3g'g'' + c^2 \neq 0.$$

Suppose that $LN - M^2 > 0$ (we have the same result if $LN - M^2 < 0$).

By a straightforward computation, the Laplacian Δ^H of the second fundamental form II on M with the help of (8) and (6) turns out to be

$$\begin{aligned} \Delta^H &= \frac{W}{R} \left(-u^2g' \frac{\partial^2}{\partial u^2} + ug'' \frac{\partial^2}{\partial v^2} + 2c \frac{\partial^2}{\partial u \partial v} \right) \\ &\quad - \frac{W}{2R^2} (g'(g'g''' - g''^2)u^4 - g'^2g''u^3 - 2c^2g''u - 4c^2g')u \frac{\partial}{\partial u} \\ &\quad + \frac{W}{2R^2} ((g'g''' + g''^2)u + 3g'g'')cu^2 \frac{\partial}{\partial v}, \end{aligned}$$

where $R(u) = R = u^3g'g'' + c^2$.

Accordingly, we get

$$\Delta^H r(u, v) = \begin{pmatrix} ug'A(u) \sinh v - cA(u) \cosh v \\ ug'A(u) \cosh v - cA(u) \sinh v \\ ug'^2A(u) + u^2B(u) \end{pmatrix}, \quad (9)$$

where

$$\begin{aligned} A(u) &= \frac{W}{2R^2} (-(g''^2 + g'g''')u^4 + g'g''u^3 + 4c^2) \\ B(u) &= \frac{W}{2R^2} (4g'^2g''^2u^3 + c^2(g''^2 + g'g''')u + 7c^2g'g''). \end{aligned}$$

Remark 2.1. We observe that

$$c^2A(u) + u^3B(u) = 2W. \quad (10)$$

Equation (1) by means of (7) and (9) gives rise to the following system of ordinary differential equations

$$\begin{cases} u(g'A(u) - a_{11}) \sinh v - (cA(u) + a_{12}u) \cosh v = a_{13}(cv + g) \\ u(g'A(u) - a_{22}) \cosh v - (cA(u) + a_{21}u) \sinh v = a_{23}(cv + g) \\ a_{32}u \cosh v + a_{31}u \sinh v = ug'^2A(u) + u^2B(u) - a_{33}(cv + g), \end{cases} \quad (11)$$

where a_{ij} ($i, j = 1, 2, 3$) denote the components of the matrix A given by (1).

However, $\sinh v$ and $\cosh v$ are linearly independent functions of v , so we finally obtain $a_{32} = a_{31} = a_{33} = a_{13} = a_{23} = 0$. We put $a_{11} = a_{22} = \lambda$ and $a_{12} = a_{21} = \mu$, $\lambda, \mu \in \mathbb{R}$.

Therefore, this system of equations is equivalently reduced to

$$\begin{cases} g'A(u) = \lambda \\ cA(u) = -\mu u \\ g'^2 A(u) + uB(u) = 0. \end{cases} \quad (12)$$

Now, let us examine the system of equations (12) according to the values of the constants λ and μ .

Case 1. Let $\lambda = 0$ and $\mu \neq 0$. In this case the system (12) is reduced equivalently to

$$\begin{cases} g' = 0 \\ cA(u) = -\mu u \\ B(u) = 0. \end{cases} \quad (13)$$

Differentiating (13), we obtain $A''(u) = 0$, which is a contradiction. Hence, there are no helicoidal surfaces of $\mathbb{E}_1^3$ in this case which satisfy the condition (1).

Case 2. Let $\lambda \neq 0$ and $\mu = 0$. In this case the system (12) is reduced equivalently to

$$\begin{cases} g'A(u) = \lambda \\ A(u) = 0. \end{cases}$$

However, this is not possible. So, in this case there are no helicoidal surfaces of $\mathbb{E}_1^3$.

Case 3. Let $\lambda = \mu = 0$. In this case the system (12) is reduced equivalently to

$$\begin{cases} A(u) = 0 \\ B(u) = 0. \end{cases}$$

From (10) we have $W = 0$, which is a contradiction. Consequently, there are no helicoidal surfaces of $\mathbb{E}_1^3$ in this case.

Case 4. Let $\lambda \neq 0$ and $\mu \neq 0$. In this case the system (12) is reduced equivalently to

$$g(u) = -\frac{\lambda c}{\mu} \ln(u) + k, \quad k \in \mathbb{R}.$$

Therefore, the helicoidal surface is given by

$$r(u, v) = \left(u \sinh v, u \cosh v, cv - \frac{\lambda c}{\mu} \ln(u) + k \right), \quad c \in \mathbb{R}^+, k \in \mathbb{R}.$$

Case 5. If $\lambda = \pm\mu \neq 0$, then $g(u) = \pm c \ln(u) + k$. Therefore, $H = 0$, which means that the surfaces are minimal. We have similar results for the helicoidal surfaces given by (7). Consequently, we have the following result.

THEOREM 2.2. *Let $r : M \rightarrow \mathbb{E}_1^3$ be an isometric immersion given by (7). Then $\Delta^H r = Ar$ if and only if the following statements hold true:*

- (1) *M has zero mean curvature;*
- (2) *M is parametrized as*

$$r(u, v) = \left(u \sinh v, u \cosh v, cv - \frac{\lambda c}{\mu} \ln(u) + k \right), \quad c \in \mathbb{R}^+, k \in \mathbb{R}.$$

3. Helicoidal surfaces of type III

In this section, we study the case of helicoidal surfaces M in $\mathbb{E}_1^3$ of type III. Suppose that M is given by (4), or equivalently by

$$r(u, v) = (cv + g(u), u \cos v, u \sin v). \quad (14)$$

The coefficients of the first and the second fundamental form are:

$$\begin{aligned} E &= 1 - g'^2, & F &= -cg', & G &= u^2 - c^2, \\ L &= \frac{ug''}{W}, & M &= -\frac{c}{W}, & N &= \frac{u^2g'}{W}, \end{aligned} \quad (15)$$

where $W = \sqrt{\varepsilon g_L(r_u \wedge_L r_v, r_u \wedge_L r_v)} = \sqrt{\varepsilon(u^2(1 - g'^2) - c^2)}$.

By a direct computation, we can see that the Gauss curvature K_G and the mean curvature H of M are given by

$$K_G = \frac{1}{W^4}(u^3g'g'' - c^2)$$

and

$$\begin{aligned} H &= \frac{1}{2W^3}(u^2g'(1 - g'^2) - 2c^2g' - ug''(c^2 - u^2)) \\ &= \frac{1}{2u} \left(\frac{u^2g'}{W} \right)'. \end{aligned}$$

If a surface M in $\mathbb{E}_1^3$ has no parabolic points, then we have

$$u^3g'g'' - c^2 \neq 0.$$

Suppose that $LN - M^2 > 0$ (we have the same result if $LN - M^2 < 0$).

By a straightforward computation, the Laplacian Δ^H of the second fundamental form II on M with the help of (6) and (15) turns out to be

$$\begin{aligned} \Delta^H &= -\frac{W}{R} \left(u^2g' \frac{\partial^2}{\partial u^2} + ug'' \frac{\partial^2}{\partial v^2} + 2c \frac{\partial^2}{\partial u \partial v} \right) \\ &\quad - \frac{W}{2R^2} (g'(g''^2 - g'g''')u^4 + g'^2g''u^3 - 2c^2g''u - 4c^2g')u \frac{\partial}{\partial u} \\ &\quad + \frac{W}{2R^2} ((g'g''' + g''^2)u + 3g'g'')cu^2 \frac{\partial}{\partial v}, \end{aligned}$$

where $R(u) = R = u^3g'g'' - c^2$.

Accordingly, we get

$$\Delta^H r(u, v) = \begin{pmatrix} ug'^2A(u) + u^2B(u) \\ ug'A(u) \cos v - cA(u) \sin v \\ ug'A(u) \sin v + cA(u) \cos v \end{pmatrix}, \quad (16)$$

where

$$\begin{aligned} A(u) &= \frac{W}{2R^2} ((g''^2 + g'g''')u^4 - g'g''u^3 + 4c^2) \\ B(u) &= \frac{W}{2R^2} (-4g'^2g''^2u^3 + c^2(g''^2 + g'g''')u + 7c^2g'g''). \end{aligned}$$

Remark 3.1. We observe that

$$c^2 A(u) - u^3 B(u) = 2W. \quad (17)$$

Equation (1) by means of (14) and (16) gives rise to the following system of ordinary differential equations

$$\begin{cases} a_{12}u \cos v + a_{13}u \sin v = ug'^2 A(u) + u^2 B(u) - a_{11}(cv + g) \\ (ug' A(u) - a_{22}u) \cos v - (cA(u) + a_{23}u) \sin v = a_{21}(cv + g) \\ (ug' A(u) - a_{33}u) \sin v + (cA(u) - a_{32}u) \cos v = a_{31}(cv + g). \end{cases} \quad (18)$$

From (18) we easily deduce that $a_{11} = a_{12} = a_{13} = a_{21} = a_{31} = 0$, $a_{22} = a_{33}$ and $a_{32} = -a_{23}$. We put $a_{22} = a_{33} = \lambda$ and $a_{32} = -a_{23} = \mu$, $\lambda, \mu \in \mathbb{R}$. Therefore, this system of equations is equivalently reduced to

$$\begin{cases} g' A(u) = \lambda \\ cA(u) = \mu u \\ g'^2 A(u) + uB(u) = 0. \end{cases} \quad (19)$$

We discuss four cases according to the constants λ and μ .

Case 1. Let $\lambda = 0$ and $\mu \neq 0$. From this system we get $A''(u) = 0$, which is a contradiction. Hence, there are no helicoidal surfaces of $\mathbb{E}_1^3$ in this case.

Case 2. Let $\lambda \neq 0$ and $\mu = 0$. In this case the system (19) is reduced equivalently to

$$\begin{cases} g' A(u) = \lambda \neq 0 \\ A(u) = 0. \end{cases}$$

However, this is not possible. So, in this case there are no helicoidal surfaces of $\mathbb{E}_1^3$.

Case 3. Let $\lambda = \mu = 0$. In this case the system (19) is reduced equivalently to

$$\begin{cases} A(u) = 0 \\ B(u) = 0. \end{cases}$$

From (17) we have $W = 0$, which is a contradiction. So, in this case there are no helicoidal surfaces of $\mathbb{E}_1^3$.

Case 4. Let $\lambda \neq 0$ and $\mu \neq 0$. In this case the system (19) is reduced equivalently to

$$g(u) = \frac{\lambda c}{\mu} \ln(u) + k, \quad k \in \mathbb{R}.$$

Therefore, the helicoidal surface is given by

$$r(u, v) = \left(cv + \frac{\lambda c}{\mu} \ln(u) + k, u \cos v, u \sin v \right).$$

THEOREM 3.2. Let $r : M \rightarrow \mathbb{E}_1^3$ be an isometric immersion given by (7). Then $\Delta^H r = Ar$ if and only if M is parametrized as

$$r(u, v) = \left(cv + \frac{\lambda c}{\mu} \ln(u) + k, u \cos v, u \sin v \right).$$

4. Helicoidal surfaces of type IV

In this section, we study the case of helicoidal surfaces M in $\mathbb{E}_1^3$ of type IV. Let M be the helicoidal surfaces given by (5).

We assume that $f(u) = u$. Then (5) reduces to

$$r(u, v) = \left(u - \frac{v^2}{2}h(u) + cv, \left(1 - \frac{v^2}{2} \right)g(u) + \frac{v^2}{2}u + cv, -vh(u) \right), \quad c \in \mathbb{R}, \quad (20)$$

where $h(u) = g(u) - u \neq 0, \forall u \in I$.

The coefficients of the first and the second fundamental form are

$$\begin{aligned} E &= -1 + g'^2, & F &= ch', & G &= h^2, \\ L &= \frac{hh''}{W}, & M &= \frac{ch'^2}{W}, & N &= \frac{h^2h'}{W}, \end{aligned} \quad (21)$$

where $W = \sqrt{\varepsilon g_L(r_u \wedge_L r_v, r_u \wedge_L r_v)} = \sqrt{\varepsilon(h'((h^2 - c^2)h' + 2h^2))}$.

From these we find that the curvature K_G and the mean curvature H of (20) are given by

$$K_G = \frac{h'}{W^4}(h^3h'' - c^2h'^3)$$

and

$$\begin{aligned} H &= \frac{1}{2W^3}(h'^3(h^2 - 2c^2) + h^2(2h'^2 + hh'')) \\ &= \frac{1}{hh'} \left(\frac{h^2h'}{W} \right)'. \end{aligned} \quad (22)$$

If a surface M in $\mathbb{E}_1^3$ has no parabolic points, then we have

$$h'(h^3h'' - c^2h'^3) \neq 0.$$

Suppose that $LN - M^2 > 0$ (we have the same result if $LN - M^2 < 0$).

By a straightforward computation, the Laplacian Δ^II of the second fundamental form II on M with the help of (6) and (21) turns out to be

$$\begin{aligned} \Delta^II &= \frac{W}{R} \left(-h^2h' \frac{\partial^2}{\partial u^2} - hh'' \frac{\partial^2}{\partial v^2} + 2ch'^2 \frac{\partial^2}{\partial u \partial v} \right) \\ &\quad - \frac{W}{2R^2} (h^4h'^2 + 2c^2h'^3hh'' - h^4h'h''' + h^3h'^2h'' - 4c^2h'^5)hh' \frac{\partial}{\partial u} \\ &\quad - \frac{W}{2R^2} (-3hh''^2 + 3h'^2)h'' + hh'h''' ch^2h'^2 \frac{\partial}{\partial v}. \end{aligned} \quad (23)$$

where $R(u) = R = h'(h^3h'' - c^2h'^3)$.

Then by using (20) and (23) we get

$$\Delta^II r(u, v) = \begin{pmatrix} h' \left(A(u) \left(h + h' \left(cv - \frac{v^2}{2}h \right) \right) + h^2B(u) \right) \\ h' \left(A(u) \left(hg' + h' \left(cv - \frac{v^2}{2}h \right) \right) + h^2B(u) \right) \\ h'^2(c - v h A(u)) \end{pmatrix}, \quad (24)$$

where

$$A(u) = \frac{W}{2R^2}((-3h''^2 + h'h''')h^4 - h'^2h''h^3 + 4c^2h'^5)$$

$$B(u) = \frac{W}{2R^2}(4h''^2h^3 + (3c^2h''^2 - c^2h'h''')h'h - 7c^2h'^3h'').$$

Remark 4.1. We observe that

$$c^2h'^3A(u) + h'^2h^3B(u) = 2W. \quad (25)$$

Equation (1) by means of (5) and (24) gives rise to the following system of ordinary differential equations

$$\begin{cases} (h'hA(u) + h^2h'B(u) - a_{11}u - a_{12}g) + (ch'^2A(u) - ca_{11} - ca_{12} + ha_{13})v \\ \quad + h(-h'^2A(u) + a_{11} + a_{12})\frac{v^2}{2} = 0 \\ (h'hg'A(u) + h^2h'B(u) - a_{21}u - a_{22}g) + (ch'^2A(u) - ca_{21} - ca_{22} + ha_{23})v \\ \quad + h(-h'^2A(u) + a_{21} + a_{22})\frac{v^2}{2} = 0 \\ (ch'^2A(u) - a_{31}u - a_{32}g) + (-hh'^2A(u) - ca_{31} - ca_{32} + ha_{33})v \\ \quad - h(a_{31} + a_{32})\frac{v^2}{2} = 0. \end{cases} \quad (26)$$

So we can regard the above equations as polynomials with variable v and from the coefficients we get

$$h'hA(u) + h^2h'B(u) - a_{11}u - a_{12}g = 0 \quad (27)$$

$$c(h'^2A(u) - a_{11} - a_{12}) + ha_{13} = 0 \quad (28)$$

$$-h'^2A(u) + a_{11} + a_{12} = 0 \quad (29)$$

$$h'hg'A(u) + h^2h'B(u) - a_{21}u - a_{22}g = 0 \quad (30)$$

$$c(h'^2A(u) - a_{21} - a_{22}) + ha_{23} = 0 \quad (31)$$

$$-h'^2A(u) + a_{21} + a_{22} = 0 \quad (32)$$

$$ch'^2A(u) - a_{31}u - a_{32}g = 0 \quad (33)$$

$$-hh'^2A(u) - c(a_{31} + a_{32}) + ha_{33} = 0 \quad (34)$$

$$a_{31} + a_{32} = 0. \quad (35)$$

From (28), (29) and (31) we have $a_{13} = a_{23} = 0$. Using (32) in (29), we obtain

$$a_{21} + a_{22} = a_{11} + a_{12}. \quad (36)$$

On the other hand, from (34) and (35) we get that

$$h'^2A(u) = a_{33}. \quad (37)$$

Using (35) in (33), we obtain

$$ch'^2A(u) = a_{32}h. \quad (38)$$

From (37) and (38) we have

$$a_{32}h = ca_{33}. \quad (39)$$

Differentiating (39) with respect to u we have $a_{32} = 0$ (because $h' \neq 0$). On the other hand, from (35) and (39) we get that

$$a_{33} = a_{31} = 0.$$

From (27) and (30) we have

$$h'A(u)(1 - g') = -h'^2 A(u) = a_{12} - a_{22} = 0. \quad (40)$$

Comparing now equations (40) and (36), we deduce that $a_{12} = a_{22}$ and $a_{21} = a_{11}$. From equation (32) we easily deduce that $a_{21} = -a_{22}$. Putting $a_{12} = a_{22} = \lambda = -a_{21} = -a_{11}$ we see that the system (26) reduces now to the following equations

$$\begin{cases} A(u) = 0 \\ hh'B(u) = \lambda. \end{cases} \quad (41)$$

We distinguish between the following cases.

Case 1. Let $\lambda = 0$. Then (41) reduces to $A(u) = B(u) = 0$. From (25) we have $W = 0$, which is a contradiction. So, in this case there are no helicoidal surfaces of $\mathbb{E}_1^3$.

Case 2. Let $\lambda \neq 0$. From (41) we get $B(u) = \lambda/hh'$. Substituting this value of $B(u)$ in (25) we obtain

$$\frac{h'h^2}{W} = \frac{2}{\lambda}. \quad (42)$$

Then, equation (42) gives rise to $H = 0$, which means that the surfaces are minimal.

THEOREM 4.2. *Let $r : M \rightarrow \mathbb{E}_1^3$ be an isometric immersion given by (7). Then $\Delta^H r = Ar$ if and only if M has zero mean curvature.*

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