REALIZATION OF HOMOGENEOUS CONES THROUGH ORIENTED GRAPHS

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Abstract. In this paper, we realize any homogeneous cone by assembling uniquely determined subcones. These subcones are realized in the cones of positive-definite real symmetric matrices of minimal possible sizes. The subcones are found through the oriented graphs drawn by using the data of the given homogeneous cones. We also exhibit several interesting examples of our realizations of homogeneous cones. These are of rank 5, of dimension 19, of dimension 11 of continuously many inequivalent homogeneous cones, and some of the low-dimensional homogeneous cones.

1. Introduction

The open convex cone $\mathcal{P}(N, \mathbb{R})$ of positive-definite real symmetric matrices of $N$th order is a typical example of Riemannian symmetric space, and appears in many areas of mathematics. The group $\text{GL}(N, \mathbb{R})$ acts on $\mathcal{P}(N, \mathbb{R})$ transitively by $g^t x g$, where $g \in \text{GL}(N, \mathbb{R})$ and $x \in \mathcal{P}(N, \mathbb{R})$. If we restrict $g$ to the subgroup $H^+(N, \mathbb{R})$ of lower triangular matrices in $\text{GL}(N, \mathbb{R})$ with positive diagonal entries, then the action is simply transitive, that is, the stabilizer at any reference point is trivial. Thus, $\mathcal{P}(N, \mathbb{R})$ is described as

$$\mathcal{P}(N, \mathbb{R}) = \{g^t g \mid g \in H^+(N, \mathbb{R})\},$$

which is the orbit of $H^+(N, \mathbb{R})$ through the identity matrix.

Homogenous cones are generalization of $\mathcal{P}(N, \mathbb{R})$ by focusing on homogeneity, and no longer symmetric spaces in general. Let $V$ be a finite-dimensional real vector space with an inner product, and $\Omega$ an open convex cone in $V$ containing no entire line. When the linear group $G(\Omega)$ defined by

$$G(\Omega) := \{g \in \text{GL}(V) \mid g(\Omega) = \Omega\}$$

acts on $\Omega$ transitively, we say that $\Omega$ is a homogenous cone. Vinberg’s paper [23] published about a half-century ago has laid the foundation for the theory of homogeneous cones. He introduced, among other algebras, a non-associative matrix algebra with an involutive anti-automorphism $*$, called a $T$-algebra, and established that any homogeneous cone is described as the set of $T$-algebra products $hh^*$, where $h$ runs through the group of lower triangular matrices with positive diagonal entries in the $T$-algebra. This is clearly a theoretically

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beautiful analogue of (1.1). However, in practice, it requires some expertise to treat such non-associative matrix algebras. In view of recent applications [1], [4] and [12] of homogeneous cones to statistics, it is desirable to have a description of homogeneous cones with easier access. This is the first motivation of our study. Another motivation is that we wanted to write up basics of homogeneous cones thoroughly in terms of clans, which, introduced also by Vinberg [23], are left-symmetric algebras (LSAs) with additional conditions (see Section 2.1 for the precise definition). LSAs are more widely known compared with the other two algebras, $T$-algebras and $N$-algebras, introduced by Vinberg [23] as is seen from the survey articles of Burde [5] and Manchon [19]. Moreover, an LSA $\mathfrak{A}$ is Lie-admissible in the sense of Albert (cf. Myung [20, Ch. 1] for Lie-admissible algebras), that is, the bracket product $[x, y] := xy - yx$ defines a Lie algebra structure in $\mathfrak{A}$, although $\mathfrak{A}$ is not necessarily associative.

A natural way of describing a homogeneous cone by matrices is to represent it in $\mathcal{P}(N, \mathbb{R})$ by choosing an appropriate $N$. Such an idea dates back to Rothaus’ paper [22]. A similar idea is found in Graczyk and Ishi [12] by using some system of vector spaces with bigraded indices. Another way of such representation as a slice of $\mathcal{P}(N, \mathbb{R})$ is due to Chua [8, Corollary 4.3] as well as Xu [25, Theorem 4.12], Ishi and Nomura [16, Proposition 5.3] (see also Ishi [15, Theorem 4]). However, in general, these descriptions demand bigger matrices than actually needed, and some parameters appear with unnecessarily more repetitions. Let us describe these circumstances with an example. We take the non-symmetric cone $\Omega_{\text{Vin}}$ of dimension five, which is nowadays called the Vinberg cone. We quote, but rewritten in accordance with our framework such that the s imply transitive action is given by lower triangular matrices, the original description of $\Omega_{\text{Vin}}$ in Vinberg [23, p. 397]: $\Omega_{\text{Vin}}$ is the cone consisting of ordered pairs

\[
\left\{ \begin{pmatrix} \lambda_1 & x_{21} \\ x_{21} & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & x_{31} \\ x_{31} & \lambda_3 \end{pmatrix} \right\}
\] (1.2)

of $2 \times 2$ positive-definite real symmetric matrices with common $(1, 1)$ entry $\lambda_1$. We can imagine that two copies of $\mathcal{P}(2, \mathbb{R})$ are stapled at the $(1, 1)$ entry. The style of description in [22] and [12] is given by (4.2) of this paper, and realizes $\Omega_{\text{Vin}}$ in $\mathcal{P}(4, \mathbb{R})$. It avoids making a pair of matrices, but the cost is that we need bigger matrices with many zeros inside. It also gives the impression that all such zeros play the same role. The method of [16] (or [15]) is to realize in a direct sum space, and requires still bigger matrices:

\[
\Omega_{\text{Vin}} \cong \{ \text{diag} \left[ \begin{pmatrix} \lambda_1 & x_{21} \\ x_{21} & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & x_{31} \\ x_{31} & \lambda_3 \end{pmatrix} \right] \} \subset \mathcal{P}(5, \mathbb{R}).
\] (1.3)

The presentation of [25] amounts to lining up three matrices in (1.3) instead of (1.2). In [8] some flexibility is given. One can choose (1.3) or the following:

\[
\Omega_{\text{Vin}} \cong \{ \text{diag} \left[ \begin{pmatrix} \lambda_1 & x_{21} \\ x_{21} & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & x_{31} \\ x_{31} & \lambda_3 \end{pmatrix} \right] \} \subset \mathcal{P}(4, \mathbb{R}).
\] (1.4)

Note that multiplying the matrix in (1.4) by the $4 \times 4$ matrix $\text{diag} \left[ (1), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (1) \right]$ from the left and the right, we see that the description (1.4) is transformed to (4.2). Needless to say, the presentation (1.4) looks like that of (1.2), and is better than that of (1.3). The choice (1.3) or
(1.4) depends on the choice of index set under what we call Chua’s condition (see Remark 4.5 of this paper). We prefer the original description (1.2) in that the matrices which appear are smaller in size with far fewer zeros. Figuratively speaking, (1.2) is a folding pocket atlas, while the others are one big sheet of map. We also would like to minimize the number of repeated parameters, and to clarify from where such repeated parameters come.

The purpose of this paper is to realize any homogeneous cone as in the original description (1.2) of $\Omega_{V\text{in}}$. Thus, we show that every homogeneous cone $\Omega$ is assembled from uniquely determined pieces of subcones $\Omega_k$ of $\mathcal{P}(N_k, \mathbb{R})$ ($k = 1, 2, \ldots$) with $N_k$ the minimal possible for $\Omega_k$, and stapled appropriately. Let us explain our realization in more detail. We extract such ‘component cones’ by drawing an oriented graph from the given homogeneous cone $\Omega$. Let $(V, \Delta)$ be the clan corresponding to $\Omega$. Suppose that the rank of $V$ is $r$, so that we have the corresponding normal decomposition (2.1). To $V$, we assign a directed graph (digraph for short) $\Gamma = \Gamma(V)$ of order $r$ by defining the vertex set $V$ and the arc set $A$ as

$$V := \{1, \ldots, r\}, \quad A := \{[j \rightarrow i] \mid j > i, \ \dim V_{ji} > 0\},$$

where $[j \rightarrow i]$ denotes the arc leaving $j$ and entering $i$. In Section 2.4, we list up the technical terms in graph theory that we borrow in this paper. Clearly our digraph $\Gamma$ is oriented, that is, we have no pairs $(i, j)$ such that both $[j \rightarrow i]$ and $[i \rightarrow j]$ are in $A$. We first collect the vertices which have no incoming arcs. Such vertices are called sources, and we denote by $S$ the set of sources of $\Gamma$. Note that we always have $r \in S$, so that $S \neq \emptyset$. For each $\omega \in S$, we pick up the vertices which are heads of arcs leaving $\omega$. These vertices are called out-neighbors of $\omega$, and the totality is denoted by $N^{\text{out}}(\omega)$. We set $N^{\text{out}}[\omega] := \{\omega\} \cup N^{\text{out}}(\omega)$. Then, $V_{[\omega]} := \bigoplus_{i \leq j} V_{ji}$, the summation being over the indices $i \leq j$ in $N^{\text{out}}[\omega]$, is a subclan of $V$. Let $\Omega_{[\omega]}$ be the corresponding homogeneous cone in $V_{[\omega]}$. We call $\Omega_{[\omega]}$ the source homogeneous cone corresponding to $\omega \in S$. The subspace $E_{[\omega]} := \bigoplus_{j} V_{\omega j}$, where the summation variable $j$ also runs only over $N^{\text{out}}[\omega]$, is a two-sided ideal of $V_{[\omega]}$. For each $x \in V_{[\omega]}$, let $\varphi_{[\omega]}(x)$ be the operator on $E_{[\omega]}$ defined by the right multiplication:

$$\varphi_{[\omega]}(x)\eta := \eta \Delta x \quad (\eta \in E_{[\omega]}).$$

After a minor modification of the inner product of $E_{[\omega]}$ inherited from $V$, we see that the map $\varphi_{[\omega]} : x \mapsto \varphi_{[\omega]}(x)$ is a representation of the clan $V_{[\omega]}$ in the sense of Ishi [15] such that all the operators $\varphi_{[\omega]}(x)$ are selfadjoint. Moreover, $\varphi_{[\omega]}$ is faithful, and it turns out that $\Omega_{[\omega]}$ is the set of $x \in V_{[\omega]}$ such that $\varphi_{[\omega]}(x)$ is positive-definite. In addition, $\varphi_{[\omega]}$ is minimal in the sense that if we have an injective LSA homomorphism from $V_{[\omega]}$ to the symmetric matrices $\text{Sym}(N, \mathbb{R})$, then it holds that $N \geq \dim E_{[\omega]}$.

If $S = \{\omega\}$ ($\omega = r$), we are done, that is, $\Omega_{[\omega]} := \varphi_{[\omega]}(\Omega_{[\omega]})$ is our realization. If $S = \{\omega_1, \ldots, \omega_s\}$ ($s > 1$), then $\Omega$ is assembled from $\Omega_{[\omega_1]}^0, \ldots, \Omega_{[\omega_s]}^0$. To describe how we assemble these source homogeneous cones, we first need to express $V_{[\omega]} \cap V_{[\omega']}$ for distinct $\omega, \omega' \in S$. Consider $\mathcal{J}(\omega, \omega') := N^{\text{out}}[\omega] \cap N^{\text{out}}[\omega']$. We call $\mathcal{J}(\omega, \omega')$ the junction set for $\omega, \omega'$. Then, we have $V_{[\omega]} \cap V_{[\omega']} = \bigoplus_{i \leq j} V_{ji}$, where the summation is over the indices $i \leq j$ in $\mathcal{J}(\omega, \omega')$. When $\mathcal{J}(\omega, \omega') = \emptyset$, the right-hand side is understood to be $\{0\}$. If $\mathcal{J}(\omega, \omega') \neq \emptyset$, then the homogeneous cone $\Omega_{[\omega']}$ corresponding to the clan $V_{[\omega]} \cap V_{[\omega']}$ is realized in two ways: one in $V_{[\omega]}^0 := \varphi_{[\omega]}(V_{[\omega]})$, and the other in $V_{[\omega']}^0 := \varphi_{[\omega']}(V_{[\omega']}).$ By a natural identification of $\varphi_{[\omega]}(V_{[\omega]} \cap V_{[\omega']})$ with $\varphi_{[\omega']}(V_{[\omega]} \cap V_{[\omega']})$, we identify two cones.
homogenous cones. Our example shows that this possibility actually occurs.

We consider the multiplication structure of \( V^0_{[\omega]} \) and \( V^0_{[\omega']} \). If \( \mathcal{J}(\omega, \omega') = \emptyset \), we do nothing. We call this process the stapling of two vector spaces \( V^0_{[\omega]} \) and \( V^0_{[\omega']} \), and of two cones \( \Omega^0_{[\omega]}, \Omega^0_{[\omega']} \). We do stapling for any distinct pair \( \omega_i, \omega_j \in S \) to assemble \( \Omega^0_{[\omega_i]} \cdot \ldots \cdot \Omega^0_{[\omega_j]} \). The above is a rough sketch of our realization process of \( \Omega \). What is interesting is that the original Vinberg’s idea to present \( \Omega_{\text{Vin}} \) is enough to realize any homogeneous cone after all. We note that use of oriented graphs is not new. It already appeared in the papers of Kaneyuki and Tsuji [17], Pjasecki [21], and Boutouria and Hassairi [4]. In view of Cayley’s paper [6] where an LSA appears in his study of trees, the connection of homogeneous cones with oriented graphs is not quite surprising.

We organize this paper as follows. In Section 2, we collect definitions of the basic objects which we treat in this paper. For completeness and readability of this paper, we write down proofs of some of the basic facts which play important roles later.

Section 3 is devoted to the study of the out-neighbor homogeneous cone \( \Omega_{[k]} \) corresponding to a vertex \( k \) of \( \Gamma \). The source homogeneous cone \( \Omega_{[\omega]} \) \((\omega \in S)\) which appeared above is the out-neighbor homogeneous cone corresponding to \( \omega \). We study not only with sources but also with general vertices for the technical needs in Section 4. The out-neighbor subclan \( V_{[k]} \) is formed by summing up \( V_{ji} \) \((i \leq j)\) only over the indices \( i \leq j \) in \( N_{\text{out}}[k] \) consisting of the out-neighbors of the vertex \( k \) together with \( k \) itself. The out-neighbor homogeneous cone \( \Omega_{[k]} \) in \( V_{[k]} \) is the homogeneous cone corresponding to \( V_{[k]} \). The two-sided ideal \( E_{[k]} \) in \( V_{[k]} \) is obtained by summing up \( V_{kj} \) with \( j \in N_{\text{out}}[k] \), and we have a faithful representation \( \varphi_{[k]} \) of the clans \( V_{[k]} \) on \( E_{[k]} \) by right multiplications. A minor modification of the inner product of \( E_{[k]} \) yields that every operator \( \varphi_{[k]}(x) \) is selfadjoint.

Our Theorem 3.5 says that \( x \in V_{[k]} \) belongs to \( \Omega_{[k]} \) if and only if \( \varphi_{[k]}(x) \) is positive-definite. Moreover, Proposition 3.9 proves that \( \Omega^0_{[k]} := \varphi_{[k]}(\Omega_{[k]}) \) is of minimal size among realizations of \( \Omega_{[k]} \). We call \( \Omega^0_{[k]} \) the canonical minimal realization of \( \Omega_{[k]} \). In our realization process, we also show that \( \varphi_{[k]} \) is equivariant for the simply transitive group actions.

In Section 4 we give the realization of \( \Omega \) outlined above in Theorem 4.13. Our realization is equivariant for the simply transitive group actions. We look carefully at the relations between the basic relative invariants of the original cone \( \Omega \) and of the out-neighbor homogeneous cone \( \Omega_{[k]} \) (Proposition 4.2). We verify Chua’s condition closely in Remarks 4.5, 4.7 and 4.16, and conclude that our realization is the proper generalization of the original description (1.2) of \( \Omega_{\text{Vin}} \). Moreover, we show in Proposition 4.8 that increase of the matrix size and presence of many zeros inside are inevitable if one persists in a realization of \( \Omega \) by one-sheet positive-definite matrices even in the case where at least two sources exist. Uniqueness of our realization of \( \Omega \) is discussed in Remark 4.15. At the end of Section 4, we give two examples of rank-five homogeneous cones to exhibit concretely some features of our realization.

In Section 5, we present a 19-dimensional example of two linearly inequivalent homogeneous cones in which all of the source cones are linearly equivalent. The point is that in Section 4, we have started with \( \Omega \) (or with \( V \)), and have assembled the source homogeneous cones. The multiplication structure of \( V \) is built into this process. In contrast, if we only have pieces of homogenous cones, then different assemblages of them may lead to inequivalent homogeneous cones. Our example shows that this possibility actually occurs.

Section 6 is devoted to a study of rank-three homogenous cones. The main objective is to present concretely continuously many linearly inequivalent 11-dimensional homogeneous
cones. We determine their linear equivalence classes by showing directly where the parameter comes from. We also compare our parameter with that of Kaneyuki and Tsuji [17] which was given in the N-algebra language.

In the final section, Section 7, we first reveal in Proposition 7.1 the clan language interpretation of the second condition of the skeleton given in Kaneyuki and Tsuji [17] described in the N-algebra language. We conclude this paper by exhibiting some of the low-dimensional homogeneous cones left unrealized in [17].

Notation. For a real symmetric matrix, or a selfadjoint linear operator $A$ on a Euclidean vector space, we write $A \gg 0$ if $A$ is positive-definite. For a subspace $X$ of the vector space $\text{Sym}(N, \mathbb{R})$ of real symmetric matrices, let $X_{++} := X \cap P(N, \mathbb{R})$. If $X$ is expressed in such a way that $X = \{A\}$, a set of real symmetric matrices $A$, then we write $\{A\}_{++}$ for $X_{++}$. Moreover, if $X = \{(a_{ji})\}$, a subspace of matrices $(a_{ji})$ in $\text{Sym}(N, \mathbb{R})$, where some entries may be identically zero, then we write $\{(a_{ji})\}_{++}$ for $X_{++}$.

2. Preliminaries

2.1. Clans

Let $V$ be a finite-dimensional real vector space with a bilinear product $\triangle$. Although we do not assume that the product $\triangle$ possesses the associative law, we say that $(V, \triangle)$ or simply $V$ is an algebra, and we write $x \triangle y = L(x)y = R(y)x$. Thus, $L(x)$ is the left multiplication operator by $x$, and $R(y)$ the right multiplication operator by $y$. The algebra $V$ is called a clan if the following three conditions are satisfied:

(C1) $[L(x), L(y)] = L(x \triangle y - y \triangle x)$ for any $x, y \in V$;
(C2) there exists $s_0 \in V^*$ such that $s_0(x \triangle y)$ defines an inner product in $V$;
(C3) for each $x \in V$, the operator $L(x)$ has only real eigenvalues.

It is easy to see that (C1) is equivalent to the following $(C1)'$:

(C1)' $[x, y, z] = [y, x, z]$ for any $x, y, z \in V$,

where $[x, y, z] := x \triangle (y \triangle z) - (x \triangle y) \triangle z$ is a trilinear map called the associator. If we only have (C1), then $V$ is called an LSA in view of (C1)'. Linear forms $s_0$ with the property (C2) are said to be admissible. In this paper, we always assume that a clan has a unit element.

Now let $(V, \triangle)$ be a clan with unit element $e_0$, and fix an inner product $\langle x | y \rangle := s_0(x \triangle y)$ by taking an admissible linear form $s_0$. We denote by $r$ the rank of $V$. This means that there is a complete system $c_1, \ldots, c_r$ of orthogonal primitive idempotents which yields a decomposition

$$V = \bigoplus_{1 \leq j \leq k \leq r} V_{kj},$$

where $V_{kk} = \mathbb{R}c_k$ ($k = 1, 2, \ldots, r$), and for $1 \leq j < k \leq r$ we have

$$V_{kj} := \{x \in V \mid L(c_i)x = 2^{-1}(\delta_{ji} + \delta_{ki})x, \ R(c_i)x = \delta_{ji}x \ (i = 1, \ldots, r)\}.$$  

The decomposition (2.1) is called the normal decomposition of $V$ relative to $c_1, \ldots, c_r$, and we fix it once and for all. In this paper, we call $V_{kj}$ ($j < k$) the off-diagonal subspaces of the
normal decomposition (2.1). We write every $x \in V$ as
\[ x = \sum_{j=1}^{r} \lambda_j c_j + \sum_{1 \leq j < k \leq r} x_{kj} \quad (\lambda_j \in \mathbb{R}, \; x_{kj} \in V_{kj}). \tag{2.3} \]

The multiplication rules between $V_{kj}$ are as follows:
\[ V_{ji} \vartriangle V_{lk} = \{0\} \quad (\text{if } i \neq k, l), \quad V_{kj} \vartriangle V_{ji} \subset V_{ki}, \tag{2.4} \]
\[ V_{ji} \vartriangle V_{ki} \subset V_{jk} \quad (\text{according to } j \geq k \text{ or } j \leq k). \]

By introducing the lexicographic order among $V_{kj}$ in (2.1), we see that every left multiplication operator $L(x)$ of $V$ is simultaneously represented by a lower triangular matrix.

The following lemma will be used frequently without explicit quotation.

**Lemma 2.1.** If $x, y \in V_{kj}$ ($j < k$), one has $x \vartriangle y = s_0(c_k)^{-1} (x \mid y) c_k$.

**Proof.** Since $x \vartriangle y \in V_{kk}$ by (2.4), we have $x \vartriangle y = \alpha c_k$ for some $\alpha \in \mathbb{R}$. To get $\alpha$, it is enough to apply $s_0$ to both sides. Note that $s_0(c_k) = s_0(c_k \vartriangle c_k) = \|c_k\|^2 > 0$. \qed

Suppose that $x = \sum_j \lambda_j c_i + \sum_{j<k} x_{kj}$ and $y = \sum_i \mu_i c_i + \sum_{j<k} y_{kj}$ are as in (2.3). Since $s_0(V_{kj}) = \{0\}$ for any $j < k$ by formula (29) of Vinberg [23, p. 374], a straightforward calculation using Lemma 2.1 and multiplication rules (2.4) yields
\[ s_0(x \vartriangle y) = \sum_{i=1}^{r} \lambda_i \mu_i s_0(c_i) + \sum_{1 \leq j < k \leq r} (x_{kj} \mid y_{kj}). \]

In particular, we see that the decomposition (2.1) of $V$ is an orthogonal one. For every $\nu > 0$, we say that $s_0$ is $\nu$-normalized if
\[ s_0(c_1) = \cdots = s_0(c_r) = \nu. \tag{2.5} \]

Usually it is natural to take $\nu = 1$ as in the case of the trace function of a Euclidean Jordan algebra (see [9]). However, it is sometimes convenient to take $\nu = 2^{-1}$, when we define an LSA product in terms of basis vectors in view of Proposition 2.4 below. For the moment we do not normalize $s_0$ in any way to keep generality.

The next lemma and its corollary are useful in defining an LSA product in later examples. We write down a proof here for completeness.

**Lemma 2.2.** Suppose $i < j < k$. If $x \in V_{kj}$, $y \in V_{ji}$ and $z \in V_{ki}$, then one has
\[ \langle x \vartriangle y \mid z \rangle = \langle x \mid y \vartriangle z \rangle = \langle x \mid z \vartriangle y \rangle. \]

**Proof.** For the first equality, we note $y \vartriangle x = 0$ and $x \vartriangle z = 0$ by (2.4). Then we have by (C1) that
\[ \langle x \vartriangle y \mid z \rangle = s_0(L(x \vartriangle y)z) = s_0(L(x \vartriangle y - y \vartriangle x)z) = s_0([L(x), L(y)]z) = s_0(x \vartriangle (y \vartriangle z)) = \langle x \mid y \vartriangle z \rangle. \]

For the second, (C1) together with $L(z)x = 0$ and $L(y)x = 0$ leads us to
\[ L(y \vartriangle z - z \vartriangle y)x = [L(y), L(z)]x = 0. \]

Application of $s_0$ yields $\langle y \vartriangle z \mid x \rangle = \langle z \vartriangle y \mid x \rangle$. \qed
COROLLARY 2.3. Suppose $i < j < k$. If $y \in V_{ji}$ and $z \in V_{ki}$, then it holds that $y \triangle z = z \triangle y$. Moreover, $y \triangle z$ is uniquely determined by the multiplication $V_{kj} \ni x \mapsto x \triangle y \in V_{ki}$.

The formula (2.6) in the next proposition plays a basic role in this paper. The proof below using two formulae in Lemma 2.2 is a little shorter than the original proof given in Vinberg [23, p. 378].

PROPOSITION 2.4. Suppose $i < j < k$. If $x \in V_{kj}$ and $y \in V_{ji}$, then one has
\[ \| x \triangle y \| = 2^{-1/2} s_0(c_j)^{-1/2} \| x \| \| y \|. \] (2.6)

Proof. Putting $z = x \triangle y \in V_{ki}$ in Lemma 2.2, we obtain
\[ \| x \triangle y \|^2 = \langle x \mid (x \triangle y) \triangle y \rangle = \langle x \mid y \triangle (x \triangle y) \rangle. \] (2.7)

Here (C1) yields
\[ y \triangle (x \triangle y) = x \triangle (y \triangle y) + (y \triangle x) \triangle y - (x \triangle y) \triangle y. \]

On the right-hand side, Lemma 2.1, (2.2) and (2.4) tell us that the first term equals $s_0(c_j)^{-1} \| y \|^2 x$, and the second term equals zero. Then, taking the inner product with $x$, we obtain from (2.7)
\[ \| x \triangle y \|^2 = \langle x \mid (x \triangle y) \triangle y \rangle = 2^{-1} s_0(c_j)^{-1} \| x \|^2 \| y \|^2. \] \[ \square \]

The following linearized form of (2.6) is also used later:
\[ \langle x \triangle y \mid x' \triangle y' \rangle + \langle x \triangle y' \mid x' \triangle y \rangle = s_0(c_j)^{-1} \langle x \mid x' \rangle \langle y \mid y' \rangle \] (2.8)

for $x, x' \in V_{kj}$ and $y, y' \in V_{ji}$. Although the next proposition will be quoted in this paper only once in the proof of Proposition 5.1, it restricts the actual possibilities of $\varphi(c_k)$ in Lemma 2.6 of the next section. See Proposition 9 in Vinberg [23, p. 376] for a proof.

PROPOSITION 2.5. Any idempotent in $V$ is of the form
\[ c_{i_1} + \cdots + c_{i_k} \quad (i_1 < \cdots < i_k). \]

2.2. Representations of clans

Let $E$ be a real vector space with inner product $\langle \cdot \mid \cdot \rangle_E$. The vector space of all linear operators on $E$ is denoted by $L(E)$. Let $V$ be a clan of rank $r$ with unit element $e_0$. We keep to the notation in Section 2.1. In particular, we are fixing $c_1, \ldots, c_r$, so that we have the corresponding normal decomposition (2.1). Let $\varphi : V \to L(E)$ be a linear map. For $x \in V$ as in (2.3) we set
\[
\varphi(x) := \frac{1}{2} \sum_{j=1}^{r} \lambda_j \varphi(c_j) + \sum_{j<k} \varphi(c_k) \varphi(x_{kj}) \varphi(c_j),
\]
\[
\varphi(x) := \frac{1}{2} \sum_{j=1}^{r} \lambda_j \varphi(c_j) + \sum_{j<k} \varphi(c_j) \varphi(x_{kj}) \varphi(c_k). \] (2.9)
The operator \( \overline{\varphi}(x) \) (respectively \( \overline{\varphi}(x) \)) is called the lower triangular part (respectively upper triangular part) of \( \varphi(x) \) relative to \( c_1, \ldots, c_r \). With definition (2.9) we call \( \varphi : V \to \mathcal{L}(E) \) a representation of the clan \( V \) if
\[
\varphi(x \Delta y) = \varphi(x)\varphi(y) + \varphi(y)\overline{\varphi}(x) \quad \text{for all } x, y \in V.
\] (2.10)
We always require that the unit element \( e_0 \) is sent to the identity operator on \( E \), that is, \( \varphi(e_0) = I_E \). Thus, putting \( y = e_0 \) in (2.10), we obtain \( \varphi(x) = \overline{\varphi}(x) + \overline{\varphi}(x) \). If every representation operator \( \varphi(x) \) is selfadjoint, then we say that \( \varphi \) is a selfadjoint representation. In this case we have \( \overline{\varphi}(x) = i(\varphi(x)) \) for any \( x \in V \).

Let \( (\varphi, E) \) be a selfadjoint representation of \( V \). We first note the following fact.

**Lemma 2.6.** The operators \( \varphi(c_1), \ldots, \varphi(c_r) \) form a complete orthogonal system of orthogonal projectors on \( E \).

**Proof.** By (2.9) we have \( \varphi(c_i) = 2^{-1}\varphi(c_i) \), so that
\[
2\delta_{ij}\varphi(c_j) = 2\varphi(c_i \Delta c_j) = \varphi(c_i)\varphi(c_j) + \varphi(c_j)\varphi(c_i). \tag{2.11}
\]
Letting \( j = i \) in (2.11), we see that \( \varphi(c_i) \) is an orthogonal projector. Suppose next \( i \neq j \). Then (2.11) tells us that
\[
\varphi(c_i)\varphi(c_j) = -\varphi(c_j)\varphi(c_i). \tag{2.12}
\]
Multiply (2.12) by \( \varphi(c_j) \) from the left, and apply (2.12). Then we get
\[
\varphi(c_i)\varphi(c_j) = -\varphi(c_i)\varphi(c_j)\varphi(c_i) = \varphi(c_j)\varphi(c_i).
\]
This together with (2.12) implies \( \varphi(c_i)\varphi(c_j) = 0 \). Since \( \varphi(c_1) + \cdots + \varphi(c_r) = \varphi(e_0) = I_E \), the proof is complete. \( \square \)

**Remark 2.7.** Based on Lemma 2.6, we choose an orthonormal basis of \( E \), first from the range \( \varphi(c_1)E \) of \( \varphi(c_1) \), then from \( \varphi(c_2)E, \ldots, \) and finally from \( \varphi(c_r)E \). With this choice of orthonormal basis of \( E \), it is clear that the operators \( \varphi(x) \) are all represented by lower-triangular matrices. This is the reason why we call \( \overline{\varphi}(x) \) the lower triangular part of \( \varphi(x) \). A similar remark also applies to the upper triangular part \( \overline{\varphi}(x) \).

**Remark 2.8.** We would like to note here that our definition of selfadjoint clan representation is essentially the same as the definition given in Ishi \([15]\). First of all, we recall that Sym(\( N, \mathbb{R} \)) has a natural clan structure as follows. We express \( A \in \text{Sym}(N, \mathbb{R}) \) uniquely as \( A = X + T + iX \) with \( X \) strictly lower triangular and \( T \) diagonal (Gauss decomposition). Then the lower triangular part \( \underline{A} \) of \( A \) is defined to be \( \underline{A} := X + 2^{-1}T \). Clearly we have \( \underline{A} + i\underline{A} = A \), and the clan structure of \( \text{Sym}(N, \mathbb{R}) \) is given by \( \underline{A} \Delta \underline{B} := \underline{A}B + B^\dagger(\underline{A}) \) with an admissible linear form \( s_{\text{Tr}} \) defined by \( s_{\text{Tr}}(A) = \text{Tr}(A) \). Evidently, \( s_{\text{Tr}} \) is 1-normalized. Then, Ishi’s definition of a clan representation \( \phi \) is the linear map \( \phi : V \to \text{Sym}(N, \mathbb{R}) \) such that
\[
\phi(x \Delta y) = \phi(x)\phi(y) + \phi(y)^\dagger(\phi(x)) \quad \text{for all } x, y \in V. \tag{2.13}
\]
In other words, the requirement is that \( \phi \) is an LSA homomorphism of \( V \to \text{Sym}(N, \mathbb{R}) \). Now if a linear map \( \varphi \) from \( V \) to the vector space of selfadjoint operators on a Euclidean vector space \( E \) satisfies (2.10), then we take, as described in Remark 2.7, an orthonormal basis of \( E \) through which we express every operator \( \varphi(x) \) by a real symmetric matrix \( \phi(x) \). Then, by Remark 2.7 we see that \( \phi \) satisfies (2.13).
Now we return to the situation where \( \varphi \) is a selfadjoint representation of \( V \) on \( E \). We have the following useful formula.

**Lemma 2.9.** Suppose \( j < k \) and \( j < m \). If \( x \in V_{kj} \) and \( y \in V_{mj} \), then one has

\[
\varphi(x \triangle y) = \varphi(x)^{(y)} + \varphi(y)^{(x)}.
\]

**Proof.** By definition (2.9), we have

\[
\varphi(x) = \varphi(c_k)\varphi(x)\varphi(c_j), \quad \varphi^{(x)}(y) = \varphi(c_j)\varphi(x)\varphi(c_k) = \varphi(y),
\]

and \( \varphi(x) = \varphi(x) + \varphi^{(x)}(y) \). We also have similar formulae for \( y \). Then, by Lemma 2.6 we obtain

\[
\varphi(x)\varphi(y) = \varphi(x)^{(y)} + \varphi(y)^{(x)}.
\]

Since \( \varphi(x \triangle y) = \varphi(x)\varphi(y) + \varphi(y)^{(x)} \), the lemma follows.  

**Corollary 2.10.** If \( x \in V_{kj} \) \( (j < k) \), one has \( 2\varphi(x)^{(y)}(x) = s_0(c_k)^{-1}\|x\|^2 \varphi(c_k) \).

**Proof.** This follows from Lemmas 2.1 and 2.9.  

### 2.3. Vinberg’s polynomials and basic relative invariants

For every \( j = 1, \ldots, r \), we put

\[
V^{[j]} := \bigoplus_{m \geq k \geq j} V_{mk}. \tag{2.14}
\]

Multiplication rules (2.4) tell us that \( V^{[j]} \) is a subclan of \( V \). Given \( x \in V \) with (2.3), we define elements

\[
x^{[j]} = \sum_{k=j}^{r} \lambda^{[j]}_k c_k + \sum_{m \geq k \geq j} x^{[j]}_{mk} \in V^{[j]} \quad (j = 1, 2, \ldots, r) \tag{2.15}
\]

inductively in the following manner. We start by defining \( x^{[1]} := x \in V^{[1]} = V \). When \( x^{[j]} \in V^{[j]} \) is defined, we define \( x^{[j+1]} \) by

\[
\lambda^{[j+1]}_k := \lambda^{[j]}_k\lambda^{[j]}_{kj} - 2^{-1}s_0(c_k)^{-1}\|x^{[j]}_{kj}\|^2 \quad (k = j + 1, \ldots, r), \tag{2.16}
\]

\[
x^{[j+1]}_{mk} := \lambda^{[j]}_j x^{[j]}_{mk} - \lambda^{[j]}_{kj} x^{[j]}_{mj} \Delta x^{[j]}_{kj} \quad (j + 1 \leq k < m \leq r). \tag{2.17}
\]

Then, Vinberg’s polynomials \( D_j(x) \) are defined as

\[
D_j(x) := \lambda^{[j]}_j \quad (j = 1, 2, \ldots, r). \tag{2.18}
\]

See Vinberg \[23, (25), p. 385\] (see also Ishi [14]).

On the other hand, by (C1) and (C3), the space \( \mathfrak{h} := \{L(x) \mid x \in V \} \) of left multiplication operators of \( V \) is a split solvable Lie algebra. Let \( H := \exp \mathfrak{h} \) be the connected and simply connected Lie group corresponding to \( \mathfrak{h} \). The homogeneous cone \( \Omega \) corresponding to the clan \( V \) is the \( H \)-orbit \( He_0 \) through the unit element \( e_0 \) of \( V \). The \( H \)-action on \( \Omega \) is simply transitive.

A function \( f \) on \( \Omega \) is said to be relatively \( H \)-invariant if there is a one-dimensional representation \( \chi \) of \( H \) by which we have \( f(hx) = \chi(h)f(x) \) for all \( h \in H \) and \( x \in \Omega \).
Vinberg’s polynomials $D_j(x)$ are relatively $H$-invariant (Gindikin [11, Section 2]), and $\Omega$ is described as
\[ \Omega = \{ x \in V \mid D_1(x) > 0, \ldots, D_r(x) > 0 \}. \tag{2.19} \]
Moreover, by Ishi [14, Theorem 2.2], there exist relatively $H$-invariant irreducible polynomial functions $\Delta_1, \ldots, \Delta_r$ such that every relatively $H$-invariant polynomial function $P$ on $V$ is written as
\[ P(x) = \text{const} \cdot \Delta_1(x)^{n_1} \cdots \Delta_r(x)^{n_r} \quad (n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}). \]
These polynomials $\Delta_1(x), \ldots, \Delta_r(x)$ are called the basic relative invariants of $\Omega$. We know that $\Delta_j(x)$ is an irreducible factor of $D_j(x)$ extracted inductively. By Ishi [14, Proposition 2.3] we have another description of $\Omega$:
\[ \Omega = \{ x \in V \mid \Delta_1(x) > 0, \ldots, \Delta_r(x) > 0 \}. \tag{2.20} \]

In later sections, we also need an explicit expression of elements in $H$. Recalling that $H$ is split solvable, we express every $h \in H$ uniquely in the following way (cf. Ishi [13, Proposition 2.1(ii)]):
\[ h = (\exp T_1)(\exp L_1)(\exp T_2) \cdots (\exp L_{r-1})(\exp T_r), \tag{2.21} \]
where $T_j := (2 \log h_j)L(c_j)$ with $h_j > 0$ ($j = 1, \ldots, r$) and
\[ L_j := \sum_{k>j} L(v_{kj}) \quad \text{with } v_{kj} \in V_{kj} \quad (1 \leq j < k \leq r). \tag{2.22} \]
Vinberg’s polynomials $D_j(x)$ appear in the unique solution of $he_0 = x$ for a given $x \in \Omega$. Indeed, by Vinberg [23, Section III-3], the numbers $h_j > 0$ of the unique solution $h \in H$ are given by
\[ h_j^2 = D_1(x), \quad h_j^2 = D_1(x)^{-1} \cdots D_{j-1}(x)^{-1} D_j(x) \quad (j = 2, \ldots, r). \]
This explains the description (2.19) of $\Omega$.

### 2.4. Terminology from graph theory

In the next section we assign an oriented graph to a given clan. To do so, we need to borrow terminology from graph theory. For readers’ convenience, we list up here the technical terms which will be used later. Our references are the books [3] and [7].

First of all, we recall the definition of directed graphs. By a directed graph (or simply digraph) $\Gamma = (V, A)$, we mean a pair of a finite set $V = V(\Gamma)$ of elements called vertices of $\Gamma$ and a set $A = A(\Gamma)$ of ordered pairs, called arcs, of distinct vertices of $\Gamma$. The sets $V$ and $A$ are called the vertex set and the arc set of $\Gamma$, respectively. The cardinality of $V$ is called the order of $\Gamma$.

For an arc $(u, v)$, the vertex $u$ is called its tail and $v$ its head. In this paper we write $u \rightarrow v$ or $[u \rightarrow v]$ for the arc $(u, v)$ to express visually that the arc $u \rightarrow v$ leaves $u$ and enters $v$.

Let $\Gamma = (V, A)$ be a digraph. For a vertex $v$ of $\Gamma$, we set
\[ N^\text{out}(v) := \{ u \in V \setminus \{ v \} \mid [v \rightarrow u] \in A \}, \]
\[ N^\text{in}(v) := \{ w \in V \setminus \{ v \} \mid [w \rightarrow v] \in A \}. \]
Elements of $N^{\text{out}}(v)$ are called \textit{out-neighbors} of $v$. Similarly, elements of $N^{\text{in}}(v)$ are called \textit{in-neighbors} of $v$. A vertex $v$ is called a \textit{source} (respectively \textit{sink}) if $N^{\text{in}}(v) = \emptyset$ (respectively $N^{\text{out}}(v) = \emptyset$).

A \textit{weighted digraph} is a digraph $\Gamma$ with a function $c : A \rightarrow \mathbb{R}$, where $c$ is called the \textit{capacity} function of $\Gamma$. The value $c([u \rightarrow v])$ is called the \textit{capacity} of the arc $u \rightarrow v$. Digraphs which concern us in this paper have the property that whenever we have $[u \rightarrow v] \in A$, it holds that $[v \rightarrow u] \notin A$. These digraphs are called \textit{oriented graphs}.

3. \textbf{Out-neighbor homogeneous cone corresponding to a vertex}

Let $V$ be a clan of rank $r$ with unit element $e_0$. Fix a complete system $c_1, \ldots, c_r$ of orthogonal primitive idempotents so that we have the normal decomposition (2.1). We continue to use the same notation introduced in Section 2. For $1 \leq i < j \leq r$ we put $d_{ji} := \dim V_{ji}$. To $V$ we assign a digraph $\Gamma = \Gamma(V)$ of order $r$ by defining the vertex set $V$ and the arc set $A$ respectively as

$$V := \{1, 2, \ldots, r\}, \quad A := \{[j \rightarrow i] \mid j > i \text{ and } d_{ji} > 0\}.$$  

We just note that the inequality symbol can be viewed as the head of arrow. Clearly $\Gamma$ is an oriented graph. Moreover, the formula (2.6) forces that $\Gamma$ is \textit{transitive}, that is, if $[k \rightarrow j] \in A$ and $[j \rightarrow i] \in A$, then we have $[k \rightarrow i] \in A$. Defining the capacity function $c$ by $c([j \rightarrow i]) = d_{ji}$ for $[j \rightarrow i] \in A$, we thus obtain a transitive weighted oriented graph $\Gamma$ from $V$. For simplicity, we call this transitive $\Gamma$ the \textit{weighted oriented graph} of $V$, and denoted by $\Gamma(V)$.

For every $k = 1, \ldots, r$, we put $N^{\text{out}}[k] := \{k\} \cup N^{\text{out}}(k)$. Using the symbol $\oplus^{[k]}$ for a direct sum only over indices belonging to $N^{\text{out}}[k]$, we define

$$V_k := \bigoplus_{i \leq j}^{[k]} V_{ji}, \quad E_k := \bigoplus_{i}^{[k]} V_{ki}. \quad (3.1)$$

We emphasize here that none of the direct summands $V_{ki}$ appearing in the definition of $E_k$ reduces to $\{0\}$. By multiplication rules (2.4), it is clear that $V_k$ is a subclan of $V$, and $E_k$ is a two-sided ideal of $V_k$. Moreover, $e_k := \sum_i^{[k]} c_i$ is the unit element of $V_k$, where $\sum_i^{[k]}$ indicates that the summation is only over the indices $i$ in $N^{\text{out}}[k]$. We call $V_k$ the \textit{out-neighbor subclan} of $V$ corresponding to the vertex $k$.

For each $x \in V_k$ we define operators $\varphi_k(x)$ and $\psi_k(x)$ on $E_k$ by

$$\varphi_k(x)\eta := \eta \Delta x, \quad \psi_k(x)\eta := x \Delta \eta \quad (\eta \in E_k).$$

Clearly both $\varphi_k(e_k)$ and $\psi_k(e_k)$ are equal to the identity operator on $E_k$. In a completely parallel way as above, we set

$$V_k := \bigoplus_{i \leq j < k}^{[k]} V_{ji}, \quad E_k := \bigoplus_{i < k}^{[k]} V_{ki}.$$  

Then we have

$$V_k = V_k \oplus E_k \oplus \mathbb{R}c_k, \quad E_k = E_k \oplus \mathbb{R}c_k.$$
Moreover, $V(k)$ is a subclan of $V$, and we have

$$E(k) \triangle V(k) \subset E(k), \quad V(k) \triangle E(k) \subset E(k).$$

Then for each $x' \in V(k)$ we define operators $\varphi'(x')$ and $\psi'(x')$ on $E' := E(k)$ by

$$\varphi'(x') \eta' := \eta' \triangle x', \quad \psi'(x') \eta' := x' \triangle \eta' \quad (\eta' \in E').$$

We fix $k$ until the end of this section, and write every $x \in V[k]$ as

$$x = x' + \xi' + \lambda c_k \quad (x' \in V(k), \xi' \in E', \lambda \in \mathbb{R}).$$

Similarly we write every $\eta \in E[k]$ as $\eta = \eta' + \mu c_k$ ($\eta' \in E'$, $\mu \in \mathbb{R}$). We redefine an inner product $\langle \cdot | \cdot \rangle_0$ on $E[k]$ by putting

$$\langle \eta_1' + \mu_1 c_k | \eta_2' + \mu_2 c_k \rangle_0 := \langle \eta_1' | \eta_2' \rangle + 2^{-1} \mu_1 \mu_2 \langle c_k \rangle_0.$$

The adjoint $^tT$ of operators $T$ on $E[k]$ is always taken with respect to this inner product $\langle \cdot | \cdot \rangle_0$. The proof of the following proposition largely follows that of [16, Lemma 4.2] except that we have rewritten it by operators not by matrices.

**Proposition 3.1.** One has

$$\varphi[k](x) = \psi[k](x) + ^t(\psi[k](x)) - s_0(c_k)^{-1} \langle x | c_k \rangle I[k] \quad (x \in V[k]), \quad (3.2)$$

where $I[k]$ is the identity operator on $E[k]$.

**Proof.** Let $x = x' + \xi' + \lambda c_k \in V[k]$ and $\eta = \eta' + \mu c_k \in E[k]$. Since multiplication rules (2.4) tell us that $c_k \triangle x' = \eta' \triangle c_k = 0$ and that $\eta' \triangle \xi' \in V_{kk}$, we have

$$\varphi[k](x) \eta = \eta' \triangle x' + \eta' \triangle \xi' + \mu c_k \triangle \xi' + \lambda(\mu c_k)$$

$$= \varphi'(x') \eta' + s_0(c_k)^{-1} \langle \eta' | \xi' \rangle c_k + 2^{-1} \mu \xi' + \lambda(\mu c_k).$$

A similar computation shows

$$\psi[k](x) \eta = \psi'(x') \eta' + s_0(c_k)^{-1} \langle \xi' | \eta' \rangle c_k + 2^{-1} \lambda \eta' + \lambda(\mu c_k).$$

Hence, $\varphi[k](x)$ and $\psi[k](x)$ are written as the following operator matrices:

$$\varphi[k](x) = \begin{pmatrix} \varphi'(x') & 2^{-1} \xi' \\ s_0(c_k)^{-1} \langle \cdot | \xi' \rangle c_k & \lambda \end{pmatrix},$$

$$\psi[k](x) = \begin{pmatrix} \psi'(x') + 2^{-1} \lambda I' & 0 \\ s_0(c_k)^{-1} \langle \cdot | \xi' \rangle c_k & \lambda \end{pmatrix}, \quad (3.3)$$

where $I'$ denotes the identity operator on $E'$. Since

$$\langle 2^{-1} \xi' | \eta' \rangle_0 = 2^{-1} \langle \xi' | \eta' \rangle = \langle c_k | s_0(c_k)^{-1} \langle \eta' | \xi' \rangle c_k \rangle_0, \quad (3.4)$$

the upper right block of the operator matrix of $^t(\psi[k](x))$ equals $2^{-1} \xi'$, and we see that (3.2) holds for $x = \xi' + \lambda c_k$. To see that (3.2) holds also for $x = x' \in V(k)$, we recall (C1) to get $[L(x'), L(\eta_1')] = L(\psi'(x') \eta_1' - \varphi'(x') \eta_1')$ for any $\eta_1' \in E'$. Apply this operator equality to
any $\eta_2' \in E'$. Then we obtain

$$x' \triangle (\eta_1' \triangle \eta_2') = \eta_1' \triangle \psi'(x') \eta_2' = (\psi'(x') \eta_1') \triangle \eta_2' - (\psi'(x') \eta_1') \triangle \eta_2'. $$

Applying the admissible linear form $s_0$, we are led to

$$\langle x' \mid \eta_1' \triangle \eta_2' \rangle - \langle \eta_1' \mid \psi'(x') \eta_2' \rangle = \langle \psi'(x') \eta_1' \mid \eta_2' \rangle - \langle \psi'(x') \eta_1' \mid \eta_2' \rangle.$$

Since $\eta_1' \triangle \eta_2' \in V_{kk}$, we have $\langle x' \mid \eta_1' \triangle \eta_2' \rangle = 0$. Then the remaining three inner products are all within $E'$, so that we arrive at $\varphi'(x') = \psi'(x') + \mathring{\psi}(\psi'(x))$. Since $\langle x' \mid c_k \rangle = 0$, we see that (3.2) holds also for $x = x'$. Now both of the maps $x \mapsto \varphi_{[k]}(x)$, $x \mapsto \psi_{[k]}(x)$ are linear, and we have finished the proof. \hfill $\square$

Proposition 3.1 shows, in particular, that every $\varphi_{[k]}(x)$ is a selfadjoint operator. Before proceeding further, we need the following lemma.

**Lemma 3.2.** For any $x \in V_{[k]}$, one has

$$\varphi_{[k]}(x) = \psi_{[k]}(x) - 2^{-1}s_0(c_k)^{-1}\langle x \mid c_k \rangle I_{[k]}.$$  \hfill (3.5)

**Proof.** Let $\eta = \sum_j^{[k]} \eta_j \in E_{[k]}$ with $\eta_j \in V_{kj}$. If $j \in N_{\text{out}}[k]$, then (2.9) gives

$$\varphi_{[k]}(c_j)\eta = 2^{-1}\varphi_{[k]}(c_j)\eta = 2^{-1}\eta_j.$$

On the other hand, we have

$$\psi_{[k]}(c_j)\eta = 2^{-1}\eta_j \quad (j \in N_{\text{out}}(k)), \quad \psi_{[k]}(c_k)\eta = 2^{-1}\eta + 2^{-1}\eta_k.$$

Thus, (3.5) holds for $x = c_j$ $(j \in N_{\text{out}}[k])$. Next suppose $x' \in V_{ji}$ for $i < j$ in $N_{\text{out}}(k)$. Then (2.9) yields

$$\varphi_{[k]}(x')\eta = \varphi_{[k]}(c_j)\varphi_{[k]}(x')\varphi_{[k]}(c_j)\varphi_{[k]}(x')\eta = \varphi_{[k]}(c_j)\varphi_{[k]}(x')\eta_i = \varphi_{[k]}(x')\eta_i,$$

where the last equality follows from $\varphi_{[k]}(x')\eta_i \in V_{kj}$ by (2.4). Now Corollary 2.3 tells us that $\varphi_{[k]}(x')\eta_i = \psi_{[k]}(x')\eta_i = \psi_{[k]}(x')\eta$. Hence, (3.5) holds also for $x = x'$. The case $x = \xi' \in V_{ki}$ $(i \in N_{\text{out}}(k))$ follows in the same way except that we use Lemma 2.1 instead of Corollary 2.3. Since both sides of (3.5) are linear in $x$, the proof is completed. \hfill $\square$

**Corollary 3.3.** One has $\varphi_{[k]}(x) = \varphi_{[k]}(x) + \mathring{\psi}(\varphi_{[k]}(x))$. Moreover, the operators $\varphi_{[k]}(x)$ $(x \in V_{[k]})$ are simultaneously represented as lower triangular matrices.

**Proof.** The formula follows immediately from Proposition 3.1 and Lemma 3.2. For the remaining assertion, recall that we are ordering the subspaces $V_{kj}$ in (2.1) so that the left multiplications are simultaneously represented as lower triangular matrices. Hence, so are the operators $\psi_{[k]}(x)$ $(x \in V_{[k]})$. \hfill $\square$

**Proposition 3.4.** The linear map $\varphi_{[k]} : x \mapsto \varphi_{[k]}(x)$ is a faithful selfadjoint representation of the clan $V_{[k]}$.

**Proof.** We begin the proof by recalling the following formula, a variant of (C1) (see formula (6) of Vinberg [23, p. 361]):

$$[L(x), R(y)] = R(x \triangle y) - R(y)R(x) \quad \text{for any} \ x, y \in V. \quad (3.6)$$
Then \((3.6)\) gives for \(x, y \in V[k]\)
\[
\varphi_{[k]}(x \triangle y) = \psi_{[k]}(x)\varphi_{[k]}(y) + \varphi_{[k]}(y)(\varphi_{[k]}(x) - \psi_{[k]}(x)).
\]

This is further rewritten by using Proposition 3.1 and \((3.5)\) as
\[
\varphi_{[k]}(x \triangle y) = \psi_{[k]}(x)\varphi_{[k]}(y) + \varphi_{[k]}(y)(\varphi_{[k]}(x)) - s_0(c_k)^{-1}(x \mid c_k)I_{[k]}) = \varphi_{[k]}(x)\varphi_{[k]}(y) + \varphi_{[k]}(y)(\varphi_{[k]}(x)).
\]  
\[ (3.7) \]

Since we know already that each \(\varphi_{[k]}(x)\) is a selfadjoint operator, \((3.7)\) implies that \(\varphi_{[k]}\) is a selfadjoint representation of the clan \(V[k]\).

To show that \(\varphi_{[k]}\) is faithful we suppose that \(\varphi_{[k]}(x)\) is the zero operator for \(x = x' + \xi' + \lambda c_k\). Then by \((3.3)\), we have \(\xi' = 0, \lambda = 0\) and \(\varphi'(x') = 0\). Hence, for any \(\eta' \in E'\), we obtain \(\eta' \triangle x' = 0\). Let us write
\[
x' = \sum_{j<k}^{[k]} \lambda'_{j} c_j + \sum_{a<j<k}^{[k]} x'_{ja} \quad (\lambda'_{j} \in \mathbb{R}, x'_{ja} \in V_{ja}).
\]

Fix arbitrary \(i \in N^{\text{out}}(k)\), and choose non-zero \(\eta' \in V_{ki}\). Then, we obtain
\[
0 = \eta' \triangle x' = \lambda'_{i} \eta' + \sum_{a<i}^{[k]} \eta' \triangle x'_{ia} + \sum_{k>j>i}^{[k]} \eta' \triangle x'_{ji}
\]
\[ \in V_{ki} + \bigoplus_{a<i}^{[k]} V_{ka} + \bigoplus_{k>j>i}^{[k]} V_{kj}. \]  
\[ (3.8) \]

Hence, \(\lambda'_{i} = 0\). Moreover, the equation \(\eta' \triangle x'_{ia} = 0\) together with \((2.6)\) yields \(x'_{ia} = 0\). This discussion is valid for any \(a < i\) such that \(a \in N^{\text{out}}(k)\). Letting \(i\) run over \(N^{\text{out}}(k)\), we conclude that \(x' = 0\).

Based on the above computations, we can express \(\varphi_{[k]}(x)\) in a more detailed form. For this purpose, it is convenient to suppose that \(s_0 = 2^{-1}\)-normalized. Thus, for any \(k\), we have \(\|c_k\| = 2^{-1/2}\) in \(V\), whereas \(\|c_k\|_0 = 2^{-1}\) in \(E_{[k]}\). We put \(c'_k := 2c_k\), so that \(c'_k\) is a unit vector in \(E_{[k]}\). By \((3.3)\) and \((3.4)\) it holds that for \(\xi' \in E'\)
\[
\langle \varphi_{[k]}(\xi')c'_k \mid \eta' \rangle_0 = \langle \xi' \mid \eta' \rangle \quad (\eta' \in E').
\]  
\[ (3.9) \]

Moreover, \((3.8)\) tells us that if we enumerate \(N^{\text{out}}[k]\) as
\[
N^{\text{out}}[k] : \; i_1 < \cdots < i_{t-1} < i_t = k,
\]  
\[ (3.10) \]
then for \(x' = \sum_{j=1}^t \lambda'_{ij} c_{ij} + \sum_{a<j<i} x'_{ij} a\) we have
\[
\varphi'(x') = \begin{pmatrix}
\lambda'_{i_1} I_{ki_1} & (\cdot) \triangle x'_{i_2 i_1} & \cdots & (\cdot) \triangle x'_{i_{t-1} i_1} \\
(\cdot) \triangle x'_{i_2 i_1} & \lambda'_{i_2} I_{ki_2} & \cdots & \vdots \\
\vdots & \ddots & \ddots & (\cdot) \triangle x'_{i_{t-1} i_{t-2}} \\
(\cdot) \triangle x'_{i_{t-1} i_{t-2}} & \cdots & (\cdot) \triangle x'_{i_{t-1} i_{t-2}} & \lambda'_{i_{t-1}} I_{ki_{t-1}}
\end{pmatrix},
\]  
\[ (3.11) \]
where \(I_{ki_j}\) denotes the identity operator on \(V_{ki_j}\). It is worth noting here that Lemma 2.2 also guarantees that if \(l < m\), the adjoint of the operator \((\cdot) \triangle x'_{in_i l} : V_{kin} \rightarrow V_{ki_l}\) is equal to
\( \triangle x_i' : V_{k_i} \rightarrow V_{k_i} \). Finally, we have
\[
(\varphi_{[k]}(\lambda c_k)c'_k | c'_k)_{0} = \lambda.
\] (3.12)

The formulae (3.9), (3.11) and (3.12) provide a matrix expression of \( \varphi_{[k]}(x) \) with respect to an orthonormal basis of \( E_{[k]} \).

We now restore \( s_0 \) to a general one. Let \( h_{[k]} \) be the split solvable Lie algebra consisting of left multiplication operators \( L_{[k]}(x) \) of the clan \( V_{[k]} \). The corresponding connected and simply connected Lie group will be denoted by \( H_{[k]} \). Let \( \Omega_{[k]} \) be the homogenous cone corresponding to the clan \( V_{[k]} \). It is the \( H_{[k]} \)-orbit of \( e_{[k]} \) in \( V_{[k]} \). We call \( \Omega_{[k]} \) the out-neighbor homogeneous cone corresponding to the vertex \( k \). Out-neighbor homogeneous cones have simple descriptions.

**Theorem 3.5.** Suppose \( x \in V_{[k]} \). Then, \( x \in \Omega_{[k]} \) if and only if the selfadjoint operator \( \varphi_{[k]}(x) \) is positive-definite.

**Proof.** Exponentiating (3.7), we see for any \( x, y \in V_{[k]} \) that
\[
\varphi_{[k]}((\exp L_{[k]}(x))y) = (\exp \varphi_{[k]}(x))\varphi_{[k]}(y)\exp \varphi_{[k]}(x).
\] (3.13)

Putting \( y = e_{[k]} \), we see that \( \varphi_{[k]}(z) \) is positive-definite for any \( z \in \Omega_{[k]} \).

For the converse, we change the situation in the following way to avoid heavy and complicated notation. Namely, \( V_{[k]} \) is now \( V \) itself (that is, \( k = r \)), \( \Omega_{[k]} = \Omega \) and \( N_{\text{out}}[r] = \{1, 2, \ldots, r\} \). Thus, we have \( E = V_r \oplus \cdots \oplus V_r \) with none of \( V_r \) reducing to \( 0 \), and we write simply \( \varphi \) for \( \varphi_{[r]} \). This change makes no essential difference in the proof. With this simplification of the situation, we now recall the subclan \( V_{[j]} \) introduced in (2.14) for \( j = 1, \ldots, r \). Putting \( \mathbb{Z}^{(j)} := V_{j+1} \oplus \cdots \oplus V_r \) for \( j = 1, \ldots, r - 1 \), we write every element \( x \in V_{[j]} \) as
\[
x = \lambda_j c_j + \xi_x + x' \quad (\lambda_j \in \mathbb{R}, \, \xi_x \in \mathbb{Z}^{(j)}, \, x' \in V_{[j+1]}).
\]

Consider the two-sided ideal \( E_{[j]} := V_r \oplus \cdots \oplus V_r \) of \( V_{[j]} \), and for each \( x \in V_{[j]} \) put
\[
\varphi_{[j]}(x) := \eta \triangle x \quad (\eta \in E_{[j]}).
\]

Proceeding in a parallel way to the proof of Proposition 3.4, we see that \( (\varphi_{[j]}, E_{[j]}) \) is a faithful representation of \( V_{[j]} \). Evidently we have \( E_{[j]} = V_r \oplus E_{[j+1]} \), and according to this decomposition, the operator \( \varphi_{[j]}(x) \) is expressed as the following operator matrix in a similar way to (3.3):
\[
\varphi_{[j]}(x) = \begin{pmatrix}
\lambda_j I_{j} & \rho_j(\xi) \\
\text{tr}(\rho_j(\xi)) & \varphi_{[j+1]}(x')
\end{pmatrix} \quad (\xi := \xi_x),
\] (3.14)

where we have put \( \rho_j(\xi) \eta' = \eta' \triangle \xi \quad (\eta' \in E_{[j+1]}^{(j+1)}) \) so that \( \rho_j(\xi) : E_{[j+1]} \rightarrow V_{r_j} \), and \( I_{j} \) denotes the identity operator on \( V_{r_j} \). Moreover, we have used Lemma 2.2 and (2.4) to obtain \( \text{tr}(\rho_j(\xi)) \eta' = \eta' \triangle \xi \quad (\eta' \in V_{r_j}) \). Note that \( \varphi_{[1]}(x) \) for \( x \in V \) is nothing but \( \varphi(x) \) expressed in a different way of operator matrix from (3.3). Here we need the following two lemmas.

**Lemma 3.6.** For a given \( x \in V \), let \( x_{[j]} \in V_{[j]} \) be as in (2.15) inductively defined by (2.16) and (2.17). Then, with \( \xi_{[j]} := \xi_{x_{[j]}} \) one has
\[
x_{[j]}(\varphi_{[j+1]}((x_{[j]})') - \text{tr}(\rho_j(\xi_{[j]}))\rho_j(\xi_{[j]}) = \varphi_{[j+1]}(x_{[j+1]}).
\] (3.15)
Proof. The left-hand side of (3.15) is equal to
\[ \lambda_j^{[j]} \sum_{m \geq k > j} \phi^{[j+1]}(x_{mk}) = \left( \sum_{m > j} \lambda_j \phi_j(x_{mj}^{[j]}) \right) \left( \sum_{k > j} \rho_j(x_{kj}^{[j]}) \right). \] (3.16)

For the right-hand side of (3.15), we recall that \( \phi^{[j]} \) is a representation of the clan \( V^{[j]} \), and apply Lemma 2.9 for \( x = x_{mj}^{[j]} \) and \( y = x_{kj}^{[j]} \) (\( m \geq k > j \)). Since we have
\[
\phi^{[j]}(x_{mj}^{[j]}) = \begin{pmatrix} 0 \\ \lambda_j \phi_j(x_{mj}^{[j]}) \end{pmatrix},
\]
as can be shown in a similar way to Corollary 3.3, we obtain by looking at the lower-right block
\[
\phi^{[j+1]}(x_{mj}^{[j]} \Delta x_{kj}^{[j]}) = \lambda_j \phi_j(x_{mj}^{[j]}) \rho_j(x_{kj}^{[j]}), \tag{3.17}
\]
If further \( m = k > j \), Lemma 2.1 gives
\[
s_0(c_k)^{-1} \| x_{kj}^{[j]} \| \phi^{[j+1]}(c_k) = 2 \lambda_j \phi_j(x_{kj}^{[j]}) \rho_j(x_{kj}^{[j]}). \tag{3.18}
\]

Now we rewrite the right-hand-side of (3.15) by using (2.16) and (2.17):
\[
\phi^{[j+1]}(x_{mj}^{[j+1]}) = \lambda_j \sum_{k > j} \lambda_k \phi^{[j+1]}(c_k) - \frac{1}{2} \sum_{k > j} s_0(c_k)^{-1} \| x_{kj}^{[j]} \| \phi^{[j+1]}(c_k) + \lambda_j \sum_{m > k > j} \phi^{[j+1]}(x_{mk}) - \sum_{m > k > j} \phi^{[j+1]}(x_{mj}^{[j]} \Delta x_{kj}^{[j]})\].
\]

Clearly, the first and the third summations form the first term of (3.16), and by (3.17) and (3.18) the second together with the fourth yields the second term of (3.16).

\[ \square \]

**Lemma 3.7.** Let \( T = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \) be a real symmetric matrix with square matrices \( A \) and \( C \). Then, one has \( T \gg 0 \) if and only if \( A \gg 0 \) and \( C - BA^{-1}B \gg 0 \).

**Proof.** The statement follows immediately from
\[
T = \begin{pmatrix} I & O \\ BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & O \\ O & C - BA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ O & I \end{pmatrix},
\]
where \( O \) and \( I \) are the zero matrix and the identity matrix of appropriate sizes. The matrix \( C - BA^{-1}B \) is sometimes written as \( T/A \) and called the Schur complement of \( A \) in \( T \) in the literature (cf. the book [26] for example).

Now we return to the proof of Theorem 3.5, and suppose that \( x = \lambda_1 c_1 + \xi + x' \in V \) \( (\lambda_1 \in \mathbb{R}, \xi \in E, x' \in V^{[2]} \) satisfies \( \phi(x) = \phi^{[1]}(x) \gg 0 \). Then by Lemma 3.7 applied to (3.14) for \( j = 1 \) we see that
\[ \lambda_1 > 0, \quad \lambda_1 \phi^{[2]}(x') - \lambda_j(\phi(\xi)) \rho_j(\xi) \gg 0. \]

By Lemma 3.6, the latter condition is equivalent to \( \phi^{[2]}(x^{[2]}) \gg 0 \). Then applying Lemma 3.7 to (3.14) for \( j = 2 \) and \( x = x^{[2]} \), we have
\[ \lambda_2^{[2]} > 0, \quad \lambda_2^{[2]} \phi^{[3]}((x^{[2]})') - \lambda_j(\phi(\xi^{[2]})) \rho_j(\xi^{[2]}) \gg 0. \]
so that Lemma 3.6 says $\varphi^{[3]}(x^{[3]}) \gg 0$. Continuing this process, we obtain
\[
\lambda_1^{[1]} > 0, \quad \lambda_2^{[2]} > 0, \quad \ldots, \quad \lambda_r^{[r]} > 0.
\]
By (2.18) we obtain $D_j(x) > 0$ for $j = 1, \ldots, r$, which in turn yields $x \in \Omega$ by virtue of (2.19). Hence, the proof is completed. \hfill \Box

**Remark 3.8.** The inductive structure of representations like (3.14) appeared already in the paper Rothaus [22]. We have made his discussion more explicit by using the formulae (2.16) and (2.17).

Let $\text{Sym}(E_{[k]})$ denote the vector space of all selfadjoint operators on $E_{[k]}$. Enumerating $N^\text{out}[k]$ as in (3.10), we fix an orthonormal basis of $E_{[k]}$ as in Remark 2.8, so that $\text{Sym}(E_{[k]})$ is identified with $\text{Sym}(N_{[k]}, \mathbb{R})$, where
\[
N_{[k]} := \dim E_{[k]} = d_{ki_1} + \cdots + d_{ki_{t-1}} + 1. \tag{3.19}
\]
By Corollary 3.3 and Remark 2.8, the faithful representation $\varphi_{[k]}$ of the clan $V_{[k]}$ considered in Proposition 3.4 is an LSA isomorphism of $V_{[k]}$ onto $V_{[k]}^0 := \varphi_{[k]}(V_{[k]})$, a subspace of $\text{Sym}(E_{[k]})$. Thus, $\varphi_{[k]}$ is considered as an injective LSA homomorphism $V_{[k]} \to \text{Sym}(N_{[k]}, \mathbb{R})$. We will now show that $N_{[k]}$ is the minimal possible.

**PROPOSITION 3.9.** If $\Phi$ is an injective LSA homomorphism from $V_{[k]}$ to $\text{Sym}(N, \mathbb{R})$, then one has $N \geq N_{[k]}$.

**Proof.** By Lemma 2.6, we know that $\Phi(c_{i_1}), \ldots, \Phi(c_{i_t})$ form a complete orthogonal system of orthogonal projectors on $\mathbb{R}^N$. Put $F_{ij} := \Phi(c_{i_j})(\mathbb{R}^N)$, the range of $\Phi(c_{i_j})$. We have $\dim F_{ij} > 0$ for any $j$. Fix a non-zero vector $\eta_0 \in F_k$. We will show that for any $j = 1, \ldots, t - 1$, the linear map
\[
\Psi : V_{ki_j} \ni x \mapsto \Phi(x)^{ki_j} \eta_0 \in F_{ij}
\]
is injective, so that $d_{ki_j} \leq \dim F_{ij}$. This implies $N_{[k]} \leq N$.

Now Corollary 2.10 tells us that if $x \in V_{ki_j}$, then
\[
s_0(c_k)^{-1} \| x \|^2 \Phi(c_k) = 2 \Phi(x)^{ki_j} \Phi(x)). \tag{3.20}
\]
Applying this operator equality to the vector $\eta_0$, we arrive at
\[
s_0(c_k)^{-1} \| x \|^2 \eta_0 = 2 \Phi(x)^{ki_j} \Phi(x)) \eta_0.
\]
This clearly implies that the linear map $\Psi$ is injective. \hfill \Box

**Remark 3.10.** In the above proof, we did not touch any orthonormal bases of $\mathbb{R}^N$. The canonical orthonormal basis of $\mathbb{R}^N$, which defines the canonical clan structure of $\text{Sym}(N, \mathbb{R})$, may be incompatible with the orthonormal basis taken in the way of Remark 2.7 for $\Phi(V_{[k]})$. It is worth observing this subtlety here.

By (3.13), we see that $\varphi_{[k]}$ is equivariant, that is, $\varphi_{[k]}$ intertwines the action of $H_{[k]} = \exp L_{[k]}(V_{[k]})$ on $V_{[k]}$ and the action of $H_{[k]}^0 := \exp L(V_{[k]}^0)$ on $V_{[k]}^0$. Summarizing all of the above, we obtain the following.
THEOREM 3.11. The linear map \( \varphi[k] \) is an equivariant linear embedding of \( \Omega[k] \) into \( \text{Sym}(E[k]) \), and minimal among such embeddings into real symmetric matrices.

We call \( \Omega^0_k := \varphi[k](\Omega[k]) \) the canonical minimal realization of \( \Omega[k] \) by real symmetric matrices.

4. Realization of homogeneous cones

The purpose of this section is to assemble out-neighbor homogeneous cones to realize the original \( \Omega \). To do so, we fix a vertex \( k \) of \( \Gamma = \Gamma(V) \) for a while and observe the relation between the \( H \)-action on \( V \) and the \( H[k] \)-action on \( V[k] \). Let \( \pi[k] \) be the orthogonal projector \( V \to V[k] \). Let us look closely at the action of \( \pi[k] \).

Proof. It is clear that \( \pi[k] \) leaves \( V[k] \) invariant and \( h[k]|_{V[k]} \in H[k] \).

PROPOSITION 4.1. Let \( h \in H \) and \( x \in V \). Then \( \pi[k](hx) = h[k](\pi[k](x)) \).

Proof. Let us look closely at the action of \( L_j := \sum_{k' > j} L(v_{k'j}) \) in (2.22) on \( V \). Suppose first \( k' \notin N^\text{out}[k] \) for the above summation variable \( k' \) in \( L_j \). Then, for any \( q \geq p \), the multiplication rules (2.4) tell us that \( v_{k'j} \Delta V_{qp} \subset (V[k])^\perp \), the orthogonal complement of \( V[k] \) in \( V \). Next we suppose \( k' \in N^\text{out}[k] \), and consider the case \( V_{k'j} \neq \{0\} \). Then we have the arcs \( k \to k' \) and \( k' \to j \) in \( \Gamma \), so that transitivity of \( \Gamma \) implies \( j \in N^\text{out}[k] \). In this case, (2.4) says that \( L(v_{k'j}) \) leaves both \( V[k] \) and \( (V[k])^\perp \) invariant. These observations lead us to \( \pi[k](L_j x) = 0 \) if \( j \notin N^\text{out}[k] \), whereas \( \pi[k](L_{jq} x) = L'_{iq}(\pi[k](x)) \) for \( s = 1, \ldots, t - 1 \). Exponentiation gives the formula in the proposition for \( h = \exp L_j (j = 1, \ldots, r - 1) \). Since the action of \( \exp T_j \) on \( V \) is diagonal for any \( j \) with respect to the normal decomposition (2.1), the proposition holds for general \( h \in H \).

We now study the relation between the basic relative invariants of \( \Omega \) and of the out-neighbor homogeneous cone \( \Omega[k] \). Let \( \Delta^V_1(x), \ldots, \Delta^V_t(x) \) be the basic relative invariants of \( \Omega[k] \). If \( i \in N^\text{out}[k] \), the basic relative invariant \( \Delta_i(x) \) of \( \Omega \) comes from one of \( \Delta^V_i \) as is shown by the following proposition.

PROPOSITION 4.2. Enumerate \( N^\text{out}[k] \) as in (3.10). Then \( \Delta_i(x) = \Delta^V_i(\pi[k](x)) \) for any \( x \in V \).

Before proving Proposition 4.2, we parametrize the one-dimensional representations \( \chi \) of \( H \). We know that such \( \chi \) is determined by the values on the abelian subgroup \( A := \exp \sum_{j=1}^r \mathbb{R}c_j \) of \( H \). Thus, for every \( \sigma = (\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r \), we put

\[
\chi_\sigma \left( \exp \sum_{j=1}^r t_j L(c_j) \right) = \exp(\sigma_1 t_1 + \cdots + \sigma_r t_r) \quad (t_1, \ldots, t_r \in \mathbb{R}).
\]
If $h$ is expressed as in (2.21), then we have $\chi_{\sigma}(h) = h_1^{2\alpha_1} \cdots h_r^{2\alpha_r}$. Now Ishi [14, Theorem 2.2(i)] tells us that if $\chi_{\sigma(i)}$ denotes the one-dimensional representation of $H$ corresponding to $\Delta_i(x)$, then we have $\sigma(i)_i = 1$ and $\sigma(i)_m = 0$ for $m = i + 1, \ldots, r$. We now prove Proposition 4.2.

**Proof.** Put $P(x) := \Delta_j^{[i]}(\pi_k(x))$ ($x \in V$). By Proposition 4.1, we see that $P$ is a relatively $H$-invariant polynomial function on $V$, and that the corresponding one-dimensional representation $\chi_{\sigma}$ of $H$ comes from that of $H[k]$. This together with the remark made just before the proof yields that there is $j$ ($1 \leq j \leq t$) such that $\sigma_i = 0$ if $i \neq p$ ($1 \leq p \leq j$), and $\sigma_{ij} = 1$. On the other hand, since $P(x)$ is irreducible, $P(x)$ must be equal to one of $\Delta_i(x)$, so that we have $\sigma = \sigma(i)$ for some $i$. This $i$ should coincide with $i_j$ by the same remark above. $\square$

We are now going to minimize the number of out-neighbor homogeneous cones to be assembled.

**Lemma 4.3.** Suppose that there is an arc $l \rightarrow k$. Then one has $V[l] \supset V[k]$. Moreover, if $\varphi_{\pi_{k}}(x) \gg \varphi_{\pi_{l}}(x)$ ($x \in V[l]$), then it holds that $\varphi_{\pi_{k}}(\pi_{\pi_{l}}(x)) \gg 0$.

**Proof.** Transitivity of $\Gamma$ tells us that if $i \in N^{out}(k)$, then it holds that $i \in N^{out}(l)$. The lemma follows from this immediately. $\square$

Lemma 4.3 suggests us that if we have a path $k = i_1 \rightarrow \cdots \rightarrow i_2 \rightarrow i_1$ in $\Gamma$ from $k$ to a sink $i_1$, then we trace it back to a source, though not necessarily uniquely determined. Therefore, we consider the set $S$ of sources of $\Gamma$. Note that since $r \in S$, we have $S \neq \emptyset$.

**Lemma 4.4.** Let $x \in V$. Then, $\varphi_{\pi_{\omega}}(\pi_{\pi_{\omega}}(x)) = 0$ for any $\omega \in S$ if and only if $x = 0$.

**Proof.** We express $x$ as in (2.3) and put $x_{jj} := \lambda_{j} c_{j}$ ($j = 1, \ldots, r$). Suppose that $V_{kj} \varphi x_{kj} \neq 0$ for some $j \leq k$. This implies $j \in N^{out}[k]$, and we choose $\omega \in S$ such that $\omega = \cdots \rightarrow k \rightarrow j$. Transitivity of $\Gamma$ implies $k, j \in N^{out}[\omega]$, and we have $x_{kj} \in V[\omega]$. Proposition 3.4 says that $\varphi_{\pi_{\omega}}(\pi_{\pi_{\omega}}(x_{kj})) \neq 0$, so that $\varphi_{\pi_{\omega}}(\pi_{\pi_{\omega}}(x)) \neq 0$. This shows the only if part of the lemma. Since the if part is trivial, the lemma follows. $\square$

**Remark 4.5.** (1) Let $I$ be a set of vertices in $\Gamma$. We say that $I$ satisfies Chua’s condition if for any vertex $i$, there exists $j \in I$ such that $j \geq i$ and $d_{ij} > 0$. This condition is considered in Chua [8, p. 504]. By transitivity of $\Gamma$, we see easily that $I$ satisfies Chua’s condition if and only if $I \supset S$.

(2) Consideration of the set of sources appears also in [4, p. 5] as the set $\wp$. However, their interests are in computing gamma functions and Riesz measures for homogeneous cones.

Let $\omega \in S$, and consider the corresponding out-neighbor homogeneous cone $\Omega_{\omega}$. We call $\Omega_{\omega}$ the source homogeneous cone for $\Omega$ corresponding to the source $\omega$. Accordingly we call $V[\omega]$ the source subclan of $V$ corresponding to $\omega$.

**Theorem 4.6.** Let $x \in V$. Then, $x \in \Omega$ if and only if $\pi_{\omega}(x) \in \Omega_{\omega}$, for any $\omega \in S$.

**Proof.** The necessity is clear from Proposition 4.2 and (2.20). For the sufficiency let $i$ be a vertex in $\Gamma$. Then either $i \in S$ or we can trace back from $i$ to some $\omega \in S$ as $\omega \rightarrow \cdots \rightarrow i$. In the latter case, transitivity of $\Gamma$ implies that all of the vertices in-between as well as $i$ are in $N^{out}[\omega] := \{i_1 < \cdots < i_u = \omega\}$. In any case, we have $i = i_j$ for some $j$ ($1 \leq j \leq u$). Then Proposition 4.2 tells us that $\Delta_i(x) = \Delta_j^{[\pi_{\omega}]}(\pi_{\pi_{\omega}}(x))$. Now the theorem follows from (2.20). $\square$
Remark 4.7. (1) Let \( I \) be a set of vertices in \( \Gamma \) satisfying Chua’s condition (see Remark 4.5). Chua’s description of \( \Omega \) in [8] corresponding to our Theorem 4.6 amounts to
\[
\Omega = \{ x \in V \mid \varphi_{i|j}(\pi_{i|j}(x)) \gg 0 \text{ for any } i \in I \},
\]
although unlike us he did not exclude beforehand the zero subspaces \( V_{i\alpha} \) (\( i < \alpha \)), if any, for \( i \in I \). Note that Lemma 4.4 and (1) of Remark 4.5 tell us that Chua’s condition is necessary to ensure that \( \varphi_{i|j}(\pi_{i|j}(x)) = 0 \) for any \( i \in I \) implies \( x = 0 \). However, Lemma 4.3 shows that the condition \( \varphi_{i|j}(\pi_{i|j}(x)) \gg 0 \) is redundant for any \( i \in I \setminus S \). In this sense, our description of \( \Omega \) in Theorem 4.6 is minimal.

(2) The easiest choice of \( I \) satisfying Chua’s condition is of course \( I = \{ 1, \ldots, r \} \). The descriptions of \( \Omega \) given in [16, Proposition 5.3] (or [15, Theorem 4]) and [25, Theorem 4.12] can be considered as a special case of Chua’s description for this choice of \( I = V \). As already said in (1) above, this choice gives rise to superfluous conditions except for the case where the arc set \( A \) is empty, that is, except for the trivial case where \( \Omega \) is the direct product \((\mathbb{R}_{>0})^r\) of \( r \) half-lines.

If \( S = \{ r \} \), a single element set, then by Theorem 4.6, the canonical minimal realization \( \Omega^0_{\{r\}} \) of the source cone \( \Omega_{\{r\}} \) is our realization of \( \Omega \). Before proceeding to the general case of \( S \), we recall here the following realization of the Vinberg cone \( \Omega^0_{\text{Vin}} \) given in Graczyk and Ishi [12, (3.6)]:
\[
\begin{bmatrix}
\lambda_1 & 0 & x_{21} & 0 \\
0 & \lambda_1 & 0 & x_{31} \\
x_{21} & 0 & \lambda_2 & 0 \\
0 & x_{31} & 0 & \lambda_3
\end{bmatrix}.
\]

One can obtain the realization (4.2) also by starting with the identical representation of the two-dimensional homogeneous cone \((\mathbb{R}_{>0})^2\) in the manner of Rothaus [22]. While it is nice in a certain respect to have this kind of realization by one-sheet positive-definite matrices for homogeneous cones as described in Section 3.2 of Graczyk and Ishi [12], its cost is that we have to treat bigger matrices than actually needed. This is already obvious for \( \Omega^0_{\text{Vin}} \) in (4.2) in contrast to the original description (1.2) of \( \Omega^0_{\text{Vin}} \). Also we observe many zeros in (4.2). We are going to show that these compensations are inevitable if one persists in a realization of \( \Omega \) by one-sheet positive-definite matrices even for the case where the cardinality of \( S \) is greater than 1.

Suppose therefore that we have \( \omega, \omega' \in S \) such that \( \omega > \omega' \). Then \( \omega' \notin N^{\text{out}}(\omega) \). Suppose moreover that there is a vertex \( i \notin S \) such that we have paths \( \omega \rightarrow \cdots \rightarrow i \) and \( \omega' \rightarrow \cdots \rightarrow i \). Transitivity of \( \Gamma \) says that \( i \in N^{\text{out}}(\omega) \cap N^{\text{out}}(\omega') \). In this situation, we consider an injective LSA homomorphism \( \Phi : V \rightarrow \text{Sym}(N, \mathbb{R}) \). As in the proof of Proposition 3.9, the operators \( \Phi(c_1), \ldots, \Phi(c_r) \) form a complete orthogonal system of orthogonal projectors acting on \( \mathbb{R}^N \). We put \( F_j := \Phi(c_j)(\mathbb{R}^N) \), the range of \( \Phi(c_j) \).

**Proposition 4.8.** One has \( \dim F_i \geq \dim F_\omega + \dim F_{\omega'} \).

**Proof.** Noting that \( V_{\omega} \neq \{ 0 \} \), we fix a vector \( x_{\omega} \in V_{\omega} \) such that \( \| x_{\omega} \|^2 = 2s_0(c_\omega) \). Similarly we fix \( x_{\omega'} \in V_{\omega'} \) such that \( \| x_{\omega'} \|^2 = 2s_0(c_{\omega'}) \). Let us put \( T_\omega := \Phi(x_{\omega}) \) and \( T_{\omega'} := \Phi(x_{\omega'}) \) for brevity. Then by (3.20) we have
\[
T_\omega^{-1}T_\omega = 2^{-1}s_0(c_k)^{-1}\| x_{\omega} \|^2 \Phi(c_\omega) = \Phi(c_\omega).
\]
This implies that $T_\omega$ is partially isometric with the initial subspace $F_\omega$ and the final subspace $M_\omega := T_\omega(F_\omega) \subset F_i$, see [18, V.2.2] for example. Hence, $T_\omega$ is also partially isometric with the initial subspace $M_\omega$ and the final subspace $\bar{F}_\omega$ (see [18]). In particular, Ker$(T_\omega) = (M_\omega)^\perp$.

Similarly $T_\omega'$ is partially isometric with the initial subspace $M_{\omega'} := T_\omega(F_{\omega'}) \subset F_i$ and the final subspace $F_{\omega'}$.

On the other hand, since $V_{\omega\omega'} = \{0\}$, it holds that $x_{o\delta} \triangle x_{o\delta i} = 0$. Application of $\Phi$ together with Lemma 4.3 gives $T_\omega T_\omega' + T_\omega' T_\omega = 0$. Since the initial subspaces $F_\omega$ and $F_{\omega'}$ of $T_\omega$ and $T_\omega'$, respectively, are orthogonal, we obtain both $T_\omega T_\omega' = 0$ and $T_\omega' T_\omega = 0$. The first equality implies Range $(T_\omega') \subset$ Ker $(T_\omega)$, that is, $M_{\omega'} \subset (M_\omega)^\perp$. Hence, $M_\omega \cap M_{\omega'} = \{0\}$, and thus

$$
\text{dim } F_i \geq \text{dim } M_\omega + \text{dim } M_{\omega'} = \text{dim } F_\omega + \text{dim } F_{\omega'}.
$$

The proof is now complete. \qed

Remark 4.9. The fact $M_\omega \perp M_{\omega'}$ in the proof of Proposition 4.8 also explains the existence of zeros in the (4.1) and (3.2) (hence (1.4) and (2.3)) entries in the realization (4.2) of $\Omega_{\text{Vin}}$. Furthermore, zero in the (2, 1) (hence (1, 2)) entry is unavoidable due to Proposition 4.8.

Instead of dealing with large matrices, we are going to make the size of matrices more adequate in our realizations of $\Omega$ by tying up some ‘components’ as is done in the original description (1.2) of $\Omega_{\text{Vin}}$. For distinct $\omega$, $\omega' \in S$, let us set

$$
\mathcal{J}(\omega, \omega') := N^{\text{out}}[\omega] \cap N^{\text{out}}[\omega'],
$$

and call it the junction set for $\omega$, $\omega'$. Elements of $\mathcal{J}(\omega, \omega')$ are called junctions for $\omega$, $\omega'$.

By the observation made just before Proposition 4.8, we see that if two paths leaving $\omega$ and $\omega'$ meet at $i \notin S$, then we have $i \in N^{\text{out}}[\omega] \cap N^{\text{out}}[\omega']$. Conversely if $i \in N^{\text{out}}[\omega] \cap N^{\text{out}}[\omega']$, then we have two arcs $\omega \to i$ and $\omega' \to i$. Let us consider the oriented subgraph $\Gamma_{\mathcal{J}(\omega, \omega')}$ of $\Gamma$ formed by $\mathcal{J}(\omega, \omega')$. The set of sources of $\Gamma_{\mathcal{J}(\omega, \omega')}$ is called the reduced junction set for $\omega$, $\omega'$, which we denote by $\mathcal{J}_0(\omega, \omega')$.

Proposition 4.10. We have $V[\omega] \cap V[\omega'] = \sum_{k \in \mathcal{J}_0(\omega, \omega')} V[k]$.

Remark 4.11. The right-hand side is just a sum of vector subspaces, not necessarily a direct sum. We will see a non-direct sum case in Example 4.18 below.

Proof. By Lemma 4.3, we see that $V[\omega] \cap V[\omega']$ contains the right-hand side subspace. To prove the converse, we note first that definition (3.1) of out-neighbor subclans and definition (4.3) of junction sets imply

$$
V[\omega] \cap V[\omega'] = \bigoplus_{i \in \mathcal{J}(\omega, \omega')} \mathbb{R}c_i \bigoplus_{j < k} \bigoplus_{j, k \in \mathcal{J}(\omega, \omega')} V_{kj}.
$$

If $V_{kj} \neq \{0\}$ for $j < k$ and $j, k \in \mathcal{J}(\omega, \omega')$, then we have $V_{kj} \subset V[k]$. If $k \notin \mathcal{J}_0(\omega, \omega')$, then it suffices to trace back from $k$ to some $l \in \mathcal{J}_0(\omega, \omega')$. Thus, we obtain $V_{kj} \subset V[l]$ by Lemma 4.3. Since $c_i \in V[i]$, the same discussion yields that for $i \in \mathcal{J}(\omega, \omega')$ we have $c_i \in V[j]$ for some $j \in \mathcal{J}_0(\omega, \omega')$. \qed

Remark 4.12. Equation (4.4) is the normal decomposition of the clan $V[\omega] \cap V[\omega']$. 

We write linear form \( s_0 \) is \( 2^{-1} \)-normalized. As a basis of \( V \), we take \( c_1, \ldots, c_r \) from \( V_{11}, \ldots, V_{rr} \), respectively, and orthonormal bases of \( V_{kj} \) for \( j < k \). We introduce the lexicographic order among \( V_{kj} \) \( (j < k) \), so that the left multiplication operators in the clan \( V \) are simultaneously lower triangular. We order the basis of \( V \) taken above in a compatible manner with this lexicographic order. Thus, we obtain a coordinate system in \( V \). As in (2.3), the coordinate for \( c_i \) is written by \( \lambda_i \) \( (i = 1, \ldots, r) \). For every vertex \( k \), the out-neighbor subclan \( V_{[k]} \) is now mapped by \( \varphi_{[k]} \) bijectively to a subclan \( V_{[k]}^0 \) of real symmetric matrices \( \text{Sym}(N_{[k]}, \mathbb{R}) \), where \( N_{[k]} \) is as in (3.19).

We next suppose that distinct \( \omega, \omega' \in \mathcal{S} \) have junctions, that is, \( \mathcal{J}(\omega, \omega') \neq \emptyset \). We put \( V_{[\omega]}^0 := \varphi_{[\omega]}(V_{[\omega]}) \) and similarly \( V_{[\omega']}^0 := \varphi_{[\omega']}(V_{[\omega']}) \). Let us consider the outer direct sum vector space \( V_{[\omega]}^0 \oplus V_{[\omega']}^0 \). Put \( W := V_{[\omega]} \cap V_{[\omega']} \) for simplicity. We consider the following vector subspace of \( V_{[\omega]}^0 \oplus V_{[\omega']}^0 \):

\[
\{(\varphi_{[\omega]}(w + x), \varphi_{[\omega']}((w + y)) \mid w \in W, x \in V_{[\omega]} \cap W_\perp, y \in V_{[\omega']} \cap W_\perp \}. \tag{4.5}
\]

We denote this subspace by \([V_{[\omega]}^0, V_{[\omega']}^0]\), and call it the vector subspace of \( V_{[\omega]}^0 \oplus V_{[\omega']}^0 \) obtained by stapling \( V_{[\omega]}^0 \) and \( V_{[\omega']}^0 \). The stapling \([\varphi_{[\omega]}, \varphi_{[\omega']}]\) of two linear maps \( \varphi_{[\omega]} \) and \( \varphi_{[\omega']} \) is obviously defined: it is a linear bijection from \( V_{[\omega]} \oplus V_{[\omega']} \) onto \([V_{[\omega]}^0, V_{[\omega']}^0]\) defined by

\[
[\varphi_{[\omega]}, \varphi_{[\omega']}](w + x + y) := (\varphi_{[\omega]}(w + x), \varphi_{[\omega']}((w + y)),
\]

where \( w, x, y \) are as in (4.5). We write this equality simply as

\[
[\varphi_{[\omega]}, \varphi_{[\omega']}](v + v') = [\varphi_{[\omega]}(v), \varphi_{[\omega']}((v')) \quad (v \in V_{[\omega]}, v' \in V_{[\omega']}).
\]

In the case \( \mathcal{J}(\omega, \omega') = \emptyset \), the vector space \([V_{[\omega]}^0, V_{[\omega']}^0]\) is nothing other than \( V_{[\omega]}^0 \oplus V_{[\omega']}^0 \).

Now let \( \mathcal{S} := \{\omega_1, \ldots, \omega_s\} \), and put \( V_{[\omega]}^0 := \varphi_{[\omega]}(V_{[\omega]}) \subset \text{Sym}(N_{[\omega]}, \mathbb{R}) \). We now staple these \( V_{[\omega]}^0 \). To do so, we first observe that (4.5) is rewritten as

\[
[V_{[\omega]}^0, V_{[\omega']}^0] = \{(X, Y) \in V_{[\omega]}^0 \oplus V_{[\omega']}^0 \mid \pi_{[\omega']} \circ \varphi^{-1}_{[\omega]}(X) = \pi_{[\omega]} \circ \varphi^{-1}_{[\omega']}(Y)\},
\]

where, for example, recall that \( \pi_{[\omega]} \) is the orthogonal projector \( V \to V_{[\omega]} \). Thus, considering the outer direct sum vector space \( V' := V_{[\omega_1]}^0 \oplus \cdots \oplus V_{[\omega_s]}^0 \), we now define \( V_{[\mathcal{S}]}^0 \) to be the vector subspace of \( V' \) consisting of the elements \( X = (X_1, \ldots, X_s) \) satisfying

\[
\pi_{[\omega_i]} \circ \varphi^{-1}_{[\omega_i]}(X_i) = \pi_{[\omega_j]} \circ \varphi^{-1}_{[\omega_j]}(X_j) \quad \text{for any} \ i \neq j.
\]

We write \( V_{[\mathcal{S}]}^0 \) as

\[
V_{[\mathcal{S}]}^0 = [V_{[\omega_1]}^0, \ldots, V_{[\omega_s]}^0]. \tag{4.6}
\]

Accordingly we define \( \varphi_{[\mathcal{S}]} := [\varphi_{[\omega_1]}, \ldots, \varphi_{[\omega_s]}] \). By Lemma 4.4, the linear map \( \varphi_{[\mathcal{S}]} \) gives a linear isomorphism \( V \to V_{[\mathcal{S}]}^0 \), and we have \( \dim V_{[\mathcal{S}]}^0 = \dim V \). In fact, since \( V = \sum_{i=1}^s V_{[\omega_i]} \), the dimension counting corresponds to the following well-known formula:

\[
\dim V = \sum_{p=1}^s (-1)^{p-1} \sum_{1 \leq t_1 < \cdots < t_p \leq s} \dim(V_{[\omega_{t_1}]} \cap \cdots \cap V_{[\omega_{t_p}]}). \]
It is clear that $V^0_{\{\omega\}_{\{\omega\}'}}$ has a natural LSA structure with unit element. Similarly by Remark 4.12, we have for distinct $\omega, \omega' \in S$

$$V_{\omega} \cap V_{\omega'} \cong V^0_{\{\omega\}_{\{\omega\}'}} := [V^0_{j_1}, \ldots, V^0_{j_t}],$$

(4.7)

where $\mathcal{J}_0(\omega, \omega')$ is enumerated as $j_1 < \cdots < j_t$. Using the terminology introduced so far, we say that $V^0_{\{\omega\}_{\{\omega\}'}}$ is assembled from $V^0_{\{\omega\}_1}, \ldots, V^0_{\{\omega\}_s}$ by stapling at $V^0_{\{\mathcal{J}_0(\omega, \omega')\}}$ for any pair of distinct sources $\omega_i, \omega_j$.

On the other hand, by Theorems 3.5 and 4.6, the original homogeneous cone $\Omega$ is described as

$$\Omega = \{ x \in V \mid \varphi_{\{\omega\}}(\pi_{\{\omega\}}(x)) \gg 0 \} \quad (j = 1, \ldots, s).$$

(4.8)

Let $\Omega_{\{\omega\}_1}, \ldots, \Omega_{\{\omega\}_s}$ be the source homogeneous cones for $\Omega$, and take their canonical minimal realizations $\Omega^0_{\{\omega\}_j}$ given in Theorem 3.11:

$$\Omega^0_{\{\omega\}_j} := \varphi_{\{\omega\}}(\Omega_{\{\omega\}_j}) \subset V^0_{\{\omega\}_j} \subset \text{Sym}(\mathcal{N}_{\{\omega\}_j}, \mathbb{R}).$$

Each $\Omega^0_{\{\omega\}_j}$ is the homogeneous cone of positive-definite matrices in $V^0_{\{\omega\}_j}$.

We staple $\Omega^0_{\{\omega\}_1}, \ldots, \Omega^0_{\{\omega\}_s}$ according to the stapling of their ambient vector spaces (4.6), and obtain

$$\Omega^0_{\{\omega\}_j} := [\Omega^0_{\{\omega\}_1}, \ldots, \Omega^0_{\{\omega\}_s}].$$

(4.9)

We also staple the simply transitive matrix groups $H^0_{\{\omega\}_j} := \exp L(V^0_{\{\omega\}_j})$ following (4.6), and obtain a split solvable Lie group

$$H^0_{\{\omega\}_j} := [H^0_{\{\omega\}_1}, \ldots, H^0_{\{\omega\}_s}].$$

Thanks to Proposition 4.1, the action of $H^0_{\{\omega\}_j}$ on $\Omega^0_{\{\omega\}_j}$ is the extension compatible with the stapling process (4.9) of each action of $H^0_{\{\omega\}_j}$ on $\Omega^0_{\{\omega\}_j}$.

We now arrive at the following theorem.

**THEOREM 4.13.** One has an equivariant linear equivalence $\Omega \cong \Omega^0_{\{\omega\}_j}$.

**Proof.** It is clear that by restriction $\varphi_{\{\omega\}_j}$ yields a bijection of $\Omega$ to $\Omega^0_{\{\omega\}_j}$. The fact that $\varphi_{\{\omega\}_j}$ is equivariant follows also from Proposition 4.1. □

We call $\Omega^0_{\{\omega\}_j}$ the realization of $\Omega$ by source homogeneous cones.

Let $\omega, \omega' \in S$ be distinct, and enumerate $\mathcal{J}_0(\omega, \omega')$ as in (4.7). We denote by $\Omega_{\{\omega\omega'\}}$ the homogeneous cone corresponding to the clan $V_{\{\omega\}} \cap V_{\{\omega'\}}$. Using the equivariance property of the orthogonal projector $\pi_{\{\omega\omega'\}} : V_{\{\omega\}} \rightarrow V_{\{\omega\}} \cap V_{\{\omega'\}}$ as in Proposition 4.1, we see that $\Omega_{\{\omega\omega'\}}$ coincides with $\pi_{\{\omega\omega'\}}(\Omega_{\{\omega\}})$, and with $\pi_{\{\omega'\omega\}}(\Omega_{\{\omega'\}})$. We have the following corollary.

**COROLLARY 4.14.** Putting $\Omega^0_{\{\mathcal{J}_0(\omega, \omega')\}} := [\Omega^0_{\{j_1\}}, \ldots, \Omega^0_{\{j_t\}}]$, one has

$$\Omega_{\{\omega\omega'\}} \cong \Omega^0_{\{\mathcal{J}_0(\omega, \omega')\}}.$$

Based on this corollary, we say that the homogeneous cone $\Omega^0_{\{\omega\}_j}$ in (4.9) is assembled from $\Omega^0_{\{\omega\}_1}, \ldots, \Omega^0_{\{\omega\}_s}$ by stapling at $\Omega^0_{\{\mathcal{J}_0(\omega, \omega')\}}$ with $\omega_i, \omega_j$ running over all distinct sources.
Remark 4.15. Here we would like to discuss the uniqueness of our realization of a given homogeneous cone $\Omega$ of rank $r$. We first note that the numbering of the complete system of orthogonal primitive idempotents $c_1, \ldots, c_r$ is not necessarily unique, and this leads to the non-uniqueness of the corresponding oriented graph $\Gamma$. Nevertheless, the set of the source cones for $\Omega$ remains the same. To see this, let $c'_1, \ldots, c'_r$ be another numbering of $c_1, \ldots, c_r$, and $V = \bigoplus_{i \leq j} V_{ji}$ the corresponding normal decomposition of $V$. We denote by $\Gamma'$ the oriented graph drawn by the data derived from $c'_1, \ldots, c'_r$. Then, Proposition 10 of Vinberg [23, p. 377] says that there is a permutation $\sigma \in S_r$ such that $c_i = c'_{\sigma(i)}$ holds for any $i = 1, \ldots, r$. In $\sigma$, inversions occur for the pairs $i < j$ such that $V_{ji} = V'_{\sigma(i)\sigma(j)} = \{0\}$. If a pair $i < j$ is not inverted by $\sigma$, we have $V_{ji} = V'_{\sigma(j)\sigma(i)}$. Hence, $\sigma$ is a source of $\Gamma$ if and only if $\sigma(\omega)$ is a source of $\Gamma'$. Then, the corresponding source cones $\Omega_{[\omega]}$ and $\Omega'_{[\sigma(\omega)]}$ are linearly equivalent. Thus, the set of source homogeneous cones is uniquely determined up to a linear equivalence, and in this sense our realization of $\Omega$ is unique.

Remark 4.16. Let $I$ be a set of vertices in $\Gamma$ satisfying Chua’s condition (see Remark 4.7). Chua’s realization in [8] is merely in the outer direct sum space $\bigoplus_{i \in I} V_{[i]}$, and does not take the actual overlapping parameters into consideration. With the above stapling process to gather up the overlaps in the direct sum vector space, our realization $\Omega^0_{[S]}$ of $\Omega$ in Theorem 4.13 is considered as the proper generalization of the original description (1.2) of $\Omega_{\text{Vin}}$ given in the introduction. Moreover, consideration of the source homogeneous cones through the oriented graph makes it clear beforehand where the parameters of $V_{k,j}$ appear in the realization.

The following example exhibits general features of Theorem 4.13.

Example 4.17. We consider the following direct sum vector space:

$$
V := \bigoplus_{1 \leq i \leq j \leq 5} V_{ji}, \quad \begin{cases}
V_{ji} := \mathbb{R} e_{ji}^1 \quad (j, i) \neq (2, 1), (5, 1), (5, 4), \\
V_{51} := \mathbb{R} e_{51}^1 \oplus \mathbb{R} e_{51}^2, \quad V_{21} = V_{54} = \{0\},
\end{cases}
$$

(4.10)

where $e_{ji}^p$ are basis vectors. Putting $c_j := e_{ji}^1 \quad (i = 1, \ldots, 5)$, we now introduce a clan structure in $V$ by which $c_1, \ldots, c_5$ form a complete system of orthogonal primitive idempotents, and (4.10) is the corresponding normal decomposition of $V$. We will take the $2^{-1}$-normalized admissible linear form $s_0$. By (2.2), it is enough to define products between the off-diagonal subspaces, and in view of (2.4) and Corollary 2.3, we define products between the basis vectors as follows:

$$
e_{kj} \triangle e_{ji}^1 = e_{ki}^1 \quad (i < j < k), \quad e_{kj}^p \triangle e_{kj}^q = 2\delta_{pq}c_k \quad (j < k),
$$

$$
e_{ji}^1 \triangle e_{ki}^1 = e_{ji}^1 \quad (j < k),
$$

(4.11)

where on the left-hand sides we only consider those $e_{ij}^p$ such that $V_{i,j} \neq \{0\}$, and we understand $e_{21}^1 = e_{34}^1 = 0$ if either of the two appears on the right-hand side of the third formula. Any product not mentioned above is equal to zero. We make two intervene in the second formula of (4.11), and by Lemma 2.1 this forces $\|e_{kj}^1\| = 1$ for any $j < k$ concerned, and $e_{51}^1, e_{51}^2$ are orthonormal. We also note the compatibility of (2.6) for $2^{-1}$-normalized $s_0$ with our multiplication rules (4.11). The reader may verify without difficulty that $V$ is actually a clan by using (C1)'. The weighted oriented graph $\Gamma$ of $V$ is the left end of the following three graphs:
The capacity of the arc $5 \to 1$ is $2$, and all other capacities not mentioned are equal to $1$. We have $S = \{4, 5\}$, and $N^{\text{out}}[4] = \{1, 2, 3, 4\}$, $N^{\text{out}}[5] = \{1, 2, 3, 5\}$. The weighted oriented subgraphs $\Gamma_{[4]}$ and $\Gamma_{[5]}$ formed by $N^{\text{out}}[4]$ and $N^{\text{out}}[5]$, respectively, are both of the type $S^5_4$ in the notation used by Kaneyuki and Tsuji [17, p. 14]. We note here that to translate what they call skeletons to our weighted oriented graphs, we first replace each vertex name $j$ with $r + 1 - j$ ($r$ is the order of the graph) in the skeleton, and we add arrows to the edges by obeying our ordering†. Let $\Omega$ be the homogeneous cone corresponding to $V$. The source homogeneous cones $\Omega_{[4]}$ and $\Omega_{[5]}$ are nine-dimensional and 10-dimensional, respectively, and their unique existence is guaranteed by [17]. According to (3.3), (3.9), (3.11) and (4.11), we have

$$\Omega^0_{[4]} = \begin{bmatrix} \lambda_1 & 0 & x_{31} & x_{41} \\ 0 & \lambda_2 & x_{32} & x_{42} \\ x_{31} & x_{32} & \lambda_3 & x_{43} \\ x_{41} & x_{42} & x_{43} & \lambda_4 \end{bmatrix}$$

and

$$\Omega^0_{[5]} = \begin{bmatrix} \lambda_1 I_2 & 0 & x_{31} e_1 & x_{51} \\ 0 & \lambda_2 & x_{32} & x_{52} \\ x_{31} e_1 & x_{32} & \lambda_3 & x_{53} \\ x_{51} & x_{52} & x_{53} & \lambda_5 \end{bmatrix}$$

(4.12)

where $I_2$ is the $2 \times 2$ identity matrix, $\mathbf{0}_2 \in \mathbb{R}^2$ is the column zero vector, $e_1 = \begin{pmatrix} 1 \end{pmatrix} \in \mathbb{R}^2$, and $x_{51} = \begin{pmatrix} x_{51}^1 \\ x_{51}^2 \end{pmatrix} \in \mathbb{R}^2$. Since $J(4, 5) = \{1, 2, 3\}$, and since $J_0(4, 5) = \{3\}$, we see that our realization $\Omega^0_{[4]}$ of $\Omega$ is the cone assembled from $\Omega^0_{[4]}$ and $\Omega^0_{[5]}$ by stapling at $\Omega^0_{[3]}$, that is, at the shaded blocks in (4.12). Note that the sizes of the shaded matrices are different in $\Omega^0_{[4]}$ and $\Omega^0_{[5]}$, although they have the same parameters. In $\Omega^0_{[4]}$ the shaded block is the canonical minimal realization of the dual Vinberg cone, whereas in $\Omega^0_{[5]}$ it is not. We also have to make the vector $e_1$ intervene in $\Omega^0_{[5]}$ to reflect the multiplication rules. Moreover, we observe paths in $\Gamma$ which leave distinct sources 4 and 5, meet at the vertex 3 and go to distinct sinks 1 and 2.

On the other hand, consider the following sets of lower triangular matrices:

$$H^0_{[4]} := \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ x_{31} & x_{32} & \lambda_3 & 0 \\ x_{41} & x_{42} & x_{43} & \lambda_4 \end{bmatrix}$$

and

$$H^0_{[5]} := \begin{bmatrix} \lambda_1 I_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \lambda_2 & \mathbf{0}_2 \\ x_{31} e_1 & x_{32} & \lambda_3 \\ x_{51} & x_{52} & x_{53} & \lambda_5 \end{bmatrix}$$

(4.13)

where $\{h_{ji}\}_{(h_{ji} > 0)}$} for lower triangular matrices $(h_{ji})$ denotes the set of matrices $(h_{ji})$ with the relevant off-diagonal parameters running over $\mathbb{R}$ while all the diagonal parameters

†We will give the definition of skeleton in accordance with our framework in Section 7.
h_{jj} being positive. We see easily that both $H^0_{[4]}$ and $H^0_{[5]}$ are groups by the ordinary matrix multiplication, and $H^0_{[4]}$ (respectively $H^0_{[5]}$) acts on $\Omega^0_{[4]}$ (respectively on $\Omega^0_{[5]}$) simply transitively by
\[ H^0_{[j]} \times \Omega^0_{[j]} \ni (h, X) \mapsto hXh^{-1} \in \Omega^0_{[j]} \quad (j = 4, 5). \]
Observe that the actions of $H^0_{[j]}$ on the shaded blocks in $\Omega^0_{[j]}$ shown in (4.12) are identical for $j = 4, 5$ as far as the parameters are concerned, and that the same is true for the group multiplications of $H^0_{[S]}$ of $\Omega^0_{[S]}$ is a subgroup of the direct product group $H^0_{[4]} \times H^0_{[5]}$ obtained by stapling the shaded blocks in (4.13):
\[ H^0_{[S]} = [H^0_{[4]}, H^0_{[5]}]. \]
Then, our realization $\Omega \cong \Omega^0_{[S]} = [\Omega^0_{[4]}, \Omega^0_{[5]}]$ intertwines the simply transitive actions of $H = \exp \mathfrak{h}$ and $H^0_{[S]}$.

**Example 4.18.** Here we present another example in which the original homogeneous cone is assembled from two homogenous cones by stapling at the Vinberg cone. Consider the following weighted oriented graph $\Gamma$, where the capacities not mentioned are equal to 1:

![Graph](image)

We have $S = \{4, 5\}$, $J(4, 5) = \{1, 2, 3\}$ and $J_0(4, 5) = \{2, 3\}$. If there is a clan $V$ such that $\Gamma(V) = \Gamma$, then Proposition 4.10 tells us that $V_{[4]} \cap V_{[5]} = V_{[2]} + V_{[3]}$. Hence $V_{[4]} \cap V_{[5]}$ is LSA isomorphic to the Vinberg clan $V_{\text{Vin}}$, the clan corresponding to $\Omega_{\text{Vin}}$. Therefore, the homogeneous cone $\Omega_{[45]}$ corresponding to the clan $V_{[4]} \cap V_{[5]}$ will be linearly equivalent to $\Omega_{\text{Vin}}$.

Let us now define a clan $V$ such that $\Gamma(V) = \Gamma$. We consider
\[ V = \bigoplus_{1 \leq i \leq j \leq 5} V_{ji}, \]
\[ V_{ji} := \mathbb{R}e_{ji}^1 \quad ((j, i) \neq (3, 2), (4, 1), (5, 1), (5, 4)), \]
\[ V_{k1} := \mathbb{R}e_{k1}^1 \oplus \mathbb{R}e_{k1}^2 \quad (k = 4, 5), \quad V_{54} = V_{32} = \{0\}, \]
where $e_{ji}^p$ are basis vectors. We put $c_i := e_{1i}^1$ ($i = 1, \ldots, 5$), and these $c_i$ will be a complete system of orthogonal primitive idempotents by which (4.14) is the corresponding normal decomposition. We will take the $2^{-1}$-normalized admissible linear form $\varpi_0$. By (2.2), it is sufficient to introduce products between the above basis vectors of the off-diagonal subspaces.
We see that \( S = \{3, 4\} \), and \( N_{\text{out}}[3] = \{1, 2, 3\} \), \( N_{\text{out}}[4] = \{1, 2, 4\} \). The capacities are indicated on each of the arcs. In the notation of [17], the weighted oriented graph \( \Gamma \) is of the type \( S_4^{k*} \), and both of the oriented subgraphs \( \Gamma_{[3]} \) and \( \Gamma_{[4]} \) formed by \( N_{\text{out}}[3] \) and \( N_{\text{out}}[4] \),
respectively, are of the type $S_3^2$. What we intend to do first is to define non-isomorphic clans $V$ and $W$ of rank four such that $\Gamma(V) = \Gamma(W) = \Gamma$.

First of all, let $V_{ii}$ be the one-dimensional vector spaces $\mathbb{R}e_{ii}^1$ ($i = 1, 2, 3, 4$). Let $d_{ji}$ be the capacity of the arc $j \to i$ indicated in $\Gamma$ above. For convenience we set $d_{ii} = 1$. Consider

$$V = \bigoplus_{1 \leq i \leq j \leq 4} V_{ji}, \quad \left\{ \begin{array}{l} V_{ji} = \mathbb{R}e_{ji}^p \quad ((j, i) \neq (4, 3)), \\ V_{43} = \{0\}, \end{array} \right.$$ (5.1)

where $e_{ji}^p$ are basis vectors. We will define an LSA product by which the vectors $e_{11}, \ldots, e_{44}$ will be a complete system orthogonal primitive idempotents and (5.1) will be the corresponding normal decomposition. We will take the $2^{-1}$-normalized admissible linear form $s_0$. Multiplications between the basis vectors $e_{ji}^p$ of the off-diagonal subspaces are defined as follows:

$$\begin{array}{c|c|c|c}
 e_{21}^1 & e_{21}^2 & e_{21}^3 \\
 e_{42}^1 & e_{41}^2 & e_{41}^3 \\
 e_{42}^1 & e_{41}^2 & -e_{41}^1 & e_{41}^4 \\
 e_{41}^3 & 0 & 0 & e_{42}^1 \\
 e_{41}^4 & 0 & 0 & e_{42}^2 \\

e_{21}^1 & e_{21}^2 & e_{21}^3 \\
 e_{32}^1 & e_{31}^2 & e_{31}^3 \\
 e_{32}^1 & e_{31}^2 & -e_{31}^1 & e_{31}^4 \\
 e_{31}^3 & 0 & 0 & e_{32}^1 \\
 e_{31}^4 & 0 & 0 & e_{32}^2 \\
\end{array}$$ (5.2)

The tables in (5.2) show, for example, $e_{42}^1 \Delta e_{21}^3 = e_{41}^3$. Next, Corollary 2.3 determines the following products from (5.2):

$$\begin{array}{c|c|c|c}
 e_{21}^1 & e_{21}^2 & e_{21}^3 \\
 e_{41}^1 & e_{42}^1 & -e_{42}^2 & 0 \\
 e_{42}^2 & e_{41}^2 & e_{41}^1 & 0 \\
 e_{41}^3 & 0 & 0 & e_{42}^1 \\
 e_{41}^4 & 0 & 0 & e_{42}^2 \\

e_{21}^1 & e_{21}^2 & e_{21}^3 \\
 e_{31}^1 & e_{32}^1 & -e_{32}^2 & 0 \\
 e_{32}^2 & e_{31}^2 & e_{31}^1 & 0 \\
 e_{31}^3 & 0 & 0 & e_{32}^1 \\
 e_{31}^4 & 0 & 0 & e_{32}^2 \\
\end{array}$$ (5.3)

In the tables in (5.3), the products are commutative by Corollary 2.3. For example, we have $e_{41}^1 \Delta e_{21}^3 = e_{21}^1 \Delta e_{41}^1 = e_{42}^1$. Moreover, $e_{41}^p \Delta e_{42}^q := 2\delta_{pq}e_{kk}^1$ for $j < k$ just as in (4.11), so that the basis vectors will be orthonormal. We leave it to the reader to verify that $V$ is actually a clan.

In a similar way, we define another clan $W$. We denote the basis vectors $f_{ii}^1$ and $f_{ji}^p$, and the corresponding $d_{ji}$-dimensional subspaces are written as $W_{ji}$. Non-obvious multiplications between basis vectors of the off-diagonal subspaces similar to $V$ are as follows:

$$\begin{array}{c|c|c|c}
 f_{21}^1 & f_{21}^2 & f_{21}^3 \\
 f_{42}^1 & f_{41}^2 & f_{41}^3 \\
 f_{42}^1 & f_{41}^2 & -f_{41}^1 & f_{41}^4 \\
 f_{42}^4 & 0 & 0 & f_{42}^4 \\
\end{array}$$
We necessarily have the following commutative products:

<table>
<thead>
<tr>
<th></th>
<th>$f_{21}^1$</th>
<th>$f_{21}^2$</th>
<th>$f_{21}^3$</th>
<th>$f_{21}^4$</th>
<th>$f_{21}^1$</th>
<th>$f_{21}^2$</th>
<th>$f_{21}^3$</th>
<th>$f_{21}^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{21}^1$</td>
<td>$f_{21}^1$</td>
<td>$-f_{21}^2$</td>
<td>$0$</td>
<td>$f_{21}^1$</td>
<td>$f_{21}^1$</td>
<td>$f_{21}^2$</td>
<td>$f_{21}^3$</td>
<td>$f_{21}^4$</td>
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With the $2^{-1}$-normalized admissible $s'_0 \in W^*$ one can show that $W$ is a clan.

**PROPOSITION 5.1.** We have that $V$ and $W$ are not isomorphic as LSAs.

**Proof.** We prove the proposition by contradiction. Thus, we suppose that there is a surjective LSA isomorphism $\varphi : V \rightarrow W$. Obviously $\varphi(e_{11}^1), \ldots, \varphi(e_{14}^4)$ form a complete orthogonal system of primitive idempotents in $W$, so that by Proposition 2.5 there is a permutation $\sigma \in \mathfrak{S}_4$ of four letters 1, 2, 3, 4 such that $\varphi(e_{ii}^1) = f_{\sigma(i)\sigma(i)}^1$ for all $i = 1, 2, 3, 4$. Then proceeding as in Proposition 10 of Vinberg [23, p. 377], we see that $\sigma$ satisfies:

(i) if $i \leq j$ and $\sigma(i) \leq \sigma(j)$, then we have $W_{ji} = \varphi(V_{\sigma(j)\sigma(i)})$;

(ii) if $i < j$ and $\sigma(i) > \sigma(j)$, then we have $W_{ji} = \{0\}, V_{\sigma(i)\sigma(j)} = \{0\}$.

Since no $W_{kj}$ other than $W_{43}$ reduces to $\{0\}$, we conclude that $(3, 4)$ is the only possible pair that admits the inversion in $\sigma$. Hence, $\sigma$ is the identity permutation $\iota$ or the transposition $\tau_{(3,4)}$ of 3, 4.

(a) The case $\sigma = \iota$. For $(k, j) = (2, 1), (4, 1), (3, 1)$, we extract two-dimensional subspaces $V_{kj}^2 := \mathbb{R}e_{kj}^1 \oplus \mathbb{R}e_{kj}^2$ of $V_{kj}$, and consider

$$V^2 := \left( \bigoplus_{i=1}^4 V_{ii} \right) \oplus V_{42} \oplus V_{32} \oplus V_{21}^2 \oplus V_{41}^2 \oplus V_{31}^2.$$  

By the multiplication tables (5.2) and (5.3), we see that $V^2$ is a subalgebra of $V$. Let $W' := \varphi(V^2)$. Then $W'$ is a subalgebra of $W$. Here we observe the following (1), (2) and (3) by (i) above:

(1) $\dim(W_{21} \cap W') = \dim(V_{21} \cap V^2) = 2$;
(2) $W_{42} \triangle (W_{21} \cap W') \subset W_{41} \cap W'$;
(3) $\dim(W_{41} \cap W') = \dim(V_{41} \cap V^2) = 2$.

Suppose here that $W_{21} \cap W'$ contains a vector $w$ of the form $w = af_{21}^1 + bf_{21}^2 + cf_{21}^3$ ($a, b \in \mathbb{R}$). Then (2) and (3) together with the multiplication tables imply that

$$W_{41} \cap W' = \mathbb{R}(af_{41}^1 + bf_{41}^2 + cf_{41}^3) \oplus \mathbb{R}(-bf_{41}^1 + af_{41}^2 + df_{41}^3).$$  

(5.4)

On the other hand, by (1) and by a subtraction of an appropriate scalar multiple of $w$, if necessary, we see that the subspace $W_{21} \cap W'$ contains also a non-zero vector of the form $cf_{21}^3 + df_{21}^3$. This together with (2) leads us to $W_{41} \cap W' \ni cf_{41}^3 + df_{41}^3 \neq 0$, which contradicts (5.4). Hence, we conclude that $W_{21} \cap W' = \mathbb{R}f_{21}^1 \oplus \mathbb{R}f_{21}^2$. A similar argument by considering $W_{21}, W_{31}$ and $W_{32}$ yields $W_{21} \cap W' = \mathbb{R}f_{21}^1 \oplus \mathbb{R}f_{21}^2$, which is not compatible with the preceding conclusion.
(b) The case $\sigma = \tau(3,4)$. In view of the multiplication tables, we can argue in quite a parallel way to the case (a) to arrive at a contradiction.

By (a) and (b), the proof is now complete. \hfill $\square$

Therefore, the homogeneous cones $\Omega^V$ and $\Omega^W$ corresponding to the clans $V$ and $W$ respectively are linearly inequivalent. On the other hand, let $\Omega^V_{[j]}$ and $\Omega^W_{[j]}$ ($j = 3, 4$) be the source homogeneous cones of $\Omega^V$ and $\Omega^W$, respectively. The corresponding source subclans are denoted by $V_{[j]}$ and $W_{[j]}$. By multiplication tables, $V_{[4]}$ and $W_{[4]}$ are evidently isomorphic as LSAs, so that $\Omega^V_{[4]}$ and $\Omega^W_{[4]}$ are linearly equivalent homogeneous cones. For $V_{[3]}$ and $W_{[3]}$, we consider the linear map $\Phi : V_{[3]} \to W_{[3]}$ such that

$$\Phi(e_2^2) = f_2^3, \quad \Phi(e_3^2) = f_2^3, \quad \Phi(e_{ij}^p) = f_{ij}^3 \quad (\text{otherwise}).$$

It is clear from the multiplication tables that $\Phi$ gives an LSA isomorphism of $V_{[3]}$ with $W_{[3]}$. Therefore, $\Omega^V_{[3]}$ and $\Omega^W_{[3]}$ are linearly equivalent. In conclusion, $\Omega^V$ and $\Omega^W$ are linearly inequivalent homogeneous cones such that the source homogeneous cones are all linearly equivalent. This example shows that it is not allowed to forget the above isomorphism $\Phi$ in assembling the source homogeneous cones.

6. Rank-three homogeneous cones

Let $\Omega$ be a homogeneous cone, and $V$ the corresponding clan. We denote by $\Gamma$ the weighted oriented graph of $V$. We say that $\Gamma$ is connected if the underlying undirected graph $G$ of $\Gamma$ is connected, that is, if for any distinct vertices $i$, $j$, there is a path $i = i_1 \circ o \cdot \cdot \cdot \circ o i_t = j$ connecting $i$ and $j$ in $G$. Asano [2, Theorem 4] tells us that $\Omega$ is irreducible if and only if $\Gamma$ is connected. In this section, homogeneous cones are all irreducible, and accordingly the corresponding digraphs are all connected.

The rank-one homogeneous cone is the half-line $\mathbb{R}_{>0}$, and the corresponding $\Gamma$ is just a one vertex. The only connected oriented graph of order 2 consists of two vertices 1, 2 and just one arc $2 \to 1$. The capacity of the arc $2 \to 1$ can be an arbitrary positive integer $n$, and the corresponding irreducible homogenous cone $\Omega$ is nothing other than the Lorentz cone $\Lambda_{n+2}$ of dimension $n + 2$. Theorem 4.13 gives the following realization of $\Lambda_{n+2}$. Let

$$A(\lambda, \mu, x; n) : = \begin{pmatrix} \lambda I_n & x \\ x & \mu \end{pmatrix} \quad (\lambda, \mu \in \mathbb{R}, \ x \in \mathbb{R}^n). \quad (6.1)$$

Then we have $\Lambda_{n+2} \cong \{ A(\lambda, \mu, x; n) \}_{++}.$

For irreducible homogeneous cones $\Omega$ of rank three, we have the following three types of $\Gamma$ in the notation of [17]:

![Diagram](image-url)
Realization of homogeneous cones

For $S_3^1$ and $S_3^2$, we have $S = \{3\}$, and no stapling is needed. Therefore, by Theorem 4.13, the canonical minimal realization $\Omega_{[3]}^0$ of the unique source homogeneous cone $\Omega_{[3]}$ provides our realization of $\Omega$ with $\Gamma(V)$ of the type $S_3^1$ or $S_3^2$. For instance, $\Omega \cong \Omega_{[3]}^0$ for the case $S_3^1$ is described as

$$\Omega \cong \{B(\lambda_1, \lambda_2, \lambda_3, x_{31}, x_{32}; d_{31}, d_{32})\}_{++}$$

(6.2)

by using the symmetric matrix $B(a, b, c, x, y; m, n)$ given by

$$B(a, b, c, x, y; m, n) := \begin{pmatrix} aI_m & O & x \\ 1O & bI_n & y \\ x & y & c \end{pmatrix},$$

(6.3)

where $a, b, c \in \mathbb{R}, x \in \mathbb{R}^m, y \in \mathbb{R}^n$, and moreover $I_m$ (respectively $I_n$) stands for the identity matrix of order $m$ (respectively of order $n$), $O$ is the $m \times n$ zero matrix. Actually the cone on the right-hand side of (6.2) is the generalized dual Vinberg cone, and the dual Vinberg cone corresponds to the case $d_{31} = d_{32} = 1$.

For the remaining case $S_3^1$, we have $S = \{3, 2\}$, and $\mathcal{J}(3, 2) = \mathcal{J}_0(3, 2) = \{1\}$. In this case, let the capacity of the arc $j \to 1$ be $d_{j1}$ ($j = 2, 3$). As noted in [23, p. 396], the positive integers $d_{j1}$ can be arbitrary. The realization $\Omega_{[3]}^0$ of $\Omega$ is assembled from two Lorentz cones $\Lambda_{d_{21}}$ and $\Lambda_{d_{31}}$ by stapling them at $\Omega_{[1]} \cong \mathbb{R}_{>0}$. Hence, with the matrices defined in (6.1), we obtain

$$\Omega_{[3]}^0 = \{A(\lambda_1, \lambda_2, x_{21}; d_{21})\}_{++}, \{A(\lambda_1, \lambda_3, x_{31}; d_{31})\}_{++}.$$

Note that $\Omega_{[3]}^0$ is the generalized Vinberg cone, and the case $d_{21} = d_{31} = 1$ gives the Vinberg cone $\Omega_{\text{Vin}}$. We remark that the right-hand side cone has appeared already in Geatti [10].

Our main objective in this section is to realize continuous one-parameter family of mutually linearly inequivalent homogeneous 11-dimensional cones such that the corresponding oriented graphs are all of the type $S_3^2$ with the capacities given by

$$d_{31} := c((3 \to 1)) = 4, \quad d_{32} := c((3 \to 2)) = 2, \quad d_{21} := c((2 \to 1)) = 2.$$  

(6.4)

The existence of these homogeneous cones has been already alluded to in Vinberg [23, p. 397]. Although the description at the $N$-algebra level can be found in Kaneyuki and Tsuji [17, Proposition 6.4], we present here our realization of the homogeneous cones following Theorem 4.13. Our description of the linear equivalence classes is given in a direct way of the clan language.

Let $\Gamma_0$ be the weighted oriented graph $S_3^2$ with the capacity function (6.4). We first suppose that there exists an 11-dimensional clan $V$ such that $\Gamma(V) = \Gamma_0$. Thus, with

$$V_{ii} := \mathbb{R}c_i (i = 1, 2, 3), \quad V_{kj} := \bigoplus_{p=1}^{d_{kj}} \mathbb{R}e_{kj}^p (1 \leq j < k \leq 3),$$

(6.5)

we have the corresponding normal decomposition $V = \bigoplus_{1 \leq j \leq k \leq 3} V_{kj}$. Moreover, we suppose that the admissible linear form $s_0$ is $2^{-1}$-normalized, and that the family of basis vectors $E := \bigcup_{1 \leq j < k \leq 3} \{e_{kj}^p \mid p = 1, \ldots, d_{kj}\}$ is orthonormal.
LEMMA 6.1. The quantity $|\langle e_{32}^1 \triangle e_{21}^1 | e_{32}^2 \triangle e_{21}^2 \rangle|$ is independent of the choice $E$ of orthonormal bases of the subspaces $V_{kj}$.

Proof. Let $F = \{ f^p_{kj} \}$ be another choice of orthonormal bases. Then, we can find orthogonal matrices $U := (a_{ij})$ and $U' := (a'_{ij})$ such that

$$ (f^1_{32}, f^2_{32}) = (e_{32}^1, e_{32}^2)U, \quad (f^1_{21}, f^2_{21}) = (e_{21}^1, e_{21}^2)U'. \quad (6.6) $$

We will compute $F := \langle f^p_{32} \triangle f^q_{21} | f^r_{32} \triangle f^s_{21} \rangle$ by using (6.6). The formulae (2.6) and (2.8) give us (recall that $s_0$ is $2^{-1}$-normalized)

$$ \langle e_{32}^p \triangle e_{21}^q | e_{32}^r \triangle e_{21}^s \rangle = \delta_{qs}, \quad \langle e_{32}^p \triangle e_{21}^q | e_{32}^r \triangle e_{21}^q \rangle = \delta_{pr}, \quad \langle e_{32}^p \triangle e_{21}^q | e_{32}^r \triangle e_{21}^1 \rangle = -\langle e_{32}^1 \triangle e_{21}^q | e_{32}^r \triangle e_{21}^1 \rangle. \quad (6.7) $$

By using these formulae, a straightforward computation leads us to

$$ F = (ab + cd)(a'b' + c'd') + (\det U)(\det U')(e_{32}^1 \triangle e_{21}^1 | e_{32}^2 \triangle e_{21}^2) $$

$$ = \pm (e_{32}^1 \triangle e_{21}^1 | e_{32}^2 \triangle e_{21}^2). $$

Hence, we obtain the lemma. \qed

Based on Lemma 6.1, we put

$$ \lambda_V := |\langle e_{32}^1 \triangle e_{21}^1 | e_{32}^2 \triangle e_{21}^2 \rangle| $$

for any clan $V$ with $2^{-1}$-normalized admissible linear form such that $\Gamma(V) = \Gamma_0$. Then, by Schwarz inequality and (2.6), we see that $0 \leq \lambda_V \leq 1$. Let $W$ be another such clan and suppose that $W$ is LSA isomorphic to $V$. Thus, we have an isometric LSA isomorphism $T : V \to W$. Let $W = \bigoplus_{j \leq k} W_{kj}$ be the normal decomposition of $W$. Since no $W_{kj}$ reduces to $\{0\}$, we necessarily have $W_{kj} = T(V_{kj})$ for any $1 \leq j \leq k \leq 3$ (cf. the discussion in the proof of Proposition 5.1). Then we have

$$ \lambda_V = |\langle T(e_{32}^1) \triangle T(e_{21}^1) | T(e_{32}^2) \triangle T(e_{21}^2) \rangle| = |\langle T(e_{32}^1) \triangle T(e_{21}^1) | T(e_{32}^2) \triangle T(e_{21}^2) \rangle|. $$

By Lemma 6.1, the last term is equal to $\lambda_W$, so that we obtain $\lambda_V = \lambda_W$. This implies that the map $\Lambda : V \mapsto \lambda_V$ factors through the LSA equivalence classes

$$ \mathcal{E} := \{ [V] | \Gamma(V) = \Gamma_0 \} $$

of such clans with the $2^{-1}$-normalized admissible linear forms.

PROPOSITION 6.2. The map $\Lambda$ induces a bijection from $\mathcal{E}$ onto the closed interval $[0, 1]$.

Proof. Let $\lambda \in [0, 1]$, and put $\mu := \sqrt{1 - \lambda^2}$. Supposing that we have vector spaces $V_{ii}$ and $V_{kj}$ as in (6.5), we shall define an LSA structure in $V = \bigoplus_{1 \leq j \leq k \leq 3} V_{kj}$ for this to be the corresponding normal decomposition. As before, it is enough to define products between the basis vectors of the off-diagonal subspaces. We first set

$$ e_{32}^1 \triangle e_{21}^1 = e_{31}^1, \quad e_{32}^1 \triangle e_{21}^2 = e_{31}^2, \quad e_{32}^2 \triangle e_{21}^1 = -\lambda e_{31}^2 + \mu e_{31}^4, \quad e_{32}^2 \triangle e_{21}^2 = \lambda e_{31}^2 + \mu e_{31}^4. \quad (6.8) $$

$$ e_{32}^1 \triangle e_{21}^1 = e_{31}^1, \quad e_{32}^1 \triangle e_{21}^2 = e_{31}^2, \quad e_{32}^2 \triangle e_{21}^1 = -\lambda e_{31}^2 + \mu e_{31}^4, \quad e_{32}^2 \triangle e_{21}^2 = \lambda e_{31}^2 + \mu e_{31}^4. \quad (6.9) $$


Corollary 2.3 determines the products $e^p_{31} \triangle e^q_{21} = e^r_{21} \triangle e^s_{31}$ as follows:

\[
\begin{array}{ccc|cc|cc|}
 & e^1_{31} & e^1_{32} & e^2_{31} & e^2_{32} & e^3_{31} & e^3_{32} \\
\hline
e^1_{31} & e^1_{32} & \lambda e^2_{32} & 0 & 0 & 0 & 0 \\
e^2_{31} & -\lambda e^2_{32} & e^1_{32} & 0 & 0 & 0 & 0 \\
e^3_{31} & 0 & \mu e^2_{32} & 0 & 0 & 0 & 0 \\
e^4_{31} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{6.10}
\]

Finally we set $e^p_{kj} \triangle e^q_{kj} = 2\delta_{pq}c_k$ for $(k, j) = (2, 1), (3, 1), (3, 2)$, so that the family $E$ of basis vectors $e_{kj}$ is orthonormal. The multiplications between basis vectors not mentioned so far are to be zero. Then with $2^{-1}$-normalized $s_0 \in V^*$, we see that $V$ is actually a clan such that $\Gamma(V) = \Gamma_0$. By definition we have $(e^1_{32} \triangle e^1_{21} | e^2_{32} \triangle e^2_{21}) = \lambda$, so that $\lambda_V = \lambda$. Hence, $\Lambda$ is surjective.

Suppose next that $V$ and $W$ are clans with $\Gamma(V) = \Gamma(W) = \Gamma_0$ such that $\lambda_V = \lambda_W = \lambda$. We may assume that $V$ is the clan defined above, so that we have the above multiplication rules for $E$. Let $W = \bigoplus_{1 \leq j \leq k \leq 3} W_{kj}$ be the normal decomposition of $W$, and take an orthonormal basis $\{f^p_{kj}\}_p$ of $W_{kj}$ $(1 \leq j < k \leq 3)$. We also assume that the admissible linear form on $W$ is $2^{-1}$-normalized. Now by interchanging $f^p_{21}$ and $f^q_{21}$ if necessary, we may further assume by the third formula of (6.7) with $e^p_{kj}$ replaced with $f^p_{kj}$ that $\lambda = (f^2_{31} \triangle f^1_{21} | f^2_{32} \triangle f^1_{21})$. We are now going to retake an orthonormal basis of $W_{31}$, so that we have the multiplication rules for the new orthonormal basis exactly the same as the rules for $\{e^p_{kj}\}$. By (2.6) and (2.8), we see that $f^1_{32} \triangle f^1_{21}$ and $f^1_{32} \triangle f^2_{21}$ are orthonormal vectors in $W_{31}$, so that we define, in view of (6.8),

\[g^1_{31} := f^1_{32} \triangle f^1_{21}, \quad g^2_{31} := f^1_{32} \triangle f^2_{21}.\]

For the same reason, $f^2_{32} \triangle f^2_{21}$ and $f^2_{32} \triangle f^1_{21}$ are orthonormal vectors in $W_{31}$. Moreover, we have by (2.8)

\[(f^2_{32} \triangle f^1_{21} | g^1_{31}) = (f^2_{32} \triangle f^1_{21} | f^1_{32} \triangle f^1_{21}) = 0.\]

A similar computation shows $(f^2_{32} \triangle f^2_{21} | g^2_{31}) = 0$. Finally we have by definition

\[(f^2_{32} \triangle f^1_{21} | g^1_{31}) = -\lambda, \quad (f^2_{32} \triangle f^2_{21} | g^2_{31}) = \lambda. \tag{6.11}\]

Suppose first that $\lambda = 1$. Then (6.11) and the equality condition of Schwarz inequality tell us that $f^2_{32} \triangle f^1_{21} = -g^1_{31}$ and $f^2_{32} \triangle f^2_{21} = g^1_{31}$. In this case we choose orthonormal $g^3_{31}, g^4_{31}$ from $(\mathbb{R}g^1_{31} \oplus \mathbb{R}g^2_{31})^\perp$ arbitrarily. Suppose next that $\lambda \neq 1$, and let $\mu := \sqrt{1 - \lambda^2}$ as before. Then as can be verified without difficulty by using the above formulae, two vectors

\[\tilde{g}^3_{31} := f^2_{32} \triangle f^1_{21} + \lambda g^2_{31}, \quad \tilde{g}^4_{31} := f^2_{32} \triangle f^2_{21} - \lambda g^1_{31}\]

are orthogonal in $(\mathbb{R}g^1_{31} \oplus \mathbb{R}g^2_{31})^\perp$ with norm $\mu > 0$. Hence, we put $g^p_{31} := \mu^{-1}\tilde{g}^p_{31}$ for $p = 3, 4$ in this case. Thus, we have obtained an orthonormal basis $\{g^p_{31}\}_{p=1}^4$ of $W_{31}$. Then, putting $g^p_{kj} := f^p_{kj}$ for $(k, j) = (2, 1), (3, 2)$ and $p = 1, 2$, we now obtain an orthonormal bases $\{g^p_{kj}\}$ of the off-diagonal subspaces $W_{kj}$ of $W$ for which the multiplication rules (6.8) and (6.9) with $e^p_{kj}$ replaced by $g^p_{kj}$ hold. Since the commutative products in (6.10) for $\{g^p_{kj}\}$ are uniquely determined from the products already described by means of Corollary 2.3, it is now clear that $W$ is LSA isomorphic to $V$. \[\square\]
By Theorem 4.13 and Proposition 6.2, we have the following realization of mutually linearly inequivalent 11-dimensional homogeneous cones \( \Omega_\lambda \) with parameter \( \lambda \in [0, 1] \):

\[
\Omega_\lambda \cong \begin{cases} 
1 \lambda_1 I_4 & X_\lambda(x_{21}) \\
1 \lambda_2 I_2 & x_{31} \\
1 \lambda_3 I_2 & x_{32} \\
1 \lambda_4 I_2 & t_3 
\end{cases} 
\]

where \( I_k \) is the \( k \times k \) identity matrix \( (k = 2, 4) \), \( x_{kj} \in \mathbb{R}^{dkj} \) (column vectors), and

\[
X_\lambda(x_{21}) := \begin{pmatrix}
\lambda x_{21}^2 & -\lambda x_{21}^1 \\
x_{21}^2 & 0 \\
\sqrt{1 - \lambda^2} x_{21}^1 & 0 \\
\sqrt{1 - \lambda^2} x_{21}^2 & 0
\end{pmatrix}
\]

for \( x_{21} = \begin{pmatrix} x_{21}^1 \\ x_{21}^2 \end{pmatrix} \).

Remark 6.3. In Kaneyuki and Tsuji [17, Proposition 6.4], we take the inner product \( \langle \cdot | \cdot \rangle_0 \) with respect to which the vectors \( \{ e_{ij}^p \}_{p=1}^2 \) for \( (i, j) = (1, 2), (2, 3) \), and \( \{ e_{12}^0, e_{23}^0 \}_{p=1}^4 \) are orthonormal. Then we have \( |\langle e_{12}^0 e_{12}^1 \rangle_0| = 3t \) in terms of their parameter \( t \in [0, \frac{1}{3}] \). Hence, we see that their parameter \( t \) coincides with \( 3 - \frac{1}{\lambda} \) for our parameter \( \lambda \in [0, 1] \) by changing their basis \( \{ e_{ij}^p \} \) appropriately (cf. the latter half of the proof of our Proposition 6.2).

7. Low-dimensional homogeneous cones

Irreducible homogeneous cones of dimension \( \leq 10 \) are classified by Kaneyuki and Tsuji [17]. They have made use of a kind of weighted graph, which they called a skeleton. Let us recall it here by using our terminology. A transitive weighted oriented graph \( \Gamma \) with the vertex set \( V = \{ 1, \ldots, r \} \) and the capacity function \( c_{kj} := c(k \rightarrow j) \) is called a skeleton in the sense* of Kaneyuki and Tsuji [17] if \( c \) is \( \mathbb{Z}_{>0} \)-valued and satisfies the following two conditions.

(S1) Suppose \( i < j < k \). If there is a path \( k \rightarrow j \rightarrow i \), then one has

\[
\max(c_{kj}, c_{ji}) \leq c_{ki}.
\]

(S2) Suppose \( i < j < k < l \). If there are two paths \( l \rightarrow k \rightarrow i \) and \( l \rightarrow j \rightarrow i \) with \( j \notin N_{\text{out}}(k) \), then one has

\[
c_{ij} \geq \max(c_{lk}, c_{ki}) + \max(c_{lj}, c_{ji}).
\]

Let \( V \) be a clan of rank \( r \) with the normal decomposition (2.1), and \( \Gamma \) the weighted oriented graph of \( V \). Then \( \Gamma \) satisfies (S1) by virtue of Proposition 2.4. The property (S2) reflects the following orthogonality relation which is behind the proof given by Kaneyuki and Tsuji [17, Lemma 3.1] treating the four cases separately in the \( N \)-algebra language.

Proposition 7.1. Suppose \( i < j < k < l \). If there are two paths \( l \rightarrow k \rightarrow i \) and \( l \rightarrow j \rightarrow i \) with \( j \notin N_{\text{out}}(k) \), then one has

\[
\langle V_{lk} \Delta V_{ki} | V_{lj} \Delta V_{ji} \rangle = \{ 0 \},
\]

where for example, \( V_{lk} \Delta V_{ki} \) denotes the subspace of \( V \) spanned by the vectors \( x \Delta y \) \( (x \in V_{lk}, y \in V_{ki}) \).

* A different use of the term skeleton is found in [21, 8.10].
Proof. Note that we have $V_{kj} = \{0\}$ by assumption. Lemma 2.2 shows
\[
(V_{ik} \triangle V_{ij} | V_{ij} \triangle V_{ji}) = (V_{ik} | V_{ki} \triangle (V_{lj} \triangle V_{ji})).
\] (7.1)
Here (C1) and multiplication rules (2.4) together with $V_{ki} \triangle V_{ji} \subset V_{kj} = \{0\}$ yield
\[
(V_{ki} \triangle (V_{lj} \triangle V_{ji}) \subset V_{lj} \triangle (V_{ki} \triangle V_{ji}) + (V_{ki} \triangle V_{lj} + V_{lj} \triangle V_{ki}) \triangle V_{ji} = \{0\}.
\]
Hence, the right-hand side of (7.1) equals $\{0\}$.

Since both of $V_{ik} \triangle V_{ki}$ and $V_{lj} \triangle V_{ji}$ are contained in $V_{ii}$ by (2.4), Proposition 7.1 implies
\[
\dim V_{ii} \geq \dim(V_{ik} \triangle V_{ki}) + \dim(V_{lj} \triangle V_{ji}) \\
\geq \max(\dim V_{ik}, \dim V_{ki}) + \max(\dim V_{lj}, \dim V_{ji}).
\]
where the last inequality follows from Proposition 2.4. Hence, we have (S2).

We see in the graph $\Gamma_{[4]}$ in Example 4.18 the situation in which the assumption of (S2) occurs, and therefore the capacity of the arc $4 \to 1$ there cannot be 1 by the conclusion of (S2).

The weighted oriented graph of the type $S_2^3$ appearing at the beginning of Section 6 with capacities $c_{21} = c_{31} = c_{32} = d$ ($d \neq 1, 2, 4, 8$) is clearly a skeleton, but it does not correspond to any clan due to Proposition 3 in Vinberg [24, p. 73]. Moreover, we have also seen in Section 6 that $S_3^2$ with capacities $c_{21} = c_{32} = 2$, $c_{31} = 4$ does correspond to continuously many mutually non-isomorphic clans. Therefore, the assignment $V \mapsto \Gamma(V)$ from clans to skeletons is neither surjective nor injective even if we consider appropriate isomorphism classes in both objects. Nevertheless, the skeletons work well in the classification of clans (hence of homogeneous cones) for dimension $\leq 10$, since the restriction of the map to $V$’s of $\dim V \leq 10$ is bijective up to isomorphisms. This is the crucial point of the paper Kaneyuki and Tsuji [17]. However, for dimensions 8, 9, 10, their presentation remained at the $N$-algebra level, and an actual description of the homogeneous cones of such dimensions was not given. Our way of realization of such homogeneous cones is clear from Theorem 4.13. But, since cones with two source cones stapled have already appeared in Examples 4.17 and 4.18 (though not dimension $\leq 10$), we will restrict ourselves to the cases where three or four source cones are stapled.

The weighted oriented graphs which concern us first are the following three types:

![Graphs](image)

The restrictions on $d_{kj}$ are due to the condition that the resulting clans are of dimension $\leq 10$, and the capacities not mentioned are equal to one. In each of the three cases, we have
$S = \{3, 4, 5\}$. Through the graphs, we see oriented subgraphs of the type $S^1_4$ that appeared at the beginning of Section 6 leaving some two sources. Let $\Omega$ be the homogeneous cone corresponding to the weighted oriented graph considered. Recall the matrices $A(\lambda, \mu, x; n)$ introduced in (6.1) and $B(a, b, c, x, y; m, n)$ in (6.3).

1. The case $S^2_5$. We have

$$\Omega^0_{[3]} = \{A(\lambda_1, \lambda_2, \lambda_3; d_{31})\}_{++}, \quad \Omega^0_{[4]} = \{A(\lambda_2, \lambda_4, x_{41}; 1)\}_{++},$$

$$\Omega^0_{[5]} = \{B(\lambda_1, \lambda_2, \lambda_5; x_{51}, x_{52}; d_{51}, 1)\}_{++}.$$ 

Hence, $\Omega \cong [\Omega^0_{[3]}, \Omega^0_{[4]}, \Omega^0_{[5]}]$. Since $J_0(3, 4) = \emptyset$, $J_0(3, 5) = \{1\}$, $J_0(4, 5) = \{2\}$, the staplings occur at $\{\lambda_1 > 0\}$ between $\Omega^0_{[3]}$ and $\Omega^0_{[5]}$, at $\{\lambda_2 > 0\}$ between $\Omega^0_{[4]}$ and $\Omega^0_{[5]}$ as indicated by the same letters in the parameters.

2. The case $S^3_5$. We have

$$\Omega^0_{[3]} = \{A(\lambda_1, \lambda_3, x_{31}; 1)\}_{++}, \quad \Omega^0_{[4]} = \{A(\lambda_1, \lambda_4, x_{41}; 1)\}_{++},$$

$$\Omega^0_{[5]} = \{B(\lambda_1, \lambda_2, \lambda_5; x_{51}, x_{52}; d_{51}, 1)\}_{++}.$$ 

Hence, $\Omega \cong [\Omega^0_{[3]}, \Omega^0_{[4]}, \Omega^0_{[5]}]$. Since $J_0(3, 4) = J_0(3, 5) = J_0(4, 5) = \{1\}$, the stapling occurs at $\{\lambda_1 > 0\}$ for all $\Omega^0_{[k]} (k = 3, 4, 5)$ after all.

3. The case $S^4$. We have

$$\Omega^0_{[3]} = \{A(\lambda_1, \lambda_3, x_{31}; 1)\}_{++}, \quad \Omega^0_{[4]} = \{B(\lambda_1, \lambda_2, \lambda_4, x_{41}, x_{42}; 1, 1)\}_{++},$$

$$\Omega^0_{[5]} = \{B(\lambda_1, \lambda_2, \lambda_5; x_{51}, x_{52}; 1, 1)\}_{++}.$$ 

Hence, $\Omega \cong [\Omega^0_{[3]}, \Omega^0_{[4]}, \Omega^0_{[5]}]$. Since $J_0(3, 4) = J_0(3, 5) = \{1\}$, $J_0(4, 5) = \{1, 2\}$, and since $V_{[1]} \cap V_{[2]} = \emptyset$, the staplings occur at $\{\lambda_1 > 0\}$ between $\Omega^0_{[3]}$ and $\Omega^0_{[4]}$ as well as between $\Omega^0_{[3]}$ and $\Omega^0_{[5]}$, and at

$$\Omega^0_{[4]} \cup J_0(4, 5) = [\Omega^0_{[1]}, \Omega^0_{[2]}] = \{\lambda_1 > 0\} \oplus \{\lambda_2 > 0\}$$

between $\Omega^0_{[4]}$ and $\Omega^0_{[5]}$.

We conclude this paper by presenting homogeneous cones of dimension $\leq 10$ which look like bunches of Lorentz cones. These cones can be also considered as generalizations of $\Omega_{Vin}$. The weighted oriented graphs concerned are the following:
As in the examples so far, the capacity not written means that it equals one.

(4) The case \( S_4^{1*} \). We have
\[
\Omega_{[k]}^0 = \{ A(\lambda_1, \lambda_k, x_{k1}; d_{k1}) \}_{++} \quad (k = 2, 3, 4),
\]
and \( \Omega \cong [\Omega_{[2]}^0, \Omega_{[3]}^0, \Omega_{[4]}^0, \Omega_{[4]}^0] \). The stapling occurs at \( \{ \lambda_1 > 0 \} \) for the three source homogeneous cones.

(5) The case \( S_3^{1*} \). We have
\[
\Omega_{[k]}^0 = \{ A(\lambda_1, \lambda_k, x_{k1}; 1) \}_{++} \quad (k = 2, 3, 4), \quad \Omega_{[5]}^0 = \{ A(\lambda_1, \lambda_5, x_{51}; d_{51}) \}_{++},
\]
and \( \Omega \cong [\Omega_{[2]}^0, \Omega_{[3]}^0, \Omega_{[4]}^0, \Omega_{[5]}^0] \). The stapling occurs at \( \{ \lambda_1 > 0 \} \) for the four source homogeneous cones after all.

(6) The case \( S_5^{1*} \). We have
\[
\Omega_{[k]}^0 = \{ A(\lambda_1, \lambda_k, x_{k1}; 1) \}_{++} \quad (k = 3, 4), \quad \Omega_{[5]}^0 = \begin{pmatrix}
\lambda_1 & x_{21} & x_{51} \\
x_{21} & \lambda_2 & x_{52} \\
x_{51} & x_{52} & \lambda_5
\end{pmatrix} ++,
\]
and \( \Omega \cong [\Omega_{[3]}^0, \Omega_{[4]}^0, \Omega_{[5]}^0] \). In this case also, the stapling occurs at \( \{ \lambda_1 > 0 \} \) for the three source homogeneous cones. We call this cone a bunch of positive-definite real symmetric matrices rather than of Lorentz cones. This applies also to \( S_5^{1*} \) with \( d_{51} = 1 \), and to \( S_4^{1*} \) with \( d_{21} = d_{31} = d_{41} = 1 \).

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