KNOPP-TYPE IDENTITIES FOR GENERALIZED MULTIPLE DEDEKIND-TYPE SUMS

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Abstract. In this paper, for any multivariable function with a periodicity and a certain distribution relation, we define a multiple Dedekind-type sum. Then by a combinatorial–geometric method, we study generalizations of Knopp’s formula for the classical Dedekind sums. The main theorem contains many of the preceding results concerning generalizations of Knopp’s formula.

1. Introduction

For $h \in \mathbb{Z}$ and $k \in \mathbb{N}$, the classical Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{\alpha \bmod k} \left(\left(\frac{\alpha}{k} \right) \right) \left(\left(\frac{\alpha h}{k} \right) \right), \quad (1.1)$$

where

$$\left((x) \right) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

In this paper, we study generalizations of the following result.

THEOREM 1.1. *For any $N \in \mathbb{N}$, we have*

$$\sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} s(ah + kb, dk) = \sigma(N)s(h, k), \quad (1.2)$$

where $\sigma(N) = \sum_{\delta|N} \delta$ and

$$\sum_{b=0}^{N-1} s(h + kb, Nk) = \sum_{d|N} \mu(d)\sigma(N/d)s(dh, k), \quad (1.3)$$

where μ is the Möbius’ function. Furthermore (1.2) and (1.3) are equivalent.

The formula (1.2) was obtained by Knopp in [13] by making use of Hecke operators, and the formula (1.3) by Subrahmanyam in [23] by an elementary method. Note that in the special case that N is a prime number, (1.2) was already known to Dedekind [11].

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In [19], Parson and Rosen generalized (1.2) to higher-order Dedekind sums defined by substituting Bernoulli functions of higher degree for $((*)$) in (1.1). As for the relation between (1.2) and (1.3), Goldberg showed in [12] that (1.2) is easily deduced from (1.3). In [2], Apostol and Vu generalized the result of Goldberg to Dedekind type sums. In [21], Redmond *et al* clarified that the generalization of (1.2) to Dedekind type sums and that of (1.3) are equivalent. In [14], [15] and [18], for inhomogeneous Dedekind sums attached to Dirichlet characters, Nagasaka and the author studied generalizations of (1.2) and (1.3) and their equivalence.

In [24], Zheng extended (1.2) to homogeneous Dedekind-type sums and in [4], Beck generalized the result of Zheng to homogeneous multivariable cases.

Recently, by a combinatorial–geometric method, Beck gave new proofs of a well-known reciprocity formula for (1.1) and some of its generalizations [5, 6, 17]. The method is based on the work of Carlitz [9] and deeply connected with the theory of lattice points in polytopes [3, 7].

In [16], the author made use of the method and gave a new proof of a generalization of (1.2) obtained in [15].

The purpose of this paper is to extend the method in [16] and generalize Theorem 1.1 to more widely generalized Dedekind-type sums.

Let us give a description of each section.

In Section 2, for a multivariable function having a periodicity and a certain distribution relation attached to Dirichlet characters, we define a multiple Dedekind-type sum and show that the sum contains some of the generalized Dedekind sums defined by means of Bernoulli functions. Then, we state the main theorem of this paper.

As in [16], in order to provide a good overview, we first prove the main theorem for a special simple case in Section 3. A complete proof for the general case is given in Section 4.

2. Definitions and the main theorem

Let R be a ring extension of $\mathbb{Q}$ containing all of the roots of unity. Let $m, n \in \mathbb{N}$ and let $\varphi_1, \dots, \varphi_{m+n}$ be Dirichlet characters (not necessarily primitive) defined modulo $N_{\varphi_1}, \dots, N_{\varphi_{m+n}}$, respectively. Consider a map

$$F : \mathbb{Q}^{m+n} \rightarrow R$$

satisfying the following conditions (i) and (ii).

(i) For any $\mathbf{x} \in \mathbb{Q}^{m+n}$ and $\mathbf{l} \in \mathbb{Z}^{m+n}$, we have

$$F(\mathbf{x} + \mathbf{l}) = F(\mathbf{x}). \quad (2.1)$$

(ii) For each $1 \leq \rho \leq m+n$, there is a constant $v_\rho \in \mathbb{Z}$ such that

$$\begin{aligned} \sum_{\lambda \bmod M} F\left(x_1, \dots, x_{\rho-1}, x_\rho + \frac{\lambda}{M}, x_{\rho+1}, \dots, x_{m+n}\right) \\ = M^{-v_\rho} \varphi_\rho^{-1}(M) F(x_1, \dots, x_{\rho-1}, Mx_\rho, x_{\rho+1}, \dots, x_{m+n}) \end{aligned} \quad (2.2)$$

holds for any $M \in \mathbb{N}$ with $\gcd\{M, N_{\varphi_\rho}\} = 1$.

By condition (i) we can also regard F as a map from $(\mathbb{Q}/\mathbb{Z})^{m+n}$ to R .

Let H be an $m \times n$ matrix with components in $\mathbb{Z}$. In what follows, we consider any $\mathbf{x} \in \mathbb{Q}^m$ as a row vector, so that $\mathbf{x}H$ is defined in $\mathbb{Q}^n$ and $F(\mathbf{x} \ \mathbf{x}H)$ in R . Let $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{N}^m$ and let $\mathcal{R}_m(\mathbf{k}) \subset \mathbb{Z}^m$ denote an arbitrarily fixed complete set of representatives of the residue class group $(\mathbb{Z}/k_1\mathbb{Z}) \times \dots \times (\mathbb{Z}/k_m\mathbb{Z})$. For $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Q}^m$, we put

$$\frac{\mathbf{x}}{\mathbf{k}} = \left(\frac{x_1}{k_1}, \dots, \frac{x_m}{k_m} \right).$$

In what follows we suppose that $\gcd\{k_i, N_{\varphi_i}\} = \gcd\{k_i, N_{\varphi_{m+j}}\} = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We now define a generalized Dedekind-type sum for F , H and $\mathbf{k}$ as

$$\mathcal{S}(F : H, \mathbf{k}) = \sum_{\mathbf{x} \in \mathcal{R}_m(\mathbf{k})} F\left(\frac{\mathbf{x}}{\mathbf{k}} \ \frac{\mathbf{x}}{\mathbf{k}}H\right). \quad (2.3)$$

In order to show some examples, we recall some definitions and basic properties concerning Bernoulli functions. As usual, we define the p th Bernoulli polynomial $B_p(X)$ by

$$\frac{te^{tX}}{e^t - 1} = \sum_{p=0}^{\infty} B_p(X) \frac{t^p}{p!}$$

and put $B_p = B_p(0)$, the p th Bernoulli number. For any $x \in \mathbb{Q}$, we write $x = [x] + \{x\}$ with $[x] \in \mathbb{Z}$ and $0 \leq \{x\} < 1$ and define the p th Bernoulli function as $\tilde{B}_p(x) = B_p(\{x\})$, which is periodic of period one and satisfies a distribution relation such as

$$\sum_{\lambda \bmod M} \tilde{B}_p\left(x + \frac{\lambda}{M}\right) = M^{-p+1} \tilde{B}_p(Mx) \quad (2.4)$$

for any $M \in \mathbb{N}$ and $x \in \mathbb{Q}$ (Raabe's theorem [20]). Note that $((x)) = \tilde{B}_1(x)$ or zero according as $x \notin \mathbb{Z}$ or $x \in \mathbb{Z}$.

For any primitive Dirichlet character χ , we denote by f_χ the conductor of χ . For any $x \in \mathbb{Q}$ with denominator relatively prime to f_χ , we can define the value $\chi(x)$ by multiplicativity. As in [22], we define the twisted p th Bernoulli function $\tilde{B}_{p,\chi}(x)$ by

$$\sum_{a=0}^{f_\chi-1} \frac{\chi(\{x\} + a)te^{(\{x\}+a)t}}{e^{f_\chi t} - 1} = \sum_{p=0}^{\infty} \tilde{B}_{p,\chi}(x) \frac{t^p}{p!},$$

or, equivalently,

$$\tilde{B}_{p,\chi}(x) = f_\chi^{p-1} \sum_{a \bmod f_\chi} \chi(x+a) \tilde{B}_p\left(\frac{x+a}{f_\chi}\right)$$

(cf. p. 301 of [22]). Unless the denominator of $x \in \mathbb{Q}$ is relatively prime to f_χ , we put $\tilde{B}_{p,\chi}(x) = 0$. Then, it is easily verified that $\tilde{B}_{p,\chi}(x)$ is also periodic of period one and that the distribution relation (2.4) is extended as

$$\sum_{\lambda \bmod M} \tilde{B}_{p,\chi}\left(x + \frac{\lambda}{M}\right) = M^{-p+1} \chi^{-1}(M) \tilde{B}_{p,\chi}(Mx)$$

for any $M \in \mathbb{N}$ with $\gcd\{M, f_\chi\} = 1$ and any $x \in \mathbb{Q}$.

On the basis of the periodicity and the distribution relations as above, let us show some examples of (2.3).

Example 1. The classical Dedekind sum $s(h, k)$ is the sum (2.3) in the case that $m = n = 1$, $F(x, y) = ((x))((y))$, $H = (h)$, and $\mathbf{k} = (k)$.

Example 2. Let $m = n = 1$, $H = (h)$ and $\mathbf{k} = (k)$. For $p, q \in \mathbb{Z}_{\geq 0}$ and primitive Dirichlet characters χ and ψ , put $F(x, y) = \tilde{B}_{p, \chi}(x) \tilde{B}_{q, \psi}(y)$. Then

$$S(F : H, \mathbf{k}) = \sum_{\alpha \bmod k} \tilde{B}_{p, \chi}\left(\frac{\alpha}{k}\right) \tilde{B}_{q, \psi}\left(\frac{\alpha h}{k}\right), \quad (2.5)$$

which is a higher-order Dedekind sum attached to χ and ψ . In the case of $p = 1$ and $\chi = \psi = 1$, the sum (2.5) is essentially the same as (1.3) in [1] and the generalization of (1.2) was studied in [19]. The case where $\chi = 1$ appeared in [22]. In [14], the author studied a generalizations of Theorem 1.1 for (2.5).

Example 3. Let $m = 1$, $n = 2$, $H = (h_1, h_2)$ and $\mathbf{k} = (k)$. For $q_1, q_2 \in \mathbb{Z}_{\geq 0}$, put $F(x, y_1, y_2) = \tilde{B}_{q_1}(y_1) \tilde{B}_{q_2}(y_2)$ (which is constant with respect to x). Then

$$S(F : H, \mathbf{k}) = \sum_{\alpha \bmod k} \tilde{B}_{q_1}\left(\frac{\alpha h_1}{k}\right) \tilde{B}_{q_2}\left(\frac{\alpha h_2}{k}\right), \quad (2.6)$$

which is what is called a homogeneous Dedekind sum. In [24], Zheng studied an extension of (1.2) for (2.6).

Example 4. As a generalization of (2.6), we can consider the case where $m = 1$ and in addition to conditions (2.1) and (2.2), $F(x_1, y_1, \dots, y_n)$ is constant with respect to x_1 . Put $F_1(y_1, \dots, y_n) = F(x_1, y_1, \dots, y_n)$ for any $x_1 \in \mathbb{Q}$. Then for $H = (h_1, \dots, h_n)$ and $\mathbf{k} = (k)$, we have

$$S(F : H, \mathbf{k}) = \sum_{\alpha \bmod k} F_1\left(\frac{\alpha h_1}{k}, \dots, \frac{\alpha h_n}{k}\right).$$

This includes the sum defined as in Definition 2 of Beck [4].

Example 5. Let

$$n = 1, \quad H = \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix}, \quad p_1, \dots, p_m, q \in \mathbb{Z}_{\geq 0}$$

and let $\varphi_1, \dots, \varphi_m, \psi$ be primitive Dirichlet characters. Put $F(x_1, \dots, x_m, y) = (\prod_{i=1}^m \tilde{B}_{p_i, \varphi_i}(x_i)) \tilde{B}_{q, \psi}(y)$. Then

$$S(F : H, \mathbf{k}) = \sum_{\alpha_i \bmod k_i \ (1 \leq i \leq m)} \left(\prod_{i=1}^m \tilde{B}_{p_i, \varphi_i}\left(\frac{\alpha_i}{k_i}\right) \right) \tilde{B}_{q, \psi}\left(\frac{\alpha_1 h_1}{k_1} + \dots + \frac{\alpha_m h_m}{k_m}\right).$$

The special case of this was studied by Carlitz in [8] and [10] with rather modified forms. In [15] and [16], under an additional condition that $k_1 = \dots = k_m$, the author studied a generalization of Theorem 1.1.

In what follows, we write ψ_j and v'_j for φ_{m+j} and v_{m+j} , respectively, for $1 \leq j \leq n$. Put

$$\varphi = \prod_{i=1}^m \varphi_i, \quad \psi = \prod_{j=1}^n \psi_j, \quad v = \sum_{i=1}^m v_i \quad \text{and} \quad v' = \sum_{j=1}^n v'_j. \quad (2.7)$$

For any Dirichlet character χ and any $l, N \in \mathbb{N}$, we put

$$\sigma_{l,\chi}(N) = \sum_{\delta|N} \chi(\delta) \delta^l.$$

For any $d \in \mathbb{N}$, let $\mathcal{I}_d$ be a set of $m \times n$ matrices such as

$$\mathcal{I}_d = \left\{ B = (b_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \mid b_{ij} \in \mathbb{Z}, \quad 0 \leq b_{ij} \leq d-1 \text{ for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n \right\}.$$

For $\mathbf{k} = (k_1, \dots, k_m)$ and $B \in \mathcal{I}_d$, we put

$$\mathbf{k} \circ B = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_m \end{pmatrix} B,$$

which is a matrix obtained from B by multiplying the i th row by k_i for each $1 \leq i \leq m$. Now we state our main theorem.

THEOREM 2.1. *Let $N \in \mathbb{N}$ with $\gcd\{N, N_{\varphi_i}\} = \gcd\{N, N_{\psi_j}\} = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Then we have*

$$\begin{aligned} N^v \varphi(N) \sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d} d^{v'-(m-1)n} \psi(d) \mathcal{S}(F : aH + \mathbf{k} \circ B, d\mathbf{k}) \\ = \sigma_{v+v'+n, \varphi\psi}(N) \mathcal{S}(F : H, \mathbf{k}) \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} N^{v+v'-(m-1)n} (\varphi\psi)(N) \sum_{B \in \mathcal{I}_N} \mathcal{S}(F : H + \mathbf{k} \circ B, N\mathbf{k}) \\ = \sum_{d|N} \mu(d) d^v \varphi(d) \sigma_{v+v'+n, \varphi\psi}(N/d) \mathcal{S}(F : dH, \mathbf{k}). \end{aligned} \quad (2.9)$$

Furthermore (2.8) and (2.9) are equivalent.

3. Proof of Theorem 2.1 in a special case

In this section, we consider the sum (2.3) in the special case where $m = n = 1$ and $\varphi_1 = \psi_1 = 1$. In this case, the conditions (2.1) and (2.2) for the map $F : \mathbb{Q}^2 \rightarrow R$ in the previous section reduce to the following (3.1) and (3.2), respectively:

$$F(x + l_1, y + l_2) = F(x, y) \quad \text{for any } x, y \in \mathbb{Q} \text{ and } l_1, l_2 \in \mathbb{Z}. \quad (3.1)$$

$$\begin{aligned} \sum_{\lambda \bmod M} F\left(x + \frac{\lambda}{M}, y\right) &= M^{-v} F(Mx, y) \quad \text{and} \\ \sum_{\lambda \bmod M} F\left(x, y + \frac{\lambda}{M}\right) &= M^{-v'} F(x, My) \quad \text{for any } x, y \in \mathbb{Q} \text{ and } M \in \mathbb{N}. \end{aligned} \quad (3.2)$$

For $h \in \mathbb{Z}$ and $k \in \mathbb{N}$, we put

$$s(F : h, k) = \sum_{\alpha \bmod k} F\left(\frac{\alpha}{k}, \frac{\alpha h}{k}\right). \quad (3.3)$$

For $l, N \in \mathbb{N}$, we put $\sigma_l(N) = \sum_{\delta|N} \delta^l$. Then the formulas (2.8) and (2.9) in Theorem 2.1 reduce to the following (3.4) and (3.5), respectively:

$$N^v \sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} d^{v'} s(F : ah + kb, dk) = \sigma_{v+v'+1}(N) s(F : h, k), \quad (3.4)$$

$$N^{v+v'} \sum_{b=0}^{N-1} s(F : h + kb, Nk) = \sum_{d|N} \mu(d) d^v \sigma_{v+v'+1}(N/d) s(F : dh, k). \quad (3.5)$$

The purpose of this section is to prove (3.4) and (3.5) and their equivalence. In the first place, we prepare some lemmas.

LEMMA 3.1. *For any $c \in \mathbb{N}$ we have*

$$s(F : ch, ck) = c^{-v} s(F : h, k).$$

Proof. Let $\mathcal{R}(k) \subset \mathbb{Z}$ (respectively $\mathcal{R}(c) \subset \mathbb{Z}$) be an arbitrarily fixed complete system of representatives of the residue class group $\mathbb{Z}/k\mathbb{Z}$ (respectively $\mathbb{Z}/c\mathbb{Z}$). Then the set $\{\alpha + kl \mid \alpha \in \mathcal{R}(k), l \in \mathcal{R}(c)\}$ is a complete system of representatives of the residue class group $\mathbb{Z}/ck\mathbb{Z}$. It follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned} s(F : ch, ck) &= \sum_{\alpha \in \mathcal{R}(k)} \sum_{l \in \mathcal{R}(c)} F\left(\frac{\alpha + kl}{ck}, \frac{(\alpha + kl)ch}{ck}\right) \\ &= \sum_{\alpha \in \mathcal{R}(k)} \sum_{l \in \mathcal{R}(c)} F\left(\frac{\alpha}{ck} + \frac{l}{c}, \frac{\alpha h}{k}\right) \\ &= \sum_{\alpha \in \mathcal{R}(k)} c^{-v} F\left(\frac{\alpha}{k}, \frac{\alpha h}{k}\right) = c^{-v} s(F : h, k). \end{aligned}$$

This completes the proof. $\square$

In [16], we introduced an additive subgroup of $\mathbb{Z}^2$ such as

$$\mathcal{A}(a, d : b) = (a, -b)\mathbb{Z} + (0, d)\mathbb{Z}$$

for $a, d \in \mathbb{N}$ and $b \in \mathbb{Z}$, and as a necessary and sufficient condition for $(l_1, l_2) \in \mathbb{Z}^2$ to be in $\mathcal{A}(a, d : b)$, we proved the following.

LEMMA 3.2. *Let $N = ad$ with $a, d \in \mathbb{N}$ and $b \in \mathbb{Z}$ and let $(l_1, l_2) \in \mathbb{Z}^2$. Put $d_1 = \gcd\{l_1, N\}$, $d_2 = \gcd\{l_1, l_2, N\}$, $l' = l_1/d_1$ and $N' = N/d_1$. Then, we have $(l_1, l_2) \in \mathcal{A}(a, d : b)$ if and only if the following three conditions hold:*

- (i) $a|d_1$;
- (ii) $(d_1/a)|d_2$;
- (iii) $l'b \equiv -(al_2/d_1) \pmod{N'}$.

As a consequence of Lemma 3.2, we extend Corollary 3.3 of [16] as follows.

LEMMA 3.3. *Let $\mathcal{F}$ be a map from $\mathbb{Z}^2$ to R and suppose that except for finitely many $(l_1, l_2) \in \mathbb{Z}^2$, $\mathcal{F}(l_1, l_2) = 0$. Then for any $N \in \mathbb{N}$, we have*

$$\sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} \sum_{(l_1, l_2) \in \mathcal{A}(a, d : b)} \mathcal{F}(l_1, l_2) = \sum_{\delta|N} \delta \sum_{(l_1, l_2) \in (\delta\mathbb{Z})^2} \mathcal{F}(l_1, l_2) \quad (3.6)$$

and

$$\sum_{b=0}^{N-1} \sum_{(l_1, l_2) \in \mathcal{A}(1, N; b)} \mathcal{F}(l_1, l_2) = \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta \sum_{(l_1, l_2) \in d\mathbb{Z} \times \delta\mathbb{Z}} \mathcal{F}(l_1, l_2). \quad (3.7)$$

Furthermore (3.6) and (3.7) are equivalent.

Proof. We use the same notation as in Lemma 3.2. Note that $\gcd\{l', N'\} = 1$. Suppose that a satisfies conditions (i) and (ii) in Lemma 3.2. Then condition (iii) shows that

$$\sharp\{b \in \mathbb{Z} \mid 0 \leq b \leq d-1, (l_1, l_2) \in \mathcal{A}(a, d : b)\} = \frac{d}{N'} = \frac{dd_1}{N} = \frac{d_1}{a}.$$

This shows that for any fixed $(l_1, l_2) \in \mathbb{Z}^2$, the coefficient of $\mathcal{F}(l_1, l_2)$ in the left-hand side of (3.6) is $\sum_{a|d_1, (d_1/a)|d_2} d_1/a$. By putting $\delta = d_1/a$, this coefficient is equal to $\sum_{\delta|d_2} \delta$, which is just the coefficient of $\mathcal{F}(l_1, l_2)$ in the right-hand side of (3.6). This proves (3.6).

Next, by considering the case of $a = 1$ and $d = N$ in Lemma 3.2, a similar argument as in the above proof of (3.6) shows that for any fixed $(l_1, l_2) \in \mathbb{Z}^2$, the coefficient of $\mathcal{F}(l_1, l_2)$ in the left-hand side of (3.7) is d_1 or zero according to whether $d_1 = d_2$ or $d_1 > d_2$. As for the right-hand side of (3.7), we have $(l_1, l_2) \in d\delta\mathbb{Z} \times \delta\mathbb{Z}$ for $d|N$ and $\delta|(N/d)$ if and only if $d\delta|d_1$ and $\delta|d_2$. Hence, the coefficient of $\mathcal{F}(l_1, l_2)$ in the right-hand side of (3.7) is $\sum_{\delta|d_2} \sum_{d|d_1} \delta \mu(d) = \sum_{\delta|d_2} \delta \sum_{d|(d_1/\delta)} \mu(d)$. Recall that, in general, we have $\sum_{d|n} \mu(d) = 1$ or 0 according to whether $n = 1$ or $n > 1$. It follows that $\sum_{\delta|d_2} \delta \sum_{d|(d_1/\delta)} \mu(d) = d_1$ or 0 according to whether $d_1 = d_2$ or $d_1 > d_2$. Thus, we obtain (3.7).

In order to prove the equivalence between (3.6) and (3.7), we put $\mathcal{F}_1(l_1, l_2) = \mathcal{F}(Nl_1, l_2)$ for $(l_1, l_2) \in (1/N)\mathbb{Z} \times \mathbb{Z}$. Note that if $N = ad$ with $a, d \in \mathbb{N}$, the map from $\mathbb{Z}^2$ to $\mathbb{Z}^2$ defined by mapping $(l_1, l_2) \in \mathbb{Z}^2$ onto $(al_1, l_2) = ((N/d)l_1, l_2) \in \mathbb{Z}^2$ gives a bijection from $\mathcal{A}(1, d : b)$ to $\mathcal{A}(a, d : b)$. Hence, (3.6) is expressed as

$$\sum_{d|N} \sum_{b=0}^{d-1} \sum_{(l_1, l_2) \in \mathcal{A}(1, d; b)} \mathcal{F}_1(l_1/d, l_2) = \sum_{\delta|N} \delta \sum_{(l_1, l_2) \in (\delta\mathbb{Z})^2} \mathcal{F}_1(l_1/N, l_2).$$

By the Möbius inversion formula, this is equivalent to

$$\sum_{b=0}^{N-1} \sum_{(l_1, l_2) \in \mathcal{A}(1, N; b)} \mathcal{F}_1(l_1/N, l_2) = \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta \sum_{(l_1, l_2) \in (\delta\mathbb{Z})^2} \mathcal{F}_1\left(\frac{l_1}{N/d}, l_2\right),$$

which is easily transformed into (3.7). This completes the proof. $\square$

Now let us proceed to prove (3.4) and (3.5). We consider a half-open parallelogram

$$\mathcal{P} = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x < Nk, \frac{hx}{k} \leq y < \frac{hx}{k} + N \right\}$$

and put $D = \mathcal{P} \cap \mathbb{Z}^2$, the set of lattice points in $\mathcal{P}$. Let us take

$$\mathcal{F}(l_1, l_2) = \begin{cases} F\left(\frac{l_1}{Nk}, \frac{1}{N}\left(\frac{l_1 h}{k} - l_2\right)\right) & \text{if } (l_1, l_2) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Let $N = ad$ with $a, d \in \mathbb{N}$. For any $b \in \mathbb{Z}$ with $0 \leq b \leq d-1$ and for any $(l_1, l_2) \in \mathcal{A}(a, d : b)$, express

$$(l_1, l_2) = r_1(a, -b) + r_2(0, d) = (ar_1, -br_1 + dr_2)$$

with $r_1, r_2 \in \mathbb{Z}$. Then $(l_1, l_2) \in D$ holds if and only if r_1 and r_2 satisfy

$$0 \leq ar_1 < Nk \quad \text{and} \quad \frac{ar_1 h}{k} \leq -br_1 + dr_2 < \frac{ar_1 h}{k} + N,$$

namely

$$0 \leq r_1 < dk \quad \text{and} \quad 0 \leq r_2 - \frac{r_1}{dk}(ah + kb) < a.$$

Note that for each $0 \leq r_1 < dk$, the set of the integers r_2 satisfying the above condition constitutes a complete system of representatives of the residue class group $\mathbb{Z}/a\mathbb{Z}$. It follows from (3.1) and (3.2) that the left-hand side of (3.6) equals

$$\begin{aligned} & \sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} \sum_{r_1=0}^{dk-1} \sum_{r_2 \bmod a} F\left(\frac{ar_1}{Nk}, \frac{1}{N}\left(\frac{ar_1 h}{k} + br_1 - dr_2\right)\right) \\ &= \sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} \sum_{r_1=0}^{dk-1} \sum_{r_2 \bmod a} F\left(\frac{r_1}{dk}, \frac{r_1(ah + kb)}{Nk} - \frac{r_2}{a}\right) \\ &= \sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} \sum_{r_1=0}^{dk-1} a^{-v'} F\left(\frac{r_1}{dk}, \frac{r_1(ah + kb)}{dk}\right) \\ &= N^{-v'} \sum_{\substack{ad=N \\ d>0}} d^{v'} \sum_{b=0}^{d-1} s(F : ah + kb, dk). \end{aligned} \tag{3.8}$$

On the other hand, the right-hand side of (3.6) equals

$$\begin{aligned} & \sum_{\delta|N} \delta \sum_{\substack{(l_1, l_2) \in \mathbb{Z}^2 \\ \text{with } \delta(l_1, l_2) \in D}} F\left(\frac{\delta l_1}{Nk}, \frac{1}{N}\left(\frac{\delta l_1 h}{k} - \delta l_2\right)\right) \\ &= \sum_{\delta|N} \delta \sum_{l_1 \bmod Nk/\delta} \sum_{l_2 \bmod N/\delta} F\left(\frac{\delta l_1}{Nk}, \frac{\delta l_1 h}{Nk} - \frac{\delta l_2}{N}\right) \\ &= \sum_{\delta|N} \delta \sum_{l_1 \bmod Nk/\delta} (N/\delta)^{-v'} F\left(\frac{l_1}{Nk/\delta}, \frac{l_1 Nh/\delta}{Nk/\delta}\right) \\ &= N^{-v'} \sum_{\delta|N} \delta^{v'+1} s(F : Nh/\delta, Nk/\delta). \end{aligned}$$

By Lemma 3.1, this equals

$$N^{-v'} \sum_{\delta|N} \delta^{v'+1} (N/\delta)^{-v} s(F : h, k) = N^{-v-v'} \sigma_{v+v'+1}(N) s(F : h, k). \tag{3.9}$$

Since (3.8) equals (3.9), we obtain (3.4).

Similarly, the left-hand side of (3.7) equals

$$\begin{aligned}
 & \sum_{b=0}^{N-1} \sum_{(l_1, l_2) \in \mathcal{A}(1, N; b) \cap D} F\left(\frac{l_1}{Nk}, \frac{1}{N}\left(\frac{l_1 h}{k} - l_2\right)\right) \\
 &= \sum_{b=0}^{N-1} \sum_{r_1=0}^{Nk-1} F\left(\frac{r_1}{Nk}, \frac{1}{N}\left(\frac{r_1 h}{k} + br_1\right)\right) = \sum_{b=0}^{N-1} \sum_{r_1=0}^{Nk-1} F\left(\frac{r_1}{Nk}, \frac{r_1}{Nk}(h + kb)\right) \\
 &= \sum_{b=0}^{N-1} s(F : h + kb, Nk). \tag{3.10}
 \end{aligned}$$

On the other hand, the right-hand side of (3.7) equals

$$\begin{aligned}
 & \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta \sum_{\substack{(l_1, l_2) \in \mathbb{Z}^2 \\ \text{with } (\delta dl_1, \delta dl_2) \in D}} F\left(\frac{\delta dl_1}{Nk}, \frac{1}{N}\left(\frac{\delta dl_1 h}{k} - \delta l_2\right)\right) \\
 &= \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta \sum_{l_1 \bmod Nk/(\delta d)} \sum_{l_2 \bmod N/\delta} F\left(\frac{\delta dl_1}{Nk}, \frac{\delta dl_1 h}{Nk} - \frac{\delta l_2}{N}\right) \\
 &= \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta \sum_{l_1 \bmod Nk/(\delta d)} (N/\delta)^{-v'} F\left(\frac{l_1}{Nk/(\delta d)}, \frac{l_1 Nh/\delta}{Nk/(\delta d)}\right) \\
 &= N^{-v'} \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^{v'+1} s(F : Nh/\delta, Nk/(\delta d)) \\
 &= N^{-v'} \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^{v'+1} (N/\delta d)^{-v} s(F : dh, k) \quad \text{by Lemma 3.1} \\
 &= N^{-v-v'} \sum_{d|N} \mu(d) d^v \sigma_{v+v'+1}(N/d) s(F : dh, k). \tag{3.11}
 \end{aligned}$$

Since (3.10) equals (3.11), we obtain (3.5).

Now let us prove the equivalence between (3.4) and (3.5). We first note that (3.4) is equivalent to

$$\sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} d^{v+v'} s(F : h + dkb, d^2 k) = \sigma_{v+v'+1}(N) s(F : h, Nk). \tag{3.12}$$

In fact, replacing k by Nk in (3.4) and making use of Lemma 3.1, we obtain (3.12) from (3.4). Conversely, (3.4) is deduced from (3.12) by replacing h by Nh and making use of Lemma 3.1. Similarly (3.5) is equivalent to

$$N^{v+v'} \sum_{b=0}^{N-1} s(F : h + Nkb, N^2 k) = \sum_{d|N} \mu(d) \sigma_{v+v'+1}(N/d) s(F : h, Nk/d). \tag{3.13}$$

Then the equivalence between (3.12) and (3.13) is derived from the Möbius inversion formula. This completes the proof. $\square$

As special cases, formula (3.4) includes (1.2) of [13], Theorem 3.4 of [19] and Theorem 2 of [2]. Formula (3.5) includes Theorem 3 of [23] and Theorem 1 of [2]. The equivalence between (3.4) and (3.5) includes Theorem 5 of [21].

4. Proof of Theorem 2.1 in the general case

In this section, we extend the method of the previous section and prove our main theorem completely. We use the same notation as in Section 2.

In addition to $\mathcal{R}_m(\mathbf{k})$ defined in Section 2, we denote by $\mathcal{R}_m(c)$ for $c \in \mathbb{N}$ an arbitrarily fixed complete set of representatives of the residue class group $(\mathbb{Z}/c\mathbb{Z})^m$.

LEMMA 4.1. *If $c \in \mathbb{N}$ with $\gcd\{c, N_{\varphi_i}\} = \gcd\{c, N_{\psi_j}\} = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, then we have*

$$\mathcal{S}(F : cH, c\mathbf{k}) = c^{-v} \varphi^{-1}(c) \mathcal{S}(F : H, \mathbf{k}).$$

Proof. For $\mathbf{k} = (k_1, \dots, k_m)$, $\mathbf{l} = (l_1, \dots, l_m) \in \mathbb{Q}^m$, we put $\mathbf{k} * \mathbf{l} = (k_1 l_1, \dots, k_m l_m)$. Then the set $\{\mathbf{x} + \mathbf{k} * \mathbf{l} \mid \mathbf{x} \in \mathcal{R}_m(\mathbf{k}), \mathbf{l} \in \mathcal{R}_m(c)\}$ is a complete set of representatives of the residue class group $(\mathbb{Z}/ck_1\mathbb{Z}) \times \dots \times (\mathbb{Z}/ck_m\mathbb{Z})$. Hence, by definition (2.3) and by applying (2.1) and (2.2) repeatedly, we have

$$\begin{aligned} \mathcal{S}(F : cH, c\mathbf{k}) &= \sum_{\mathbf{x} \in \mathcal{R}_m(\mathbf{k})} \sum_{\mathbf{l} \in \mathcal{R}_m(c)} F\left(\frac{\mathbf{x} + \mathbf{k} * \mathbf{l}}{c\mathbf{k}} \mid \frac{\mathbf{x} + \mathbf{k} * \mathbf{l}}{c\mathbf{k}} cH\right) \\ &= \sum_{\mathbf{x} \in \mathcal{R}_m(\mathbf{k})} \sum_{\mathbf{l} \in \mathcal{R}_m(c)} F\left(\frac{\mathbf{x}}{c\mathbf{k}} + \frac{\mathbf{l}}{c} \mid \frac{\mathbf{x}}{\mathbf{k}} H\right) \\ &= \sum_{\mathbf{x} \in \mathcal{R}_m(\mathbf{k})} c^{-v} \varphi^{-1}(c) F\left(\frac{\mathbf{x}}{\mathbf{k}} \mid \frac{\mathbf{x}}{\mathbf{k}} H\right) = c^{-v} \varphi^{-1}(c) \mathcal{S}(F : H, \mathbf{k}). \end{aligned}$$

This completes the proof. $\square$

Let $a, d \in \mathbb{N}$ and let B be an $m \times n$ matrix with components in $\mathbb{Z}$. Note that the additive group $\mathcal{A}(a, d : b)$ in the previous section is expressed as

$$\mathcal{A}(a, d : b) = \left\{ \begin{pmatrix} r_1 & r_2 \end{pmatrix} \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix} \mid r_1, r_2 \in \mathbb{Z} \right\}.$$

Generalizing this, we define an additive subgroup of $\mathbb{Z}^{m+n}$ as

$$\mathcal{A}(a, d : B) = \left\{ \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{pmatrix} \begin{pmatrix} aE_m & -B \\ O & dE_n \end{pmatrix} \mid \mathbf{r}_1 \in \mathbb{Z}^m, \mathbf{r}_2 \in \mathbb{Z}^n \right\},$$

where E_m and E_n are unit matrices of degrees m and n , respectively. Note that in the case of $n = 1$, this group coincides with $\mathcal{A}(a, d : B)$ in Section 4 of [16]. The next lemma is a generalization of Lemma 3.2 and also of Lemma 4.2 of [16].

LEMMA 4.2. *Let $N = ad$ with $a, d \in \mathbb{N}$ and B an $m \times n$ matrix with components in $\mathbb{Z}$. Let $(\mathbf{l}_1 \mid \mathbf{l}_2) \in \mathbb{Z}^{m+n}$ with $\mathbf{l}_1 = (l_{11}, \dots, l_{1m})$ and $\mathbf{l}_2 = (l_{21}, \dots, l_{2n})$. Put $d_1 = \gcd\{l_{11}, \dots, l_{1m}, N\}$, $d_2 = \gcd\{d_1, l_{21}, \dots, l_{2n}, N\}$, $\mathbf{l}'_1 = \mathbf{l}_1/d_1$ and $N' = N/d_1$. Then, we have $(\mathbf{l}_1 \mid \mathbf{l}_2) \in \mathcal{A}(a, d : B)$ if and only if the following three conditions hold:*

- (i) $a|d_1$;
- (ii) $(d_1/a)|d_2$;
- (iii) $\mathbf{l}'_1 B \equiv -(a\mathbf{l}_2/d_1) \pmod{N'\mathbb{Z}^n}$.

Proof. Let $(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathcal{A}(a, d : B)$ and express

$$(\mathbf{l}_1 \ \mathbf{l}_2) = (\mathbf{r}_1 \ \mathbf{r}_2) \left(\frac{aE_m}{O} \middle| \frac{-B}{dE_n} \right) \quad (4.1)$$

with $\mathbf{r}_1 \in \mathbb{Z}^m$ and $\mathbf{r}_2 \in \mathbb{Z}^n$. Then we have $\mathbf{l}_1 = a\mathbf{r}_1$, which shows the condition (i). Further, we also have $\mathbf{l}_2 = -\mathbf{r}_1 B + d\mathbf{r}_2$. Multiplying a/d_1 on both sides, we obtain

$$\frac{1}{d_1/a} \mathbf{l}_2 = -\frac{a}{d_1} \mathbf{r}_1 B + \frac{N}{d_1} \mathbf{r}_2 = -\mathbf{l}'_1 B + N' \mathbf{r}_2 \in \mathbb{Z}^n,$$

which shows conditions (ii) and (iii). Conversely under conditions (i), (ii) and (iii), we can easily deduce (4.1) with $\mathbf{r}_1 \in \mathbb{Z}^m$ and $\mathbf{r}_2 \in \mathbb{Z}^n$. This completes the proof. $\square$

LEMMA 4.3. *Let $\mathcal{F}$ be a map from $\mathbb{Z}^{m+n}$ to R and suppose that except for finitely many $(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathbb{Z}^{m+n}$, $\mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2) = 0$. Then for any $N \in \mathbb{N}$, we have*

$$\begin{aligned} & \sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d} \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathcal{A}(a, d : B)} a^{(m-1)n} \mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2) \\ &= N^{(m-1)n} \sum_{\delta|N} \delta^n \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in (\delta\mathbb{Z})^{m+n}} \mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2), \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} & \sum_{B \in \mathcal{I}_N} \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathcal{A}(1, N : B)} \mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2) \\ &= N^{(m-1)n} \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^n \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in d\delta\mathbb{Z}^m \times \delta\mathbb{Z}^n} \mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2). \end{aligned} \quad (4.3)$$

Furthermore (4.3) and (4.4) are equivalent.

Proof. Let $\mathbf{l}_1 \in \mathbb{Z}^m$ and $\mathbf{l}_2 \in \mathbb{Z}^n$ and we use the same notation as in Lemma 4.2. Put $\mathbf{l}'_1 = (l'_{11}, \dots, l'_{1m})$. Let $N = ad$ with $a, d \in \mathbb{N}$ and suppose that a satisfies conditions (i) and (ii). Let us consider the number of $B \in \mathcal{I}_d$ satisfying condition (iii). Let $B = (b_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ with $0 \leq b_{ij} \leq d-1$. Then condition (iii) means that for each $1 \leq j \leq n$, the transposition of j th column of B , namely $(b_{1j}, \dots, b_{mj})$, is one of the solutions $(x_1, \dots, x_m)$ of the congruence

$$l'_{11}x_1 + \dots + l'_{1m}x_m \equiv -\frac{al_{2j}}{d_1} \pmod{N'}. \quad (4.4)$$

Note that the map from $\mathbb{Z}^m$ to $\mathbb{Z}$ defined by mapping $(x_1, \dots, x_m)$ onto $l'_{11}x_1 + \dots + l'_{1m}x_m$ induces a map from $(\mathbb{Z}/N'\mathbb{Z})^m$ to $\mathbb{Z}/N'\mathbb{Z}$, which is surjective because $\gcd\{l'_{11}, \dots, l'_{1m}, N'\} = 1$. Hence, the number of m -tuples $(x_1, \dots, x_m) \pmod{N'} \in (\mathbb{Z}/N'\mathbb{Z})^m$ satisfying (4.4) is $\sharp(\mathbb{Z}/N'\mathbb{Z})^m / \sharp(\mathbb{Z}/N'\mathbb{Z}) = N'^{m-1}$. Note that $d/N' = dd_1/N = d_1/a \in \mathbb{N}$ and it follows that for each $1 \leq j \leq n$, the number of m -tuples $(b_{1j}, \dots, b_{mj}) \in \mathbb{Z}^m$ which satisfies $0 \leq b_{ij} \leq d-1$ and congruence (4.4) by taking $(x_1, \dots, x_m) = (b_{1j}, \dots, b_{mj})$ is

$$N'^{m-1} \left(\frac{d}{N'} \right)^m = \frac{d^m}{N'} = \frac{d^{m-1}d_1}{a}.$$

Hence, the number of $B \in \mathcal{I}_d$ satisfying condition (iii) is $(d^{m-1}d_1/a)^n$. This shows that for any fixed $(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathbb{Z}^{m+n}$, the coefficient of $\mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2)$ in the left-hand side of (4.2) is

$$\sum_{\substack{a|d_1 \\ (d_1/a)|d_2}} a^{(m-1)n} \left(\frac{d^{m-1}d_1}{a} \right)^n = N^{(m-1)n} \sum_{\substack{a|d_1 \\ (d_1/a)|d_2}} \left(\frac{d_1}{a} \right)^n.$$

By putting $\delta = d_1/a$, this coefficient is equal to $N^{(m-1)n} \sum_{\delta|d_2} \delta^n$, which is just the coefficient of $\mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2)$ in the right-hand side of (4.2). This proves (4.2).

Next, by considering the case of $a = 1$ and $d = N$ in Lemma 4.2, similar argument as in the above proof of (4.2) shows that for any fixed $(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathbb{Z}^{m+n}$, the coefficient of $\mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2)$ in the left-hand side of (4.3) is $N^{(m-1)n} d_1^n$ or zero according as $d_1 = d_2$ or $d_1 > d_2$. In the similar way as in the proof of (3.7) of Lemma 3.3, we can also deduce the coefficient in the right-hand side of (4.3) and verify the formula (4.3).

In order to prove the equivalence between (4.2) and (4.3), we put

$$\mathcal{F}_1(\mathbf{l}_1 \ \mathbf{l}_2) = \mathcal{F}(N\mathbf{l}_1 \ \mathbf{l}_2) \quad \text{for } (\mathbf{l}_1 \ \mathbf{l}_2) \in \left(\frac{1}{N}\mathbb{Z} \right)^m \times \mathbb{Z}^n.$$

Note that if $N = ad$ with $a, d \in \mathbb{N}$, the map from $\mathbb{Z}^{m+n}$ to $\mathbb{Z}^{m+n}$ defined by mapping $(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathbb{Z}^{m+n}$ onto $(a\mathbf{l}_1 \ \mathbf{l}_2) = ((N/d)\mathbf{l}_1 \ \mathbf{l}_2)$ gives a bijection from $\mathcal{A}(1, d : B)$ to $\mathcal{A}(a, d : B)$. Hence, (4.2) is expressed as

$$\begin{aligned} & \sum_{d|N} \sum_{B \in \mathcal{I}_d} \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathcal{A}(1, d : B)} \left(\frac{N}{d} \right)^{(m-1)n} \mathcal{F}_1 \left(\frac{\mathbf{l}_1}{d} \ \mathbf{l}_2 \right) \\ &= N^{(m-1)n} \sum_{\delta|N} \delta^n \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in (\delta\mathbb{Z})^{m+n}} \mathcal{F}_1 \left(\frac{\mathbf{l}_1}{N} \ \mathbf{l}_2 \right), \end{aligned}$$

namely

$$\sum_{d|N} \sum_{B \in \mathcal{I}_d} \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathcal{A}(1, d : B)} \frac{1}{d^{(m-1)n}} \mathcal{F}_1 \left(\frac{\mathbf{l}_1}{d} \ \mathbf{l}_2 \right) = \sum_{\delta|N} \delta^n \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in (\delta\mathbb{Z})^{m+n}} \mathcal{F}_1 \left(\frac{\mathbf{l}_1}{N} \ \mathbf{l}_2 \right).$$

By the Möbius inversion formula, this is equivalent to

$$\begin{aligned} & \sum_{B \in \mathcal{I}_N} \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathcal{A}(1, N : B)} \frac{1}{N^{(m-1)n}} \mathcal{F}_1 \left(\frac{\mathbf{l}_1}{N} \ \mathbf{l}_2 \right) \\ &= \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^n \sum_{(\mathbf{l}_1 \ \mathbf{l}_2) \in (\delta\mathbb{Z})^{m+n}} \mathcal{F}_1 \left(\frac{d\mathbf{l}_1}{N} \ \mathbf{l}_2 \right), \end{aligned}$$

which is easily transformed into (4.3). This completes the proof. $\square$

Proof of Theorem 2.1. For $l \in \mathbb{N}$ and $\mathbf{c} = (c_1, \dots, c_l) \in \mathbb{N}^l$, we put

$$\mathcal{J}_l(\mathbf{c}) = \{ (x_1, \dots, x_l) \in \mathbb{R}^l \mid 0 \leq x_i < c_i \text{ for each } 1 \leq i \leq l \}.$$

If $\mathbf{c} = (c, \dots, c) \in \mathbb{N}^l$, we put $\mathcal{J}_l(c) = \mathcal{J}_l(\mathbf{c})$.

We put

$$D = \left\{ (\mathbf{l}_1 \ \mathbf{l}_2) \in \mathbb{Z}^{m+n} \mid \mathbf{l}_1 \in \mathcal{J}_m(N\mathbf{k}), \ \mathbf{l}_2 - \frac{\mathbf{l}_1}{\mathbf{k}} H \in \mathcal{J}_n(N) \right\}.$$

and take

$$\mathcal{F}(\mathbf{l}_1 \ \mathbf{l}_2) = \begin{cases} F\left(\frac{\mathbf{l}_1}{N\mathbf{k}} \ \frac{1}{N}\left(\frac{\mathbf{l}_1}{\mathbf{k}}H - \mathbf{l}_2\right)\right) & \text{if } (\mathbf{l}_1 \ \mathbf{l}_2) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

For $N = ad$ with $a, d \in \mathbb{N}$ and $B \in \mathcal{I}_d$, let $(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathcal{A}(a, d : B)$ and express

$$(\mathbf{l}_1 \ \mathbf{l}_2) = (\mathbf{r}_1 \ \mathbf{r}_2) \left(\frac{aE_m}{O} \middle| \frac{-B}{dE_n} \right) = (a\mathbf{r}_1 - \mathbf{r}_1 B + d\mathbf{r}_2)$$

with $\mathbf{r}_1 \in \mathbb{Z}^m$ and $\mathbf{r}_2 \in \mathbb{Z}^n$. Then $(\mathbf{l}_1 \ \mathbf{l}_2) \in D$ holds if and only if we have

$$a\mathbf{r}_1 \in \mathcal{J}_m(N\mathbf{k}) \quad \text{and} \quad (-\mathbf{r}_1 B + d\mathbf{r}_2) - \frac{a\mathbf{r}_1}{\mathbf{k}} H \in \mathcal{J}_n(N),$$

which are equivalent to

$$\mathbf{r}_1 \in \mathcal{J}_m(d\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 - \frac{\mathbf{r}_1}{d\mathbf{k}}(aH + \mathbf{k} \circ B) \in \mathcal{J}_n(a),$$

respectively. Note that for each $\mathbf{r}_1 \in \mathcal{J}_m(d\mathbf{k})$, the set of $\mathbf{r}_2 \in \mathbb{Z}^n$ satisfying the above condition constitutes a complete system of representatives of the residue class group $(\mathbb{Z}/a\mathbb{Z})^n$. Applying (2.1) and (2.2) repeatedly, and recalling the notation (2.7), we see that the left-hand side of (4.2) equals

$$\begin{aligned} & \sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d} \sum_{\mathbf{r}_1 \in \mathcal{J}_m(d\mathbf{k}) \cap \mathbb{Z}^m} \sum_{\mathbf{r}_2 \in (\mathbb{Z}/a\mathbb{Z})^n} a^{(m-1)n} F\left(\frac{a\mathbf{r}_1}{N\mathbf{k}} \ \frac{\mathbf{r}_1}{N\mathbf{k}}(aH + \mathbf{k} \circ B) - \frac{\mathbf{r}_2}{a}\right) \\ &= \sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d} \sum_{\mathbf{r}_1 \in \mathcal{J}_m(d\mathbf{k}) \cap \mathbb{Z}^m} a^{(m-1)n} a^{-v'} \psi^{-1}(a) F\left(\frac{\mathbf{r}_1}{d\mathbf{k}} \ \frac{\mathbf{r}_1}{d\mathbf{k}}(aH + \mathbf{k} \circ B)\right) \\ &= N^{(m-1)n-v'} \psi^{-1}(N) \sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d} d^{v'-(m-1)n} \psi(d) \mathcal{S}(F : aH + \mathbf{k} \circ B, d\mathbf{k}). \end{aligned} \quad (4.5)$$

On the other hand, the right-hand side of (4.2) equals

$$\begin{aligned} & N^{(m-1)n} \sum_{\delta|N} \delta^n \sum_{\substack{(\mathbf{l}_1 \ \mathbf{l}_2) \in \mathbb{Z}^m \times \mathbb{Z}^n \\ \text{with } \delta(\mathbf{l}_1 \ \mathbf{l}_2) \in D}} F\left(\frac{\delta\mathbf{l}_1}{N\mathbf{k}} \ \frac{\delta\mathbf{l}_1}{N\mathbf{k}}H - \frac{\delta\mathbf{l}_2}{N}\right) \\ &= N^{(m-1)n} \sum_{\delta|N} \delta^n \sum_{\mathbf{l}_1 \in \mathcal{R}_m(N\mathbf{k}/\delta)} \sum_{\mathbf{l}_2 \in \mathcal{R}_n(N/\delta)} F\left(\frac{\delta\mathbf{l}_1}{N\mathbf{k}} \ \frac{\delta\mathbf{l}_1}{N\mathbf{k}}H - \frac{\delta\mathbf{l}_2}{N}\right) \\ &= N^{(m-1)n} \sum_{\delta|N} \delta^n \sum_{\mathbf{l}_1 \in \mathcal{R}_m(N\mathbf{k}/\delta)} (N/\delta)^{-v'} \psi^{-1}(N/\delta) F\left(\frac{\mathbf{l}_1}{(N/\delta)\mathbf{k}} \ \frac{\mathbf{l}_1}{(N/\delta)\mathbf{k}}(N/\delta)H\right) \\ &= N^{(m-1)n-v'} \psi^{-1}(N) \sum_{\delta|N} \delta^{n+v'} \psi(\delta) \mathcal{S}(F : (N/\delta)H, (N/\delta)\mathbf{k}) \\ &= N^{(m-1)n-v'} \psi^{-1}(N) \sum_{\delta|N} \delta^{n+v'} \psi(\delta) (N/\delta)^{-v} \varphi^{-1}(N/\delta) \mathcal{S}(F : H, \mathbf{k}) \quad \text{by Lemma 4.1} \\ &= N^{(m-1)n-v-v'} (\varphi\psi)^{-1}(N) \sigma_{v+v'+n, \varphi\psi}(N) \mathcal{S}(F : H, \mathbf{k}). \end{aligned} \quad (4.6)$$

Since (4.5) equals (4.6), we obtain (2.8).

Similarly, the left-hand side of (4.3) equals

$$\begin{aligned} & \sum_{B \in \mathcal{I}_N} \sum_{r_1 \in \mathcal{J}_m(N\mathbf{k}) \cap \mathbb{Z}^m} F\left(\frac{r_1}{N\mathbf{k}} \frac{r_1}{N\mathbf{k}}(H + \mathbf{k} \circ B)\right) \\ &= \sum_{B \in \mathcal{I}_N} \mathcal{S}(F : H + \mathbf{k} \circ B, N\mathbf{k}). \end{aligned}$$

The right-hand side of (4.3) equals

$$\begin{aligned} & N^{(m-1)n} \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^n \sum_{(I_1 \ I_2) \in \mathbb{Z}^{m+n}} \mathcal{F}(\delta d I_1 \ \delta I_2) \\ &= N^{(m-1)n} \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^n \sum_{\substack{(I_1 \ I_2) \in \mathbb{Z}^{m+n} \\ \text{with } (\delta d I_1 \ \delta I_2) \in D}} F\left(\frac{\delta d I_1}{N\mathbf{k}} \frac{\delta d I_1}{N\mathbf{k}} H - \frac{\delta I_2}{N}\right) \\ &= N^{(m-1)n} \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^n \sum_{I_1 \in \mathcal{J}_m(N\mathbf{k}/(\delta d)) \cap \mathbb{Z}^m} \sum_{I_2 \in \mathcal{R}_n(N/\delta)} F\left(\frac{\delta d I_1}{N\mathbf{k}} \frac{\delta d I_1}{N\mathbf{k}} H - \frac{\delta I_2}{N}\right) \\ &= N^{(m-1)n} \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^n (N/\delta)^{-v'} \psi^{-1}(N/\delta) \\ &\quad \times \sum_{I_1 \in \mathcal{R}_m(N\mathbf{k}/(\delta d))} F\left(\frac{I_1}{(N/(\delta d))\mathbf{k}} \frac{I_1}{(N/(\delta d))\mathbf{k}} (N/\delta) H\right) \\ &= N^{(m-1)n-v'} \psi^{-1}(N) \sum_{d|N} \mu(d) \sum_{\delta|(N/d)} \delta^{v'+n} \psi(\delta) \mathcal{S}(F : (N/\delta)H, (N/(\delta d))\mathbf{k}). \end{aligned}$$

By Lemma 4.1, this also equals

$$N^{(m-1)n-v-v'} (\varphi\psi)^{-1}(N) \sum_{d|N} \mu(d) d^v \varphi(d) \sigma_{v+v'+n, \varphi\psi}(N/d) \mathcal{S}(F : dH, \mathbf{k})$$

and thereby we obtain (2.9).

Finally let us prove the equivalence between (2.8) and (2.9). As in the proof of the equivalence between (3.4) and (3.5) in Section 3, we can replace (2.8) (respectively (2.9)) by the following equivalent formula (4.7) (respectively (4.8)) derived from Lemma 4.1:

$$\begin{aligned} & \sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d} d^{v+v'-(m-1)n} (\varphi\psi)(d) \mathcal{S}(F : H + d\mathbf{k} \circ B, d^2\mathbf{k}) \\ &= \sigma_{v+v'+n, \varphi\psi}(N) \mathcal{S}(F : H, N\mathbf{k}). \end{aligned} \tag{4.7}$$

$$\begin{aligned} & N^{v+v'-(m-1)n} (\varphi\psi)(N) \sum_{B \in \mathcal{I}_N} \mathcal{S}(F : H + N\mathbf{k} \circ B, N^2\mathbf{k}) \\ &= \sum_{d|N} \mu(d) \sigma_{v+v'+n, \varphi\psi}(N/d) \mathcal{S}(F : H, N\mathbf{k}/d). \end{aligned} \tag{4.8}$$

Then (4.7) and (4.8) are obviously equivalent by the Möbius inversion formula. This completes the proof. $\square$

As special cases, our main theorem includes Theorems 1, 2 and 3 of [24], Theorem 11 of [4], Theorem 2.3 of [14] and Theorem 4.1 of [15].

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