CONVOLUTION IDENTITIES FOR CAUCHY NUMBERS OF THE SECOND KIND

Takao KOMATSU

(Received 17 March 2014 and revised 30 June 2014)

Abstract. Let $\hat{c}_n$ be the n th Cauchy number of the second kind, defined by the generating function $x/(1+x) \ln(1+x) = \sum_{n=0}^{\infty} \hat{c}_n x^n/n!$. We obtain explicit expressions for $(\hat{c}_l + \hat{c}_m)^n := \sum_{j=0}^n \binom{n}{j} \hat{c}_{l+j} \hat{c}_{m+n-j}$ with arbitrary fixed integers $l, m \geq 0$. The corresponding expression for Bernoulli numbers has been studied by Agoh and Dilcher, and that for Cauchy numbers of the first kind by the author.

1. Introduction

The Cauchy numbers of the second kind $\hat{c}_n$ ($n \geq 0$) are defined by

$$\hat{c}_n = \int_0^1 (-x)(-x-1) \cdots (-x-n+1) dx$$

and the generating function of $\hat{c}_n$ is given by

$$\frac{x}{(1+x) \ln(1+x)} = \sum_{n=0}^{\infty} \hat{c}_n \frac{x^n}{n!} \quad (|x| < 1)$$

(see [4, 10]). Several initial values are

$$\begin{aligned} \hat{c}_0 = 1, \quad \hat{c}_1 = -\frac{1}{2}, \quad \hat{c}_2 = \frac{5}{6}, \quad \hat{c}_3 = -\frac{9}{4}, \quad \hat{c}_4 = \frac{251}{30}, \\ \hat{c}_5 = -\frac{475}{12}, \quad \hat{c}_6 = \frac{19\,087}{84}, \quad \hat{c}_7 = -\frac{36\,799}{24}. \end{aligned}$$

With the classical umbral calculus notation (see, e.g., [11]), define $(\hat{c}_l + \hat{c}_m)^n$ for $l, m, n \geq 0$ by

$$(\hat{c}_l + \hat{c}_m)^n := \sum_{j=0}^n \binom{n}{j} \hat{c}_{l+j} \hat{c}_{m+n-j}.$$

The similar concepts about the Bernoulli numbers B_n and the Cauchy numbers of the first kind have been studied by extensively Agoh and Dilcher [1, 2, 5] and the present author [9]. The Bernoulli numbers B_n ($n \geq 0$) are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).$$

2010 Mathematics Subject Classification: Primary 11B75; Secondary 05A15, 05A40.

Keywords: Cauchy numbers of the second kind; convolution identities; Bernoulli numbers; Cauchy numbers of the first kind.

If we define

$$(B_l + B_m)^n := \sum_{j=0}^n \binom{n}{j} B_{l+j} B_{m+n-j},$$

then the famous Euler's formula can be written as

$$(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n \quad (n \geq 1). \quad (1)$$

In [1] an expression for $(B_l + B_m)^n$ was found. In addition, some initial cases were listed, including, for example,

$$\begin{aligned} (B_0 + B_1)^n &= -\frac{1}{2}(n+1)B_n - \frac{1}{2}nB_{n+1}, \\ (B_0 + B_2)^n &= -\frac{1}{6}(n-1)B_n - \frac{1}{2}nB_{n+1} - \frac{1}{3}nB_{n+2}, \\ (B_1 + B_1)^n &= \frac{1}{6}(n-1)B_n - B_{n+1} - \frac{1}{6}(n+3)B_{n+2}. \end{aligned}$$

The Cauchy numbers of the first kind c_n ($n \geq 0$) are defined by

$$c_n = \int_0^1 x(x-1) \cdots (x-n+1) dx$$

and the generating function of c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \quad (|x| < 1)$$

(see [4, 10]). Note that the Cauchy numbers c_n can be expressed in terms of the unsigned Stirling numbers of the first kind and $\left[\begin{smallmatrix} n \\ i \end{smallmatrix} \right]$:

$$c_n = \sum_{i=0}^n \frac{(-1)^{n-i}}{i+1} \left[\begin{smallmatrix} n \\ i \end{smallmatrix} \right] \quad (2)$$

(see [8, 10]), where the unsigned Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ i \end{smallmatrix} \right]$ arise as coefficients of the rising factorial

$$x(x+1) \cdots (x+n-1) = \sum_{i=0}^n \left[\begin{smallmatrix} n \\ i \end{smallmatrix} \right] x^i$$

(see, e.g., [7]). If we define $(c_l + c_m)^n$ by

$$(c_l + c_m)^n := \sum_{j=0}^n \binom{n}{j} c_{l+j} c_{m+n-j},$$

an analogous formula to (1) is expressed by

$$(c_0 + c_0)^n = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \geq 0) \quad (3)$$

(see [12]). In [9], the present author gave a general explicit formula for $(c_l + c_m)^n$ ($n \geq 0$). Some initial cases include

$$(c_0 + c_1)^n = -\frac{1}{2}(n+1)(n-1)c_n - \frac{1}{2}nc_{n+1}, \quad (4)$$

$$(c_0 + c_2)^n = \frac{n!}{6} \sum_{k=0}^n \frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6}n(2n+1)c_{n+1} - \frac{1}{3}nc_{n+2}, \quad (5)$$

$$(c_1 + c_1)^n = -\frac{n!}{6} \sum_{k=0}^n \frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6}n(n+5)c_{n+1} - \frac{1}{6}(n+3)c_{n+2}. \quad (6)$$

In this paper we give expression formulae for $(\hat{c}_l + \hat{c}_m)^n$. Some of the initial cases are

$$(\hat{c}_0 + \hat{c}_0)^n = n! \sum_{k=0}^n (-1)^{n-k} \frac{\hat{c}_k}{k!} - n\hat{c}_n, \quad (7)$$

$$(\hat{c}_0 + \hat{c}_1)^n = -\frac{(n+1)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\hat{c}_k}{k!} - \frac{1}{2}n\hat{c}_{n+1}, \quad (8)$$

$$\begin{aligned} (\hat{c}_0 + \hat{c}_2)^n &= \frac{n!}{12} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} (2k(n+2k-2) + 5(n-k+2)(n-k+1))\hat{c}_k \\ &\quad - \frac{n}{3}\hat{c}_{n+2} - \frac{n}{2}\hat{c}_{n+1}, \end{aligned} \quad (9)$$

$$(\hat{c}_1 + \hat{c}_1)^n = \frac{n!}{12} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} ((n+1)(n+2) + k(8n-9k+19))\hat{c}_k - \hat{c}_{n+1} - \frac{n+3}{6}\hat{c}_{n+2}. \quad (10)$$

2. Fundamental results

Euler's identity (1) is an easy consequence of the formula

$$b(x)^2 = (1-x)b(x) - xb'(x),$$

where $b(x) = x/(e^x - 1)$ (see, e.g., [6]). Similarly, the identity (3) is obtained because $c(x) = x/\ln(1+x)$ satisfies the formula

$$c(x)^2 = (1+x)c(x) - (1+x)xc'(x). \quad (11)$$

Since $\hat{c}(x) = x/(1+x) \ln(1+x)$ satisfies the formula

$$\hat{c}(x)^2 = -x\hat{c}'(x) + \frac{1}{1+x}\hat{c}(x), \quad (12)$$

by $1/(1+x) = \sum_{i=0}^{\infty} (-1)^i x^i$ and the fact

$$x^i \hat{c}^{(v)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} \hat{c}_{n+v-i} \frac{x^n}{n!} \quad (i, v \geq 0), \quad (13)$$

we have

$$(\hat{c}_0 + \hat{c}_0)^n = -n\hat{c}_n + n! \sum_{k=0}^n (-1)^{n-k} \frac{\hat{c}_k}{k!} \quad (n \geq 0),$$

which is the formula (7). Differentiating both sides of (12) by x and dividing them by 2, we obtain

$$\hat{c}(x)\hat{c}'(x) = -\frac{1}{2}x\hat{c}''(x) - \frac{x}{2(1+x)}\hat{c}'(x) - \frac{1}{2(1+x)^2}\hat{c}(x). \quad (14)$$

Then by (13) and for $k \geq 1$,

$$\frac{1}{(1+x)^k} = \sum_{i=0}^{\infty} (-1)^i \binom{k+i-1}{i} x^i, \quad (15)$$

we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} (\hat{c}_0 + \hat{c}_1)^n \frac{x^n}{n!} &= -\frac{1}{2} \sum_{n=0}^{\infty} n \hat{c}_{n+1} \frac{x^n}{n!} - \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \sum_{n=0}^{\infty} \frac{n!}{(n-i-1)!} \hat{c}_{n-i} \frac{x^n}{n!} \\
 &\quad - \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i (i+1) \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} \hat{c}_{n-i} \frac{x^n}{n!} \\
 &= -\frac{1}{2} \sum_{n=0}^{\infty} n \hat{c}_{n+1} \frac{x^n}{n!} - \frac{1}{2} \sum_{i=0}^n (-1)^i \sum_{n=0}^{\infty} \frac{(n+1)!}{(n-i)!} \hat{c}_{n-i} \frac{x^n}{n!},
 \end{aligned}$$

yielding the formula (8).

In general, by differentiating both sides of (12) by x at μ times, we have the following.

PROPOSITION 1. For $\mu \geq 0$,

$$\sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \hat{c}^{(\kappa)}(x) \hat{c}^{(\mu-\kappa)}(x) = \sum_{v=0}^{\mu} (-1)^{\mu-v} \frac{\mu! \hat{c}^{(v)}(x)}{v!(1+x)^{\mu-v+1}} - \mu \hat{c}^{(\mu)}(x) - x \hat{c}^{(\mu+1)}(x).$$

Proof. The left-hand side is due to the general Leibniz's rule. The right-hand side can be proved by induction. If we differentiate the right-hand side by x , then

$$\begin{aligned}
 &\sum_{v=0}^{\mu} \left((-1)^{\mu-v+1} (\mu - v + 1) \frac{\mu! \hat{c}^{(v)}(x)}{v!(1+x)^{\mu-v+2}} + (-1)^{\mu-v} \frac{\mu! \hat{c}^{(v+1)}(x)}{v!(1+x)^{\mu-v+1}} \right) \\
 &\quad - \mu \hat{c}^{(\mu+1)}(x) - \hat{c}^{(\mu+1)}(x) - x \hat{c}^{(\mu+2)}(x) \\
 &= \sum_{v=0}^{\mu} (-1)^{\mu-v+1} \frac{(\mu+1)! \hat{c}^{(v)}(x)}{v!(1+x)^{\mu-v+2}} - \sum_{v=1}^{\mu} (-1)^{\mu-v+1} \frac{\mu! \hat{c}^{(v)}(x)}{(v-1)!(1+x)^{\mu-v+2}} \\
 &\quad + \sum_{v=1}^{\mu+1} (-1)^{\mu-v+1} \frac{\mu! \hat{c}^{(v)}(x)}{(v-1)!(1+x)^{\mu-v+2}} - (\mu+1) \hat{c}^{(\mu+1)}(x) - x \hat{c}^{(\mu+2)}(x) \\
 &= \sum_{v=0}^{\mu+1} (-1)^{\mu-v+1} \frac{(\mu+1)! \hat{c}^{(v)}(x)}{v!(1+x)^{\mu-v+2}} - (\mu+1) \hat{c}^{(\mu+1)}(x) - x \hat{c}^{(\mu+2)}(x). \quad \square
 \end{aligned}$$

By applying (15) and (13) in Proposition 1, we obtain the following result.

THEOREM 1. For $\mu \geq 0$ and $n \geq 0$ we have

$$\sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} (\hat{c}_{\kappa} + \hat{c}_{\mu-\kappa})^n = -(n+\mu) \hat{c}_{n+\mu} + \sum_{v=0}^{\mu} \frac{\mu! n!}{v!} \sum_{i=0}^n (-1)^{\mu-v+i} \binom{\mu-v+i}{i} \frac{\hat{c}_{n+v-i}}{(n-i)!}.$$

Examples. If we put $\mu = 0, 1$ in Theorem 1, we have the identities (7) and (8), respectively. If we put $\mu = 2, 3, 4$ in Theorem 1, we obtain

$$(\hat{c}_0 + \hat{c}_2)^n + (\hat{c}_1 + \hat{c}_1)^n = \frac{(n+2)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\hat{c}_k}{k!} - \frac{n+2}{2} \hat{c}_{n+1} - \frac{n+1}{2} \hat{c}_{n+2}, \quad (16)$$

$$\begin{aligned}
 &(\hat{c}_0 + \hat{c}_3)^n + 3(\hat{c}_1 + \hat{c}_2)^n \\
 &= -\frac{(n+3)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\hat{c}_k}{k!} + \frac{(n+2)(n+3)}{2} \hat{c}_{n+1} - \frac{n+3}{2} \hat{c}_{n+2} - \frac{n+2}{2} \hat{c}_{n+3}, \quad (17)
 \end{aligned}$$

$$\begin{aligned}
& (\hat{c}_0 + \hat{c}_4)^n + 4(\hat{c}_1 + \hat{c}_3)^n + 3(\hat{c}_2 + \hat{c}_2)^n \\
&= \frac{(n+4)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\hat{c}_k}{k!} - \frac{(n+2)(n+3)(n+4)}{2} \hat{c}_{n+1} + \frac{(n+3)(n+4)}{2} \hat{c}_{n+2} \\
&\quad - \frac{n+4}{2} \hat{c}_{n+3} - \frac{n+3}{2} \hat{c}_{n+4}.
\end{aligned} \tag{18}$$

Next, note that $\hat{c}(x)$ satisfies the identity

$$\hat{c}(x)\hat{c}''(x) = -\frac{x}{3}\hat{c}^{(3)}(x) - \frac{x}{2(1+x)}\hat{c}''(x) + \frac{x}{6(1+x)^2}\hat{c}'(x) + \frac{5}{6(1+x)^3}\hat{c}(x).$$

By (13) and (15)

$$\begin{aligned}
(\hat{c}_0 + \hat{c}_2)^n &= -\frac{n}{3}\hat{c}_{n+2} - \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \frac{n!}{(n-i-1)!} \hat{c}_{n-i+1} + \frac{1}{6} \sum_{i=0}^{\infty} (-1)^i \frac{(i+1)n!}{(n-i-1)!} \hat{c}_{n-i} \\
&\quad + \frac{5}{6} \sum_{i=0}^{\infty} (-1)^i \frac{(i+2)(i+1)}{2} \frac{n!}{(n-i)!} \hat{c}_{n-i} \\
&= -\frac{n}{3}\hat{c}_{n+2} - \frac{n}{2}\hat{c}_{n+1} + \frac{n!}{2} \sum_{i=0}^{n-2} \frac{(-1)^i}{(n-i-1)!} \hat{c}_{n-i} \\
&\quad + \frac{n!}{6} \sum_{i=0}^{n-1} \frac{(-1)^i (i+1)}{(n-i-1)!} \hat{c}_{n-i} + \frac{5n!}{12} \sum_{i=0}^{n-2} \frac{(-1)^i (i+2)(i+1)}{(n-i)!} \hat{c}_{n-i} \\
&= -\frac{n}{3}\hat{c}_{n+2} - \frac{n}{2}\hat{c}_{n+1} + \frac{n!}{12} \sum_{i=0}^n \frac{(-1)^i}{(n-i)!} \\
&\quad \times (6(n-i)(n-i-1) + 2(i+1)(n-i) + 5(i+2)(i+1)) \hat{c}_{n-i},
\end{aligned}$$

yielding the identity (9). In addition, by the identities (9) and (16), we have

$$\begin{aligned}
(\hat{c}_1 + \hat{c}_1)^n &= \frac{n!}{12} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} ((n+1)(n+2) + k(8n-9k+19)) \hat{c}_k - \hat{c}_{n+1} \\
&\quad - \frac{n+3}{6} \hat{c}_{n+2},
\end{aligned}$$

which is the identity (10).

PROPOSITION 2. For $m \geq 0$ we have

$$\begin{aligned}
\hat{c}(x)\hat{c}^{(m)}(x) &= -\frac{x}{m+1}\hat{c}^{(m+1)}(x) - \sum_{k=0}^{m-1} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{x}{(1+x)^{k+1}} \hat{c}^{(m-k)}(x) \\
&\quad + \frac{\hat{c}_m}{(1+x)^{m+1}} \hat{c}(x).
\end{aligned} \tag{19}$$

LEMMA 1. For $n \geq 0$ we have

$$\hat{c}^{(n)}(x) = \left(\frac{-1}{1+x} \right)^{n+1} \sum_{j=0}^n \frac{\hat{g}_{n,j}}{(\ln(1+x))^{j+1}},$$

where

$$\hat{g}_{n,j} = j! \left(n \begin{bmatrix} n \\ j+1 \end{bmatrix} - \begin{bmatrix} n \\ j \end{bmatrix} x \right).$$

Proof. Since $\hat{g}_{1,0} = 1$ and $\hat{g}_{1,1} = -x$, for $n = 1$ we have

$$\begin{aligned} \frac{d}{dx} \frac{x}{(1+x) \ln(1+x)} &= \frac{\ln(1+x) - x}{(1+x)^2 (\ln(1+x))^2} \\ &= \frac{1}{(1+x)^2} \left(\frac{\hat{g}_{1,0}}{\ln(1+x)} + \frac{\hat{g}_{1,1}}{(\ln(1+x))^2} \right). \end{aligned}$$

Assume that the result holds for n . Then

$$\begin{aligned} &\frac{d^{n+1}}{dx^{n+1}} \frac{x}{(1+x) \ln(1+x)} \\ &= (n+1) \left(\frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{\hat{g}_{n,j}}{(\ln(1+x))^{j+1}} \\ &\quad + \left(\frac{-1}{1+x} \right)^{n+1} \sum_{j=0}^n \frac{-j! \begin{bmatrix} n \\ j \end{bmatrix} (\ln(1+x))^{j+1} - \hat{g}_{n,j} (j+1) (\ln(1+x))^j (1/(1+x))}{(\ln(1+x))^{2j+2}} \\ &= n \left(\frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{\hat{g}_{n,j}}{(\ln(1+x))^{j+1}} + \left(\frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{\hat{g}_{n,j}}{(\ln(1+x))^{j+1}} \\ &\quad + \left(\frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{j! \begin{bmatrix} n \\ j \end{bmatrix} (1+x)}{(\ln(1+x))^{j+1}} + \left(\frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{\hat{g}_{n,j} (j+1)}{(\ln(1+x))^{j+2}} \\ &= \left(\frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{n \hat{g}_{n,j} + j! \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}}{(\ln(1+x))^{j+1}} + \left(\frac{-1}{1+x} \right)^{n+2} \sum_{j=1}^{n+1} \frac{j \hat{g}_{n,j-1}}{(\ln(1+x))^{j+1}} \\ &= \left(\frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^{n+1} \frac{\hat{g}_{n+1,j}}{(\ln(1+x))^{j+1}}. \quad \square \end{aligned}$$

LEMMA 2. For integers m and j with $m \geq j \geq 0$ we have

$$\hat{g}_{m+1,j} - (m+1) \hat{g}_{m,j-1} = \sum_{k=0}^{m-j} \binom{m+1}{k+1} (-1)^k c_{k+1} \hat{g}_{m-k,j}.$$

Proof. It is enough to prove

$$\begin{bmatrix} m+1 \\ j+1 \end{bmatrix} - \frac{m}{j} \begin{bmatrix} m \\ j \end{bmatrix} = \sum_{k=0}^{m-j} \binom{m+1}{k+1} (-1)^k c_{k+1} \frac{m-k}{m+1} \begin{bmatrix} m-k \\ j+1 \end{bmatrix} \quad (20)$$

and

$$\begin{bmatrix} m+1 \\ j \end{bmatrix} - \frac{m+1}{j} \begin{bmatrix} m \\ j-1 \end{bmatrix} = \sum_{k=0}^{m-j} \binom{m+1}{k+1} (-1)^k c_{k+1} \begin{bmatrix} m-k \\ j \end{bmatrix}. \quad (21)$$

The identity (20) holds because by (2) and [7, Table 241],

$$\begin{aligned}
 & \sum_{k=0}^{m-j+1} \binom{m+1}{k} (-1)^k c_k \frac{m-k+1}{m+1} \left[\begin{matrix} m-k+1 \\ j+1 \end{matrix} \right] \\
 &= \sum_{k=0}^{m-j+1} \binom{m+1}{k} (-1)^k \sum_{i=0}^k \frac{(-1)^{k-i}}{i+1} \left[\begin{matrix} k \\ i \end{matrix} \right] \frac{m-k+1}{m+1} \left[\begin{matrix} m-k+1 \\ j+1 \end{matrix} \right] \\
 &= \sum_{i=0}^{m-j+1} \frac{(-1)^i}{i+1} \sum_{k=i}^{m-j+1} \binom{m+1}{k} \left[\begin{matrix} k \\ i \end{matrix} \right] \frac{m-k+1}{m+1} \left[\begin{matrix} m-k+1 \\ j+1 \end{matrix} \right] \\
 &= \sum_{i=0}^{m-j+1} \frac{(-1)^i}{i+1} \binom{i+j}{j} \left[\begin{matrix} m+1 \\ i+j+1 \end{matrix} \right] \\
 &= \frac{m}{j} \left[\begin{matrix} m \\ j \end{matrix} \right].
 \end{aligned}$$

Similarly, the identity (21) holds because

$$\begin{aligned}
 & \sum_{k=0}^{m-j+1} \binom{m+1}{k} (-1)^k c_k \left[\begin{matrix} m-k+1 \\ j \end{matrix} \right] \\
 &= \sum_{k=0}^{m-j+1} \binom{m+1}{k} (-1)^k \sum_{i=0}^k \frac{(-1)^{k-i}}{i+1} \left[\begin{matrix} k \\ i \end{matrix} \right] \left[\begin{matrix} m-k+1 \\ j \end{matrix} \right] \\
 &= \sum_{i=0}^{m-j+1} \frac{(-1)^i}{i+1} \sum_{k=i}^{m-j+1} \binom{m+1}{k} \left[\begin{matrix} k \\ i \end{matrix} \right] \left[\begin{matrix} m-k+1 \\ j \end{matrix} \right] \\
 &= \sum_{i=0}^{m-j+1} \frac{(-1)^i}{i+1} \binom{i+j}{j} \left[\begin{matrix} m+1 \\ i+j \end{matrix} \right] \\
 &= \frac{m+1}{j} \left[\begin{matrix} m \\ j-1 \end{matrix} \right].
 \end{aligned}$$

□

We also need the following relation [10, Theorem 2.4(2.1)] in order to prove Proposition 2.

LEMMA 3. For $m \geq 0$,

$$(-1)^m \frac{\hat{c}_m}{m!} = \sum_{k=0}^m (-1)^k \frac{c_k}{k!}.$$

Proof of Proposition 2. The identity (19) holds for $m = 0$ and $m = 1$ because of (12) and (14), respectively. Let $m \geq 2$. By Lemma 1 with $\hat{g}_{n,n} = -n!x$ ($n \geq 0$), $\hat{g}_{n,0} = n!$ ($n \geq 1$) and $\hat{g}_{0,0} = -x$ together with Lemmata 2 and 3

$$\begin{aligned}
 & \hat{c}(x) \hat{c}^{(m)}(x) + \frac{x}{m+1} \hat{c}^{(m+1)}(x) \\
 &= \frac{x}{m+1} \left(\frac{-1}{1+x} \right)^{m+2} \left(\sum_{j=0}^{m-1} \frac{\hat{g}_{m+1,j+1} - (m+1) \hat{g}_{m,j}}{(\ln(1+x))^{j+2}} + \frac{(m+1)!}{\ln(1+x)} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{x}{m+1} \left(\frac{-1}{1+x} \right)^{m+2} \left(\sum_{j=0}^{m-1} \frac{1}{(\ln(1+x))^{j+2}} \sum_{k=0}^{m-j-1} \binom{m+1}{k+1} (-1)^k c_{k+1} \hat{g}_{m-k,j+1} \right. \\
&\quad \left. + \frac{(m+1)!}{\ln(1+x)} \sum_{k=0}^{m-1} (-1)^k \frac{c_{k+1}}{(k+1)!} + \frac{(-1)^m (m+1) \hat{c}_m}{\ln(1+x)} \right) \\
&= \frac{x}{m+1} \left(\frac{-1}{1+x} \right)^{m+2} \left(\sum_{j=0}^m \frac{m+1}{(\ln(1+x))^{j+1}} \sum_{k=0}^{m-j} \frac{c_{k+1}}{k+1} \binom{m}{k} (-1)^k \hat{g}_{m-k,j} \right. \\
&\quad \left. + \frac{(-1)^m}{\ln(1+x)} (c_{m+1}x + (m+1)\hat{c}_m) \right) \\
&= \left(\frac{-1}{1+x} \right)^{m+2} \sum_{k=0}^m \binom{m}{k} \frac{c_{k+1}}{k+1} (-1)^k \sum_{j=0}^{m-k} \frac{x \hat{g}_{m-k,j}}{(\ln(1+x))^{j+1}} \\
&\quad + \left(\frac{-1}{1+x} \right)^{m+2} \frac{x}{\ln(1+x)} (-1)^m \left(\frac{c_{m+1}}{m+1} x + \hat{c}_m \right) \\
&= - \sum_{k=0}^{m-1} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{x}{(1+x)^{k+1}} \hat{c}^{(m-k)}(x) + \frac{\hat{c}_m}{(1+x)^{m+1}} \hat{c}(x). \quad \square
\end{aligned}$$

By Proposition 2, we obtain the following theorem.

THEOREM 2. For $m \geq 0$ and $n \geq 0$ we have

$$\begin{aligned}
(\hat{c}_0 + \hat{c}_m)^n &= -\frac{n}{m+1} \hat{c}_{n+m} \\
&\quad - \frac{1}{m+1} \sum_{k=0}^{m-1} \binom{m+1}{k+1} c_{k+1} \sum_{i=0}^{n-1} (-1)^i \binom{k+i}{i} \frac{n!}{(n-i-1)!} \hat{c}_{n+m-k-i-1} \\
&\quad + \hat{c}_m \sum_{i=0}^n (-1)^i \binom{m+i}{i} \frac{n!}{(n-i)!} \hat{c}_{n-i}.
\end{aligned}$$

Examples. When $m=0$, $m=1$ and $m=2$, we have the formulae (7), (8) and (9), respectively. If $m=3$, we have

$$\begin{aligned}
(\hat{c}_0 + \hat{c}_3)^n &= -\frac{(n+1)!}{8} \sum_{k=0}^n \frac{(-1)^{n-k} (2(2n^2 - 6n + 9) - (n-k)(n+9k-27)) \hat{c}_k}{k!} \\
&\quad + \frac{n(2n-1)\hat{c}_{n+1}}{4} - \frac{n\hat{c}_{n+2}}{2} - \frac{n\hat{c}_{n+3}}{4}.
\end{aligned}$$

Proof. By (13),

$$x \hat{c}^{(m+1)}(x) = \sum_{n=0}^{\infty} n \hat{c}_{n+m} \frac{x^n}{n!}.$$

By (15) we get

$$\begin{aligned}
& \sum_{k=0}^{m-1} \binom{m+1}{k+1} \frac{c_{k+1}x}{(1+x)^{k+1}} \hat{c}^{(m-k)}(x) \\
&= \sum_{k=0}^{m-1} \binom{m+1}{k+1} c_{k+1} \sum_{i=0}^{\infty} (-1)^i \binom{k+i}{i} x^{i+1} \hat{c}^{(m-k)}(x) \\
&= \sum_{k=0}^{m-1} \binom{m+1}{k+1} c_{k+1} \sum_{i=0}^{n-1} (-1)^i \binom{k+i}{i} \sum_{n=0}^{\infty} \frac{n!}{(n-i-1)!} \hat{c}_{n+m-k-i-1} \frac{x^n}{n!}.
\end{aligned}$$

Finally, for $m \geq 0$

$$\frac{\hat{c}_m}{(1+x)^{m+1}} \hat{c}(x) = \hat{c}_m \sum_{i=0}^n (-1)^i \binom{m+i}{i} \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} \hat{c}_{n-i} \frac{x^n}{n!}.$$

□

PROPOSITION 3. For $m \geq 1$ we have

$$\begin{aligned}
\hat{c}'(x) \hat{c}^{(m)}(x) &= -\frac{x}{(m+2)(m+1)} \hat{c}^{(m+2)}(x) - \frac{1}{m+1} \hat{c}^{(m+1)}(x) \\
&+ \sum_{k=0}^{m-2} \binom{m}{k} \left(\frac{c_{k+2}}{k+2} + \frac{kc_{k+1}}{k+1} \right) \frac{x \hat{c}^{(m-k)}(x)}{(1+x)^{k+2}} \\
&- \sum_{k=0}^{m-2} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{\hat{c}^{(m-k)}(x)}{(1+x)^{k+2}} \\
&+ \frac{(c_{m+1} + (m-1)c_m)x + (\hat{c}_m - c_m)}{(1+x)^{m+1}} \hat{c}'(x) - \frac{\hat{c}_{m+1} + (m+1)\hat{c}_m}{(1+x)^{m+2}} \hat{c}(x).
\end{aligned} \tag{22}$$

Proof. By Proposition 2 for $m \geq 1$ we have

$$\begin{aligned}
\hat{c}'(x) \hat{c}^{(m)}(x) &= (\hat{c}(x) \hat{c}^{(m)}(x))' - \hat{c}(x) \hat{c}^{(m+1)}(x) \\
&= -\frac{1}{m+1} \hat{c}^{(m+1)}(x) - \frac{x}{m+1} \hat{c}^{(m+2)}(x) \\
&+ \sum_{k=0}^{m-1} \binom{m}{k} \frac{kc_{k+1}}{k+1} \frac{x}{(1+x)^{k+2}} \hat{c}^{(m-k)}(x) \\
&- \sum_{k=0}^{m-1} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{1}{(1+x)^{k+2}} \hat{c}^{(m-k)}(x) \\
&- \sum_{k=0}^{m-1} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{x}{(1+x)^{k+1}} \hat{c}^{(m-k+1)}(x) \\
&- \frac{(m+1)\hat{c}_m}{(1+x)^{m+2}} \hat{c}(x) + \frac{\hat{c}_m}{(1+x)^{m+1}} \hat{c}'(x)
\end{aligned}$$

$$\begin{aligned}
& + \frac{x}{m+2} \hat{c}^{(m+2)}(x) + \sum_{k=0}^m \binom{m+1}{k} \frac{c_{k+1}}{k+1} \frac{x}{(1+x)^{k+1}} \hat{c}^{(m-k+1)}(x) \\
& - \frac{\hat{c}_{m+1}}{(1+x)^{m+2}} \hat{c}(x) \\
& = - \frac{x}{(m+2)(m+1)} \hat{c}^{(m+2)}(x) - \frac{1}{m+1} \hat{c}^{(m+1)}(x) \\
& + \sum_{k=0}^{m-2} \binom{m}{k} \frac{kc_{k+1}}{k+1} \frac{x}{(1+x)^{k+2}} \hat{c}^{(m-k)}(x) + (m-1)c_m \frac{x}{(1+x)^{m+1}} \hat{c}'(x) \\
& - \sum_{k=0}^{m-2} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{1}{(1+x)^{k+2}} \hat{c}^{(m-k)}(x) - \frac{c_m}{(1+x)^{m+1}} \hat{c}'(x) \\
& - \sum_{k=1}^{m-1} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{x}{(1+x)^{k+1}} \hat{c}^{(m-k+1)}(x) \\
& + \sum_{k=1}^{m-1} \binom{m+1}{k} \frac{c_{k+1}}{k+1} \frac{x}{(1+x)^{k+1}} \hat{c}^{(m-k+1)}(x) + \frac{c_{m+1}x}{(1+x)^{m+1}} \hat{c}'(x) \\
& + \frac{\hat{c}_m}{(1+x)^{m+1}} \hat{c}'(x) - \frac{\hat{c}_{m+1} + (m+1)\hat{c}_m}{(1+x)^{m+2}} \hat{c}(x) \\
& = - \frac{x}{(m+2)(m+1)} \hat{c}^{(m+2)}(x) - \frac{1}{m+1} \hat{c}^{(m+1)}(x) \\
& + \sum_{k=0}^{m-2} \binom{m}{k} \left(\frac{c_{k+2}}{k+2} + \frac{kc_{k+1}}{k+1} \right) \frac{x \hat{c}^{(m-k)}(x)}{(1+x)^{k+2}} \\
& - \sum_{k=0}^{m-2} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{\hat{c}^{(m-k)}(x)}{(1+x)^{k+2}} \\
& + \frac{(c_{m+1} + (m-1)c_m)x + (\hat{c}_m - c_m)}{(1+x)^{m+1}} \hat{c}'(x) - \frac{\hat{c}_{m+1} + (m+1)\hat{c}_m}{(1+x)^{m+2}} \hat{c}(x). \quad \square
\end{aligned}$$

By Proposition 3 we have the following.

THEOREM 3. For $m \geq 1$ and $n \geq 0$ we have

$$\begin{aligned}
& (\hat{c}_1 + \hat{c}_m)^n \\
& = - \frac{(n+m+2)\hat{c}_{n+m+1}}{(m+2)(m+1)} \\
& + \sum_{k=0}^{m-2} \binom{m}{k} \left(\frac{c_{k+2}}{k+2} + \frac{kc_{k+1}}{k+1} \right) \sum_{i=0}^{n-1} (-1)^i \binom{k+i+1}{i} \frac{n!}{(n-i-1)!} \hat{c}_{n+m-k-i-1} \\
& - \sum_{k=0}^{m-2} \binom{m}{k} \frac{c_{k+1}}{k+1} \sum_{i=0}^n (-1)^i \binom{k+i+1}{i} \frac{n!}{(n-i)!} \hat{c}_{n+m-k-i} \\
& + (c_{m+1} + (m-1)c_m) \sum_{i=0}^{n-1} (-1)^i \binom{m+i}{i} \frac{n!}{(n-i-1)!} \hat{c}_{n-i}
\end{aligned}$$

$$\begin{aligned}
& + (\hat{c}_m - c_m) \sum_{i=0}^n (-1)^i \binom{m+i}{i} \frac{n!}{(n-i)!} \hat{c}_{n-i+1} \\
& - (\hat{c}_{m+1} + (m+1)\hat{c}_m) \sum_{i=0}^n (-1)^i \binom{m+i+1}{i} \frac{n!}{(n-i)!} \hat{c}_{n-i}.
\end{aligned}$$

Examples. If $m = 1$, we have the identity (10). If $m = 2$, we have

$$\begin{aligned}
(\hat{c}_1 + \hat{c}_2)^n &= -\frac{(n+1)!}{24} \sum_{k=0}^n \frac{(-1)^{n-k} \hat{c}_k}{k!} ((n+2)(n+3) + k(8n-9k+27)) \\
&+ \frac{(11n+12)\hat{c}_{n+1}}{12} - \frac{\hat{c}_{n+2}}{2} - \frac{(n+4)\hat{c}_{n+3}}{12}.
\end{aligned}$$

By $\hat{c}''(x)\hat{c}^{(m)}(x) = (\hat{c}'(x)\hat{c}^{(m)}(x))' - \hat{c}'(x)\hat{c}^{(m+1)}(x)$ in Proposition 3, we have the following.

PROPOSITION 4. For $m \geq 2$ we have

$$\begin{aligned}
& \hat{c}''(x)\hat{c}^{(m)}(x) \\
&= -\frac{2x}{(m+3)(m+2)(m+1)}\hat{c}^{(m+3)}(x) - \frac{2}{(m+2)(m+1)}\hat{c}^{(m+2)}(x) \\
&- \sum_{k=0}^{m-3} \binom{m}{k} \left(\frac{c_{k+3}}{k+3} + \frac{2(k+1)c_{k+2}}{k+2} + kc_{k+1} \right) \frac{x}{(1+x)^{k+3}} \hat{c}^{(m-k)}(x) \\
&+ 2 \sum_{k=0}^{m-3} \binom{m}{k} \left(\frac{c_{k+2}}{k+2} + c_{k+1} \right) \frac{1}{(1+x)^{k+3}} \hat{c}^{(m-k)}(x) \\
&- \frac{1}{2}((m-2)c_{m+1} + 2(m-1)^2c_m + m(m-1)(m-2)c_{m-1}) \frac{x}{(1+x)^{m+1}} \hat{c}''(x) \\
&+ ((m-1)c_m + m(m-1)c_{m-1} + \hat{c}_m) \frac{1}{(1+x)^{m+1}} \hat{c}''(x) \\
&- (c_{m+2} + 2mc_{m+1} + m(m-1)c_m) \frac{x}{(1+x)^{m+2}} \hat{c}'(x) \\
&+ 2(c_{m+1} + mc_m - \hat{c}_{m+1} - (m+1)\hat{c}_m) \frac{1}{(1+x)^{m+2}} \hat{c}'(x) \\
&+ (\hat{c}_{m+2} + 2(m+2)\hat{c}_{m+1} + (m+2)(m+1)\hat{c}_m) \frac{1}{(1+x)^{m+3}} \hat{c}(x). \tag{23}
\end{aligned}$$

By Proposition 4, we have the following result.

THEOREM 4. For $m \geq 2$ and $n \geq 0$,

$$\begin{aligned}
& (\hat{c}_2 + \hat{c}_m)^n \\
&= -\frac{2(n+m+3)}{(m+3)(m+2)(m+1)} \hat{c}_{n+m+2}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{k=0}^{m-3} \binom{m}{k} \left(\frac{c_{k+3}}{k+3} + \frac{2(k+1)c_{k+2}}{k+2} + kc_{k+1} \right) \\
& \times \sum_{i=0}^{n-1} (-1)^i \binom{k+i+2}{i} \frac{n!}{(n-i-1)!} \hat{c}_{n+m-k-i-1} \\
& + 2 \sum_{k=0}^{m-3} \binom{m}{k} \left(\frac{c_{k+2}}{k+2} + c_{k+1} \right) \sum_{i=0}^n (-1)^i \binom{k+i+2}{i} \frac{n!}{(n-i)!} \hat{c}_{n+m-k-i} \\
& - \frac{1}{2} ((m-2)c_{m+1} + 2(m-1)^2 c_m + m(m-1)(m-2)c_{m-1}) \\
& \times \sum_{i=0}^{n-1} (-1)^i \binom{m+i}{i} \frac{n!}{(n-i-1)!} \hat{c}_{n-i+1} \\
& + ((m-1)c_m + m(m-1)c_{m-1} + \hat{c}_m) \sum_{i=0}^n (-1)^i \binom{m+i}{i} \frac{n!}{(n-i)!} \hat{c}_{n-i+2} \\
& - (c_{m+2} + 2mc_{m+1} + m(m-1)c_m) \sum_{i=0}^{n-1} (-1)^i \binom{m+i+1}{i} \frac{n!}{(n-i-1)!} \hat{c}_{n-i} \\
& + 2(c_{m+1} + mc_m - \hat{c}_{m+1} - (m+1)\hat{c}_m) \sum_{i=0}^n (-1)^i \binom{m+i+1}{i} \frac{n!}{(n-i)!} \hat{c}_{n-i+1} \\
& + (\hat{c}_{m+2} + 2(m+2)\hat{c}_{m+1} + (m+2)(m+1)\hat{c}_m) \sum_{i=0}^n (-1)^i \binom{m+i+2}{i} \frac{n!}{(n-i)!} \hat{c}_{n-i}.
\end{aligned}$$

Example. If $m = 2$, we have

$$\begin{aligned}
(\hat{c}_2 + \hat{c}_2)^n &= \frac{n!}{6!} \sum_{k=0}^n \frac{\hat{c}_k}{k!} (-1)^{n-k} (n-k+3)(11(n-k+4)(n-k+2)(n-k+1) \\
& + 4k(15(9n-9k+38)(k-1) + (19n-19k+79)(n-k+2))) \\
& - \frac{(29n+4)\hat{c}_{n+1}}{6} + \frac{5\hat{c}_{n+2}}{3} - \frac{(n+5)\hat{c}_{n+4}}{30}.
\end{aligned}$$

3. The general case

For general integers l and m , we obtain the following result.

PROPOSITION 5. *Let l, m be fixed non-negative integers with $m \geq l \geq 1$. Then*

$$\begin{aligned}
\hat{c}^{(l)}(x) \hat{c}^{(m)}(x) &= - \frac{l!m!}{(l+m+1)!} x \hat{c}^{(l+m+1)}(x) - \frac{l!m!}{(l+m)!} \hat{c}^{(l+m)}(x) \\
& + \sum_{k=0}^{m-l-1} \frac{a_{l,m,m-k}x + b_{l,m,m-k}}{(1+x)^{l+k+1}} \hat{c}^{(m-k)}(x) \\
& + \sum_{k=0}^l \frac{a_{l,m,l-k}x + b_{l,m,l-k}}{(1+x)^{m+k+1}} \hat{c}^{(l-k)}(x),
\end{aligned}$$

where for $l + 1 \leq r = m - k \leq m$,

$$a_{l,m,r} = (-1)^{l+1} \binom{m}{k} \sum_{i=0}^l i! \binom{l}{i} \binom{l+k-1}{i} \frac{c_{l+k-i+1}}{l+k-i+1},$$

$$b_{l,m,r} = (-1)^l l \binom{m}{k} \sum_{i=0}^{l-1} i! \binom{l-1}{i} \binom{l+k-1}{i} \frac{c_{l+k-i}}{l+k-i},$$

and for $1 \leq r = l - k \leq l$,

$$a_{l,m,r} = \sum_{j=0}^{r-2} \frac{(-1)^{l+j+1}}{r} \binom{l}{j} \binom{m}{r-j-1} \binom{r-1}{j}^{-1}$$

$$\times \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1}$$

$$+ \frac{(-1)^{l+r}}{r} \binom{l}{r-1} \sum_{i=0}^{l-r+1} i! \binom{l-r+1}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1}$$

with $a_{l,m,0} = 0$ and for $0 \leq r = l - k \leq l$,

$$b_{l,m,r} = \sum_{j=1}^r (-1)^{l+j+1} \binom{l}{j} \binom{m}{r-j} \binom{r}{j}^{-1} \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-1}{i} c_{l+m-i-r}$$

$$+ (-1)^{l+r} \binom{l}{r} \sum_{i=0}^{l-r} i! \binom{l-r}{i} \binom{l+m-r}{i} \hat{c}_{l+m-i-r}.$$

Proof. Note that $a_{l,m+1,m+1} = a_{l,m,m}$, $b_{l,m+1,m+1} = b_{l,m,m}$, and for $k = 0, 1, \dots, m - l - 1$,

$$-(l+k)a_{l,m,m-k} + a_{l,m,m-k-1} - a_{l,m+1,m-k} = a_{l+1,m,m-k}, \quad (24)$$

$$a_{l,m,m-k} - (l+k+1)b_{l,m,m-k} - b_{l,m+1,m-k} + b_{l,m,m-k-1} = b_{l+1,m,m-k},$$

and for $k = 0, 1, \dots, l - 1$,

$$-(m+k)a_{l,m,l-k} + a_{l,m,l-k-1} - a_{l,m+1,l-k} = a_{l+1,m,l-k}, \quad (25)$$

$$a_{l,m,l-k} - (m+k+1)b_{l,m,l-k} + b_{l,m,l-k-1} - b_{l,m+1,l-k} = b_{l+1,m,l-k},$$

and $-(l+m+1)b_{l,m,0} - b_{l,m+1,0} = b_{l+1,m,0}$. By using these relations, we have

$$(\hat{c}^{(l)}(x)\hat{c}^{(m)}(x))' - \hat{c}^{(l)}(x)\hat{c}^{(m+1)}(x)$$

$$= -\frac{l!m!}{(l+m+1)!} \hat{c}^{(l+m+1)}(x) - \frac{l!m!}{(l+m+1)!} x \hat{c}^{(l+m+2)}(x) - \frac{l!m!}{(l+m)!} \hat{c}^{(l+m+1)}(x)$$

$$+ \frac{l!(m+1)!}{(l+m+2)!} x \hat{c}^{(l+m+2)}(x) + \frac{l!(m+1)!}{(l+m+1)!} \hat{c}^{(l+m+1)}(x)$$

$$- \sum_{k=0}^{m-l-1} (l+k) \frac{a_{l,m,m-k}x}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) + \sum_{k=0}^{m-l-2} \frac{a_{l,m,m-k-1}x}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x)$$

$$\begin{aligned}
& - \sum_{k=0}^{m-l-1} \frac{a_{l,m+1,m-k}x}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) - \sum_{k=0}^{m-l-1} (l+k+1) \frac{b_{l,m,m-k}}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) \\
& + \sum_{k=0}^{m-l-2} \frac{b_{l,m,m-k-1}}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) + \sum_{k=0}^{m-l-1} \frac{a_{l,m,m-k}}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) \\
& - \sum_{k=0}^{m-l-1} \frac{b_{l,m+1,m-k}}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) - \sum_{k=0}^l (m+k) \frac{a_{l,m,l-k}x}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) \\
& + \sum_{k=0}^{l-1} \frac{a_{l,m,l-k-1}x}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) - \sum_{k=0}^l \frac{a_{l,m+1,l-k}x}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) \\
& - \sum_{k=0}^l (m+k+1) \frac{b_{l,m,l-k}}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) + \sum_{k=0}^{l-1} \frac{b_{l,m,l-k-1}}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) \\
& + \sum_{k=0}^l \frac{a_{l,m,l-k}}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) - \sum_{k=0}^l \frac{b_{l,m+1,l-k}}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) \\
& + \frac{a_{l,m,l}x + b_{l,m,l}}{(1+x)^{m+1}} \hat{c}^{(l+1)}(x) \\
& = - \frac{(l+1)!m!}{(l+m+2)!} x \hat{c}^{(l+m+2)}(x) - \frac{(l+1)!m!}{(l+m+1)!} \hat{c}^{(l+m+1)}(x) \\
& + \sum_{k=0}^{m-l-2} \frac{a_{l+1,m,m-k}x}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) + \sum_{k=0}^{m-l-2} \frac{b_{l+1,m,m-k}}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) \\
& - (m-1) \frac{a_{l,m,l+1}x}{(1+x)^{m+1}} \hat{c}^{(l+1)}(x) - \frac{a_{l,m+1,l+1}x}{(1+x)^{m+1}} \hat{c}^{(l+1)}(x) \\
& + \frac{a_{l,m,l+1}}{(1+x)^{m+1}} \hat{c}^{(l+1)}(x) - m \frac{b_{l,m,l+1}x}{(1+x)^{m+1}} \hat{c}^{(l+1)}(x) - \frac{b_{l,m+1,l+1}x}{(1+x)^{m+1}} \hat{c}^{(l+1)}(x) \\
& + \frac{a_{l,m,l}x + b_{l,m,l}}{(1+x)^{m+1}} \hat{c}^{(l+1)}(x) + \sum_{k=0}^{l-1} \frac{a_{l+1,m,l-k}x}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) \\
& + \sum_{k=0}^{l-1} \frac{b_{l+1,m,l-k}}{(1+x)^{m+k+2}} \hat{c}^{(l-k)}(x) + \frac{b_{l+1,m,0}}{(1+x)^{l+m+2}} \hat{c}(x) \\
& = - \frac{(l+1)!m!}{(l+m+2)!} x \hat{c}^{(l+m+2)}(x) - \frac{(l+1)!m!}{(l+m+1)!} \hat{c}^{(l+m+1)}(x) \\
& + \sum_{k=0}^{m-l-2} \frac{a_{l+1,m,m-k}x}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) + \sum_{k=0}^{m-l-2} \frac{b_{l+1,m,m-k}}{(1+x)^{l+k+2}} \hat{c}^{(m-k)}(x) \\
& + \sum_{k=1}^{l+1} \frac{a_{l+1,m,l-k+1}x}{(1+x)^{m+k+1}} \hat{c}^{(l-k+1)}(x) + \sum_{k=1}^{l+1} \frac{b_{l+1,m,l-k+1}}{(1+x)^{m+k+1}} \hat{c}^{(l-k+1)}(x) \\
& + \frac{a_{l+1,m,l+1}x + b_{l+1,m,l+1}}{(1+x)^{m+1}} \hat{c}^{(l+1)}(x) \\
& = \hat{c}^{(l+1)}(x) \hat{c}^{(m)}(x).
\end{aligned}$$

Finally, we shall prove only the relations (24) and (25). Other relations can be proved similarly and omitted. The left-hand side of (24) is equal to

$$\begin{aligned}
& - (l+k)(-1)^{l+1} \binom{m}{k} \sum_{i=0}^l i! \binom{l}{i} \binom{l+k-1}{i} \frac{c_{l+k-i+1}}{l+k-i+1} \\
& + (-1)^{l+1} \binom{m}{k+1} \sum_{i=0}^l i! \binom{l}{i} \binom{l+k}{i} \frac{c_{l+k-i+2}}{l+k-i+2} \\
& - (-1)^{l+1} \binom{m+1}{k+1} \sum_{i=0}^l i! \binom{l}{i} \binom{l+k}{i} \frac{c_{l+k-i+2}}{l+k-i+2} \\
& = -(l+k)(-1)^{l+1} \binom{m}{k} \sum_{i=1}^{l+1} (i-1)! \binom{l}{i-1} \binom{l+k-1}{i-1} \frac{c_{l+k-i+2}}{l+k-i+2} \\
& \quad - (-1)^{l+1} \binom{m}{k} \sum_{i=0}^l i! \binom{l}{i} \binom{l+k}{i} \frac{c_{l+k-i+2}}{l+k-i+2} \\
& = (-1)^l \binom{m}{k} \sum_{i=0}^{l+1} i! \binom{l+1}{i} \binom{l+k}{i} \frac{c_{l+k-i+2}}{l+k-i+2} \\
& = a_{l+1,m,m-k}.
\end{aligned}$$

The left-hand side of (25) is equal to

$$\begin{aligned}
& - (l+m-r-1)a_{l,m,r} - a_{l,m+1,r} + a_{l,m,r-1} \\
& = -(l+m-r-1) \sum_{j=0}^{r-2} \frac{(-1)^{l+j+1}}{r} \binom{l}{j} \binom{m}{r-j-1} \binom{r-1}{j}^{-1} \\
& \quad \times \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-2}{i} c_{l+m-i-r+1} \\
& \quad - (l+m-r-1) \frac{(-1)^{l+r}}{r} \binom{l}{r-1} \sum_{i=0}^{l-r+1} i! \binom{l-r+1}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1} \\
& \quad + \sum_{j=0}^{r-3} \frac{(-1)^{l+j+1}}{r-1} \binom{l}{j} \binom{m}{r-j-2} \binom{r-2}{j}^{-1} \\
& \quad \times \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+2} \\
& \quad + \frac{(-1)^{l+r-1}}{r-1} \binom{l}{r-2} \sum_{i=0}^{l-r+2} i! \binom{l-r+2}{i} \binom{l+m-r}{i} c_{l+m-i-r+2} \\
& \quad - \sum_{j=0}^{r-2} \frac{(-1)^{l+j+1}}{r} \binom{l}{j} \binom{m+1}{r-j-1} \binom{r-1}{j}^{-1}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+2} \\
& - \frac{(-1)^{l+r}}{r} \binom{l}{r-1} \sum_{i=0}^{l-r+1} i! \binom{l-r+1}{i} \binom{l+m-r}{i} c_{l+m-i-r+2} \\
& = \sum_{j=0}^{r-2} \frac{(-1)^{l+j}}{r} \binom{l+1}{j} \binom{m}{r-j-1} \binom{r-1}{j}^{-1} \\
& \quad \times \sum_{i=0}^{l-j+1} i! \binom{l-j+1}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+2} \\
& \quad + \frac{(-1)^{l+r+1}}{r} \binom{l+1}{r-1} \sum_{i=0}^{l-r+2} i! \binom{l-r+2}{i} \binom{l+m-r}{i} c_{l+m-i-r+2}
\end{aligned}$$

$$= a_{l+1, m, r}.$$

□

If $l = 3$ in Proposition 5 we have

$$\begin{aligned}
& \hat{c}'''(x) \hat{c}^{(m)}(x) \\
& = -\frac{3!m!x}{(m+4)!} \hat{c}^{(m+4)}(x) - \frac{3!m!}{(m+3)!} \hat{c}^{(m+3)}(x) \\
& \quad + \sum_{k=0}^{m-4} \binom{m}{k} \left(\frac{c_{k+4}}{k+4} + \frac{3(k+2)c_{k+3}}{k+3} + 3(k+1)c_{k+2} + (k+2)kc_{k+1} \right) \\
& \quad \times \frac{x}{(1+x)^{k+4}} \hat{c}^{(m-k)}(x) \\
& \quad - 3 \sum_{k=0}^{m-4} \binom{m}{k} \left(\frac{c_{k+3}}{k+3} + 2c_{k+2} + (k+2)c_{k+1} \right) \frac{1}{(1+x)^{k+4}} \hat{c}^{(m-k)}(x) \\
& \quad + \left(\frac{m^2 - 4m + 6}{3!} c_{m+1} + \frac{(m-1)^2(m-2)}{2} c_m + \frac{m(m-1)(m-2)^2}{2} c_{m-1} \right. \\
& \quad \left. + \frac{m(m-1)^2(m-2)(m-3)}{3!} c_{m-2} \right) \frac{x}{(1+x)^{m+1}} \hat{c}'''(x) \\
& \quad - \left(\frac{(m-1)(m-2)}{2} (c_m + 2mc_{m-1} + m(m-1)c_{m-2}) - \hat{c}_m \right) \frac{1}{(1+x)^{m+1}} \hat{c}'''(x) \\
& \quad + \frac{1}{2} ((m-3)c_{m+2} + 3m(m-2)c_{m+1} + 3m(m-1)^2c_m \\
& \quad + m^2(m-1)(m-2)c_{m-1}) \frac{x}{(1+x)^{m+2}} \hat{c}''(x) \\
& \quad - \frac{3}{2} ((m-2)c_{m+1} + 2m(m-1)c_m + m^2(m-1)c_{m-1} \\
& \quad + 2(\hat{c}_{m+1} + (m+1)\hat{c}_m)) \frac{1}{(1+x)^{m+2}} \hat{c}''(x) \\
& \quad + (c_{m+3} + 3(m+1)c_{m+2} + 3(m+1)mc_{m+1} + (m+1)m(m-1)c_m)
\end{aligned}$$

$$\begin{aligned}
& \times \frac{x}{(1+x)^{m+3}} \hat{c}'(x) - 3(c_{m+2} + 2(m+1)c_{m+1} + (m+1)mc_m \\
& - (\hat{c}_{m+2} + 2(m+2)\hat{c}_{m+1} + (m+2)(m+1)\hat{c}_m)) \frac{1}{(1+x)^{m+3}} \hat{c}'(x) \\
& - (\hat{c}_{m+3} + 3(m+3)\hat{c}_{m+2} + 3(m+3)(m+2)\hat{c}_{m+1} + (m+3)(m+2)(m+1)\hat{c}_m) \\
& \times \frac{\hat{c}(x)}{(1+x)^{m+4}}. \tag{26}
\end{aligned}$$

By using Proposition 5, we obtain explicit expressions for $(\hat{c}_l + \hat{c}_m)^n$.

THEOREM 5. *Let l, m, n be integers with $m \geq l \geq 1$ and $n \geq 0$. Then*

$$\begin{aligned}
& (\hat{c}_l + \hat{c}_m)^n \\
& = -\frac{l!m!}{(l+m+1)!} (n+l+m+1) \hat{c}_{n+l+m} \\
& + \sum_{k=0}^{m-l-1} \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{l+k+i}{i} \left(\frac{i}{l+k+i} a_{l,m,m-k} - b_{l,m,m-k} \right) \hat{c}_{n+m-k-i} \\
& + \sum_{k=0}^l \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{m+k+i}{i} \left(\frac{i}{m+k+i} a_{l,m,l-k} - b_{l,m,l-k} \right) \hat{c}_{n+l-k-i}.
\end{aligned}$$

where $a_{l,m,r}$ and $b_{l,m,r}$ are defined in Proposition 5.

Proof. By (13) and (15) we have

$$\begin{aligned}
& (\hat{c}_l + \hat{c}_m)^n \\
& = -\frac{l!m!}{(l+m+1)!} n \hat{c}_{n+l+m} - \frac{l!m!}{(l+m)!} \hat{c}_{n+l+m} \\
& + \sum_{k=0}^{m-l-1} a_{l,m,m-k} \sum_{i=0}^{\infty} (-1)^i \binom{l+k+i}{i} \frac{n!}{(n-i-1)!} \hat{c}_{n+m-k-i-1} \\
& + \sum_{k=0}^{m-l-1} b_{l,m,m-k} \sum_{i=0}^{\infty} (-1)^i \binom{l+k+i}{i} \frac{n!}{(n-i)!} \hat{c}_{n+m-k-i} \\
& + \sum_{k=0}^l a_{l,m,l-k} \sum_{i=0}^{\infty} (-1)^i \binom{m+k+i}{i} \frac{n!}{(n-i-1)!} \hat{c}_{n+l-k-i-1} \\
& + \sum_{k=0}^l b_{l,m,l-k} \sum_{i=0}^{\infty} (-1)^i \binom{m+k+i}{i} \frac{n!}{(n-i)!} \hat{c}_{n+l-k-i} \\
& = -\frac{l!m!}{(l+m+1)!} (n+l+m+1) \hat{c}_{n+l+m} \\
& + \sum_{k=0}^{m-l-1} \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{l+k+i}{i} \left(\frac{i}{l+k+i} a_{l,m,m-k} - b_{l,m,m-k} \right) \hat{c}_{n+m-k-i} \\
& + \sum_{k=0}^l \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{m+k+i}{i} \left(\frac{i}{m+k+i} a_{l,m,l-k} - b_{l,m,l-k} \right) \hat{c}_{n+l-k-i}. \quad \square
\end{aligned}$$

4. Remarks

If $l = 1$ in Theorem 5, we have

$$\begin{aligned}
 (\hat{c}_1 + \hat{c}_m)^n = & -\frac{n+m+2}{(m+2)(m+1)}\hat{c}_{n+m+1} \\
 & + \sum_{k=0}^{m-2} \sum_{i=0}^{n-1} \frac{(-1)^i n!}{(n-i-1)!} \binom{k+i+1}{i} a_{1,m,m-k} \hat{c}_{n+m-k-i-1} \\
 & - \sum_{k=0}^{m-2} \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{k+i+1}{i} b_{1,m,m-k} \hat{c}_{n+m-k-i} \\
 & + \sum_{i=0}^{n-1} \frac{(-1)^i n!}{(n-i-1)!} \binom{m+i}{i} a_{1,m,1} \hat{c}_{n-i} \\
 & + \sum_{i=0}^{n-1} \frac{(-1)^i n!}{(n-i-1)!} \binom{m+i+1}{i} a_{1,m,0} \hat{c}_{n-i-1} \\
 & - \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{m+i}{i} b_{1,m,1} \hat{c}_{n-i+1} \\
 & - \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{m+i+1}{i} b_{1,m,0} \hat{c}_{n-i}.
 \end{aligned}$$

Since

$$\begin{aligned}
 a_{1,m,m-k} &= \binom{m}{k} \left(\frac{c_{k+2}}{k+2} + \frac{kc_{k+1}}{k+1} \right), \\
 b_{1,m,m-k} &= -\binom{m}{k} \frac{c_{k+1}}{k+1}, \\
 a_{1,m,1} &= c_{m+1} + (m-1)c_m, \\
 a_{1,m,0} &= 0, \\
 b_{1,m,1} &= \hat{c}_m - c_m, \\
 b_{1,m,0} &= -\hat{c}_{m+1} - (m+1)\hat{c}_m,
 \end{aligned}$$

we obtain Theorem 3.

If $l = 2$ in Theorem 5, we have

$$\begin{aligned}
 (\hat{c}_2 + \hat{c}_m)^n = & -\frac{n+m+3}{(m+3)(m+2)(m+1)}\hat{c}_{n+m+2} \\
 & + \sum_{k=0}^{m-3} \sum_{i=0}^{n-1} \frac{(-1)^i n!}{(n-i-1)!} \binom{k+i+2}{i} a_{2,m,m-k} \hat{c}_{n+m-k-i-1} \\
 & - \sum_{k=0}^{m-2} \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{k+i+2}{i} b_{2,m,m-k} \hat{c}_{n+m-k-i}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^{n-1} \frac{(-1)^i n!}{(n-i-1)!} \binom{m+i}{i} a_{2,m,2} \hat{c}_{n-i+1} \\
& + \sum_{i=0}^{n-1} \frac{(-1)^i n!}{(n-i-1)!} \binom{m+i+1}{i} a_{2,m,1} \hat{c}_{n-i} \\
& + \sum_{i=0}^{n-1} \frac{(-1)^i n!}{(n-i-1)!} \binom{m+i+2}{i} a_{2,m,0} \hat{c}_{n-i-1} \\
& - \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{m+i}{i} b_{2,m,2} \hat{c}_{n-i+2} \\
& - \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{m+i+1}{i} b_{2,m,1} \hat{c}_{n-i+1} \\
& - \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \binom{m+i+2}{i} b_{2,m,0} \hat{c}_{n-i}.
\end{aligned}$$

Since

$$\begin{aligned}
a_{2,m,m-k} &= -\binom{m}{k} \left(\frac{c_{k+3}}{k+3} + \frac{2(k+1)c_{k+2}}{k+2} + c_{k+1} \right), \\
b_{2,m,m-k} &= 2\binom{m}{k} \left(\frac{c_{k+2}}{k+2} + c_{k+1} \right), \\
a_{2,m,2} &= \left(-\frac{m}{2} + 1 \right) c_{m+1} - (m-1)^2 c_m - \frac{m(m-1)(m-2)}{2} c_{m-1}, \\
a_{2,m,1} &= -c_{m+2} - 2m c_{m+1} - m(m-1) c_m, \\
a_{2,m,0} &= 0, \\
b_{2,m,2} &= (m-1) c_m + m(m-1) c_{m-1} + \hat{c}_m, \\
b_{2,m,1} &= 2c_{m+1} + 2m c_m - 2\hat{c}_{m+1} - 2(m+1) \hat{c}_m, \\
b_{2,m,0} &= \hat{c}_{m+2} + 2(m+2) \hat{c}_{m+1} + (m+2)(m+1) \hat{c}_m,
\end{aligned}$$

we obtain Theorem 4.

REFERENCES

- [1] T. Agoh and K. Dilcher. Convolution identities and lacunary recurrences for Bernoulli numbers. *J. Number Theory* **124** (2007), 105–122.
- [2] T. Agoh and K. Dilcher. Higher-order recurrences for Bernoulli numbers. *J. Number Theory* **129** (2009), 1837–1847.
- [3] T. Agoh and K. Dilcher. Recurrence relations for Nörlund numbers and Bernoulli numbers of the second kind. *Fibonacci Quart.* **48** (2010), 4–12.
- [4] L. Comtet. *Advanced Combinatorics*. Reidel, Dordrecht, 1974.
- [5] K. Dilcher. Sums of products of Bernoulli numbers. *J. Number Theory* **60** (1996), 23–41.
- [6] I. M. Gessel. On Miki's identity for Bernoulli numbers. *J. Number Theory* **110** (2005), 75–82.
- [7] R. L. Graham, D. E. Knuth and O. Patashnik. *Concrete Mathematics*, 2nd edn. Addison-Wesley, Reading, MA, 1994.
- [8] T. Komatsu. Poly-Cauchy numbers. *Kyushu J. Math.* **67** (2013), 143–153.

- [9] T. Komatsu. Convolution identities for Cauchy numbers. *Acta Math. Hungar.* **144** (2014), 76–91.
- [10] D. Merlini, R. Sprugnoli and M. C. Verri. The Cauchy numbers. *Discrete Math.* **306** (2006), 1906–1920.
- [11] S. Roman. *The Umbral Calculus*. Dover, New York, 2005.
- [12] F.-Z. Zhao. Sums of products of Cauchy numbers. *Discrete Math.* **309** (2009), 3830–3842.

Takao Komatsu
School of Mathematics and Statistics
Wuhan University
Wuhan, 430072
China
(E-mail: komatsu@whu.edu.cn)