Abstract. Following the analogies between three-dimensional topology and number theory, we study an idèlic form of class field theory for 3-manifolds. For a certain set $\mathcal{K}$ of knots in a 3-manifold $M$, we first present a local theory for each knot in $\mathcal{K}$, which is analogous to local class field theory, and then, getting together over all knots in $\mathcal{K}$, we give an analogue of idèlic global class field theory for an integral homology sphere $M$.

1. Introduction

The analogies between knots and primes were first pointed out by Mazur [Ma] in the 1960s and, after a long silence, Kapranov and Reznikov took up the analogies between 3-manifolds and number rings again [Ka, R1, R2], and Morishita, whose work started independently, investigated the subject systematically [Mo1, Mo2, Mo3]. This new area of mathematics is now called arithmetic topology.

Here is a part of basic analogies: for a number field $k$, $\mathcal{O}_k$ stands for the ring of integers of $k$:

$$
\begin{align*}
3\text{-manifold } M & \leftrightarrow \text{ number ring } \text{Spec}(\mathcal{O}_k), \\
\text{knot } K \text{ in } M & \leftrightarrow \text{ prime } p \text{ in } \text{Spec}(\mathcal{O}_k), \\
(\text{ramified}) \text{ covering } N \rightarrow M & \leftrightarrow (\text{ramified}) \text{ extension } K / k, \\
\text{first homology group } H_1(M; \mathbb{Z}) & \leftrightarrow \text{ ideal class group } H_k.
\end{align*}
$$

(1.1)

In particular, we have the following analogy between the Hurewicz isomorphism and unramified class field theory (Artin’s isomorphism):

$$
\begin{align*}
H_1(M; \mathbb{Z}) & \cong \text{Gal}(M^{ab}/M) \leftrightarrow H_k \cong \text{Gal}((\bar{k}^{ab}/k).
\end{align*}
$$

(1.2)

Here $M^{ab}$ (respectively $\bar{k}^{ab}$) denotes the maximal Abelian covering of $M$ (respectively the maximal unramified Abelian extension of $k$).

The purpose of this paper is, following the spirit of arithmetic topology, to pursue an idèlë theoretic form of class field theory for 3-manifolds and so extend the analogy (1.2) for ramified coverings/extensions.

For this, we first develop a local theory for each knot in a 3-manifold, which is analogous to local class field theory, based on the following analogies:

$$
\begin{align*}
tubular \text{neighborhood of } K & \leftrightarrow \text{p-adic integers } \mathcal{O}_p, \\
\text{boundary of } V_K & \leftrightarrow \text{p-adic field } \text{Spec}(\mathcal{O}_p), \\
\partial V_K \cong V_K \setminus K & \leftrightarrow \text{Spec}(\mathcal{O}_p) = \text{Spec}(\mathcal{O}_p) \setminus \text{Spec}(\mathcal{O}_p/p).
\end{align*}
$$

(1.3)

2010 Mathematics Subject Classification: Primary 57M12; Secondary 11R37, 11S31.

Keywords: idèlë; class field theory; 3-manifold; arithmetic topology.
Then a topological analogue of the local reciprocity homomorphism is simply given by the Hurewicz homomorphism:

$$\rho_K : H_1(\partial V_K; \mathbb{Z}) \to \text{Gal}(\partial V_K^{ab}/\partial V_K).$$

For a certain given set $\mathcal{K}$ of knots in a 3-manifold $M$ (cf. Section 5), we introduce the idèle group $I(M; \mathcal{K})$ as a restricted product of $H_1(\partial V_K; \mathbb{Z})$ over all $K$ in $\mathcal{K}$, and getting $\rho_K$ together over all $K$ in $\mathcal{K}$, we define the homomorphism

$$\varphi(M; \mathcal{K}) : I(M; \mathcal{K}) \to \text{Gal}(M; \mathcal{K})^{ab} := \lim_{\leftarrow L} \text{Gal}(X_L^{ab}/X_L),$$

where $L$ runs over all finite subsets of $\mathcal{K}$, $X_L := M \setminus L$ and $X_L^{ab}$ is the maximal Abelian covering of $X_L$. The homomorphism $\varphi(M; \mathcal{K})$ factors through the idèle class group $C(M; \mathcal{K}) := I(M; \mathcal{K})/P(M; \mathcal{K})$, with the principal idèle group $P(M; \mathcal{K})$, and hence we obtain an analogue of the global reciprocity homomorphism

$$\rho_{(M; \mathcal{K})} : C(M; \mathcal{K}) \to \text{Gal}(M; \mathcal{K})^{ab}.$$

Then our main result (Theorem 5.9 below) is stated as follows. Suppose that $M$ is an integral homology sphere. For a finite Abelian covering $h : N \to M$ branched over a finite subset of $\mathcal{K}$, the global reciprocity homomorphism $\rho_{(M; \mathcal{K})}$ induces an isomorphism

$$\rho_{N/M} : C(M; \mathcal{K})/h_*(C_{N,h^{-1}(\mathcal{K})}) \cong \text{Gal}(N/M).$$

This result may be regarded as an analogue of the fundamental theorem in global class field theory for number fields [KKS].

We note that idèlic class field theory for 3-manifolds was firstly studied by Sikora [S1, S2]. Our approach is different from his and elementary.

This paper is organized as follows. In Section 2 we review the class field theory for algebraic fields. In Section 3 we give a description of Hilbert theory for 3-manifolds. In Section 4 we give the local class field theory for tori, and in Section 5 we present the global class field theory over an integral homology 3-sphere.

**Notation**

For a connected topological space $X$ (respectively a field $k$), we denote by $X^{ab}$ (respectively $k^{ab}$) the maximal Abelian covering of $X$ (respectively the maximal Abelian extension of $k$). We denote by $\text{Gal}(Y/X)$ (respectively $\text{Gal}(F/k)$) for the Galois group of a Galois covering $h : Y \to X$ (respectively a Galois extension $F/k$). For topological spaces $X$ and $Y$, $X \simeq Y$ means that $X$ and $Y$ are homotopy equivalent. We write $\pi_1(X)$ for the fundamental group of $X$ omitting a base point and write $H_1(X)$ simply for the homology group with coefficients in $\mathbb{Z}$.

2. **Review of class field theory in number theory**

In this section, we review local and global class field theory in number theory whose topological analogies will be studied in Sections 4 and 5. We consult [KKS] and [N] as basic references for this section.
Let $k$ be a number field of finite degree over the rational number field $\mathbb{Q}$. We denote by $\mathcal{O}_k$ the ring of integers of $k$. A prime $p$ of $k$ is a class of equivalent valuations of $k$. The finite primes belong to the maximal ideals of $\mathcal{O}_k$. The infinite primes fall into two classes, real and complex, where the real primes correspond to the embeddings $k \hookrightarrow \mathbb{R}$ and the complex primes correspond to the pairs of conjugate non-real embeddings $k \hookrightarrow \mathbb{C}$. For a finite prime $p$, let $v_p$ be the corresponding additive valuation of $k$, and set $|a|_p = (Np)^{-v_p(a)}$ for $a \in k$ where $Np = \#(\mathcal{O}_k/p)$. For a real prime $p$ with corresponding embedding $\iota : k \hookrightarrow \mathbb{R}$, set $|a|_p = |\iota(a)|$ for $a \in k$, and for a complex prime $p$ with corresponding embedding $\iota : k \hookrightarrow \mathbb{C}$, set $|a|_p = |\iota(a)|^2$ for $a \in k$.

Let $k_p$ be the local field obtained as the completion of a number field with respect to the metric $| \cdot |_p$. Suppose that $p$ is a finite prime of $k$. Then $k_p$ is non-archimedean local field, a finite extension of the $p$-adic field $\mathbb{Q}_p$ for a prime number $p$. Let $v_p : k_p^\times \rightarrow \mathbb{Z}$ be the discrete valuation normalized by $v_p(k_p^\times) = \mathbb{Z}$. We let $\mathcal{O}_p := \{a \in k \mid v_p(a) \geq 0\}$ be the valuation ring and $p = \{a \in k \mid v_p(a) > 0\}$ be the unique maximal ideal of $\mathcal{O}_p$, and let $F_p$ be the residue field $\mathcal{O}_p/p$, a finite extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We denote by $U_p$ the unit group $\mathcal{O}_p^\times$. We note that $U_p = \ker(v_p)$ and so we have the following split exact sequence

\[0 \rightarrow U_p \rightarrow k_p^\times \xrightarrow{v_p} \mathbb{Z} \rightarrow 0. \tag{2.1}\]

When $p$ is an infinite prime, we let $\mathcal{O}_p = k_p$ and $U_p = k_p^\times$ by convention.

Let $k_p^{ab}$ be the maximal Abelian extension of $k_p$. When $k_p$ is non-archimedean, we denote by $k_p^{ur}$ the maximal unramified extension of $k_p$. Note that the Galois group $\text{Gal}(k_p^{ur}/k_p)$ is identified with $\text{Gal}(\hat{\mathbb{Q}}_p/F_p) \cong \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$. A main part of local class field theory for the local field $k_p$ is stated as follows.

**Theorem 2.1. (Local class field theory)** There is a canonical homomorphism, called the local reciprocity homomorphism,

\[\rho_{k_p} : k_p^\times \rightarrow \text{Gal}(k_p^{ab}/k_p)\]

which satisfies the following properties.

1. For any finite Abelian extension $F/k_p$, $\rho_{k_p}$ induces the isomorphism

\[\rho_{F/k_p} : k_p^\times / N_{F/k_p}(F^\times) \cong \text{Gal}(F/k_p)\]

where $N_{F/k_p}$ denotes the norm map for $F/k_p$.

2. When $k_p$ is non-archimedean, we have the following commutative diagram with exact horizontal sequences:

\[
\begin{array}{cccccccc}
0 & \rightarrow & U_p & \rightarrow & k_p^\times & \xrightarrow{v_p} & \mathbb{Z} & \rightarrow & 0 \\
\downarrow{\rho_{k_p}\mid_{U_p}} & & \downarrow{\rho_{k_p}} & & \downarrow{\rho_{k_p}} & & \downarrow & \\
0 & \rightarrow & \text{Gal}(k_p^{ab}/k_p^{ur}) & \rightarrow & \text{Gal}(k_p^{ab}/k_p) & \rightarrow & \text{Gal}(\hat{\mathbb{F}}_p/\mathbb{F}_p) & \rightarrow & 0.
\end{array}
\]

**Corollary 2.2.** There is the one-to-one correspondence between finite unramified extensions of $k_p$ and open subgroups of finite index of $k_p^\times$ containing $U_p$. 

...
Now, let \( k \) be a number field. We define the idèle group \( I_k \) of \( k \) by the following restricted product of \( k_p^\times \) with respect to \( U_p \) over all primes \( p \) of \( k \):

\[
I_k := \left\{ (a_p)_p \in \prod_{p \text{ prime}} k_p^\times \; \middle| \; v_p(a_p) = 0 \text{ for almost all finite prime } p \right\}.
\]

Since, for \( a \in k^\times \), we have \( v_p(a) = 0 \) for almost all finite prime \( p \), \( k^\times \) is embedded into \( I_k \) diagonally. We let \( P_k \) be the image of \( k^\times \) in \( I_k \) and call it the group of principal idèles. We then define the idèle class group of \( k \) by

\[
C_k := I_k / k^\times.
\]

We recall that the homomorphism

\[
\varphi : I_k \to \bigoplus_{p \text{ prime}} \mathbb{Z}; \quad (a_p)_p \mapsto \prod_p p^{v_p(a_p)}
\]

induces the isomorphism

\[
I_k / (U \cdot P_k) \cong H_k
\]

where \( U = \text{Ker}(\varphi) = \prod_p U_p \), and \( H_k \) denotes the ideal class group of \( k \).

Let \( k^{ab} \) be the maximal Abelian extension of \( k \). Here is a global class field theory for \( k \) (cf. [N]). A main part of global class field theory is summarized as follows.

**Theorem 2.3.** (Global class field theory) There is a canonical homomorphism, called the global reciprocity map,

\[
\rho_k : C_k \to \text{Gal}(k^{ab}/k)
\]

which has the following properties:

1. For any finite Abelian extension \( F/k \), \( \rho_k \) induces the isomorphism

\[
C_k / N_{F/k}(C_F) \cong \text{Gal}(F/k)
\]

where \( N_{F/k} \) denotes the norm map on the idèle groups.

2. For a prime \( p \) of \( k \), we have the following commutative diagram

\[
\begin{array}{ccc}
    k_p^\times & \xrightarrow{\rho_{k_p}} & \text{Gal}(k_p^{ab}/k_p) \\
    \downarrow{\iota_p} & & \downarrow{\circ} \\
    C_k & \xrightarrow{\rho_k} & \text{Gal}(k^{ab}/k)
\end{array}
\]

where \( \iota_p \) is the map induced by the natural inclusion \( k_p^\times \to I_k \).

By class field theory, we obtain the following proposition.

**Proposition 2.4.** For a finite Abelian extension \( F/k \), let \( \rho_{F/k} : C_k \to \text{Gal}(F/k) \) be the homomorphism defined by composing \( \rho_k \) with the natural projection \( \text{Gal}(k^{ab}/k) \to \text{Gal}(F/k) \). Then we have:

1. \( p \) is completely decomposed in \( F/k \) if and only if \( \rho_{F/k} \circ \iota_p(k_p^\times) = \{1\} \);
2. \( p \) is unbranched in \( F/k \) if and only if \( \rho_{F/k} \circ \iota_p(U_p) = \{1\} \).
3. Hilbert theory for 3-manifolds

In this section, we review a Hilbert theory for 3-manifolds according to [Mo3, Ch. 5]. We also show a relation between the linking number and the decomposition law of a knot in a finite Abelian covering, which generalizes a result in [Mo3, Ch. 5].

Let \( M \) be an integral homology 3-sphere, namely \( M \) be a oriented closed 3-manifold and \( H_i(M) \cong H_i(S^3) \) for each \( i \in \mathbb{Z} \), and let \( h : N \rightarrow M \) be a finite Galois covering of connected oriented closed 3-manifolds branched over a link \( L \subset M \). Let \( X_L := M \setminus L, Y_L := N \setminus h^{-1}(L) \), and let \( n \) denote the covering degree of \( Y_L \) over \( X_L \) so that \( n = \#\text{Gal}(Y_L/X_L) = \#\text{Gal}(N/M) \). Let \( K \) be a knot in \( M \) which is a component of \( L \) or disjoint from \( L \), and suppose \( h^{-1}(K) = K_1 \cup \cdots \cup K_r \). Then \( \text{Gal}(N/M) \) acts transitively on the set of knots \( S_K := \{K_1, \ldots, K_r\} \) lying over \( K \). We call the stabilizer \( D_{K_i} \) of \( K_i \) the decomposition group of \( K_i \):

\[
D_{K_i} := \{g \in \text{Gal}(N/M) \mid g(K_i) = K_i\}.
\]

Since we obtain the bijection \( \text{Gal}(N/M)/D_{K_i} \cong S_K \) for each \( i \), \( #D_{K_i} = n/r \) is independent of \( K_i \).

Since each \( g \in \text{Gal}(N/M) \) induces a homeomorphism \( g|_{\partial V_{K_i}} : \partial V_{K_i} \rightarrow \partial V_{g(K_i)}, \) \( g|_{\partial V_{K_i}} \) is a covering transformation of \( \partial V_{K_i} \) over \( \partial V_K \), so we have following isomorphism,

\[
D_{K_i} \cong \text{Gal}(\partial V_{K_i}/\partial V_K).
\]

The Fox completion of the subcovering space of \( Y_L \) over \( X_L \) corresponding to \( D_{K_i} \) is called the decomposition covering space of \( K_i \) and this space is denoted by \( Z_{K_i} \). The map \( g \mapsto \tilde{g} := g|_{\partial V_{K_i}} \) induces the homomorphism

\[
D_{K_i} \rightarrow \text{Gal}(K_i/K)
\]

whose kernel is called the inertia group of \( K_i \) and is denoted by \( I_{K_i} \):

\[
I_{K_i} := \{g \in D_{K_i} \mid \tilde{g} = \text{id}_{K_i}\}.
\]

If \( K_j = g(K_i) \ (g \in \text{Gal}(N/M)) \), we obtain \( I_{K_j} = gI_{K_i}g^{-1} \) and, hence, \( #I_{K_i} \) is independent of \( K_i \). If \( e = e_{K_i} := #I_{K_i} \). The Fox completion of the subcovering space of \( Y_L \) over \( X_L \) corresponding to \( I_{K_i} \) is called the inertia covering space of \( K_i \) and denoted by \( T_{K_i} \):

\[
N \quad \xrightarrow{1} \quad T_{K_i} \quad \xrightarrow{\phi} \quad Z_{K_i} \quad \xrightarrow{\rho} \quad M
\]

\[
\begin{array}{ccc}
1 & \xrightarrow{e} & I_{K_i} & \xrightarrow{f} & D_{K_i} & \xrightarrow{r} & \text{Gal}(N/M).
\end{array}
\]

By the isomorphism \( D_{K_i} \cong \text{Gal}(\partial V_{K_i}/\partial V_K) \), we see that the homomorphism \( D_{K_i} \rightarrow \text{Gal}(K_i/K) \) is surjective:

\[
1 \rightarrow I_{K_i} \rightarrow D_{K_i} \rightarrow \text{Gal}(K_i/K) \rightarrow 1 \quad \text{(exact)}.
\]

Then we have the equalities

\[
#D_{K_i} = ef, \quad #I_{K_i} = e, \quad #\text{Gal}(K_i/K) =: f,
\]

where \( f \) is called the covering degree of \( K \).

Suppose \( h : N \rightarrow M \) is an Abelian covering. Then \( D_{K_i} \) and \( I_{K_i} \) are independent of \( K_i \) lying over \( K \) and so we denote them by \( D_K \) and \( I_K \) respectively.
THEOREM 3.1. [Mo3, Ch. 5] Let the notation be as above and suppose \( h : N \to M \) is an Abelian covering. Then we have the exact sequence

\[
1 \to I_K \to D_K \to \text{Gal}(K_i/K) \to 1
\]

and the equality

\[ n = efr. \]

Finally, let us extend the relation between the linking number and the decomposition law of a knot in a finite Abelian covering. In this paper, a meridian of \( K \) is a closed oriented essential loop which is the boundary of a proper embedded disk \( D^2 \) in \( V_K \). A longitude of \( K \) is a closed loop on \( \partial V_K \) which intersects with a meridian at one point and is null-homologous in \( X_K \).

PROPOSITION 3.2. Let \( L := K_1 \cup \cdots \cup K_r \) be an \( r \)-component link in an integral homology 3-sphere \( M \). For given integers \( n_i \geq 2 \), let \( \psi : \pi_1(X_L) \to \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} \) be the homomorphism sending a each meridian of \( K_i \) to \( (0, \ldots, 0, \hat{i}, 0, \ldots, 0) \). Let \( Y_L \to X_L \) be the covering corresponding to \( \text{Ker}(\psi) \), whose covering degree is \( n := n_1n_2 \cdots n_r \), and let \( h : N \to M \) be its Fox completion. Then, for a knot \( K \) in \( M \) disjoint from \( L \), the covering degree of \( K \) in \( h : N \to M \) coincides with the order of \( (\text{lk}(K, K_1) \mod n_1, \ldots, \text{lk}(K, K_i) \mod n_i, \ldots, \text{lk}(K, K_r) \mod n_r) \) in \( \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} \).

Proof. Let \( K' \) be a component of \( h^{-1}(K) \). Since \( I_{K'} = I_K = \{1\} \), by Theorem 3.1, the covering degree of \( K \) in \( h : N \to M \) is the order of a generator \( \sigma_K \) of \( \text{Gal}(K'/K) \cong D_K \) in \( \text{Gal}(N/M) \), where \( \sigma_K \) corresponds to a loop \( K \). Since \( [K] \) is sent to \( (\text{lk}(K, K_1) \mod n_1, \ldots, \text{lk}(K, K_r) \mod n_r) \) by the natural homomorphism \( H_1(X_L) \to \text{Gal}(N/M) \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} \) given by the Hurewicz map and Galois theory, our assertion follows.

In particular, suppose \( K \) is not a component of \( L \), so that \( K \) is unbranched in \( N \). Then the equality \( fr = n \) implies that \( K \) is decomposed completely in \( N \) (i.e. decomposed into an \( n \)-component link) if and only if for each \( i \), \( \text{lk}(K_i, K) \equiv 0 \mod n_i \).

4. Local class field theory for tori

In this section, we present a topological analogue of local class field theory for two-dimensional tori.

Let \( K \) be a fixed knot in an orientable 3-manifold and let \( V_K \) be a tubular neighborhood of \( K \). Let \( T_K = \partial V_K \) be the boundary of \( V_K \). Then \( T_K \) is a two-dimensional torus. According to (1.3), \( T_K \) and \( V_K \) are regarded as analogues of a \( p \)-adic local field \( k_p \) and the integer ring \( \mathcal{O}_p \). Let \( m \) and \( I \) be a meridian and a longitude on \( T_K \), respectively. The inclusion \( T_K \hookrightarrow V_K \) induces the homomorphism \( v_K : H_1(T_K) \to H_1(V_K) = \mathbb{Z}[I] \) whose kernel is \( \mathbb{Z}[m] \). Thus, we have the exact sequence

\[
0 \to \mathbb{Z}[m] \to H_1(T_K) \to \mathbb{Z}[I] \to 0
\]

which may be regarded as an analogue of the exact sequence (2.1).
Let $T_{K}^{\text{ab}}$ be the maximal Abelian covering of $T_K$ (which is the universal covering). Since $V_K \setminus K$ is homotopy equivalent to the torus $T_K$, (unramified) coverings of $T_K$ correspond to ramified covering of $V_K$ along $K$. Let $T_K^{\text{ur}}$ be the maximal covering of $T_K$ which comes from the maximal (unramified) covering of $V_K$. Then we have the following theorem which may be regarded as an analogy of Theorem 2.1.

**THEOREM 4.1.** (Local class field theory for tori) There is a canonical isomorphism

$$\rho_{T_K} : H_1(T_K) \rightarrow \text{Gal}(T_{K}^{\text{ab}} / T_K)$$

which satisfies following properties.

1. For any finite Abelian covering $h : R \rightarrow T_K$, $\rho_{T_K}$ induces the isomorphism

   $$\rho_{R/T_K} : H_1(T_K)/h_*(H_1(R)) \cong \text{Gal}(R/T_K).$$

2. We have the following commutative diagram

   $$\begin{array}{cccccc}
   0 & \rightarrow & \mathbb{Z}[m] & \rightarrow & H_1(T_K) & \stackrel{\nu_K}{\rightarrow} & \mathbb{Z}[l] & \rightarrow & 0 \\
   | & & \downarrow \rho_{T_K}[\mathbb{Z}[m]] & & \downarrow \rho_{T_K} & & | \\
   0 & \rightarrow & \text{Gal}(T_{K}^{\text{ab}} / T_K) & \rightarrow & \text{Gal}(T_{K}^{\text{ur}} / T_K) & \rightarrow & 0
   \end{array}$$

where horizontal sequences are exact and vertical maps are all isomorphisms.

**Proof.** Let $\eta_{T_K} : H_1(T_K) \rightarrow \pi_1(T_K)/[\pi_1(T_K), \pi_1(T_K)]$ be the Hurewicz isomorphism, where $[\pi_1(T_K), \pi_1(T_K)]$ is commutator subgroup of $\pi_1(T_K)$. We define $\rho_{T_K} : H_1(T_K) \rightarrow \text{Gal}(T_{K}^{\text{ab}} / T_K)$ by the composite of $\eta_{T_K}$ with the isomorphism $\pi_1(T_K)/[\pi_1(T_K), \pi_1(T_K)] \cong \text{Gal}(T_{K}^{\text{ab}} / T_K)$ coming from Galois theory of covering spaces.

1. Since $h : R \rightarrow T_K$ is the finite Abelian covering, $R$ is torus. By Galois theory we have $H_1(T_K)/h_*(\pi_1(R)) \cong \text{Gal}(R/T_K)$. Since $\pi_1(T_K)$ and $\pi_1(R)$ are Abelian groups, we have $\pi_1(T_K) = H_1(T_K)$ and $\pi_1(R) = H_1(R)$. Hence, $\rho_{T_K}$ induces the isomorphism $H_1(T_K)/h_*(H_1(R)) \cong \text{Gal}(R/T_K)$.

2. The upper horizontal exact sequence is nothing but (4.1). The lower horizontal sequence is coming from Galois theory. First, we obtain the following isomorphism:

   $$\text{Gal}(T_{K}^{\text{ur}} / T_K) \cong H_1(T_K)/h_*(H_1(T_{K}^{\text{ur}}))$$

   $$\cong (\mathbb{Z}[m] \times \mathbb{Z}[l]) / (\mathbb{Z}[m] \times 0)$$

   $$\cong \mathbb{Z}[l].$$

Then we consider following isomorphism, $\text{Gal}(T_{K}^{\text{ab}} / T_K) \cong \pi_1(T_K)/h_*(\pi_1(T_{K}^{\text{ab}}))$. Since $\pi_1(T_K)$ is Abelian group, $T_K^{\text{ab}}$ is the universal covering. Therefore, we have $\text{Gal}(T_{K}^{\text{ab}} / T_K) \cong \pi_1(T_{K}^{\text{ur}}) = H_1(T_{K}^{\text{ur}})$. By the construction of $T_{K}^{\text{ur}}$, $T_{K}^{\text{ur}}$ is homeomorphic to $S^1 \times \mathbb{R}$, whose $S^1$ corresponds to a meridian on $T_K$. Hence, $H_1(T_{K}^{\text{ur}}) \cong \mathbb{Z}[m]$. \hfill \Box

**COROLLARY 4.2.** There is the one-to-one correspondence between the set of finite unbranched coverings of $V_K$ and the set of finite index subgroups of $H_1(T_K)$ containing $\mathbb{Z}[m]$.

**Proof.** The commutative diagram of (2) implies the corollary. \hfill \Box
Definition 4.3. Since we have \( H_1(T_K) = \mathbb{Z}[m] \times \mathbb{Z}[l] \), we can write an element \( a \in H_1(T_K) \) as \( a = (q, p) \in \mathbb{Z}^2 \) if \( a = q[m] + p[l] \). We call \( q \) the meridian component of \( a \) and \( p \) the longitude component of \( a \). The longitude component represents the value of \( v_K \), namely \( v_K(q, p) = p \).

5. Global class field theory for 3-manifolds

In this section, let \( M \) be a closed orientable 3-manifold, and for a certain set \( K \) of knots in a 3-manifold \( M \), we introduce the id` ele group and id`ele class group, and the global reciprocity homomorphism, by getting the local theory in Section 4 together over all knots in \( K \). We then establish an analogue of the isomorphism theorem in global class field theory. Now, we define a set of knots \( K \).

Definition 5.1. We call a set \( K \) of knots in \( M \) admissible, if \( K \) satisfies the following conditions:

1. for each \( K_i \in K \), there exists a tubular neighborhood \( V_{K_i} \) such that \( V_{K_i} \cap V_{K_j} = \emptyset \) if \( K_i, K_j \in K \) and \( K_i \neq K_j \);
2. \( K \) contains generators of \( H_1(M) \);
3. \#\( K \) = \#\( \mathbb{N} \).

In this paper, we fix such an admissible set \( K \) of knots in \( M \) and consider a pair \( (M; K) \).

We then establish an analogue of the isomorphism theorem in global class field theory. Now, we define a set of knots \( K \).

Definition 5.2. (Id`ele group) We define the id`ele group of \( (M; K) \) by

\[
I(M; K) := \{ (a_K)_{K} \mid a_K \in H_1(\partial V_K) \quad \text{for almost all } K \in K \}.
\]

Let \( \mathcal{L} = \{ L_\alpha \}_\alpha \) be a set of finite subsets of \( K \). We define the order for \( \mathcal{L} \) in the following way: for \( L_\alpha, L_\beta \in \mathcal{L}, L_\alpha \leq L_\beta \) if \( L_\alpha \subset L_\beta \). In this paper, for a finite subset \( L \subset K \), we denote by \( X_L \) the exterior space \( M \setminus L \). Then for each \( L_\alpha \leq L_\beta \), we consider the homomorphism

\[
\varphi_{\alpha\beta} : \text{Gal}(X^{ab}_{L_\beta}/X_{L_\alpha}) \to \text{Gal}(X^{ab}_{L_\alpha}/X_{L_\alpha})
\]

which is induced by natural inclusion \( t_{\alpha\beta} : X_{L_\beta} \hookrightarrow X_{L_\alpha} \).

When \( L_\alpha \leq L_\beta \leq L_\gamma \), it satisfies \( \varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} \) from \( t_{\alpha\gamma} = t_{\alpha\beta} \circ t_{\beta\gamma} \). Then we define \( \text{Gal}(M; K)^{ab} \) by the inverse limit of the Galois groups \( \text{Gal}(X^{ab}_{L_\alpha}/X_{L_\alpha}) \) with respect to \( \varphi_{\alpha\beta} \):

\[
\text{Gal}(M; K)^{ab} := \lim_{\alpha} \text{Gal}(X^{ab}_{L_\alpha}/X_{L_\alpha}) := \left\{ (a_\alpha)_{\alpha} \in \prod_{L_\alpha \in \mathcal{L}} \text{Gal}(X^{ab}_{L_\alpha}/X_{L_\alpha}) \mid \varphi_{\alpha\beta}(a_\beta) = a_\alpha \right\}.
\]

This group may be regarded as an analogue of the maximal Abelian Galois group \( \text{Gal}(k^{ab}/k) \) of a number field \( k \).

Now, we are going to define an analogue of the global reciprocity homomorphism

\[
\rho_M : I(M; K) \to \text{Gal}(M; K)^{ab}
\]

as follows. First, for each \( \alpha \) and \( K \in K \), the natural inclusion \( \iota_K^{\alpha} : \partial V_K \to X_{L_\alpha} \) induces the homomorphisms \( \iota_K^{\alpha} : H_1(\partial V_K) \to H_1(X_{L_\alpha}) \) and \( g_K^{\alpha} : \text{Gal}(\partial V_K/X_{L_\alpha}) \to \text{Gal}(X^{ab}_{L_\alpha}/X_{L_\alpha}) \)
which fit in the commutative diagram

\[
\begin{array}{ccc}
H_1(\partial V_K) & \xrightarrow{\rho_{\partial V_K}} & \text{Gal}(\partial V_K^{ab} / \partial V_K) \\
\xi^\alpha_{K*} & \circlearrowleft & \eta_{X_{L*}} \\
H_1(X_{L*}) & \xrightarrow{\eta_{X_{L*}}} & \text{Gal}(X_{L*}^{ab} / X_{L*})
\end{array}
\]

where \( \eta_{X_{L*}} \) is the isomorphism by the Hurewicz map. We let

\[
\lambda^\alpha_K : H_1(\partial V_K) \to \text{Gal}(X_{L*}^{ab} / X_{L*})
\]

be the composite \( g^\alpha_K \circ \rho_{\partial V_K} = \eta_{X_{L*}} \circ l^\alpha_{K*} \) and define

\[
\psi_\alpha : I_{(M; K)} \to \text{Gal}(X_{L*}^{ab} / X_{L*})
\]

by

\[
\psi_\alpha((a_K)_K) := \sum_{K \in \mathcal{K}} \lambda^\alpha_K (a_K).
\]

Here the summation over \( \mathcal{K} \) is finite, because longitude component of \( a_K \) is zero for almost all \( K \in \mathcal{K} \) and the meridian component of \( a_K \) is zero in \( H_1(X_{L*}) \) for \( K \not\in L_\alpha \).

Finally, we define the global reciprocity homomorphism \( \rho_M : I_{(M; K)} \to \text{Gal}(M; K)^{ab} \) by

\[
\rho_M((a_K)_K) := (\psi_\alpha((a_K)_K))_\alpha,
\]

noticing \( \psi_\alpha = \varphi_{\alpha \beta} \circ \psi_\beta \), which can be checked as follows:

\[
\varphi_{\alpha \beta} \circ \psi_\beta((a_K)_K) = \varphi_{\alpha \beta} \left( \sum_{K \in \mathcal{K}} \eta_{X_{L*}} \circ l^\beta_{K*} (a_K) \right)
\]

\[
= \sum_{K \in \mathcal{K}} \eta_{X_{L*}} \circ l_{\alpha \beta*} \circ l^\beta_{K*} (a_K)
\]

\[
= \sum_{K \in \mathcal{K}} \eta_{X_{L*}} \circ l^\alpha_{K*} (a_K)
\]

\[
= \psi_\alpha((a_K)_K)
\]

where each \( \iota \) is induced by following commutative diagram:

\[
\begin{array}{ccc}
\partial V_K & \xrightarrow{\iota^\beta_K} & X_{L*} \\
\iota^\alpha_K & \xrightarrow{\iota_{\alpha \beta}} & X_{L*}
\end{array}
\]

Here we note that \( \varphi_{\alpha \beta} \circ \eta_{X_{L*}} = \eta_{X_{L*}} \circ l_{\alpha \beta*} \circ l^\beta_{K*} = \iota^\alpha_{K*} \).

We define a principal idèle group, and idèle class group of manifolds as follows.
Here, by moving knots condition: (A) $F$ or any finite subset $L$ distinct, $K_{\mu}$ (Very admissible set) We call an admissible set

\[ \text{Definition 5.5.} \]

We note first that there are only countably many isotopy classes of links in $M$ and that there are only countably many closed 3-manifolds. Let $\mathcal{L}$ be the set of finite subsets of knots in $M$. For each $L \in \mathcal{L}$, there are only countably many covers of $M$ branched over $L$, say $\mathcal{C}_L = \{ h_L : N_L \to M \}$. Choose a finite set of knots in $N_L$, $\{ K_{\mu}^L \mid 1 \leq \mu \leq m_{N_L} \}$, such that their homology classes \([K_{\mu}^L]\) generate $H_1(N_L)$. We then define $\mathcal{K}$ to be the union of $h(K_{\mu}^L)$ over all $N_L \in \mathcal{C}_L$ and $L \in \mathcal{L}$:

\[ \mathcal{K} := \{ h(K_{\mu}^L) \mid [K_{\mu}^L] \text{ generate } H_1(N_L), N_L \in \mathcal{C}_L, L \in \mathcal{L} \}. \]

Here, by moving knots $K_{\mu}^L$, a little bit if necessary, we may assume that if $K_{\mu}^L$ and $K_{\nu}^{L_i}$ are distinct, $K_{\mu}^L \cap K_{\nu}^{L_i}$ are empty and that any $K_{\mu}^L$ does not intersect a branch set of $N_L$. Then the condition (A) is satisfied by our construction. 

\[ \text{Definition 5.5. (Very admissible set)} \]

We call an admissible set $\mathcal{K}$ of knots in $M$ very admissible if $\mathcal{K}$ satisfies condition (A) in Lemma 5.4.

Hereafter we fix such a good admissible set $\mathcal{K}$ of knots once and for all. So we assume that $h^{-1}(\mathcal{K})$ is an admissible set of Abelian covering space $N$, which is branched over $L \subset \mathcal{K}$. We set $\mathcal{K}_N := h^{-1}(\mathcal{K})$. And its order is induced by $\mathcal{K}$, namely if $L_{\alpha} \leq L_\beta$ in $\mathcal{K}$, then $h^{-1}(L_{\alpha}) \leq h^{-1}(L_\beta)$ in $\mathcal{K}_N$.

Next, we define the norm map $h_{N/M} : I(N;\mathcal{K}_N) \to I(M;\mathcal{K})$. Let $h^{-1}(K) = K_1 \cup K_2 \cup \cdots \cup K_r$ for each $K \in \mathcal{K}$. For a tubular neighborhood $V_K$ of $K$, let $V_{K_i}$ be a connected component of $h^{-1}(V_K)$ containing $K_i$. Let $h_{K_i} : H_1(\partial V_{K_i}) \to H_1(\partial V_K)$ be the homomorphism which is induced by $h_i := h|_{\partial V_{K_i}} : \partial V_{K_i} \to \partial V_K$. We then define

\[ h_K : \bigoplus_{i=1}^r H_1(\partial V_{K_i}) \to H_1(\partial V_K) \text{ by } h_K((a_i)_{i=1}^r) := \sum_{i=1}^r h_{K_i}(a_i) \]

and $h_{N/M} : I(N;\mathcal{K}_N) \to I(M;\mathcal{K})$ is defined by $h_{N/M} := \prod_{K \in \mathcal{K}} h_K$. The norm map $h_{N/M}$ induces the homomorphism $\tilde{h}_{N/M} : C(N;\mathcal{K}_N) \to C(M;\mathcal{K})$. We will write $\tilde{h}_{N/M}$ simply as $h_{N/M}$ when no confusion can arise.
Remark 5.6. We note that $h_{N/M} : C_{(N; K_N)} \to C_{(M; K)}$ is well defined, because the following diagram is commutative:

$$
\begin{array}{ccc}
I_{(N; K_N)} & \xrightarrow{\rho_N} & \text{Gal}(N; K_N)_{ab} \\
\downarrow & & \downarrow \\
I_{(M; K)} & \xrightarrow{\rho_M} & \text{Gal}(M; K)_{ab}
\end{array}
$$

Next, we show an analogue of the relation between idèle class group and ideal class group in our 3-manifold context. We define the following homomorphism,

$$
\xi : I_{(M; K)} \to \bigoplus_{K \in \mathcal{K}} \mathbb{Z}
$$

by

$$
\xi((a_K)_K) := (v_K(a_K))_K.
$$

Then, we denote Ker($\xi$) by $U$.

**Proposition 5.7.** Assume that a very admissible set $K$ satisfies the following condition: for any finite subset $\{K_j\}_{j \in J}$ of $\mathcal{K}$, if one has $\sum_{j \in J} c_j[K_j] = 0$ in $H_1(M)$ with $c_j \in \mathbb{Z} \setminus \{0\}$, then $[K_j] = 0$ for all $j$. Then we have

$$
H_1(M) \cong I_{(M; K)}/(U + P_{(M; K)}).
$$

**Proof.** For $K \in \mathcal{K}$, let $\iota^M_K : \partial V_K \to M$ be the inclusion map and $\iota^K_{K*} : H_1(\partial V_K) \to H_1(M)$ the induced homomorphism, and define

$$
\varphi : I_{(M; K)} \to H_1(M)
$$

by

$$
\varphi((a_K)_K) := \sum_{K \in \mathcal{K}} \iota^K_{K*}(a_K).
$$

Here the summation over $K \in \mathcal{K}$ is actually a finite sum, because $v_K(a_K) = 0$ for almost all $K \in \mathcal{K}$ and a meridian of any $K$ is null-homologous in $M$. It is easy to see that $\varphi$ is surjective, using to the condition (2) of Definition 5.1 of $\mathcal{K}$. Therefore, it suffices to show $\text{Ker}(\varphi) = U + P_{(M; K)}$.

Suppose $(a_K)_K \in U + P_{(M; K)}$. Then we can write $(a_K)_K = (b_K)_K + (c_K)_K$ with $(b_K)_K \in P_{(M; K)}$ and $(c_K)_K \in U$. By Definition 5.3 of $P_{(M; K)}$, it is easy to see $\varphi((b_K)_K) = 0$, hence $(b_K)_K \in \text{Ker}(\varphi)$. As for $(c_K)_K$, we also have $\varphi((c_K)_K) = 0$, because $v_K(c_K) = 0$ for all $K \in \mathcal{K}$ and a meridian of any $K \in \mathcal{K}$ is null homologous in $M$. Therefore, we have $(a_K)_K \in \text{Ker}(\varphi)$.

Suppose $(a_K)_K \in \text{Ker}(\varphi)$. As in Definition 4.3, we decompose $a_K$ to the meridian and longitude components:

$$
a_K = (q_K, p_K) = q_K[m_K] + p_K[l_K],
$$

where $m_K$ is a meridian of $K$, $l_K$ is a longitude of $K$, and $p_K, q_K \in \mathbb{Z}$. If $p_K = 0$ for all $K \in \mathcal{K}$, then $(a_K)_K \in U$. So we suppose there are some $K$, say $K_1, \ldots, K_n$, such that
$p_{K_1}, \ldots, p_{K_n} \neq 0$. We write

\[
\begin{aligned}
(a_K)_K &= (b_K)_K + (c_K)_K \\
(b_K)_K &= (\ldots, 0, (0, p_{K_i}), 0, \ldots, 0, (0, p_{K_n}), 0, \ldots) \\
(c_K)_K &\in U,
\end{aligned}
\]

where $0 = (0, 0)$. Then it suffices to show $(\ldots, 0, (0, p_{K_i}), 0, \ldots) \in P_{(M; K)} + U$ for each $i = 1, \ldots, n$. Since $\varphi((a_K)_K) = 0$ and $\varphi((c_K)_K) = 0$, we have

\[
0 = \varphi((b_K)_K) = \sum_{i=1}^n p_{K_i}[K_i].
\]

Since $\sum_{i=1}^n p_{K_i}[K_i] = 0$, we have $[K_i] = 0$ in $H_1(M)$ for each $i = 1, \ldots, n$ by our assumption. Therefore, there exists a surface $S_i$ such that $\partial S_i = K_i$. If there is no $K \in K$ which intersects with $S_i$, then $(\ldots, 0, (0, p_{K_i}), 0, \ldots) \in P_{(M; K)}$. Suppose there is $K_{\mu} \in K$ such that $K_{\mu}$ intersects with $S_i$. Then let $n_{i\mu} := \text{lk}(K_i, K_{\mu})$ be the linking number of $K_i$ and $K_{\mu}$ which can be defined by the existence of the surface $S_i$, and we have

\[
(\ldots, 0, (0, p_{K_i}), 0, \ldots) = (\ldots, 0, (0, p_{K_i}), 0, \ldots, 0, (-p_{K_i}n_{i\mu}, 0), 0, \ldots)
\]

\[
+ (\ldots, 0, (p_{K_i}n_{i\mu}, 0), 0, \ldots).
\]

Since the first term of the right-hand side is in $P_{(M; K)}$ and the second is in $U$, we obtain the claim.

COROLLARY 5.8. Let $M$ be an integral homology sphere. Then $I_{(M; K)} = U + P_{(M; K)}$.

Proof. Since $H_1(M) = 0$, a very admissible set $K$ of $M$ satisfies the assumption of Proposition 5.7. Then $I_{(M; K)}/(U + P_{(M; K)}) = 0$ by Proposition 5.7.

Finally, we present our main result on global class field theory for integral homology sphere.

THEOREM 5.9. (Global class field theory over an integral homology sphere) Let $M$ be an integral homology sphere and let $K$ be a very admissible set of knots in $M$. Then, there exists a homomorphism,

\[
\rho_M : C_{(M; K)} \rightarrow \text{Gal}(M; K)^{ab}
\]

which has the following properties.

1. For any finite Abelian covering $h : N \rightarrow M$ branched over $L \in \mathcal{L}$, $\rho_M$ induces the isomorphism,

\[
C_{(M; K)}/h_{N/M}(C_{(N; K_N)}) \cong \text{Gal}(N/M)
\]

where $h_{N/M}$ denotes the norm map on the idele class group.

2. For a knot $K \in K$, we have the following commutative diagram:

\[
\begin{array}{ccc}
H_1(\partial V_K) & \xrightarrow{\rho_K} & \text{Gal}(\partial V_K^{ab}/\partial V_K) \\
\downarrow^{i_K} & \circ & \downarrow^g_K \\
C_{(M; K)} & \xrightarrow{\rho_M} & \text{Gal}(M; K)^{ab}
\end{array}
\]
where \( \iota_K \) is the homomorphism induced by the natural inclusion \( H_1(\partial V_K) \to I_1(M;K) \).

Namely, \( \iota_K(a_K) = [(\ldots, 0, a_K, 0, \ldots)] \), and \( g_K \) is the homomorphism induced by \( g^\sigma_K \) in (5.2).

**Proof.** We note that a very admissible set \( \mathcal{K} \) exists by Lemma 5.4. Let \( M \setminus L = X_L \), \( N \setminus h^{-1}(L) = Y_L \), \( L = K_1 \cup \cdots \cup K_r \). Since \( h : N \to M \) is Abelian covering branched over \( L \), \( \text{Gal}(N/M) \cong H_1(X_L)/h_*(H_1(Y_L)) \).

We define \( \rho_M \) as the canonical homomorphism induced by (5.1). We note that this is well-defined by the definition of \( P(M;K) \).

(1) We have

\[
C_{(M;K)}/h_{N/M}(C(N;K_N)) = (I_{(M;K)}/P(M;K))/h_{N/M}(I(N;K_N)/P(N;K_N))
\]

\[
= I_{(M;K)}/(P(M;K) + h_{N/M}(I(N;K_N)))
\]

\[
= (P(M;K) + U)/(P(M;K) + h_{N/M}(I(N;K_N))) \quad \text{by Corollary 5.8}
\]

\[
= U/U \cap (P(M;K) + h_{N/M}(I(N;K_N))).
\]

Therefore, we need to prove \( U/U \cap (P(M;K) + h_{N/M}(I(N;K_N))) \cong \text{Gal}(N/M) \cong H_1(X_L)/h_*(H_1(Y_L)) \).

Let us define the homomorphism \( \varphi : U \to H_1(X_L)/h_*(H_1(Y_L)) \) by \( \varphi((a_K)_K) := \pi \circ \sum_{K \in \mathcal{K}} \iota_K^L(a_K) \), where the homomorphism \( \iota_K^L : H_1(T_K) \to H_1(X_L) \) is induced by the natural inclusion \( T_K \to X_L \), and \( \pi : H_1(X_L) \to H_1(X_L)/h_*(H_1(Y_L)) \) is the natural projection.

Since \( M \) is an integral homology sphere, \( H_1(X_L) \) is generated by the meridian classes of \( K_i \). Hence, \( \varphi \) is surjective by definition. Therefore, it suffices to prove \( \text{Ker}(\varphi) = U \cap (P(M;K) + h_{N/M}(I(N;K_N))) \).

Let \( (a_K)_K \) be an element of \( \text{Ker}(\varphi) \), namely \( \sum_{K \in \mathcal{K}} \iota_K^L(a_K) \in h_*(H_1(Y_L)) \). We note that the longitude component of \( a_K \) is zero for each \( K \in \mathcal{K} \). For each component \( K \in \mathcal{K} \), we consider the homomorphism \( h_K : \bigoplus_{K \in \mathcal{K}} \iota_K^L(a_K) \in h_*(H_1(Y_L)) \) if \( K \) is unbranched component, the definition of unbranched covering implies \( \text{Im}(h_K) \supset \mathbb{Z}[m_K] \times 0 \ominus a_K \).

Therefore, it suffices to check that the meridian component of branched component \( K_i \) coming from \( h_{K_i} \). Since \( H_1(X_L) \) is freely generated by the meridian classes of \( K_i \), and the longitude component of \( a_{K_i} \) is zero, \( \sum_{K \in \mathcal{K}} \iota_K^L(a_K) \in h_*(H_1(Y_L)) \) implies \( \iota_{K_i}^L(a_{K_i}) \in h_*(H_1(Y_L))) \). Indeed, since \( h_*(H_1(Y_L)) = \sum_{i=1}^n n_i \mathbb{Z}[m_{K_i}] \subset H_1(X_L) = \sum_{i=1}^n \mathbb{Z}[m_{K_i}] \cong \mathbb{Z}^n \) with some \( n_i \) and \( \iota_{K_i}^L(a_{K_i}) \in \mathbb{Z}[m_{K_i}] \), the equivalence \( \sum_{i=1}^n \iota_{K_i}^L(a_{K_i}) \equiv \sum_{K \in \mathcal{K}} \iota_K^L(a_K) \equiv 0 \mod h_*(H_1(Y_L)) \) yields that \( \iota_{K_i}^L(a_{K_i}) \in n_i \mathbb{Z}[m_{K_i}] \subset h_*(H_1(Y_L)) \). Then, the following commutative diagram

\[
\begin{array}{ccc}
H_1(T_K_i) & \longrightarrow & H_1(Y_L) \\
\downarrow \iota_{K_i}^L \downarrow & & \downarrow h_* \\
H_1(X_L) & \longrightarrow & H_1(X_L)
\end{array}
\]

and the fact that \( \iota_{K_i}^L : \mathbb{Z}[m_{K_i}] \) is injective imply that \( a_{K_i} \) is an element of \( \text{Im}(h_{K_i}) \). Therefore, \( (a_K)_K \) is an element of \( U \cap h_{N/M}(I(N;K_N)) \).

Let \( (a_K)_K + (b_K)_K \) be an element of \( U \cap (P(M;K) + h_{N/M}(I(N;K_N))) \), where \( (a_K)_K \) is an element of \( P(M;K) \), and \( (b_K)_K \) is an element of \( h_{N/M}(I(N;K_N)) \). By the definition of
\[ P(M;K), \sum_{K \in \mathcal{K}} \ell^a_{K}(a_K) = 0. \] Then \( \varphi((b_K)_K) = 0 \) by the definition of the norm map \( h_{N/M}. \) Hence, \( (a_K)_K + (b_K)_K \in \text{Ker}(\varphi). \)

Thus, we obtain \( \text{Ker}(\varphi) = U \cap (P(M;K) + h_{N/M}(I(N;K_N))). \)

(2) Our assertion follows from
\[
g_K \circ \rho_K(a, b) = (\eta_{NL} \circ t^a_{K_s}(a, b))_a
\]
and
\[
\rho_M \circ t^a_K(a, b) = \rho_M((\ldots, 0, (a, b), 0, \ldots))
\]
\[
= \left( \sum_{K^{'} \in \mathcal{K}} \eta_{N_L} \circ t^a_{K^{'}_s}((a_K^{'})) \right)_a
\]
\[
= (\eta_{NL} \circ t^a_{K_s}(a, b))_a,
\]
where \( (a_K^{'})_K^{'} = (\ldots, 0, (a, b), 0, \ldots) \).

\[ \square \]

**Examples 5.10.** Let \( M = S^3 \), we choose a link \( L = K_1 \cup \cdots \cup K_r \in \mathcal{L} \). Let \( X_L := M \setminus L \), and let \( m_i \) be a meridian class of \( K_i \). The map sending each meridian class \( m_i \) to one defines a surjective homomorphism \( \psi : \pi_1(X_L) \rightarrow \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} ; m_i \mapsto 1 \in \mathbb{Z}/n_i\mathbb{Z} \). For an Abelian covering of \( X_L \) corresponding to \( \text{Ker}(\psi) \), we have the Fox completion \( N \), which is an Abelian covering of \( S^3 \) branched over \( L \). Then, the Galois group \( \text{Gal}(N/S^3) \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} \). We are going to show that \( C_{S^3}^{L}(C(N;K_N)) \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} \) by the definition of idècle class group and Hilbert theory for 3-manifolds. By the above proof, it is sufficient to show that \( U/U \cap (P(S^3;K) + h(I(N;K_N))) \).

We define the homomorphism \( \varphi : U \rightarrow \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} \) by \( \varphi((a_K)_K) := (q_{K_1}, \ldots, q_{K_r}) \), where \( q_{K_i} \) is the meridian component of \( H_1(T_{K_i}) \).

Since \( \varphi \) is surjective by definition, we prove that \( \text{Ker}(\varphi) = U \cap (P(S^3;K) + h(I(N;K_N))) \).

Let \( (a_K)_K \) be an element of \( \text{Ker}(\varphi) \). Assume that \( K \) is an unbranched component of \( \mathcal{K} \), the meridian component of \( a_K \in H_1(T_{K_i}) \) is coming from a meridian component of some element of \( H_1(T_{h^{-1}(K_i)}) \). Therefore, we move on the branched component \( K_i \). We denote by \( K_i^{'} - 1 \) of the connected components of \( h^{-1}(K_i) \). The meridian component of \( q_{K_i} \) is a multiple of \( n_i \), so \( a_K \in H_1(T_{K_i}) \) is coming from \( H_1(T_{K_i^{'}}) \) by the fact that \( K_i \) has the branched index \( n_i \). Therefore, \( (a_K)_K \in U \cap (0 + h(I(N;K_N))) \).

Let \( (a_K)_K \) be an element of \( U \cap (P(S^3;K) + h(I(N;K_N))) \), which is written in the form
\[
(a_K)_K = (\ldots, (q_K, p_K), \ldots) + (\ldots, (q_K^{'}, -p_K), \ldots).
\]
The first term element is in \( P(S^3;K) \), and the second term element is in \( h(I(N;K_N)) \). Then we denote the first term by \( (b_K)_K \) and the second term by \( (c_K)_K \). Since \( c_K \in \text{Im}(h_{K_i}) \) and \( h_{K_i}(h_1(T_{K_i^{'}})) \subset n_i\mathbb{Z}[m_{K_i}] + \mathbb{Z}[l_{K_i}] \) for each connected component \( K_i^{'} \) of \( h^{-1}(K_i) \), \( q_K^{'} \) is a multiple of \( n_i \). Therefore we consider \( q_{K_i} \). We define the subset \( I := \{ i \in \{ 1, 2, \ldots, n \} \mid q_{K_i} \neq 0 \} \).

Since \( (b_K)_K \in P(S^3;K)_K, \sum_{K \in \mathcal{K}} \ell^a_{K_s}(b_K) = 0 \in H_1(X_K) \) for \( L_\alpha = K_i \). Hence, for each \( i \in I, \quad q_{K_i}[m_{K_i}] = \ell^a_{K_i^{*\prime}}(b_K) = -\sum_{K \neq K_i} \ell^a_{K_s}(b_K) = -\sum_{\mu} \ell^a_{K_{i\mu}^{*\prime}}(b_{K_{i\mu}}) = -\sum_{\mu} \mu \cdot p_{K_{i\mu}} n_{i\mu}[m_{K_i}] \) for some finite subset \( \{ K_{i\mu} \}_{\mu} \subset \mathcal{K} \), where \( n_{i\mu} := \text{lk}(K_i, K_{i\mu}) \). Then \( q_{K_i} = \sum_{\mu} n_{i\mu} x_{i\mu}, \) i.e.
\[
(b_K)_K = \left( \ldots, \left( \sum_{\mu} x_{i\mu} n_{i\mu} \cdot p_{K_i} \right), \ldots, (*, -x_{i\mu}), \ldots \right)
\]
where \( x_{i\mu} = -p_{K_{i\mu}}. \)
The form of \((b_K)_K\) and the fact that \((a_K)_K = (b_K)_K + (c_K)_K \in U\) imply the form of 
\((c_K)_K\), namely 
\[(c_K)_K = (\ldots, (\ast, -p_K), \ldots, (\ast, x_{i\mu}), \ldots).\]

Next, we consider \(x_{i\mu}\). We denote by \(K'_\mu\) a connected component of \(h^{-1}(K_{i\mu})\). Let 
\(f_{i\mu}\) be the covering degree of \(K'_\mu\) over \(K_{i\mu}\). By the Proposition 3.2, \(f_{i\mu}\) is the order of the element 
\((\text{lk}(K_{i\mu}, K_1), \ldots, \text{lk}(K_{i\mu}, K_r)) \in \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}\).

Hence, \(f_{i\mu}\) is the multiple of \(n_i/d_i\), where \(d_i\) is the greatest common divisor of \(n_{i\mu}\) and \(n_i\). Therefore, \(x_{i\mu}\) is the multiple of \(n_i/d_i\). This implies that 
\[q_{K_i} = \sum \mu n_{i\mu}x_{i\mu} = \sum \mu n_{i\mu}(n_i/d_i)y_{i\mu} = \sum \mu (n_{i\mu}/d_i)n_i y_{i\mu} \in n_i\mathbb{Z}.
\]

Since \(q_{K_i}\) is the multiple of \(n_i\) for each \(K_i\), \((a_K)_K\) is an element of \(K\). Thus, we obtain the isomorphism 
\[C(S^3; \mathcal{K})/h_{N/S^3}(C(N; \mathcal{K}, \mathcal{N})) \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}.
\]

Finally, we are going to introduce an analogue of Proposition 2.4.

**Proposition 5.11.** For a finite Abelian covering \(h : N \to M\) branched over \(L \in \mathcal{L}\), let \(\rho_{N/M} : \mathcal{C}_M \to \text{Gal}(N/M)\) be the homomorphism defined by composing \(\rho_M\) with the natural projection \(\text{Gal}(M; \mathcal{K})^{ab} \to \text{Gal}(N/M)\). Then we have:

1. \(K \in \mathcal{K}\) is completely decomposed in \(N\) if and only if \(\rho_{N/M} \circ \iota_K(H_1(\partial V_K)) = \{1\}\);
2. \(K \in \mathcal{K}\) is unbranched in \(N\) if and only if \(\rho_{N/M} \circ \iota_K(\mathbb{Z}[m]) = \{1\}\).

**Proof.** (1) Let \(K'\) be a connected component of \(h^{-1}(K)\). By Section 3, we identify with 
\(\text{Gal}(\partial V_{K'}/\partial V_K)\) and the decomposition group of \(K'\). By Theorem 5.9, the following diagram is commutative:

\[
\begin{array}{ccc}
H_1(\partial V_K) & \longrightarrow & \text{Gal}(\partial V_{K'}/\partial V_K) \\
\rho_{N/M} \circ \iota_K & \downarrow \circ & \downarrow \cap \\
\text{Gal}(N/M) & \longrightarrow & \text{Gal}(N/M)
\end{array}
\]

Here, \(H_1(\partial V_K) \to \text{Gal}(\partial V_{K'}/\partial V_K)\) is surjective by Theorem 4.1. Thus, from the diagram we see \(\rho_{N/M} \circ \iota_K(H_1(\partial V_K)) = \{1\} \iff \text{Gal}(\partial V_{K'}/\partial V_K) = \{1\} \iff K\) is completely decomposed in \(N\).

(2) For the proof of the if part, \(\mathbb{Z}[m]\) is zero in \(\text{Gal}(\partial V_{K'}/\partial V_K) \cong H_1(\partial V_K)/h_\ast(\text{H}_1(\partial V_K))\) by the above diagram. This implies that \(h_\ast(\text{H}_1(\partial V_{K'}))\) is containing \(\mathbb{Z}[m]\). Therefore, \(h : V_{K'} \to V_K\) is unbranched subcovering by Corollary 4.2. Thus, \(K\) is unbranched in \(N\).

For the only if part, by Corollary 4.2, there exist a subgroup \(H \subset H_1(\partial V_K)\) such that 
\(\text{Gal}(\partial V_{K'}/\partial V_K) \cong H_1(\partial V_K)/H\) and \(H\) containing \(\mathbb{Z}[m]\). By the above diagram, \(\rho_{N/M} \circ \iota_K(H) = \{1\}\). This implies that \(\rho_{N/M} \circ \iota_K(\mathbb{Z}[m]) = \{1\}\). \(\Box\)

**Acknowledgements.** I would like to thank my supervisor Professor Masanori Morishita for his advice and encouragement. I would also like to express my deep gratitude to my family for their support.

**References**


Hirofumi Niibo
Faculty of Mathematics
Kyushu University
744, Motooka, Nishi-ku
Fukuoka, 819-0395
Japan
(E-mail: h.niibo.411@s.kyushu-u.ac.jp)