HITTING TIME OF A HALF-LINE BY A TWO-DIMENSIONAL NON-SYMMETRIC RANDOM WALK

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Abstract. We consider the probability that a two-dimensional random walk starting from the origin never returns to the half-line $(-\infty, 0] \times \{0\}$ before time n . Let $X = (X_1, X_2)$ be the increment of the two-dimensional random walk. For an aperiodic random walk with moment conditions $E[X_2] = 0$ and $E[|X_1|^\delta] < \infty$, $E[|X_2|^{2+\delta}] < \infty$ for some $\delta \in (0, 1)$, we obtain an asymptotic estimate (as $n \rightarrow \infty$) of this probability by assuming the behavior of the characteristic function of X near zero.

1. Introduction and statements of results

We consider the probability P_n that a two-dimensional random walk starting at the origin never returns to the half-line $(-\infty, 0] \times \{0\}$ before time n . The purpose of this paper is to investigate the asymptotic behavior of P_n as $n \rightarrow \infty$.

For a simple random walk, $n^{1/4}P_n$ is bounded both from above and from below by positive constants. This bound was shown by Lawler [5, (2.35)]. The estimate plays an important role in computing an upper bound of a hitting distribution from infinity for a simple random walk (see [5, Proposition 2.4.10]). The upper bound gives a growth estimate for the two-dimensional diffusion limited aggregation model [4] and an estimate of the two-dimensional intersection exponent [6].

Let $S = \{S(n)\}_{n=0}^\infty$ be the two-dimensional random walk starting at $x \in \mathbb{Z}^2$ defined by

$$S(n) = x + X(1) + \cdots + X(n),$$

where the $X(j)$ are independent copies of a $\mathbb{Z}^2$ -valued random variable X . Here S is called simple if X takes the four values $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$ with equal probability. We write $X = (X_1, X_2)$ and $S(n) = (S_1(n), S_2(n))$. We denote by P_x the distribution of the two-dimensional random walk starting at x and by E_x its expectation. For a subset A of $\mathbb{Z}^2$, define

$$\tau_A = \inf\{n \geq 1 : S(n) \in A\}.$$

We also denote by U the first coordinate axis:

$$U = \{x = (x_1, x_2) \in \mathbb{Z}^2 : x_2 = 0\}$$

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and by V_- and V_+ the left half and right half of U defined by

$$V_- = \{x = (x_1, x_2) \in \mathbb{Z}^2 : x_1 \leq 0, x_2 = 0\}.$$

By replacing $\geq$ by $\leq$ in V_- , we define V_+ . Here S is called aperiodic if the smallest additive subgroup containing $\{x \in \mathbb{Z}^2 \mid P\{X = x\} > 0\}$ agrees with $\mathbb{Z}^2$.

THEOREM 1.1. *Assume that X satisfies the following:*

- (a) S is aperiodic;
- (b) $E[X_2] = 0$;
- (c) $E[|X_1|^\delta] < \infty$;
- (d) $E[|X_2|^{2+\delta}] < \infty$ for some $\delta \in (0, 1)$.

Assume that there exist constants $\alpha \in (0, 2]$, $\varepsilon > 0$ and $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$ such that, as $\vartheta \rightarrow 0$,

$$\begin{aligned} 1 - E[\cos \vartheta X_1] - \frac{(E[(\sin \vartheta X_1)X_2])^2}{2E[X_2^2]} + \frac{(E[(-1 + \cos \vartheta X_1)X_2])^2}{2E[X_2^2]} \\ = c_1|\vartheta|^\alpha + O(|\vartheta|^{\alpha+\varepsilon}) \end{aligned} \quad (1.1)$$

and that, as $\vartheta \rightarrow 0$,

$$\begin{aligned} E[\sin \vartheta X_1] - \frac{E[(\sin \vartheta X_1)X_2]E[(-1 + \cos \vartheta X_1)X_2]}{E[X_2^2]} \\ = c_2|\vartheta|^\alpha \frac{\vartheta}{|\vartheta|} + O(|\vartheta|^{\alpha+\varepsilon}). \end{aligned} \quad (1.2)$$

Then

$$P_0\{\tau_{V_-} > n\} \asymp n^{-1/4+\beta}, \quad (1.3)$$

where β is the constant given by

$$\beta = \frac{1}{\alpha\pi} \arcsin \frac{c_2}{\sqrt{2}\sqrt{c_1^2 + c_2^2} + c_1\sqrt{c_1^2 + c_2^2}},$$

and we write $a_n \asymp b_n$ if $c, \tilde{c} > 0$ such that $cb_n \leq a_n \leq \tilde{c}b_n$ for all n .

EXAMPLE 1.2. Suppose that (a), (b), (d), and (I), (II) below hold:

- (I) X_1 and X_2 are independent;
- (II) for $\alpha \in (1, 2)$,

$$P\{X_1 = n\} = \begin{cases} c_- \frac{1}{|n|^{1+\alpha}} & n \in \{-1, -2, \dots\}, \\ c_+ & n = 1, \\ 0 & \text{otherwise} \end{cases}$$

with constants c_- and c_+ satisfying

$$\sum_{n \in \mathbb{Z}} P\{X_1 = n\} = 1 \quad \text{and} \quad \sum_{n \in \mathbb{Z}} n P\{X_1 = n\} = 0.$$

Indeed, by a simple calculation, we see that X_1 satisfies (c), and (1.1) and (1.2) become

$$1 - E[\cos \vartheta X_1] = \frac{c - \pi}{2\Gamma(\alpha + 1) \cos((\alpha - 1)\pi/2)} |\vartheta|^\alpha + O(|\vartheta|^2), \quad (1.4)$$

$$E[\sin \vartheta X_1] = \frac{c - \pi}{2\Gamma(\alpha + 1) \sin((\alpha - 1)\pi/2)} |\vartheta|^\alpha \frac{\vartheta}{|\vartheta|} + O(|\vartheta|^2), \quad (1.5)$$

where Γ is the gamma function (see Appendix B). Then we deduce that $\beta = (2 - \alpha)/4\alpha$ and

$$P_0\{\tau_{V_-} > n\} \asymp n^{-(\alpha-1)/(2\alpha)}.$$

EXAMPLE 1.3. Suppose that X satisfies the same condition as Example 1.2 except with (II) replaced by $P\{X_1 = 1\} = 1$. Then it is clear that for $n \in \mathbb{N}$,

$$P_0\{\tau_{V_-} > n\} = 1.$$

(1.1) and (1.2) become

$$1 - E[\cos \vartheta X_1] = O(|\vartheta|^2), \quad E[\sin \vartheta X_1] = \vartheta + O(|\vartheta|^2).$$

Hence, we obtain

$$P_0\{\tau_{V_-} > n\} \asymp 1,$$

which is in agreement with $P_0\{\tau_{V_-} > n\} = 1$.

Remark. (1) For a two-dimensional simple random walk, (1.3) was shown by Lawler [5, (2.35)] and Kesten [4]. Bousquet-Mélou and Schaeffer [1] considered $P_0\{\tau_{V_-} > n\}$ for a two-dimensional random walk satisfying the condition that the increment X takes finite values with equal probability. Moreover, in the case of a simple random walk, they gave a sharp estimate

$$\lim_{n \rightarrow \infty} P_0\{\tau_{V_-} > n\} \Big/ \left(\frac{\sqrt{1 + \sqrt{2}}}{2\Gamma(3/4)} n^{-1/4} \right) = 1$$

by using a combinatorial argument. Isozaki [3] considered $P_0\{\tau_{V_-} > n\}$ for two-dimensional symmetric random walks.

(2) When S is aperiodic with $E[X] = 0$ and $E[|X|^\delta] < \infty$ for some $\delta > 0$, it is proved that $n^{1/4} P_0\{\tau_{V_-} > n\}$ converges to some positive constant as $n \rightarrow \infty$ (see [2]). Although Theorem 1.1 provides a weaker result for this model than that in [2], it has the advantage that it can be applied to a large class of random walks.

Let T_λ be a random variable independent of S satisfying $P\{T_\lambda = j\} = (1 - \lambda)\lambda^j$ ($j \in \{0\} \cup \mathbb{N}$). The key of the proof of Theorem 1.1 is to estimate $P_0\{T_\lambda < \tau_{V_-}\}$. In fact, by the Tauberian theorem, the asymptotic behavior of $P_0\{T_\lambda < \tau_{V_-}\}$ as $\lambda \uparrow 1$ determines that of $P_0\{\tau_{V_-} > n\}$ as $n \rightarrow \infty$, which is our target in Theorem 1.1.

In [2], we deduce the asymptotic behavior of $P_0\{T_\lambda < \tau_{V_-}\}$ as $\lambda \uparrow 1$ from that of $P_0\{T_\lambda < \tau_{V_-}\} \times P_0\{T_\lambda < \tau_{V_+}\}$ and $P_0\{T_\lambda < \tau_{V_-}\}/P_0\{T_\lambda < \tau_{V_+}\}$. The method to control $P_0\{T_\lambda < \tau_{V_-}\}$ follows from [2]. We need a new device to estimate $P_0\{T_\lambda < \tau_{V_-}\}/P_0\{T_\lambda < \tau_{V_+}\}$.

The organization of the paper is as follows. In Section 2, we quote some results from [2]. In Section 3, we state some results needed to prove Theorem 1.1. In Section 4, we give a proof of Theorem 1.1. In Appendix A, we give the proof of lemmas not proved in Section 3.

2. Fundamental relations

LEMMA 2.1. *For an arbitrary two-dimensional random walk,*

$$\begin{aligned} P_0\{T_\lambda < \tau_U\} &= P_0\{T_\lambda < \tau_{\tilde{V}_+}\}P_0\{T_\lambda < \tau_{V_-}\} \\ &= (1 - P_0\{S(\tau_{V_+}) = 0, \tau_{V_+} \leq T_\lambda\})^{-1}P_0\{T_\lambda < \tau_{V_+}\}P_0\{T_\lambda < \tau_{V_-}\}, \end{aligned}$$

where $\tilde{V}_+ = V_+ \setminus \{0\}$.

The proof of Lemma 2.1 is identical to that given for Proposition 2.4.6 in [5] and is thus omitted (see [2, Lemma 2.1]).

LEMMA 2.2. *A two-dimensional random walk satisfying (a), (b) and $E[X_2^2] < \infty$ has the property that*

$$P_0\{T_\lambda < \tau_U\} \sim (2E[X_2^2])^{1/2}(1 - \lambda)^{1/2},$$

where we write $\alpha_\lambda \sim \beta_\lambda$ if $\lim_{\lambda \uparrow 1} \alpha_\lambda / \beta_\lambda = 1$.

The proof of Lemma 2.2 is identical to that given in [5, p. 68] and is thus omitted (see [2, Proposition 3.1]).

To formulate the next lemma, we introduce some notation. For S , we introduce the following random variables:

$$\begin{aligned} \eta(1) &= \inf\{j \geq 1 \mid S(j) \in U\}, \quad \eta(k+1) = \inf\{j > \eta(k) \mid S(j) \in U\} \quad (k = 1, 2, \dots), \\ \zeta(1) &= S_1(\eta(1)), \quad \zeta(k+1) = S_1(\eta(k+1)) \quad (k = 1, 2, \dots). \end{aligned}$$

For $\lambda \in (0, 1)$, we define

$$\begin{aligned} c(\lambda) &= \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} E_0[\lambda^{\eta(k)}; \zeta(k) = 0]\right), \\ f_\infty(z; \lambda) &= \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} E_0[\lambda^{\eta(k)} z^{\zeta(k)}; \zeta(k) < 0]\right), \\ f_0(z; \lambda) &= \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} E_0[\lambda^{\eta(k)} z^{\zeta(k)}; \zeta(k) > 0]\right). \end{aligned}$$

LEMMA 2.3. *For every two-dimensional random walk satisfying (b) and $E[X_2^2] < \infty$,*

$$\begin{aligned} P_0\{T_\lambda < \tau_{V_-}\} &= c(\lambda)f_\infty(1; \lambda), \\ P_0\{T_\lambda < \tau_{V_+}\} &= c(\lambda)f_0(1; \lambda). \end{aligned}$$

Moreover, for $l \in \mathbb{Z}$,

$$E_0[\lambda^{\eta(k)}; \zeta(k) = l] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta_1 l} \left(1 - \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - \lambda\phi(\theta)} d\theta_2\right)^{-1}\right)^k d\theta_1, \quad (2.1)$$

where $\phi(\theta)$ is the characteristic function of X , that is,

$$\phi(\theta) = E[\exp(i\theta \cdot X)], \quad \theta = (\theta_1, \theta_2) \in \mathbb{R}^2.$$

The proof of Lemma 2.3 is identical to that given in Section 17, Proposition 5 of [7] and is thus omitted (see [2, Proposition 2.4]).

3. Preliminary lemmas

Let $a_\lambda(\theta_1)$ and $b_\lambda(\theta_1)$ be real numbers such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - \lambda\phi(\theta)} d\theta_2 = a_\lambda(\theta_1) + ib_\lambda(\theta_1).$$

Here we write $\theta = (\theta_1, \theta_2)$. Let

$$\begin{aligned} \varphi_\lambda(\theta) &= 1 - \lambda E[\cos \theta_1 X_1] + \lambda E[(\sin \theta_1 X_1) X_2] \theta_2 + \frac{\lambda}{2} E[X_2^2] \theta_2^2 \\ &\quad - i \{ \lambda E[\sin \theta_1 X_1] + \lambda E[(-1 + \cos \theta_1 X_1) X_2] \theta_2 \}. \end{aligned}$$

Then we have the following lemma.

LEMMA 3.1. *Assume that (a)–(d) hold. Then there exists a constant $c_3 > 0$ such that, for $\lambda \in (1/2, 1)$ and $\theta \in [-\pi, \pi]^2$,*

$$\left| \frac{\varphi_\lambda(\theta)}{1 - \lambda\phi(\theta)} - 1 \right| \leq c_3(|\theta_1|^{\hat{\delta}} + |\theta_2|^{\delta}), \quad (3.1)$$

where $\hat{\delta} = \delta^2/(2 + \delta)$. In addition, there exist $\tilde{c}_3 > 0$ and $r_0 > 0$ such that,

$$\left| \frac{\Re[1 - \lambda\phi(\theta)]}{\Re[\varphi_\lambda(\theta)]} - 1 \right| \leq \tilde{c}_3(|\theta_1|^{\hat{\delta}} + |\theta_2|^{\delta}), \quad (3.2)$$

for $\lambda \in (1/2, 1)$ and $\theta \in [-\pi, \pi]^2$ with $|\theta| < r_0$.

Proof. By definition, $1 - \lambda\phi(\theta) = 1 - \lambda E[e^{i\theta_1 X_1} e^{i\theta_2 X_2}]$. We use Taylor polynomials for $\cos x$ and $\sin x$ at $x = 0$ to obtain

$$|e^{ix} - (1 + ix - \frac{1}{2}x^2)| \leq 2 \min\{x^2, |x|^3\} \leq 2|x|^{2+\delta} \quad (x \in \mathbb{R}).$$

This inequality and condition (d) give

$$|E[e^{i\theta_1 X_1} e^{i\theta_2 X_2}] - E[e^{i\theta_1 X_1} (1 + i\theta_2 X_2 - \frac{1}{2}\theta_2^2 X_2^2)]| \leq 2E[|X_2|^{2+\delta}]|\theta_2|^{2+\delta}. \quad (3.3)$$

Using the Hölder inequality and conditions (c) and (d), we obtain

$$E[|X_1|^{\hat{\delta}} X_2^2] < \infty.$$

The inequality $|e^{ix} - 1| \leq 3 \min\{1, |x|\} \leq 3|x|^{\hat{\delta}}$ ($x \in \mathbb{R}$) gives

$$|E[e^{i\theta_1 X_1} X_2^2] - E[X_2^2]| \leq 3E[|X_1|^{\hat{\delta}} X_2^2]|\theta_1|^{\hat{\delta}}. \quad (3.4)$$

We note that

$$E[(-1 + \cos \theta_1 X_1) X_2] = E[(\cos \theta_1 X_1) X_2] \quad (3.5)$$

from condition (b). From (3.3), (3.4), (3.5), we deduce that

$$|1 - \lambda\phi(\theta) - \varphi_\lambda(\theta)| \leq c(|\theta_1|^{\hat{\delta}} + |\theta_2|^{\delta})\theta_2^2 \quad (3.6)$$

for some suitable constant $c > 0$. Since the random walk is aperiodic, there exists a constant $c_* > 0$ such that, for $\lambda \in (1/2, 1)$ and $\theta \in [-\pi, \pi]^2$,

$$|1 - \lambda\phi(\theta)| \geq \Re[1 - \lambda\phi(\theta)] \geq \lambda(1 - \Re[\phi(\theta)]) \geq \frac{1}{2}c_*|\theta|^2 \quad (3.7)$$

as shown in [7, Section 7, Proposition 5]. Equation (3.1) follows from (3.6) and (3.7).

Equation (3.6) implies that

$$|\Re[1 - \lambda\phi(\theta)] - \Re[\varphi_\lambda(\theta)]| \leq c(|\theta_1|^\delta + |\theta_2|^\delta)\theta_2^2. \quad (3.8)$$

By combining (3.8) with the last inequality in (3.7), there exists $r_0 > 0$ (small enough) such that, for $\lambda \in (1/2, 1)$ and $\theta \in [-\pi, \pi]^2$ with $|\theta| < r_0$,

$$\Re[\varphi_\lambda(\theta)] \geq \frac{1}{4}c_*\theta_2^2. \quad (3.9)$$

Equation (3.2) follows from (3.8) and (3.9). $\square$

To estimate $a_\lambda(\theta)$, we calculate

$$\frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \Re \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2,$$

where r_0 is the same constant as in Lemma 3.1. Let

$$\begin{aligned} A &= \frac{E[(\sin \theta_1 X_1)X_2]}{(1/2)E[X_2^2]}, & B &= \frac{1 - \lambda E[\cos \theta_1 X_1]}{(\lambda/2)E[X_2^2]}, \\ C &= \frac{E[(-1 + \cos \theta_1 X_1)X_2]}{(1/2)E[X_2^2]}, & D &= \frac{E[\sin \theta_1 X_1]}{(1/2)E[X_2^2]}. \end{aligned}$$

Then

$$\varphi_\lambda(\theta) = \frac{\lambda}{2} E[X_2^2] \{ \theta_2^2 + A\theta_2 + B - i(C\theta_2 + D) \}$$

and, for each θ_1 ,

$$\Re \left[\frac{1}{\varphi_\lambda(\theta)} \right] = \frac{\Re[\varphi_\lambda(\theta)]}{|\varphi_\lambda(\theta)|^2}$$

is a rational function in θ_2 .

Put

$$K = (4B - A^2 + C^2)^2 + 4(2D - AC)^2.$$

To calculate the integral of $\Re[1/\varphi_\lambda(\theta)]$ over $[-r_0/2, r_0/2]$ with respect to θ_2 , we use routine integration of a rational function using partial fractions. This calculation gives the following lemma.

LEMMA 3.2. *Suppose that $0 < E[X_2^2] < \infty$. Then there exist $c_4 > 0$, $\lambda_* \in (1/2, 1)$ and $s_* > 0$ such that, for all $\lambda \in (\lambda_*, 1)$ and $\theta_1 \in [-\pi, \pi]$ with $0 < |\theta_1| < s_*$,*

$$\left| \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \Re \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2 - \tilde{a}_\lambda(\theta_1) \right| \leq c_4, \quad (3.10)$$

where

$$\tilde{a}_\lambda(\theta_1) = \frac{\sqrt{2}}{\lambda E[X_2^2] \sqrt{K}} \sqrt{4B - A^2 + C^2 + \sqrt{K}}.$$

From (3.1), (3.6), (3.7) and (3.9), we deduce the following.

LEMMA 3.3. *Suppose that (a)–(d). Then there exist $c_5 > 0$, $\lambda_* \in (1/2, 1)$ and $s_* > 0$ such that, for $\lambda \in (\lambda_*, 1)$ and $\theta_1 \in [-\pi, \pi]$ with $0 < |\theta_1| < s_*$,*

$$\left| a_\lambda(\theta_1) - \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \Re \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2 \right| \leq c_5 (|\theta_1|^\delta + K^{\delta/4} |\log K|) K^{-1/4}.$$

LEMMA 3.4. *Suppose that $0 < E[X_2^2] < \infty$. Then there exist $c_6 > 0$, $\lambda_* \in (1/2, 1)$ and $s_* > 0$ such that, for all $\lambda \in (\lambda_*, 1)$ and $\theta_1 \in (-\pi, \pi)$ with $0 < |\theta_1| < s_*$,*

$$\left| \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \Im \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2 - \tilde{b}_\lambda(\theta_1) \right| \leq c_6 \left(\frac{|2D - AC|}{\sqrt{K}} + \frac{|C|}{K^{1/4}} \right), \quad (3.11)$$

where

$$\tilde{b}_\lambda(\theta_1) = \frac{\sqrt{2}}{\lambda E[X_2^2] \sqrt{K}} \frac{2(2D - AC)}{\sqrt{4B - A^2 + C^2 + \sqrt{K}}}.$$

LEMMA 3.5. *Suppose that (a)–(d) hold. Then there exist $c_7 > 0$, $\lambda_* \in (1/2, 1)$ and $s_* > 0$ such that, for $\lambda \in (\lambda_*, 1)$ and $\theta_1 \in [-\pi, \pi]$ with $0 < |\theta_1| < s_*$,*

$$\begin{aligned} & \left| b_\lambda(\theta_1) - \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \Im \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2 \right| \\ & \leq c_7 \{ |\theta_1|^\delta + |A| K^{(-1+\delta)/4} + (|\theta_1|^\delta + K^{\delta/4}) (|C| + |D| K^{-1/4}) K^{-1/4} \} K^{-1/4}. \end{aligned}$$

The proofs of the above four lemmas are given in Appendix A.

4. Proof of Theorem 1.1

To prove Theorem 1.1, we consider $P_0\{T_\lambda < \tau_{V_-}\} / P_0\{T_\lambda < \tau_{V_+}\}$.

LEMMA 4.1. *Suppose that (b) holds and $E[X_2^2] < \infty$. Then*

$$\frac{P_0\{T_\lambda < \tau_{V_-}\}}{P_0\{T_\lambda < \tau_{V_+}\}} = \frac{f_\infty(1; \lambda)}{f_0(1; \lambda)} = \exp \left(\lim_{L \rightarrow \infty} C_L(\lambda) \right), \quad (4.1)$$

where

$$C_L(\lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{\sin \vartheta}{1 - \cos \vartheta} (1 - \cos L\vartheta) + \sin L\vartheta \right\} \arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} d\vartheta. \quad (4.2)$$

Proof. By Lemma 2.3 and the definitions of $f_0(z; \lambda)$ and $f_\infty(z; \lambda)$, we have

$$\frac{P_0\{T_\lambda < \tau_{V_-}\}}{P_0\{T_\lambda < \tau_{V_+}\}} = \frac{f_\infty(1; \lambda)}{f_0(1; \lambda)} = \exp \left(\lim_{L \rightarrow \infty} C_L(\lambda) \right),$$

where

$$C_L(\lambda) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^L (E_0[\lambda^{\eta(k)}; \zeta(k) = l] - E_0[\lambda^{\eta(k)}; \zeta(k) = -l]).$$

To obtain an expression for $C_L(\lambda)$, we use (2.1) in Lemma 2.3 which implies that

$$C_L(\lambda) = \Re \left[\sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} e_L(\theta_1) \frac{1}{k} \left(1 - \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - \lambda \phi(\theta)} d\theta_2 \right)^{-1} \right)^k d\theta_1 \right], \quad (4.3)$$

where

$$e_L(\theta_1) = \sum_{l=1}^L (e^{-il\theta_1} - e^{il\theta_1}).$$

From the inequality [2, pp. 332–333]

$$\sup_{\theta_1 \in [-\pi, \pi]} \left| 1 - \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - \lambda\phi(\theta)} d\theta_2 \right)^{-1} \right| < 1,$$

we interchange the order of summation and integration on the right-hand side of (4.3), and then use the identity

$$-\sum_{k=1}^{\infty} \frac{z^k}{k} = \text{Log}(1 - z) \quad (|z| < 1).$$

Here, $\text{Log}(z)$ is the principal logarithm of z and we choose $|\Im[\text{Log}(z)]| < \pi$. Here $C_L(\lambda)$ reduces to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e_L(\theta_1) \text{Log} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - \lambda\phi(\theta)} d\theta_2 \right) d\theta_1.$$

By a simple calculation,

$$e_L(\theta_1) = i \left\{ \frac{\sin \theta_1}{1 - \cos \theta_1} (-1 + \cos L\theta_1) - \sin L\theta_1 \right\}.$$

If $\Re[z] > 0$, then $\Im[\text{Log}(z)] \in (-\pi/2, \pi/2)$ and

$$\Im[\text{Log}(z)] = \arcsin \frac{\Im[z]}{|z|}.$$

Recall that $a_\lambda(\theta_1)$ is the real part of $(1/(2\pi)) \int_{-\pi}^{\pi} 1/(1 - \lambda\phi(\theta)) d\theta_2$ and $b_\lambda(\theta_1)$ the imaginary part. It is easy to check that $a_\lambda(\theta_1) > 0$. Thus, we have (4.2). $\square$

Let

$$B(\vartheta) = 1 - E[\cos \vartheta X_1] - \frac{(E[(\sin \vartheta X_1)X_2])^2}{2E[X_2^2]} + \frac{(E[(-1 + \cos \vartheta X_1)X_2])^2}{2E[X_2^2]},$$

$$D(\vartheta) = E[\sin \vartheta X_1] - \frac{E[(\sin \vartheta X_1)X_2]E[(-1 + \cos \vartheta X_1)X_2]}{E[X_2^2]}$$

and

$$K_\lambda(\vartheta) = (1 - \lambda + \lambda B(\vartheta))^2 + (\lambda D(\vartheta))^2.$$

Then $\tilde{a}_\lambda$ in Lemma 3.2 can be written as

$$\tilde{a}_\lambda(\vartheta) = \frac{1}{2\sqrt{\lambda E[X_2^2]K_\lambda(\vartheta)^{1/2}}} \sqrt{1 - \lambda + \lambda B(\vartheta) + K_\lambda(\vartheta)^{1/2}},$$

and $\tilde{b}_\lambda$ in Lemma 3.4 can be written as

$$\tilde{b}_\lambda(\vartheta) = \frac{1}{2\sqrt{\lambda E[X_2^2]K_\lambda(\vartheta)^{1/2}}} \frac{\lambda D(\vartheta)}{\sqrt{1 - \lambda + \lambda B(\vartheta) + K_\lambda(\vartheta)^{1/2}}}.$$

Hence,

$$\frac{\tilde{b}_\lambda(\vartheta)}{\sqrt{\tilde{a}_\lambda^2(\vartheta) + \tilde{b}_\lambda^2(\vartheta)}} = \frac{\lambda D(\vartheta)}{\sqrt{2}\sqrt{K_\lambda(\vartheta) + (1 - \lambda + \lambda B(\vartheta))K_\lambda(\vartheta)^{1/2}}}. \quad (4.4)$$

To prove (1.3) in Theorem 1.1 for $c_1 > 0$ and $c_2 > 0$, we assume the following condition.

Assumption (A1). There exist constants $\alpha \in (0, 2]$ and $\varepsilon, c_1, c_2 > 0$ such that

$$B(\vartheta) = c_1|\vartheta|^\alpha + O(|\vartheta|^{\alpha+\varepsilon}) \quad (\vartheta \rightarrow 0)$$

and

$$D(\vartheta) = c_2|\vartheta|^\alpha \frac{\vartheta}{|\vartheta|} + O(|\vartheta|^{\alpha+\varepsilon}) \quad (\vartheta \rightarrow 0).$$

From assumption (A1) and the inequality

$$|\sqrt{x+y} - \sqrt{x'+y'}| \leq |\sqrt{x} - \sqrt{x'}| + |\sqrt{y} - \sqrt{y'}| \quad (x, y, x', y' > 0),$$

it is easy to see that there exist constants $c_8 > 0, s > 0$ such that, for $\lambda \in (1/2, 1)$ and $0 < |\vartheta| \leq s$,

$$\begin{aligned} |K_\lambda(\vartheta)^{1/2} - k_\lambda(\vartheta)^{1/2}| &= |\sqrt{(1-\lambda+\lambda B(\vartheta))^2 + (\lambda D(\vartheta))^2} - k_\lambda(\vartheta)^{1/2}| \\ &\leq |B(\vartheta) - c_1|\vartheta|^\alpha| + \left| D(\vartheta) - c_2|\vartheta|^\alpha \frac{\vartheta}{|\vartheta|} \right| \leq c_8 k_\lambda(\vartheta)^{1/2} |\vartheta|^\varepsilon, \end{aligned} \quad (4.5)$$

where

$$k_\lambda(\vartheta) = (1 - \lambda + \lambda c_1 |\vartheta|^\alpha)^2 + \left(\lambda c_2 |\vartheta|^\alpha \frac{\vartheta}{|\vartheta|} \right)^2.$$

Equation (4.5) implies the following lemma.

LEMMA 4.2. Assume that $0 < E[X_2^2] < \infty$ and (A1) hold. Then there exist $c_9 > 0, s_* > 0$ such that, for $\lambda \in (1/2, 1)$ and $0 < |\vartheta| \leq s_*$,

$$\left| \frac{\tilde{b}_\lambda(\vartheta)}{\sqrt{\tilde{a}_\lambda^2(\vartheta) + \tilde{b}_\lambda^2(\vartheta)}} - \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right| \leq c_9 |\vartheta|^\varepsilon,$$

where

$$Q_\lambda(s) = \{(1 - \lambda + \lambda c_1 s)^2 + (\lambda c_2 s)^2 + (1 - \lambda + \lambda c_1 s) \sqrt{(1 - \lambda + \lambda c_1 s)^2 + (\lambda c_2 s)^2}\}^{1/2}.$$

Proof. In view of (4.4), we use

$$\begin{aligned} &\left| \frac{z}{\sqrt{x+y}} - \frac{z'}{\sqrt{x'+y'}} \right| \\ &\leq \frac{|z - z'|}{\sqrt{x'}} + |z| \frac{|\sqrt{x} - \sqrt{x'}|}{\sqrt{x}\sqrt{x'}} + |z| \frac{|y - y'|}{\sqrt{x}x'} \quad (x, y, x', y' > 0, z, z' \in \mathbb{R}) \end{aligned}$$

to obtain

$$\begin{aligned} &\sqrt{2} \left| \frac{\tilde{b}_\lambda(\vartheta)}{\sqrt{\tilde{a}_\lambda^2(\vartheta) + \tilde{b}_\lambda^2(\vartheta)}} - \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right| \\ &= \left| \frac{\lambda D(\vartheta)}{\sqrt{K_\lambda(\vartheta) + (1 - \lambda + \lambda B(\vartheta)) K_\lambda(\vartheta)^{1/2}}} - \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{k_\lambda(\vartheta) + (1 - \lambda + \lambda c_1 |\vartheta|^\alpha) k_\lambda(\vartheta)^{1/2}}} \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{|D(\vartheta) - c_2|\vartheta|^\alpha(\vartheta/|\vartheta|)|}{k_\lambda(\vartheta)^{1/2}} + |D(\vartheta)| \frac{|K_\lambda(\vartheta)^{1/2} - k_\lambda(\vartheta)^{1/2}|}{K_\lambda(\vartheta)^{1/2}k_\lambda(\vartheta)^{1/2}} \\
&\quad + |D(\vartheta)| \frac{|(1 - \lambda + \lambda B(\vartheta))K_\lambda(\vartheta)^{1/2} - (1 - \lambda + \lambda c_1|\vartheta|^\alpha)k_\lambda(\vartheta)^{1/2}|}{K_\lambda(\vartheta)^{1/2}k_\lambda(\vartheta)}. \tag{4.6}
\end{aligned}$$

From (A1) and

$$k_\lambda(\vartheta)^{1/2} \geq \frac{1}{2}c_1|\vartheta|^\alpha \quad (\lambda \in (1/2, 1), \vartheta \in [-\pi, \pi]), \tag{4.7}$$

the first term on the right-hand side of the last inequality in (4.6) is bounded by a constant multiple of $|\vartheta|^\varepsilon$.

If $|\vartheta|$ is small enough (i.e. $c_8|\vartheta|^\varepsilon \leq 1/2$), then (4.5) gives

$$K_\lambda(\vartheta)^{1/2} \geq \frac{1}{2}k_\lambda(\vartheta)^{1/2}.$$

With the help of the above inequality and (4.7), (A1) and (4.5) imply that the second term on the right-hand side of the last inequality in (4.6) is bounded by a constant multiple of $|\vartheta|^\varepsilon$.

By using the triangle inequality, the third term on the right-hand side of the last inequality in (4.6) is bounded by

$$|D(\vartheta)| \frac{|B(\vartheta) - c_1|\vartheta|^\alpha|}{k_\lambda(\vartheta)} + |D(\vartheta)| \frac{|K_\lambda(\vartheta)^{1/2} - k_\lambda(\vartheta)^{1/2}|}{K_\lambda(\vartheta)^{1/2}k_\lambda(\vartheta)} |1 - \lambda + \lambda c_1|\vartheta|^\alpha|.$$

From (A1) and (4.7), the first term of the above is bounded by a constant multiple of $|\vartheta|^\varepsilon$. It is proved that the second term of the above is bounded by a constant multiple of $|\vartheta|^\varepsilon$ by the same argument as showing that the second term on the right-hand side of the last inequality in (4.6) is bounded by a constant multiple of $|\vartheta|^\varepsilon$.

Hence, we have the desired inequality. $\square$

From (3.9) and the definitions of A and B ,

$$\frac{1}{4}c_*|\theta_2|^2 \leq \Re[\varphi_\lambda(\theta_1, \theta_2)] = \frac{\lambda}{2}E[X_2^2] \left\{ \left(\theta_2 + \frac{A}{2} \right)^2 + B - \frac{A^2}{4} \right\}$$

for $\lambda \in (1/2, 1)$ and $|(\theta_1, \theta_2)| \leq r_0$. Since A tends to zero as $|\theta_1| \rightarrow 0$, we choose θ_1 small enough so that $|(\theta_1, -A/2)| \leq r_0$. Then the above inequality (setting $\theta_2 = -A/2$) gives

$$\frac{c_*}{2\lambda E[X_2^2]} A^2 \leq 4B - A^2,$$

and letting $\lambda \uparrow 1$,

$$\frac{c_*}{2E[X_2^2]} A^2 \leq 4 \frac{1 - E[\cos \theta_1 X_1]}{(1/2)E[X_2^2]} - A^2 \leq 8 \frac{B(\theta_1)}{E[X_2^2]}.$$

From $1 - E[\cos \theta_1 X_1] \geq (E[(\sin \theta_1 X_1)X_2])^2/(2E[X_2^2])$, it follows that

$$C^2 \leq \frac{8}{E[X_2^2]} B(\theta_1).$$

Combining the above inequalities with $2|D| \leq |2D - AC| + |A||C|$, and using (4.5), Lemmas 3.2–3.5 give the following lemma.

LEMMA 4.3. Assume that (a)–(d) and (A1) hold. Then there exist $c_{10} > 0$, $\lambda_* \in (1/2, 1)$ and $s_* \in (0, 1)$ such that, for $\lambda \in (\lambda_*, 1)$ and $0 < |\vartheta| \leq s_*$,

$$\begin{aligned} |a_\lambda(\vartheta) - \tilde{a}_\lambda(\vartheta)| &\leq c_{10}(|\vartheta|^{\hat{\delta}} + k_\lambda^{\delta/4}(\vartheta)|\log k_\lambda(\vartheta)|)k_\lambda^{-1/4}(\vartheta), \\ |b_\lambda(\vartheta) - \tilde{b}_\lambda(\vartheta)| &\leq c_{10}|\vartheta|^{\tilde{\delta}}k_\lambda^{-1/4}(\vartheta), \\ \left| \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} - \frac{\tilde{b}_\lambda(\vartheta)}{\sqrt{\tilde{a}_\lambda^2(\vartheta) + \tilde{b}_\lambda^2(\vartheta)}} \right| &\leq c_{10}|\vartheta|^{\tilde{\delta}}|\log k_\lambda(\vartheta)|, \end{aligned}$$

where $\tilde{\delta} = \min\{\hat{\delta}, \alpha\delta/2\}$.

LEMMA 4.4. Assume that (a)–(d) and (A1) hold. Then there exist $c_{11} > 0$, $\lambda^* \in (1/2, 1)$ and $s^* \in (0, 1)$ such that, for $\lambda \in (\lambda^*, 1)$ and $0 < |\vartheta| \leq s^*$,

$$\left| \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} - \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right| \leq c_{11} |\vartheta|^{\delta_0}, \quad (4.8)$$

where $\delta_0 = \min\{\varepsilon, \tilde{\delta}/2\}$.

Proof. We apply Lemmas 4.2 and 4.3 to obtain

$$\left| \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} - \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right| \leq c_9 |\vartheta|^\varepsilon + c_{10} |\vartheta|^{\tilde{\delta}} |\log k_\lambda(\vartheta)|. \quad (4.9)$$

From the definition of $k_\lambda(\vartheta)$, it is easy to see that

$$\frac{1}{4}(c_1^2 + c_2^2)|\vartheta|^{2\alpha} \leq k_\lambda(\vartheta) \leq 1 \quad (\lambda \in (1/2, 1), |\vartheta|^\alpha \leq 1/(4 \max\{c_1, c_2\})),$$

which gives

$$|\vartheta|^{\tilde{\delta}/2} |\log k_\lambda(\vartheta)| \leq |\vartheta|^{\tilde{\delta}/2} |\log \frac{1}{4}(c_1^2 + c_2^2)|\vartheta|^{2\alpha}|.$$

If ϑ is small enough, then the right-hand side of the above inequality is bounded by 1 and the second term of the right-hand side of (4.9) is bounded by $c_{10}|\vartheta|^{\tilde{\delta}/2}$. Thus, we have (4.8). $\square$

LEMMA 4.5. Assume that (a)–(d) and (A1) hold. Then there exist $c_{12} > 0$, $\lambda_0 \in (1/2, 1)$ and $s_0 \in (0, 1)$ such that, for $\lambda \in (\lambda_0, 1)$ and $0 < |\vartheta| \leq s_0$,

$$\left| \arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} - \arcsin \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right| \leq c_{12} |\vartheta|^{\delta_0} \quad (4.10)$$

Proof. By applying the mean value theorem to the inverse of the sin function, $\arcsin t$, on the closed interval with end points $\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)/(\sqrt{2} Q_\lambda(|\vartheta|^\alpha))$ and $b_\lambda(\vartheta)/\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}$, the left-hand side of (4.10) is equal to

$$\frac{1}{\sqrt{1 - t_0^2}} \left| \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} - \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right|, \quad (4.11)$$

where t_0 is a number between $\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)/(\sqrt{2} Q_\lambda(|\vartheta|^\alpha))$ and $b_\lambda(\vartheta)/\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}$.

Note that for $\vartheta \neq 0$,

$$\left| \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right| \leq \frac{1}{\sqrt{2}}. \quad (4.12)$$

If ϑ is small enough, then (4.8) implies that

$$\left| \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right| \leq \frac{1}{\sqrt{2}} + c_{11} |\vartheta|^{\delta_0} < \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right),$$

and we obtain

$$|t_0| \leq \max \left\{ \left| \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right|, \left| \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right| \right\} \leq \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) < 1.$$

Equation (4.11), together with (4.8) and the above inequality, gives the desired estimate (4.10). $\square$

To estimate $\lim_{L \rightarrow \infty} C_L(\lambda)$, we decompose $C_L(\lambda)$. With $s_0 > 0$ being the same constant as in Lemma 4.5,

$$\begin{aligned} C_L(\lambda) &= \frac{1}{\pi} \int_{-s_0}^{s_0} \frac{1}{\vartheta} \left(\arcsin \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right) d\vartheta \\ &\quad + I_1(s_0, \lambda) + I_2(\lambda, L) + I_3(\lambda, L) + I_4(s_0, \lambda), \end{aligned}$$

where

$$\begin{aligned} I_1(s_0, \lambda) &= \frac{1}{2\pi} \int_{-s_0}^{s_0} \frac{\sin \vartheta}{1 - \cos \vartheta} \left(\arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right) \\ &\quad - \frac{2}{\vartheta} \left(\arcsin \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right) d\vartheta, \\ I_2(\lambda, L) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right) \sin(L\vartheta) d\vartheta, \\ I_3(\lambda, L) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right) \frac{\sin \vartheta}{1 - \cos \vartheta} \cos(L\vartheta) d\vartheta, \\ I_4(s_0, \lambda) &= \frac{1}{2\pi} \int_{s_0 < |\vartheta| \leq \pi} \frac{\sin \vartheta}{1 - \cos \vartheta} \left(\arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right) d\vartheta. \end{aligned}$$

Since $\arcsin t \in [-\pi/2, \pi/2]$, $\arcsin(b_\lambda(\vartheta)/\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)})$ is integrable on $[-\pi, \pi]$. The Riemann–Lebesgue lemma implies that

$$\lim_{L \rightarrow \infty} I_2(\lambda, L) = 0.$$

It is easy to verify that

$$\begin{aligned} |I_4(s_0, \lambda)| &\leq \frac{1}{2\pi} \int_{s_0 < |\vartheta| \leq \pi} \left| \frac{\sin \vartheta}{1 - \cos \vartheta} \left(\arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right) \right| d\vartheta \\ &\leq \frac{\pi}{2} \frac{1}{1 - \cos s_0}. \end{aligned}$$

By applying mean value theorem to the function $\arcsin t$ on the closed interval with end points $\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)/(\sqrt{2}Q_\lambda(|\vartheta|^\alpha))$ and 0, and using (4.12),

$$\left| \arcsin \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2}Q_\lambda(|\vartheta|^\alpha)} \right| \leq \sqrt{2} \left| \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2}Q_\lambda(|\vartheta|^\alpha)} \right| \leq \frac{\lambda c_2 |\vartheta|^\alpha}{1-\lambda}.$$

This inequality and (4.10) give

$$\left| \left(\arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right) \frac{\sin \vartheta}{1 - \cos \vartheta} \right| \leq \begin{cases} \left(\frac{\lambda c_2 |\vartheta|^\alpha}{1-\lambda} + c_{12} |\vartheta|^{\delta_0} \right) \frac{\pi}{|\vartheta|} & (0 < |\vartheta| \leq s_0), \\ \frac{\pi}{2} \frac{1}{1 - \cos s_0} & (s_0 < |\vartheta| \leq \pi). \end{cases}$$

For fixed $\lambda \in (\lambda_0, 1)$, the above inequality implies that the left-hand side of the above is integrable on $[-\pi, \pi]$. By the Riemann–Lebesgue lemma,

$$\lim_{L \rightarrow \infty} I_3(\lambda, L) = 0.$$

As for $I_1(s_0, \lambda)$, we consider the absolute value of the integrand in $I_1(s_0, \lambda)$. From (4.10) and $\arcsin t \in [-\pi/2, \pi/2]$,

$$\begin{aligned} & \left| \frac{\sin \vartheta}{1 - \cos \vartheta} \left(\arcsin \frac{b_\lambda(\vartheta)}{\sqrt{a_\lambda^2(\vartheta) + b_\lambda^2(\vartheta)}} \right) - \frac{2}{\vartheta} \left(\arcsin \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2}Q_\lambda(|\vartheta|^\alpha)} \right) \right| \\ & \leq \frac{2}{|\vartheta|} c_{12} |\vartheta|^{\delta_0} + \left| \frac{\sin \vartheta}{1 - \cos \vartheta} - \frac{2}{\vartheta} \right| \frac{\pi}{2} \leq 2c_{12} |\vartheta|^{\delta_0-1} + \frac{\pi}{2} |\vartheta|. \end{aligned}$$

Hence, $I_1(s_0, \lambda)$ is bounded by a constant independent of $\lambda \in (\lambda_0, 1)$. We have the following estimate.

There exists a constant c_{13} such that, for $\lambda \in (\lambda_0, 1)$,

$$\left| \lim_{L \rightarrow \infty} C_L(\lambda) - \frac{1}{\pi} \int_{-s_0}^{s_0} \frac{1}{\vartheta} \left(\arcsin \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2}Q_\lambda(|\vartheta|^\alpha)} \right) d\vartheta \right| \leq c_{13}. \quad (4.13)$$

LEMMA 4.6. Assume that (a)–(d) and (A1) hold. Then there exists a constant $c_{14} > 0$ such that, for $\lambda \in (\lambda_0, 1)$,

$$\left| \lim_{L \rightarrow \infty} C_L(\lambda) + \frac{2}{\alpha\pi} \left(\arcsin \frac{c_2}{\sqrt{2}\sqrt{c_1^2 + c_2^2 + c_1\sqrt{c_1^2 + c_2^2}}} \right) \log(1 - \lambda) \right| \leq c_{14}. \quad (4.14)$$

Proof. Equation (4.13) reduces our problem to the estimate for

$$I_0(s_0, \lambda) := \frac{1}{\pi} \int_{-s_0}^{s_0} \frac{1}{\vartheta} \left(\arcsin \frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2}Q_\lambda(|\vartheta|^\alpha)} \right) d\vartheta.$$

To estimate this integral, we use the power series representation for $\arcsin t$:

$$\arcsin t = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{t^{2n+1}}{2n+1} \quad (-1 < t < 1).$$

By the dominated convergence theorem,

$$I_0(s_0, \lambda) = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{1}{2n+1} \frac{1}{\pi} \int_{-s_0}^{s_0} \frac{1}{\vartheta} \left(\frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2}Q_\lambda(|\vartheta|^\alpha)} \right)^{2n+1} d\vartheta. \quad (4.15)$$

We note that the function

$$\frac{1}{\vartheta} \left(\frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right)^{2n+1}$$

(on the right-hand side of (4.15)) is an even function, and perform the change of variables $s = \vartheta^\alpha$, so that

$$\begin{aligned} \int_{-s_0}^{s_0} \frac{1}{\vartheta} \left(\frac{\lambda c_2 |\vartheta|^\alpha (\vartheta/|\vartheta|)}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right)^{2n+1} d\vartheta &= 2 \int_0^{s_0} \frac{1}{\vartheta} \left(\frac{\lambda c_2 |\vartheta|^\alpha}{\sqrt{2} Q_\lambda(|\vartheta|^\alpha)} \right)^{2n+1} d\vartheta \\ &= \frac{2}{\alpha} \left(\frac{\lambda c_2}{\sqrt{2}} \right)^{2n+1} \int_0^{s_0^\alpha} \frac{s^{2n}}{(Q_\lambda(s))^{2n+1}} ds. \end{aligned} \quad (4.16)$$

Put

$$Q_\lambda^{(1)}(s) = \{\lambda^2(c_1^2 + c_2^2 + c_1\sqrt{c_1^2 + c_2^2})s^2 + (1-\lambda)\lambda(3c_1 + \sqrt{c_1^2 + c_2^2})s + 2(1-\lambda)^2\}^{1/2}$$

and

$$\begin{aligned} Q_\lambda^{(2)}(s) &= \left\{ \lambda^2(c_1^2 + c_2^2 + c_1\sqrt{c_1^2 + c_2^2})s^2 \right. \\ &\quad \left. + (1-\lambda)\lambda \left(2c_1 + \sqrt{c_1^2 + c_2^2} + \frac{c_1^2}{\sqrt{c_1^2 + c_2^2}} \right) s + (1-\lambda)^2 \left(1 + \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \right) \right\}^{1/2}. \end{aligned}$$

Here $Q_\lambda^{(1)}(s)$ is obtained by replacing $\sqrt{(1-\lambda+\lambda c_1 s)^2 + (\lambda c_2 s)^2}$ in the definition of $Q_\lambda(s)$ with $\lambda\sqrt{c_1^2 + c_2^2}s + 1-\lambda$ and $Q_\lambda^{(2)}(s)$ is obtained by replacing $\sqrt{(1-\lambda+\lambda c_1 s)^2 + (\lambda c_2 s)^2}$ in the definition of $Q_\lambda(s)$ with $\lambda\sqrt{c_1^2 + c_2^2}s + (1-\lambda)(c_1/\sqrt{c_1^2 + c_2^2})$.

From

$$\lambda\sqrt{c_1^2 + c_2^2}s + (1-\lambda)\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \leq \sqrt{(1-\lambda+\lambda c_1 s)^2 + (\lambda c_2 s)^2} \leq \lambda\sqrt{c_1^2 + c_2^2}s + 1-\lambda$$

for $s > 0$, we have

$$\int_0^{s_0^\alpha} \frac{s^{2n}}{(Q_\lambda^{(1)}(s))^{2n+1}} ds \leq \int_0^{s_0^\alpha} \frac{s^{2n}}{(Q_\lambda(s))^{2n+1}} ds \leq \int_0^{s_0^\alpha} \frac{s^{2n}}{(Q_\lambda^{(2)}(s))^{2n+1}} ds. \quad (4.17)$$

For $a, b, c \in (0, \infty)$ with $4ac - b^2 > 0$, we next evaluate the integral

$$I_n^\alpha(s_0) := \int_0^{s_0^\alpha} \frac{s^{2n}}{(as^2 + bs + c)^{(2n+1)/2}} ds \quad (n \in \{0\} \cup \mathbb{N}).$$

If $n = 0$,

$$I_0^\alpha(s_0) = \frac{1}{\sqrt{a}} \left[\log \left(2as + b + 2\sqrt{a(as^2 + bs + c)} \right) \right]_0^{s_0^\alpha}.$$

By setting $a = \lambda^2(c_1^2 + c_2^2 + c_1\sqrt{c_1^2 + c_2^2})$, $b = (1-\lambda)\lambda(3c_1 + \sqrt{c_1^2 + c_2^2})$, $c = 2(1-\lambda)^2$, this identity yields the following estimate.

There exists a constant $c_{15} > 0$ such that, for $\lambda \in (1/2, 1)$,

$$\left| \frac{\lambda c_2}{\sqrt{2}} \int_0^{s_0^\alpha} \frac{1}{Q_\lambda^{(1)}(s)} ds + \frac{c_2}{\sqrt{2} \sqrt{c_1^2 + c_2^2 + c_1 \sqrt{c_1^2 + c_2^2}}} \log(1 - \lambda) \right| \leq c_{15}. \quad (4.18)$$

If $n \geq 1$, we note that

$$\frac{d}{ds} \frac{1}{(as^2 + bs + c)^{(2n-1)/2}} = -\frac{2n-1}{2} \frac{2as + b}{(as^2 + bs + c)^{(2n+1)/2}}.$$

The formula for integration by parts gives

$$I_n^\alpha(s_0) = \frac{1}{a} I_{n-1}^\alpha(s_0) + e_n,$$

where

$$e_n = -\frac{1}{(2n-1)a} \frac{s_0^{(2n-1)\alpha}}{(as_0^{2\alpha} + bs_0^\alpha + c)^{(2n-1)/2}} - \frac{b}{2a} \int_0^{s_0^\alpha} \frac{s^{2n-1}}{(as^2 + bs + c)^{(2n+1)/2}} ds.$$

By using the above relation repeatedly, we obtain

$$I_n^\alpha(s_0) = \frac{1}{a^n} I_0^\alpha(s_0) + \sum_{j=0}^{n-1} \frac{1}{a^j} e_{n-j}. \quad (4.19)$$

Since a, b, c are positive and satisfy the inequality $4ac - b^2 > 0$,

$$\begin{aligned} |e_n| &\leq \frac{1}{a^{(2n+1)/2}} + \frac{b}{2a^{(2n+1)/2}} \int_0^{s_0^\alpha} \frac{1}{as^2 + bs + c} ds \\ &\leq \frac{1}{a^{(2n+1)/2}} \left(1 + \frac{\pi}{2} \frac{b}{\sqrt{4ac - b^2}} \right). \end{aligned} \quad (4.20)$$

With the help of (4.19) and (4.20), and by setting $a = \lambda^2(c_1^2 + c_2^2 + c_1 \sqrt{c_1^2 + c_2^2})$, $b = (1 - \lambda)\lambda(3c_1 + \sqrt{c_1^2 + c_2^2})$, $c = 2(1 - \lambda)^2$, (4.18) implies the following estimate for $k = 1$.

There exists a constant $c_{16} > 0$ such that, for $\lambda \in (1/2, 1)$ and $n \in \{0\} \cup \mathbb{N}$,

$$\begin{aligned} &\left| \left(\frac{\lambda c_2}{\sqrt{2}} \right)^{2n+1} \int_0^{s_0^\alpha} \frac{s^{2n}}{(Q_\lambda^{(k)}(s))^{2n+1}} ds + \left(\frac{c_2}{\sqrt{2} \sqrt{c_1^2 + c_2^2 + c_1 \sqrt{c_1^2 + c_2^2}}} \right)^{2n+1} \log(1 - \lambda) \right| \\ &\leq c_{16}(n+1) \left(\frac{1}{\sqrt{2}} \right)^{2n} \quad (k = 1, 2). \end{aligned} \quad (4.21)$$

The same argument as in the proof of the estimate for $k = 1$ in (4.21) gives the estimate for $k = 2$ in (4.21).

By (4.17), (4.21) also holds when the integral of $s^{2n}/(Q_\lambda^{(k)}(s))^{2n+1}$ over $[0, s_0^\alpha]$ with respect to s is replaced by the integral of $s^{2n}/(Q_\lambda(s))^{2n+1}$ over $[0, s_0^\alpha]$ with respect to s . This estimate implies (4.14) in view of (4.16) and (4.15). $\square$

We recall (4.1). It immediate from Lemma 4.6 that, for $\lambda \in (\lambda_0, 1)$,

$$(1 - \lambda)^{-2\beta} e^{-c_{14}} \leq \frac{P_0\{T_\lambda < \tau_{V_-}\}}{P_0\{T_\lambda < \tau_{V_+}\}} \leq (1 - \lambda)^{-2\beta} e^{c_{14}},$$

where β is the same constant as in Theorem 1.1. By combining the above inequalities, Lemmas 2.1 and 2.2, we have the following estimate.

PROPOSITION 4.7. *Assume that (a)–(d) and (A1) hold. Then there exist constants c_{17} , $c_{18} > 0$ and $\lambda_0^* \in (1/2, 1)$ such that, for $\lambda \in (\lambda_0^*, 1)$,*

$$c_{17}(1 - \lambda)^{-\beta+(1/4)} \leq P_0\{T_\lambda < \tau_{V_-}\} = (1 - \lambda) \sum_{n=0}^{\infty} \lambda^n P_0\{\tau_{V_-} > n\} \leq c_{18}(1 - \lambda)^{-\beta+(1/4)}.$$

This asymptotic behavior of $P_0\{T_\lambda < \tau_{V_-}\}$ implies (1.3) by the Tauberian theorem (see [5, Theorem 2.4.3]). Thus, we have (1.3) in Theorem 1.1 for $c_1 > 0$ and $c_2 > 0$.

The other proofs are similar. The proof of Theorem 1.1 is complete.

A. Appendix: Proof of Lemmas 3.2–3.5

We calculate

$$\Re \left[\frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \frac{1}{\varphi_\lambda(\theta)} d\theta \right] = \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \Re \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta,$$

where r_0 is the constant in Lemma 3.1.

Recall that

$$\begin{aligned} A &= \frac{E[(\sin \theta_1 X_1) X_2]}{(1/2)E[X_2^2]}, & B &= \frac{1 - \lambda E[\cos \theta_1 X_1]}{(\lambda/2)E[X_2^2]}, \\ C &= \frac{E[(-1 + \cos \theta_1 X_1) X_2]}{(1/2)E[X_2^2]}, & D &= \frac{E[\sin \theta_1 X_1]}{(1/2)E[X_2^2]}, \\ K &= (4B - A^2 + C^2)^2 + 4(2D - AC)^2 \end{aligned}$$

and

$$\varphi_\lambda(\theta) = \frac{\lambda}{2} E[X_2^2] \{ \theta_2^2 + A\theta_2 + B - i(C\theta_2 + D) \}.$$

Put

$$\begin{aligned} a_\pm &= A \pm \sqrt{H}, \\ b_\pm &= \begin{cases} \frac{1}{4} \left\{ A^2 + C^2 + \sqrt{K} \pm \frac{2AH - 2C(2D - AC)}{\sqrt{H}} \right\}, & 2D - AC \neq 0, \\ \frac{1}{2} \{ 2B + C^2 \pm \sqrt{4B - A^2 + C^2} |C| \}, & 2D - AC = 0, \end{cases} \end{aligned}$$

where

$$H = \frac{1}{2}(-4B + A^2 - C^2 + \sqrt{K}).$$

Then $a_\pm$, $b_\pm$ are real numbers, since $4B - A^2 - C^2 > 0$, and

$$(\theta_2^2 + A\theta_2 + B)^2 + (C\theta_2 + D)^2 = (\theta_2^2 + a_+\theta_2 + b_+)(\theta_2^2 + a_-\theta_2 + b_-).$$

In the case $2D - AC \neq 0$, it is easy to verify that

$$M := (a_+ - a_-)(a_+b_- - a_-b_+) + (b_- - b_+)^2 = \frac{1}{2}(-4B + A^2 + C^2 + \sqrt{K})\sqrt{K} \neq 0 \quad (\text{A.1})$$

using

$$(b_- - b_+)^2 = \left\{ -\frac{AH - C(2D - AC)}{\sqrt{H}} \right\}^2 = \frac{1}{4}(A^2 + C^2 + \sqrt{K})^2 - 4(B^2 + D^2). \quad (\text{A.2})$$

From the definition of $a_{\pm}$ and $b_{\pm}$

$$4b_{\pm} - a_{\pm}^2 = J_1 \pm J_2,$$

where

$$J_1 = \frac{1}{2}(4B - A^2 + 3C^2 + \sqrt{K}), \quad J_2 = -\frac{2C(2D - AC)}{\sqrt{H}}.$$

From (A.2), $4B - A^2 - C^2 > 0$ implies

$$J_1^2 - J_2^2 = \frac{1}{4}(4B - A^2 - C^2 + \sqrt{K})^2 > 0. \quad (\text{A.3})$$

Combining the above inequality with $J_1 > 0$, we have

$$|J_2| < J_1$$

and

$$4b_{\pm} - a_{\pm}^2 > 0. \quad (\text{A.4})$$

Equation (A.4) means that $\theta_2^2 + a_{\pm}\theta_2 + b_{\pm}$ is not zero for any $\theta_2 \in \mathbb{R}$.

The partial-fraction decomposition of $\Re[1/\varphi_{\lambda}(\theta)]$ is

$$\Re\left[\frac{1}{\varphi_{\lambda}(\theta)}\right] = \frac{\Re[\varphi_{\lambda}(\theta)]}{|\varphi_{\lambda}(\theta)|^2} = \frac{1}{(\lambda/2)E[X_2^2]} \left(\frac{F\theta_2 + G}{\theta_2^2 + a_+\theta_2 + b_+} + \frac{-F\theta_2 + I}{\theta_2^2 + a_-\theta_2 + b_-} \right),$$

where

$$\begin{aligned} F &= \frac{-(a_+b_- - a_-b_+) + A(b_- - b_+) + B(a_+ - a_-)}{M}, \\ G &= \frac{-b_+(b_- - b_+) - A(a_+ - a_-)b_+ + Ba_+(a_+ - a_-) + B(b_- - b_+)}{M}, \\ I &= \frac{b_-(b_- - b_+) + A(a_+ - a_-)b_- - Ba_-(a_+ - a_-) - B(b_- - b_+)}{M}. \end{aligned}$$

We turn now to the integration. In the case $2D - AC \neq 0$,

$$\begin{aligned} & \frac{\lambda E[X_2^2]}{2} \int_{-r_0/2}^{r_0/2} \Re\left[\frac{1}{\varphi_{\lambda}(\theta)}\right] d\theta_2 \\ &= \int_{-r_0/2}^{r_0/2} \frac{F\theta_2 + G}{\theta_2^2 + a_+\theta_2 + b_+} + \frac{-F\theta_2 + I}{\theta_2^2 + a_-\theta_2 + b_-} d\theta_2 \\ &= \frac{F}{2} \left[\log \frac{\theta_2^2 + a_+\theta_2 + b_+}{\theta_2^2 + a_-\theta_2 + b_-} \right]_{-r_0/2}^{r_0/2} + \frac{-a_+F + 2G}{\sqrt{4b_+ - a_+^2}} \left[\arctan \frac{2\theta_2 + a_+}{\sqrt{4b_+ - a_+^2}} \right]_{-r_0/2}^{r_0/2} \\ & \quad + \frac{a_-F + 2I}{\sqrt{4b_- - a_-^2}} \left[\arctan \frac{2\theta_2 + a_-}{\sqrt{4b_- - a_-^2}} \right]_{-r_0/2}^{r_0/2}. \end{aligned} \quad (\text{A.5})$$

In the case $2D - AC = 0$, we obtain

$$\begin{aligned} & \frac{\lambda E[X_2^2]}{2} \int_{-r_0/2}^{r_0/2} \Re \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2 \\ &= \frac{1}{\sqrt{4B - A^2 + C^2}} \left[\arctan \frac{2\theta_2 + A}{\sqrt{4B - A^2 + C^2} + |C|} \right]_{-r_0/2}^{r_0/2} \\ &+ \frac{1}{\sqrt{4B - A^2 + C^2}} \left[\arctan \frac{2\theta_2 + A}{\sqrt{4B - A^2 + C^2} - |C|} \right]_{-r_0/2}^{r_0/2}. \end{aligned}$$

Proof of Lemma 3.2. We will prove (3.10) in the case $2D - AC \neq 0$; the proof of (3.10) in the case $2D - AC = 0$ is similar and is omitted.

Assume that $2D - AC \neq 0$. From the definition of $a_\pm$, $a_\pm$ tends to 0 as $\theta_1 \rightarrow 0$ and $\lambda \uparrow 1$. In view of (A.2) and the definition of $b_\pm$, $b_\pm$ also tends to 0 as $\theta_1 \rightarrow 0$ and $\lambda \uparrow 1$. If $a_\pm$ and $b_\pm$ are small enough, then we can apply

$$\log(1+x) \leq 2|x| \quad (|x| \leq 1/2)$$

to estimate the first term on the right-hand side of the last equality in (A.5). With the help of

$$F = -\frac{\sqrt{H}}{\sqrt{K}},$$

the first term on the right-hand side of the last equality in (A.5) is bounded by a constant.

A simple calculation gives

$$-a_+ F + 2G = \frac{1}{2\sqrt{K}} \left\{ 4B - A^2 + C^2 + \sqrt{K} - \frac{2C(2D - AC)}{\sqrt{H}} \right\}, \quad (\text{A.6})$$

$$a_- F + 2I = \frac{1}{2\sqrt{K}} \left\{ 4B - A^2 + C^2 + \sqrt{K} + \frac{2C(2D - AC)}{\sqrt{H}} \right\}. \quad (\text{A.7})$$

We note that

$$\left| -\frac{2C(2D - AC)}{\sqrt{H}} \right| = |J_2| < J_1 = \frac{1}{2}(4B - A^2 + 3C^2 + \sqrt{K}),$$

as shown in the proof of (A.4). Equation (A.6) implies that $|-a_+ F + 2G| \leq 3J_1/(2\sqrt{K})$, which is bounded by a constant. The difference between the second term on the right-hand side of the last equality in (A.5) and

$$\frac{-a_+ F + 2G}{\sqrt{4b_+ - a_+^2}} \pi = \frac{\pi}{2\sqrt{J_1 + J_2}\sqrt{K}} \left\{ 4B - A^2 + C^2 + \sqrt{K} - \frac{2C(2D - AC)}{\sqrt{H}} \right\}$$

is bounded by a constant. Here we have used the inequality

$$0 \leq \frac{\pi}{2} - \arctan x \leq \frac{1}{x} \quad (x > 0).$$

Similarly, (A.7) implies that the difference between the third term on the right-hand side of the last equality in (A.5) and

$$\frac{a_- F + 2I}{\sqrt{4b_- - a_-^2}} \pi = \frac{\pi}{2\sqrt{J_1 - J_2}\sqrt{K}} \left\{ 4B - A^2 + C^2 + \sqrt{K} + \frac{2C(2D - AC)}{\sqrt{H}} \right\}$$

is bounded by a constant. Put

$$\bar{a}_\lambda(\theta_1) = \frac{1}{2\lambda E[X_2^2]\sqrt{K}} \left(\frac{J + J_2}{\sqrt{J_1 + J_2}} + \frac{J - J_2}{\sqrt{J_1 - J_2}} \right),$$

where

$$J = 4B - A^2 + C^2 + \sqrt{K}.$$

Then the difference between $(1/2\pi) \int_{-r_0/2}^{r_0/2} \Re[1/\varphi_\lambda(\theta)] d\theta_2$ and $\bar{a}_\lambda(\theta_1)$ is bounded by a constant.

It remains to show that

$$\bar{a}_\lambda(\theta_1) = \tilde{a}_\lambda(\theta_1) := \frac{\sqrt{2}}{\lambda E[X_2^2]\sqrt{K}} \sqrt{4B - A^2 + C^2 + \sqrt{K}}. \quad (\text{A.8})$$

Since $4B - A^2 - C^2 > 0$,

$$|J_2| < J_1 \leq J,$$

which implies that $\bar{a}_\lambda(\theta_1) \geq 0$. (The first inequality above has already been shown in the proof of (A.4).) We will compute $(\bar{a}_\lambda(\theta_1))^2$. By (A.3) and $J = 2J_1 - 2C^2$,

$$\begin{aligned} & \{(J + J_2)\sqrt{J_1 - J_2} + (J - J_2)\sqrt{J_1 + J_2}\}^2 \\ &= 2(J_1 + \sqrt{J_1^2 - J_2^2})J^2 + 2(J_1 - 2J - \sqrt{J_1^2 - J_2^2})J_2^2 \\ &= 8(4B - A^2 + C^2 + \sqrt{K})(J_1 - C^2)^2 - 4(4B - A^2 + \sqrt{K})J_2^2 \\ &= 8(4B - A^2 + C^2 + \sqrt{K})(J_1^2 - J_2^2) + 8(4B - A^2 + C^2 + \sqrt{K})(-2J_1 + C^2)C^2 \\ &\quad + 4(4B - A^2 + 2C^2 + \sqrt{K})J_2^2. \end{aligned} \quad (\text{A.9})$$

In view of

$$J_2^2 = J_1^2 - \frac{1}{4}(4B - A^2 - C^2 + \sqrt{K})^2 = 2(4B - A^2 + C^2 + \sqrt{K})C^2,$$

which follows from (A.3), the sum of the second and third terms on the right-hand side of the last equality in (A.9) vanish. Hence,

$$(\bar{a}_\lambda(\theta_1))^2 = (\tilde{a}_\lambda(\theta_1))^2.$$

Since $\bar{a}_\lambda(\theta_1)$ and $\tilde{a}_\lambda(\theta_1)$ are non-negative functions, the above relation yields (A.8). $\square$

To show Lemma 3.3, we require the following lemmas.

LEMMA A.1. Assume that (a)–(d) hold. Then, for $\lambda \in (1/2, 1)$ and $\theta_1 \in [-\pi, \pi]$ with $|\theta_1| \leq r_0/2$,

$$\begin{aligned} \text{(i)} \quad & \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \frac{\theta_2^2}{|\varphi_\lambda(\theta)|^2} d\theta_2 \leq \frac{16}{c_* E[X_2^2]} K^{-1/4}, \\ \text{(ii)} \quad & \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \frac{1}{|\varphi_\lambda(\theta)|^2} d\theta_2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|\varphi_\lambda(\theta)|^2} d\theta_2 \leq \frac{64}{(E[X_2^2])^2} K^{-3/4}. \end{aligned}$$

Proof. By (3.9), we obtain that, for $|\theta_1| \leq r_0/2$,

$$\int_{-r_0/2}^{r_0/2} \frac{\theta_2^2}{|\varphi_\lambda(\theta)|^2} d\theta_2 \leq \frac{4}{c_*} \int_{-r_0/2}^{r_0/2} \frac{\Re[\varphi_\lambda(\theta)]}{|\varphi_\lambda(\theta)|^2} d\theta_2 \leq \frac{4}{c_*} \int_{-\infty}^{\infty} \Re\left[\frac{1}{\varphi_\lambda(\theta)}\right] d\theta_2.$$

Assume that $2D - AC \neq 0$. In view of (A.5), the definition of $\tilde{a}_\lambda(\theta_1)$ and $4B - A^2 + C^2 \leq \sqrt{K}$ imply that

$$\int_{-\infty}^{\infty} \Re\left[\frac{1}{\varphi_\lambda(\theta)}\right] d\theta_2 = 2\pi \tilde{a}_\lambda(\theta_1) \leq \frac{4\pi}{\lambda E[X_2^2]} K^{-1/4}.$$

The above two inequalities give the desired estimate (i) in the case $2D - AC \neq 0$. The proof of (i) in the case $2D - AC = 0$ is similar and is omitted.

Since $1/|\varphi_\lambda(\theta)|^2 \geq 0$, the first inequality of (ii) is clear. To obtain the last inequality of (ii), we calculate the integral of $1/|\varphi_\lambda(\theta)|^2$ over $(-\infty, \infty)$ with respect to θ_2 . Assume $2D - AC \neq 0$. The partial-fraction decomposition of $1/|\varphi_\lambda(\theta)|^2$ is

$$\frac{1}{|\varphi_\lambda(\theta)|^2} = \frac{4}{(\lambda E[X_2^2])^2} \left(\frac{\bar{F}\theta_2 + \bar{G}}{\theta_2^2 + a_+\theta_2 + b_+} + \frac{-\bar{F}\theta_2 + \bar{I}}{\theta_2^2 + a_-\theta_2 + b_-} \right),$$

where $M = (a_+ - a_-)(a_+b_- - a_-b_+) + (b_- - b_+)^2$ as in (A.1) and

$$\begin{aligned} \bar{F} &= \frac{a_+ - a_-}{M}, \\ \bar{G} &= \frac{a_+(a_+ - a_-) + (b_- - b_+)}{M}, \\ \bar{I} &= \frac{-a_-(a_+ - a_-) - (b_- - b_+)}{M}. \end{aligned}$$

This gives the integral

$$\frac{(\lambda E[X_2^2])^2}{4} \int_{-\infty}^{\infty} \frac{1}{|\varphi_\lambda(\theta)|^2} d\theta_2 = \left(\frac{-a_+\bar{F} + 2\bar{G}}{\sqrt{4b_+ - a_+^2}} + \frac{a_-\bar{F} + 2\bar{I}}{\sqrt{4b_- - a_-^2}} \right) \pi.$$

A simple calculation and (A.1) show that the right-hand side of the above equality is equal to

$$c_\lambda(\theta_1) := \left\{ \frac{2H - J_2}{\sqrt{J_1 + J_2}} + \frac{2H + J_2}{\sqrt{J_1 - J_2}} \right\} \times \frac{2\pi}{(-4B + A^2 + C^2 + \sqrt{K})\sqrt{K}}.$$

We compute $(c_\lambda(\theta_1))^2$ by using (A.3). From this computation and $c_\lambda(\theta_1) \geq 0$, we have

$$c_\lambda(\theta_1) = \frac{4\sqrt{2}\pi}{\sqrt{K}} \frac{\sqrt{4B - A^2 + C^2 + \sqrt{K}}}{4B - A^2 - C^2 + \sqrt{K}}.$$

This implies the desired estimate (ii) in the case $2D - AC \neq 0$. The proof of (ii) in the case $2D - AC = 0$ is similar and is omitted. $\square$

LEMMA A.2. Assume that (a)–(d) hold. Then there exists a constant $c > 0$ such that, for $\lambda \in (1/2, 1)$ and $\theta_1 \in [-\pi, \pi]$ with $|\theta_1| \leq r_0/2$,

$$(iii) \quad \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \frac{|\theta_2|^{2+\delta}}{|\varphi_\lambda(\theta)|^2} d\theta_2 \leq cK^{(-1+\delta)/4}(1 + |\log K|),$$

$$(iv) \quad \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \frac{|\theta_2|^\gamma}{|\varphi_\lambda(\theta)|^2} d\theta_2 \leq c K^{(-3+\gamma)/4} \quad (\gamma = \delta, 1, 1 + \delta).$$

Proof. We prove (iii). Since $\delta \in (0, 1)$,

$$|\theta_2|^{2+\delta} \leq \begin{cases} K^{(2+\delta)/4}, & |\theta_2| \leq K^{1/4}, \\ |\theta_2|^3 K^{(-1+\delta)/4}, & |\theta_2| > K^{1/4}. \end{cases} \quad (A.10)$$

We use (A.10) to obtain that the left-hand side of (iii) is bounded by

$$\frac{K^{(2+\delta)/4}}{2\pi} \int_{-r_0/2}^{r_0/2} \frac{1}{|\varphi_\lambda(\theta)|^2} d\theta_2 + \frac{K^{(-1+\delta)/4}}{2\pi} \int_{K^{1/4} < |\theta_2| \leq r_0/2} \frac{|\theta_2|^3}{|\varphi_\lambda(\theta)|^2} d\theta_2. \quad (A.11)$$

By Lemma A.1(ii), the first term of (A.11) is bounded by a constant multiple of $K^{(-1+\delta)/4}$. From (3.9),

$$\frac{|\theta_2|^3}{|\varphi_\lambda(\theta)|^2} \leq \frac{16}{c_*^2 |\theta_2|} \quad (|\theta| < r_0, \lambda \in (1/2, 1))$$

and the integral of $1/|\theta_2|$ over $K^{1/4} < |\theta_2| \leq r_0/2$ with respect to θ_2 is bounded by $|\log(r_0/2)| + (1/4)|\log K|$. The second term of (A.11) is bounded by a constant multiple of $K^{(-1+\delta)/4}(1 + |\log K|)$. Thus, we conclude that (iii) holds.

To prove (iv), we use the inequality

$$|\theta_2|^\gamma \leq \begin{cases} K^{\gamma/4} & (|\theta_2| \leq K^{1/4}), \\ |\theta_2|^2 K^{(\gamma-2)/4} & (|\theta_2| > K^{1/4}), \end{cases}$$

instead of (A.10), and the result follows from Lemma A.1(i) and (ii). $\square$

Proof of Lemma 3.3. With $r_0 > 0$,

$$\begin{aligned} a_\lambda(\theta_1) - \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \Re \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2 &= \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \chi_\lambda(\theta) d\theta_2 \\ &\quad + \frac{1}{2\pi} \int_{r_0/2 < |\theta_2| \leq \pi} \frac{\Re[1 - \lambda\phi(\theta)]}{|1 - \lambda\phi(\theta)|^2} d\theta_2, \end{aligned} \quad (A.12)$$

where

$$\chi_\lambda(\theta) = \frac{\Re[1 - \lambda\phi(\theta)]}{|1 - \lambda\phi(\theta)|^2} - \frac{\Re[\varphi_\lambda(\theta)]}{|\varphi_\lambda(\theta)|^2}.$$

Equation (3.7) implies that, for $\lambda \in (1/2, 1)$ and $\theta \in [-\pi, \pi]^2$ with $|\theta_2| \geq r_0/2$,

$$0 \leq \frac{\Re[1 - \lambda\phi(\theta)]}{|1 - \lambda\phi(\theta)|^2} \leq \frac{1}{|1 - \lambda\phi(\theta)|} \leq \frac{2}{c_* |\theta|^2} \leq \frac{8}{c_* r_0^2}.$$

Hence, the absolute value of the second term on the right-hand side of (A.12) is bounded by a constant.

To estimate the first term on the right-hand side of (A.12), we consider $\chi_\lambda(\theta)$. By applying the estimates (3.6), (3.8) and

$$\frac{1}{|1 - \lambda\phi(\theta)|} \leq \frac{c_3(\pi^{\hat{\delta}} + \pi^\delta) + 1}{|\varphi_\lambda(\theta)|} \quad (\lambda \in (1/2, 1), \theta \in [-\pi, \pi]^2),$$

which follows from (3.1), we deduce that

$$\begin{aligned}
|\chi_\lambda(\theta)| &\leq |\Re[1 - \lambda\phi(\theta)] - \Re[\varphi_\lambda(\theta)]| \frac{1}{|1 - \lambda\phi(\theta)|^2} \\
&\quad + |\Re[\varphi_\lambda(\theta)]| \frac{|1 - \lambda\phi(\theta) - \varphi_\lambda(\theta)|}{|1 - \lambda\phi(\theta)||\varphi_\lambda(\theta)|} \left(\frac{1}{|1 - \lambda\phi(\theta)|} + \frac{1}{|\varphi_\lambda(\theta)|} \right) \\
&\leq c_{19} \frac{(|\theta_1|^\delta + |\theta_2|^\delta)\theta_2^2}{|\varphi_\lambda(\theta)|^2},
\end{aligned} \tag{A.13}$$

where c_{19} is a constant independent of $\lambda \in (1/2, 1)$ and $\theta \in [-\pi, \pi]^2$.

The following estimates complete the proof. The integral of $|\theta_2|^2/|\varphi_\lambda(\theta)|^2$ appearing on the right-hand side of the above inequality over $[-r_0/2, r_0/2]$ with respect to θ_2 is bounded by a constant multiple of $K^{-1/4}$. Moreover, the integral of $|\theta_2|^{2+\delta}/|\varphi_\lambda(\theta)|^2$ appearing on the right-hand side of the above inequality over $[-r_0/2, r_0/2]$ with respect to θ_2 is bounded by a constant multiple of $K^{(-1+\delta)/4}|\log K|$. These estimates come from Lemma A.1(i) and Lemma A.2(iii). Note that K tends to zero as $\theta_1 \rightarrow 0$ and $r \uparrow 1$, which implies that $1 \leq |\log K|$. $\square$

Proof of Lemma 3.4. Equation (3.11) can be proved by the same procedure as in Lemma 3.2. We only give a sketch of the proof of (3.11) in the case $2D - AC \neq 0$.

Assume that $2D - AC \neq 0$. We calculate

$$\Im \left[\frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \frac{1}{\varphi_\lambda(\theta)} d\theta_2 \right] = \frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \Im \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2.$$

The partial-fraction decomposition of $\Im[1/\varphi_\lambda(\theta)]$ is

$$\Im \left[\frac{1}{\varphi_\lambda(\theta)} \right] = \frac{-\Im[\varphi_\lambda(\theta)]}{|\varphi_\lambda(\theta)|^2} = \frac{1}{(\lambda/2)E[X_2^2]} \left(\frac{\tilde{F}\theta_2 + \tilde{G}}{\theta_2^2 + a_+\theta_2 + b_+} + \frac{-\tilde{F}\theta_2 + \tilde{I}}{\theta_2^2 + a_-\theta_2 + b_-} \right),$$

where $M = (a_+ - a_-)(a_+b_- - a_-b_+) + (b_- - b_+)^2$ as in (A.1) and

$$\begin{aligned}
\tilde{F} &= \frac{C(b_- - b_+) + D(a_+ - a_-)}{M}, \\
\tilde{G} &= \frac{-C(a_+ - a_-)b_+ + Da_+(a_+ - a_-) + D(b_- - b_+)}{M}, \\
\tilde{I} &= \frac{C(a_+ - a_-)b_- - Da_-(a_+ - a_-) - D(b_- - b_+)}{M}.
\end{aligned}$$

This gives the integral

$$\begin{aligned}
&\frac{\lambda E[X_2^2]}{2} \int_{-r_0/2}^{r_0/2} \Im \left[\frac{1}{\varphi_\lambda(\theta)} \right] d\theta_2 \\
&= \frac{\tilde{F}}{2} \left[\log \frac{\theta_2^2 + a_+\theta_2 + b_+}{\theta_2^2 + a_-\theta_2 + b_-} \right]_{-r_0/2}^{r_0/2} + \frac{-a_+\tilde{F} + 2\tilde{G}}{\sqrt{4b_+ - a_+^2}} \left[\arctan \frac{2\theta_2 + a_+}{\sqrt{4b_+ - a_+^2}} \right]_{-r_0/2}^{r_0/2} \\
&\quad + \frac{a_-\tilde{F} + 2\tilde{I}}{\sqrt{4b_- - a_-^2}} \left[\arctan \frac{2\theta_2 + a_-}{\sqrt{4b_- - a_-^2}} \right]_{-r_0/2}^{r_0/2}.
\end{aligned} \tag{A.14}$$

Using

$$\tilde{F} = \frac{2D - AC}{\sqrt{K}\sqrt{H}}, \quad |\log(1+x)| \leq 2|x| \quad (|x| \leq 1/2),$$

and

$$(2D - AC)^2 = \frac{1}{2}(4B - A^2 + C^2 + \sqrt{K})H, \quad (\text{A.15})$$

the first term on the right-hand side of (A.14) is bounded by a constant multiple of $|2D - AC|/\sqrt{K} + |C|$. A simple calculation gives

$$-a_+\tilde{F} + 2\tilde{G} = \frac{2D - AC}{\sqrt{K}} - \frac{C\sqrt{H}}{\sqrt{K}}, \quad a_-\tilde{F} + 2\tilde{I} = \frac{2D - AC}{\sqrt{K}} + \frac{C\sqrt{H}}{\sqrt{K}}.$$

The difference between the second term on the right-hand side of (A.14) and

$$\frac{-a_+\tilde{F} + 2\tilde{G}}{\sqrt{4b_+ - a_+^2}}\pi = \frac{\pi}{\sqrt{J_1 + J_2}\sqrt{K}}(2D - AC - C\sqrt{H})$$

is bounded by a constant multiple of $|2D - AC|/\sqrt{K} + |C|/K^{1/4}$. Similarly, the difference between the third term on the right-hand side of (A.14) and

$$\frac{a_-\tilde{F} + 2\tilde{I}}{\sqrt{4b_- - a_-^2}}\pi = \frac{\pi}{\sqrt{J_1 - J_2}\sqrt{K}}(2D - AC + C\sqrt{H})$$

is bounded by a constant multiple of $|2D - AC|/\sqrt{K} + |C|/K^{1/4}$.

Put

$$\bar{b}_\lambda(\theta_1) = \frac{1}{\lambda E[X_2^2]\sqrt{K}} \left(\frac{2D - AC - C\sqrt{H}}{\sqrt{J_1 + J_2}} + \frac{2D - AC + C\sqrt{H}}{\sqrt{J_1 - J_2}} \right).$$

Then the difference between $(1/2\pi) \int_{-r_0/2}^{r_0/2} \Im[1/\varphi_\lambda(\theta)] d\theta_2$ and $\bar{b}_\lambda(\theta_1)$ is bounded by a constant multiple of $|2D - AC|/\sqrt{K} + |C|/K^{1/4}$.

It remains to show that

$$\bar{b}_\lambda(\theta_1) = \tilde{b}_\lambda(\theta_1) := \frac{\sqrt{2}}{\lambda E[X_2^2]\sqrt{K}} \frac{2(2D - AC)}{\sqrt{4B - A^2 + C^2 + \sqrt{K}}}.$$

By using (A.3) and noticing (A.15), we have

$$\bar{b}_\lambda^2(\theta_1) = \frac{4H}{(\lambda E[X_2^2])^2 K} = \tilde{b}_\lambda^2(\theta_1). \quad (\text{A.16})$$

From the inequality

$$0 \leq C^2 H \leq \frac{1}{2}(4B - A^2 + C^2 + \sqrt{K})H = (2D - AC)^2,$$

the sign of $\tilde{b}_\lambda(\theta_1)$ is the same as that of $\bar{b}_\lambda(\theta_1)$. Equation (A.16) yields $\bar{b}_\lambda(\theta_1) = \tilde{b}_\lambda(\theta_1)$. $\square$

Proof of Lemma 3.5. Let $\hat{b}_\lambda(\theta_1)$ be the imaginary part of

$$\frac{1}{2\pi} \int_{-r_0/2}^{r_0/2} \frac{1}{\varphi_\lambda(\theta_1, \theta_2)} d\theta_2.$$

By performing the change of variables $\theta_2 = -\tilde{\theta}_2$ and using the symmetry of trigonometric functions, $b_\lambda(\theta_1) = -b_\lambda(-\theta_1)$ and $\hat{b}_\lambda(\theta_1) = -\hat{b}_\lambda(-\theta_1)$. In view of these equalities,

$$\begin{aligned} b_\lambda(\theta_1) - \hat{b}_\lambda(\theta_1) &= \frac{1}{2} \{b_\lambda(\theta_1) - b_\lambda(-\theta_1) - (\hat{b}_\lambda(\theta_1) - \hat{b}_\lambda(-\theta_1))\} \\ &= \frac{1}{4\pi} \int_{-r_0/2}^{r_0/2} \chi_\lambda^{(1)}(\theta_1, \theta_2) d\theta_2 + \frac{1}{4\pi} \int_{-r_0/2}^{r_0/2} \chi_\lambda^{(2)}(\theta_1, \theta_2) d\theta_2 \\ &\quad + \frac{1}{4\pi} \int_{r_0/2 \leq |\theta_2| \leq \pi} \chi_\lambda^{(3)}(\theta_1, \theta_2) d\theta_2, \end{aligned} \quad (\text{A.17})$$

where

$$\begin{aligned} \chi_\lambda^{(1)}(\theta_1, \theta_2) &= \lambda E[(\sin \theta_1 X_1)(\cos \theta_2 X_2)] \left(\frac{1}{|1 - \lambda\phi(\theta_1, \theta_2)|^2} + \frac{1}{|1 - \lambda\phi(-\theta_1, \theta_2)|^2} \right) \\ &\quad - \lambda E[\sin \theta_1 X_1] \left(\frac{1}{|\varphi_\lambda(\theta_1, \theta_2)|^2} + \frac{1}{|\varphi_\lambda(-\theta_1, \theta_2)|^2} \right), \\ \chi_\lambda^{(2)}(\theta_1, \theta_2) &= \lambda E[(\cos \theta_1 X_1)(\sin \theta_2 X_2)] \left(\frac{1}{|1 - \lambda\phi(\theta_1, \theta_2)|^2} - \frac{1}{|1 - \lambda\phi(-\theta_1, \theta_2)|^2} \right) \\ &\quad - \lambda E[(-1 + \cos \theta_1 X_1)X_2] \theta_2 \left(\frac{1}{|\varphi_\lambda(\theta_1, \theta_2)|^2} - \frac{1}{|\varphi_\lambda(-\theta_1, \theta_2)|^2} \right) \end{aligned}$$

and

$$\chi_\lambda^{(3)}(\theta_1, \theta_2) = \frac{\lambda E[\sin(\theta_1 X_1 + \theta_2 X_2)]}{|1 - \lambda\phi(\theta_1, \theta_2)|^2} - \frac{\lambda E[\sin(-\theta_1 X_1 + \theta_2 X_2)]}{|1 - \lambda\phi(-\theta_1, \theta_2)|^2}.$$

Using condition (c),

$$\begin{aligned} |\phi(\theta_1, \theta_2) - \phi(-\theta_1, \theta_2)| &\leq 2|E[(\sin \theta_1 X_1)(\sin \theta_2 X_2)]| + 2|E[(\sin \theta_1 X_1)(\cos \theta_2 X_2)]| \\ &\leq 4E[|\sin \theta_1 X_1|] \leq 4E[|X_1|^\delta]|\theta_1|^\delta. \end{aligned}$$

The above inequalities and (3.7) imply that, for $\lambda \in (1/2, 1)$ and $(\theta_1, \theta_2) \in [-\pi, \pi] \times [-\pi, \pi]$ with $|\theta_2| \geq r_0/2$,

$$\begin{aligned} |\chi_\lambda^{(3)}(\theta_1, \theta_2)| &\leq \frac{|E[(\sin \theta_1 X_1)(\cos \theta_2 X_2)]|}{|1 - \lambda\phi(\theta_1, \theta_2)|^2} + \frac{|E[(\sin \theta_1 X_1)(\cos \theta_2 X_2)]|}{|1 - \lambda\phi(-\theta_1, \theta_2)|^2} \\ &\quad + |E[(\cos \theta_1 X_1)(\sin \theta_2 X_2)]| \left| \frac{1}{|1 - \lambda\phi(\theta_1, \theta_2)|^2} - \frac{1}{|1 - \lambda\phi(-\theta_1, \theta_2)|^2} \right| \\ &\leq \frac{E[|X_1|^\delta]|\theta_1|^\delta}{|1 - \lambda\phi(\theta_1, \theta_2)|^2} + \frac{E[|X_1|^\delta]|\theta_1|^\delta}{|1 - \lambda\phi(-\theta_1, \theta_2)|^2} \\ &\quad + |\phi(\theta_1, \theta_2) - \phi(-\theta_1, \theta_2)| \frac{|1 - \lambda\phi(-\theta_1, \theta_2)| + |1 - \lambda\phi(\theta_1, \theta_2)|}{|1 - \lambda\phi(\theta_1, \theta_2)|^2 |1 - \lambda\phi(-\theta_1, \theta_2)|^2} \\ &\leq \frac{128E[|X_1|^\delta]}{c_*^2 r_0^4} \left(1 + \frac{32}{c_* r_0^2} \right) |\theta_1|^\delta. \end{aligned}$$

Hence, the third term on the right-hand side of the last equality in (A.17) is bounded by a constant multiple of $|\theta_1|^\delta$.

To estimate remaining terms on the right-hand side of the last equality in (A.17), we consider $\chi_\lambda^{(1)}(\theta_1, \theta_2)$ and $\chi_\lambda^{(2)}(\theta_1, \theta_2)$. With the help of

$$|E[(\sin \theta_1 X_1)(\cos \theta_2 X_2)] - E[\sin \theta_1 X_1]| \leq E[|X_1|^\delta X_2^2] |\theta_1|^\delta \theta_2^2$$

and (3.7), the same argument as shown in (A.13) implies that

$$\begin{aligned} |\chi_\lambda^{(1)}(\theta_1, \theta_2)| &\leq \frac{|E[(\sin \theta_1 X_1)(\cos \theta_2 X_2)] - E[\sin \theta_1 X_1]|}{|1 - \lambda \phi(\theta_1, \theta_2)|^2} \\ &\quad + \frac{|E[(\sin \theta_1 X_1)(\cos \theta_2 X_2)] - E[\sin \theta_1 X_1]|}{|1 - \lambda \phi(-\theta_1, \theta_2)|^2} \\ &\quad + |E[\sin \theta_1 X_1]| \frac{|1 - \lambda \phi(\theta_1, \theta_2) - \varphi_\lambda(\theta_1, \theta_2)|}{|1 - \lambda \phi(\theta_1, \theta_2)| |\varphi_\lambda(\theta_1, \theta_2)|} \\ &\quad \times \left(\frac{1}{|1 - \lambda \phi(\theta_1, \theta_2)|} + \frac{1}{|\varphi_\lambda(\theta_1, \theta_2)|} \right) \\ &\quad + |E[\sin \theta_1 X_1]| \frac{|1 - \lambda \phi(-\theta_1, \theta_2) - \varphi_\lambda(-\theta_1, \theta_2)|}{|1 - \lambda \phi(-\theta_1, \theta_2)| |\varphi_\lambda(-\theta_1, \theta_2)|} \\ &\quad \times \left(\frac{1}{|1 - \lambda \phi(-\theta_1, \theta_2)|} + \frac{1}{|\varphi_\lambda(-\theta_1, \theta_2)|} \right) \\ &\leq c_{20} \{ |D|(|\theta_1|^\delta + |\theta_2|^\delta) + |\theta_1|^\delta \theta_2^2 \} \left(\frac{1}{|\varphi_\lambda(\theta_1, \theta_2)|^2} + \frac{1}{|\varphi_\lambda(-\theta_1, \theta_2)|^2} \right), \end{aligned} \quad (\text{A.18})$$

where c_{20} is a constant independent of $\lambda \in (1/2, 1)$ and $(\theta_1, \theta_2) \in [-\pi, \pi] \times [-\pi, \pi]$.

By performing the change of variables $\theta_2 = -\tilde{\theta}_2$ and using the symmetry of trigonometric functions,

$$\int_{-r_0/2}^{r_0/2} \frac{|\theta_2|^\gamma}{|\varphi_\lambda(-\theta_1, \theta_2)|^2} d\theta_2 = \int_{-r_0/2}^{r_0/2} \frac{|\tilde{\theta}_2|^\gamma}{|\varphi_\lambda(\theta_1, \tilde{\theta}_2)|^2} d\tilde{\theta}_2 \quad (\gamma \geq 0). \quad (\text{A.19})$$

Taking note of (A.19), it follows from (A.18), Lemma A.1 and Lemma A.2(iv) that the first term on the right-hand side of the last equality in (A.17) is bounded by a constant multiple of $|\theta_1|^\delta K^{-1/4} + |D|(|\theta_1|^\delta + K^{\delta/4}) K^{-3/4}$.

From conditions (b), (c) and (d), we obtain

$$\begin{aligned} &|E[(\cos \theta_1 X_1)(\sin \theta_2 X_2)] - E[(-1 + \cos \theta_1 X_1)X_2] \theta_2| \\ &= |E[(\cos \theta_1 X_1)(\sin \theta_2 X_2)] - E[(\cos \theta_1 X_1)X_2] \theta_2| \leq E[|X_2|^{2+\delta}] |\theta_2|^{2+\delta} \end{aligned}$$

and

$$|\phi(\theta_1, \theta_2) - \phi(-\theta_1, \theta_2)| \leq 2|E[\sin \theta_1 X_1]| + 2|E[(\sin \theta_1 X_1)X_2]| |\theta_2| + 2E[|X_1|^\delta X_2^2] |\theta_1|^\delta \theta_2^2.$$

Using the above two inequalities and (3.7), the same argument as given in (A.13) implies that

$$\begin{aligned}
|\chi_\lambda^{(2)}(\theta_1, \theta_2)| &\leq \frac{|E[(\cos \theta_1 X_1)(\sin \theta_2 X_2)] - E[(-1 + \cos \theta_1 X_1)X_2]\theta_2|}{|1 - \lambda\phi(\theta_1, \theta_2)||1 - \lambda\phi(-\theta_1, \theta_2)|} \\
&\quad \times |\phi(\theta_1, \theta_2) - \phi(-\theta_1, \theta_2)| \left(\frac{1}{|1 - \lambda\phi(\theta_1, \theta_2)|} + \frac{1}{|1 - \lambda\phi(-\theta_1, \theta_2)|} \right) \\
&\quad + |E[(-1 + \cos \theta_1 X_1)X_2]\theta_2| \frac{|1 - \lambda\phi(\theta_1, \theta_2) - \varphi_\lambda(\theta_1, \theta_2)|}{|1 - \lambda\phi(\theta_1, \theta_2)||\varphi_\lambda(\theta_1, \theta_2)|} \\
&\quad \times \left(\frac{1}{|1 - \lambda\phi(\theta_1, \theta_2)|} + \frac{1}{|\varphi_\lambda(\theta_1, \theta_2)|} \right) \\
&\quad + |E[(-1 + \cos \theta_1 X_1)X_2]\theta_2| \frac{|1 - \lambda\phi(-\theta_1, \theta_2) - \varphi_\lambda(-\theta_1, \theta_2)|}{|1 - \lambda\phi(-\theta_1, \theta_2)||\varphi_\lambda(-\theta_1, \theta_2)|} \\
&\quad \times \left(\frac{1}{|1 - \lambda\phi(-\theta_1, \theta_2)|} + \frac{1}{|\varphi_\lambda(-\theta_1, \theta_2)|} \right) \\
&\leq c_{21} \{ |D||\theta_2|^\delta + (|A| + |C|)|\theta_2|^{1+\delta} + (|C||\theta_2| + |\theta_2|^2)|\theta_1|^\delta \} \\
&\quad \times \left(\frac{1}{|\varphi_\lambda(\theta_1, \theta_2)|^2} + \frac{1}{|\varphi_\lambda(-\theta_1, \theta_2)|^2} \right), \tag{A.20}
\end{aligned}$$

where c_{21} is a constant independent of $\lambda \in (1/2, 1)$ and $(\theta_1, \theta_2) \in [-\pi, \pi] \times [-\pi, \pi]$. Recalling (A.19), it follows from (A.20), Lemma A.1(i) and Lemma A.2(iv) that the second term on the right-hand side of the last equality in (A.17) is bounded by a constant multiple of $\{|\theta_1|^\delta + |A|K^{(-1+\delta)/4} + |C|(|\theta_1|^\delta + K^{\delta/4})K^{-1/4} + |D|K^{(-2+\delta)/4}\}K^{-1/4}$. Hence, we have the desired estimate. $\square$

B. Appendix: Calculation of (1.4) and (1.5)

To obtain (1.4) and (1.5), we use

$$\int_0^{+\infty} \frac{\sin ax}{x^\beta} dx = \frac{\pi a^{\beta-1}}{2\Gamma(\beta) \sin((\beta\pi)/2)} \tag{B.1}$$

for $\beta \in (0, 2)$ and $a > 0$.

From the definition of X_1 ,

$$1 - E[\cos \vartheta X_1] = c_+(1 - \cos \vartheta) + c_- \sum_{n=1}^{\infty} (1 - \cos \vartheta) \frac{1}{n^{1+\alpha}}.$$

Since the absolute value of the difference between the right-hand side of the above and

$$c_- \int_0^{+\infty} \frac{1 - \cos |\vartheta|x}{x^{1+\alpha}} dx$$

is bounded by a constant multiple of ϑ^2 , it remains to show that

$$c_- \int_0^{+\infty} \frac{1 - \cos |\vartheta|x}{x^{1+\alpha}} dx = \frac{c_- \pi}{2\Gamma(\alpha + 1) \cos((\alpha - 1)\pi/2)} |\vartheta|^\alpha. \tag{B.2}$$

We perform integration by parts for the left-hand side of (B.2) and use (B.1). Then we have (B.2).

Noting $E[X_1] = 0$,

$$E[\sin \vartheta X_1] = c_+(\sin \vartheta - \vartheta) + c_- \sum_{n=1}^{\infty} (\vartheta n - \sin \vartheta n) \frac{1}{n^{1+\alpha}}.$$

Since the absolute value of the difference between the right-hand side of the above and

$$c_- \int_0^{+\infty} \frac{\vartheta x - \sin \vartheta x}{x^{1+\alpha}} dx$$

is bounded by a constant multiple of ϑ^2 , it remains to show that

$$c_- \int_0^{+\infty} \frac{\vartheta x - \sin \vartheta x}{x^{1+\alpha}} dx = \frac{c_- \pi}{2\Gamma(\alpha + 1) \sin((\alpha - 1)\pi/2)} |\vartheta|^\alpha \frac{\vartheta}{|\vartheta|}. \quad (\text{B.3})$$

We perform integration by part twice for the left-hand side of (B.3) and use (B.1). Then we have (B.3).

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