THE INVERSE MEAN CURVATURE FLOW IN RANK ONE SYMMETRIC SPACES OF NON-COMPACT TYPE

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Abstract. In this paper, we study the inverse mean curvature flow starting from a mean convex hypersurface in rank one symmetric spaces of non-compact type. We derive a lower bound for the mean curvature along the inverse mean curvature flow starting from a strictly star-shaped mean convex hypersurface.

1. Introduction

Let $M$ be an $n$-dimensional closed smooth manifold, $(N, \bar{g})$ an $(n + 1)$-dimensional complete smooth Riemannian manifold and

$$F_t : M \hookrightarrow (N, \bar{g}) \quad (t \in [0, T])$$

a smooth family of regular and smooth closed hypersurfaces in a Riemannian manifold $N$. Define $F : M \times [0, T) \to (N, \bar{g})$ by $F(p, t) := F_t(p)$ ($(p, t) \in M \times [0, T)$) and set $M_t := F_t(M)$. The smooth family $F_t$ ($t \in [0, T)$) is called the inverse mean curvature flow (IMCF) if it satisfies the following evolution equation:

$$\frac{\partial F}{\partial t} = \frac{1}{H} \nu,$$

(IMCF)

where $H$ and $\nu$ are the mean curvature and the outer unit normal vector of $M_t$, respectively. We assume that each $M_t$ is mean convex, that is, $H > 0$ on $M \times [0, T)$. It was shown by Huisken–Polden that a short-time existence of a classical smooth solution of the IMCF for any mean convex initial data is assured. In detail, see [8, Theorem 3.1].

In the Euclidean case (i.e., $N = \mathbb{R}^{n+1}$), it was shown by Gerhardt [3] that for smooth star-shaped mean convex initial data, the IMCF has a smooth solution for all times which approach a homothetically expanding spherical solution as $t \to \infty$, where we note that he also treated the inverse curvature flow. For smooth star-shaped mean convex initial data, Huisken–Ilmanen [7] derived a lower bound for the mean curvature of the star-shaped solutions of the IMCF which is independent of the curvature of the initial hypersurface. In detail, see [7, Theorem 1.1]. On the other hand, in the hyperbolic space case, it was shown by Gerhardt [5] that the IMCF has a unique solution for all times which approach a homothetically expanding geodesic spherical solution as $t \to \infty$, where we note that he

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also treated the inverse curvature flow. Also see [4], where Gerhardt studies the inverse curvature flow. More generally, Ding [2] showed that the similar fact holds in a rotationally symmetric space. However, we cannot expect that the similar fact holds in a (Riemannian) symmetric space of non-compact type other than the hyperbolic space because the space is not rotationally symmetric. For example, in the case where the symmetric space is of rank greater than one and irreducible, a homothetically expanding geodesic spherical solution of the inverse mean curvature flow does not exist.

The main purpose of this paper is to attain the following result similar to [7, Theorem 1.1] for the inverse mean curvature in a rank one symmetric space of non-compact type.

**Theorem 1.1.** Let $M$ be an $n$-dimensional smooth closed manifold, let $G/K$ be an $(n+1)$-dimensional in rank one symmetric space $G/K$ of non-compact type and let $F : M \times [0, T) \to G/K$ be a smooth star-shaped solution of (IMCF) satisfying the estimates

$$0 < R_1 \leq w_0 \leq R_2 \quad \text{and} \quad 0 < R_3 \leq H_t \quad (0 \leq t < T)$$

for some positive constants $R_1$, $R_2$, and $R_3$, where $w_0$ is the support function of $F_0$. (See Section 3 for details about this definition.) Then the estimate

$$H \geq \frac{\sqrt{-c}}{\text{arcsinh}(e^{t/n} \sinh(\sqrt{-c} R_2))}\left(\frac{2 \sqrt{-c} \cdot C(n, r) \, \text{vol}(M_0)^{1/n} \, e^{t/n}}{\text{arcsinh}(e^{\max(2t/n, (t-1)/n)} \sinh(2\sqrt{-c} R_1))} + \max\left\{\left(\frac{t}{2}\right)^{-1/2}, 1\right\} \sqrt{\hat{C}(n, r) \left(1 + \frac{C_{A_0, R_3}}{(2r + 1)^{1/r}}\right) \text{vol}(M_0)^{1/n} e^{(1/2r + C_{A_0, R_3})t}} \right)^{-1}$$

holds on $M \times [0, T)$ and hence the estimate

$$\int_M H_t \, d\mu_t \geq \frac{\sqrt{-c}}{\text{arcsinh}(e^{t/n} \sinh(\sqrt{-c} R_2))}\left(\frac{2 \sqrt{-c} \cdot C(n, r) \, \text{vol}(M_0)^{1/n} \, e^{t/n}}{\text{arcsinh}(e^{\max(2t/n, (t-1)/n)} \sinh(2\sqrt{-c} R_1))} + \max\left\{\left(\frac{t}{2}\right)^{-1/2}, 1\right\} \sqrt{\hat{C}(n, r) \left(1 + \frac{C_{A_0, R_3}}{(2r + 1)^{1/r}}\right) \text{vol}(M_0)^{1/n} e^{(1/2r + C_{A_0, R_3})t}} \right)^{-1}$$

holds on $[0, T)$, where $d\mu_t$ is the volume element of $g_t = F_t^* \tilde{g}$, $r$ is an arbitrary fixed constant satisfying

$$r > \begin{cases} \frac{2q - 1}{q - 1} & (n = 2) \quad (q > 1 \text{ is an arbitrary fixed constant}) \\ \frac{n + 2}{2} & (n \geq 3) \end{cases}$$

and $C(n, r)$ and $\hat{C}(n, r)$ are constants depending only on $n$ and $r$, and $C_{A_0, R_3} := (\max_M \|A_0\|^2) / R_3^2$, and

$$\Psi_{2r}(t) := \int_0^t \left(\frac{\text{arcsinh}(e^{t/n} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}}\right)^2 e^{(2C_{A_0, R_3} + (n-2r)/nr)\tau} \, d\tau.$$
In Section 2, we derive the evolution equations for some geometric quantities under the IMCF.

In Section 3, we derive the evolution equations for the support function under the IMCF.

In Section 4, we prove Theorem 1.1 in conjunction with the Sobolev inequality for Riemannian submanifolds [6] and iteration scheme due to De Giorgi [1, Lemma 12.5].

In this paper, we shall use notation based on the theory of connections on the vector bundles following [10].

2. Evolution equations for the basic quantities

In this section, we consider the inverse mean curvature flow $F_t : M \hookrightarrow N (t \in [0, T))$ starting from a closed hypersurface $M$ in a general Riemannian manifold $N$ with Riemannian metric $\bar{g}$. Define $F : M \times [0, T) \to N$ by $F(p, t) := F_t(p) ((p, t) \in M \times [0, T))$. From now on, we use the notation defined in Introduction. Under the IMCF, we derive the evolution equations for some geometric quantities of $M_t$. In the sequel, we shall define some notational conventions which we use throughout this paper. Latin indices range from 1 to $n$ and Greek indices range from 0 to $n$. Denote by $\bar{\nabla}$ the Levi-Civita connection of $\bar{g}$. Denote by $\bar{\nabla} F_t$ (respectively $\bar{\nabla} F$) the pullback connection of $\bar{\nabla}$ along $F_t$ (respectively $F$). We shall indicate by suffix $t$ all quantities on $M$ induced by $F_t$, for example, by $\bar{g}_t$ the induced metric by $F_t$, by $\bar{\nabla}_t$ the induced connection of $\bar{g}_t$, by $d\mu_t$ the volume element of $\bar{g}_t$, by $h_t$ the second fundamental form with respect to $-\nu$ of $M_t$ and by $A_t$ the shape operator with respect to $-\nu$ of $M_t$. Let $(x^1, \ldots, x^n)$ be a local coordinate of $M$. Note that $(g_t)_{ij}, (h_t)_{ij}$ and $(A_t)^j_i (1 \leq i, j \leq n)$ are the components of $g_t, h_t$ and $A_t$ with respect to $(x^1, \ldots, x^n)$, respectively.

For a vector bundle $E$ over $M$, denote by $\pi^*_M(E)$ the pullback bundle of $E$ by $\pi_M$. Let $T^{(r,s)} M$ be the $(r, s)$-tensor bundle over $M$. For any $S \in \Gamma(\pi^*_M(T^{(0,2)} M))$, we define an element $\partial S/\partial t$ of $\Gamma(\pi^*_M(T^{(0,2)} M))$ by

$$\left( \frac{\partial S}{\partial t} \right)_{(p,t)} := \frac{dS(p,t)}{dt} ((p, t) \in M \times [0, T)), \quad (2.1)$$

where $dS(p,t)/dt$ is the derivative of the vector-valued function $t \mapsto S(p,t) \in T^{(0,2)}_p M$. Define $g, h \in \Gamma(\pi^*_M(T^{(0,2)} M))$ and $A \in \Gamma(\pi^*_M(T^{(1,1)} M))$ by

$$g(p,t) := (g_t)_p, \quad h(p,t) := (h_t)_p, \quad A(p,t) := (A_t)_p \quad ((p, t) \in M \times [0, T)).$$
We denote by $\tilde{X}$ the horizontal lift of $X \in \Gamma(TM)$ to $M \times [0, T)$ by $\pi_M$. For simplicity, put $\tilde{\partial}_i := \partial/\partial x^i$ ($1 \leq i \leq n$). Define a connection $\nabla$ of $\pi_M^*(TM)$ by

$$\nabla V W := \nabla^t V W(\cdot, t), \quad (\nabla \partial_t W)(p, t) := dW(p, t)/dt, \quad (V \in (\pi_M^*(TM))(p, t), W \in \Gamma(\pi_M^*(TM))),$$

(2.2)

where $dW(p, t)/dt$ is the derivative of the vector-valued function $t \mapsto W(p, t) \in TpM$. For $S \in \Gamma(\pi_M^*(T(0,2)M))$, denote by $S_{ij}$ (respectively $(\partial S/\partial t)_{ij}$) ($1 \leq i, j \leq n$) the components of $S$ (respectively $\partial S/\partial t$) with respect to $(x^1, \ldots, x^n)$. Clearly we have

$$\left(\frac{\partial S}{\partial t}\right)_{ij} = \frac{\partial}{\partial t}(S_{ij}).$$

In general, the similar relation holds for $S \in \Gamma(\pi_M^*(T(r,s)M))$. We shall derive the evolution equations for some geometric quantities along the IMCF.

**LEMMA 2.1.** The induced metric $g$ satisfies the following evolution equation:

$$\left(\frac{\partial g}{\partial t}\right)_{ij} = \frac{\partial}{\partial t}(g_{ij}) = 2H_{ij}.$$

By using Lemma 2.1, we can show the following fact.

**LEMMA 2.2.** The volume element $d\mu$ satisfies the following evolution equation:

$$\frac{d}{dt}(d\mu) = d\mu.$$

Put $e_0 := \nu$ and $e_i := dF(\tilde{\partial}_i)$ ($i = 1, 2, \ldots, n$). Then $(e_0, e_1, \ldots, e_n)$ is a local frame field along $F$. For $\tilde{S} \in F^*(T(0,5)N)$, denote by $\tilde{S}_{\alpha_1\ldots\alpha_s}$ ($0 \leq \alpha_1, \ldots, \alpha_s \leq n$) the components of $\tilde{S}$ with respect to $(e_0, e_1, \ldots, e_n)$.

Next we derive the evolution equation for the outer unit normal.

**LEMMA 2.3.** The outer unit normal $\nu$ satisfies the following evolution equation:

$$\nabla^{\bar{F}}_{\partial_t} \nu = \frac{1}{H^2} dF(\text{grad } H).$$

**Proof.** Since $\nu$ is a unit normal, we have

$$\nabla^{\bar{F}}_{\partial_t} \nu = \sum_{\alpha, \beta=0}^n \tilde{g}(\nabla^{\bar{F}}_{\partial_t} \nu, e_\alpha)\tilde{g}_{\alpha\beta} = \sum_{i,j=1}^n \tilde{g}(\nabla^{\bar{F}}_{\partial_t} \nu, e_i)e_j\tilde{g}^{ij}.$$

Also we have

$$\tilde{g}(\nabla^{\bar{F}}_{\partial_t} \nu, e_i) = -\tilde{g}(\nu, \nabla^{\bar{F}}_{\partial_t} dF(\tilde{\partial}_i)) = -\tilde{g}(\nu, \nabla_{\partial_t} \left(\frac{1}{H} \nu\right)) = \frac{1}{H^2} \nabla_i H.$$

Hence the desired relation follows. 

Taking the trace of the Simons identity [11], we can show the following relation. See also Zhu [13, Ch. 10].
**Lemma 2.4.** The following relation holds:

\[
(\Delta h)_{ij} = (\nabla \nabla H)_{ij} + H \sum_{k,l=1}^{n} h_{ik} h_{jl} g^{kl} - \|A\|^2 h_{ij} + H \bar{R}_{0ij0} - h_{ij} \bar{Ric}_{00}
\]

\[
+ \sum_{k,l,a,b=1}^{n} (\tilde{R}_{kial} h_{bj} + \tilde{R}_{kjai} h_{ib} - 2 \tilde{R}_{kija} h_{lb}) g^{kl} g^{ab}
\]

\[
+ \sum_{k,l=1}^{n} ((\tilde{\nabla} \tilde{\nabla})_{k0jii} + (\tilde{\nabla} \tilde{\nabla})_{i0klj}) g^{kl},
\]

where

\[
\tilde{R}_{a\beta\gamma}\delta := \sum_{\epsilon=0}^{n} \tilde{R}_{a\beta\gamma}^{\epsilon} \tilde{g}_{\epsilon\delta} \quad (\tilde{\tilde{R}} = \{\tilde{R}_{a\beta\gamma}^{\epsilon}\} : \text{the Riemannian curvature tensor of } \tilde{\tilde{g}})
\]

and

\[
\bar{Ric}_{00} := \sum_{k,l=1}^{n} \bar{R}_{0k0j0} g^{kl} \quad ((g^{ij}) \text{ is the inverse matrix of } (g_{ij})).
\]

**Lemma 2.5.** The second fundamental form \( h \) satisfies the following evolution equation:

\[
\left(\frac{\partial h}{\partial t}\right)_{ij} = \frac{\partial}{\partial t} h_{ij} = -\left(\nabla \frac{1}{H}\right)_{ij} + \frac{1}{H} \sum_{k,l=1}^{n} h_{jk} h_{il} g^{kl} - \frac{1}{H} \tilde{R}_{0ij0}
\]

\[
= -\frac{2}{H^3} \nabla_i H \cdot \nabla_j H + \frac{1}{H^2} \left( (\nabla \nabla H)_{ij} + H \sum_{k,l=1}^{n} h_{jk} h_{il} g^{kl} - H \bar{R}_{0ij0} \right)
\]

\[
= \frac{1}{H^2} (\Delta h)_{ij} - \frac{2}{H^3} \nabla_i H \cdot \nabla_j H - \frac{2}{H} \tilde{R}_{0ij0} + \frac{1}{H^2} (\|A\|^2 + \bar{Ric}_{00}) h_{ij}
\]

\[
- \frac{1}{H^2} \sum_{k,l,a,b=1}^{n} (\tilde{R}_{kial} h_{bj} + \tilde{R}_{kjai} h_{ib} - 2 \tilde{R}_{kija} h_{lb}) g^{kl} g^{ab}
\]

\[
- \frac{1}{H^2} \sum_{k,l=1}^{n} ((\tilde{\nabla} \tilde{\nabla})_{k0jii} + (\tilde{\nabla} \tilde{\nabla})_{i0klj}) g^{kl}.
\]

**Proof.** From \( \tilde{\tilde{\nabla}}_{\partial_i} \tilde{\tilde{g}} = 0 \), we have

\[
\left(\frac{\partial h}{\partial t}\right)_{ij} = \frac{\partial}{\partial t} h_{ij} = \frac{\partial}{\partial t} (h(\partial_i, \partial_j)) = \frac{\partial}{\partial t} (\tilde{g}(h(\partial_i, \partial_j), v, v))
\]

\[
= -\tilde{g}(\tilde{\tilde{\nabla}}_{\partial_i} (\tilde{\tilde{\nabla}}_{\partial_j} d F(\partial_j)), v) - \tilde{g}(\tilde{\tilde{\nabla}}_{\partial_j} d F(\partial_j), \tilde{\tilde{\nabla}}_{\partial_i} v).
\]

Since \([\partial_i, \tilde{\partial}_i] = [\partial_i, \tilde{\partial}_j] = 0\), we have

\[
\tilde{g}(\tilde{\tilde{\nabla}}_{\partial_i} (\tilde{\tilde{\nabla}}_{\partial_j} d F(\partial_j)), v) = \tilde{g}(\tilde{\tilde{\nabla}}_{\partial_j} (\tilde{\tilde{\nabla}}_{\partial_i} d F(\partial_i)), v) + \tilde{g}(d F(\partial_i), d F(\tilde{\partial}_i)) d F(\tilde{\partial}_j), v)
\]

\[
= \tilde{g}(\tilde{\tilde{\nabla}}_{\partial_i} (\tilde{\tilde{\nabla}}_{\partial_j} (\frac{1}{H} v)), v) + \frac{1}{H} \tilde{R}_{0ij0}
\]
\[
\begin{align*}
&= 2 H^3 \nabla_{\tilde{\alpha}} H \cdot \nabla_{\tilde{\alpha}} H - \frac{1}{H^2} \nabla_{\tilde{\alpha}} (\nabla_{\tilde{\alpha}} H) + \frac{1}{H} \tilde{g}(\nabla_{\tilde{\alpha}} dF(\tilde{\alpha} j), \nu) \\
&+ \frac{1}{H} \tilde{R}_{0ij0} \\
&= 2 H^3 \nabla_{\tilde{\alpha}} H \cdot \nabla_{\tilde{\alpha}} H - \frac{1}{H^2} \nabla_{\tilde{\alpha}} (\nabla_{\tilde{\alpha}} H) \\
&- \frac{1}{H} \sum_{k,l=1}^n h_{jk} g^{kl} h_{il} + \frac{1}{H} \tilde{R}_{0ij0}.
\end{align*}
\]

On the other hand, from Lemma 2.3, we have

\[
\tilde{g}(\nabla_{\tilde{\alpha}} dF(\tilde{\alpha} j), \nabla_{\tilde{\alpha}} dF) = \frac{1}{H^2} g(\nabla_{\tilde{\alpha}} \tilde{\alpha}_j, \text{grad } H).
\]

Combining these relations, we obtain

\[
\left( \frac{\partial h}{\partial t} \right)_{ij} = \frac{\partial}{\partial t} (h_{ij}) = -\left( \nabla\nabla \frac{1}{H} \right)_{ij} + \frac{1}{H} \sum_{k,l=1}^n h_{jk} g^{kl} h_{il} - \frac{1}{H} \tilde{R}_{0ij0}
\]

\[
= -\frac{2}{H^3} \nabla_i H \cdot \nabla_j H + \frac{1}{H^2} \left( \nabla\nabla H \right)_{ij} + H \sum_{k,l=1}^n h_{jk} g^{kl} h_{il} - \frac{1}{H} \tilde{R}_{0ij0}.
\]

From Lemma 2.4, we obtain the last equality of the relation in Lemma 2.5.

**Lemma 2.6.** The mean curvature \( H \) satisfies the following evolution equation:

\[
\frac{\partial H}{\partial t} = \frac{1}{H^2} \Delta H - \frac{2}{H^3} \|\text{grad } H\|^2 - \frac{1}{H} (\|A\|^2 + \overline{\text{Ric}}_{00}).
\]

**Proof.** Differentiating the identity \( \delta_i^j = \sum_{k=1}^n g_{ik} g^{kj} \) by \( t \), we have

\[
\frac{\partial}{\partial t} (g^{ij}) = -\frac{2}{H} \sum_{k,l=1}^n g^{ij} h_{lk} g^{kl}.
\]

From (2.3) and Lemma 2.5, we obtain

\[
\frac{\partial H}{\partial t} = \sum_{i,j=1}^n \left( \frac{\partial}{\partial t} (g^{ij}) \cdot h_{ij} + g^{ij} \cdot \frac{\partial}{\partial t} (h_{ij}) \right)
\]

\[
= -\frac{2}{H} \|A\|^2 + \sum_{i,j=1}^n g^{ij} \left( \frac{1}{H^2} (\Delta h)_{ij} - \frac{2}{H^3} \nabla_i H \cdot \nabla_j H - \frac{2}{H} \tilde{R}_{0ij0} \right)
\]

\[
+ \frac{1}{H^2} (\|A\|^2 + \overline{\text{Ric}}_{00}) h_{ij}
\]

\[
- \frac{1}{H^2} \sum_{k,l,a,b=1}^n (\tilde{R}_{kial} h_{bj} + \tilde{R}_{kjal} h_{ib} - 2 \tilde{R}_{kija} h_{lb}) g^{kl} g^{ab}
\]

\[
- \frac{1}{H^2} \sum_{k,l=1}^n (\overline{\nabla} \overline{\nabla} R)_{k0lij} + (\overline{\nabla} \overline{\nabla} R)_{i0klj}) g^{kl}.
\]

This relation implies the desired relation.
3. The evolution equation for the support function

Let $F_t : M \hookrightarrow N \ (t \in [0, T))$ be an inverse mean curvature flow starting from a closed smooth hypersurface $M$ in a complete smooth Riemannian manifold $N$ with Riemannian metric $\bar{g}$. In this section, we shall derive a lower bound for the support function of the star-shaped solution of (IMCF) in a rank one symmetric space of non-compact type. In the Euclidean case (i.e., $N = \mathbb{R}^{n+1}$), Huisken–Ilmanen [7] derived a lower bound for the mean curvature of the solution. In this case, the support function $w_t$ of $M_t$ is defined by

$$w_t(p) := \langle F(p, t), \nu(p, t) \rangle \quad (p \in M),$$

where $\langle \cdot, \cdot \rangle$ is the canonical metric on $\mathbb{R}^{n+1}$. Define a function $w$ on $M \times [0, T)$ by $w(p, t) := w_t(p) \ ((p, t) \in M \times [0, T))$. Combining the evolution equation for $w$, the Micheal–Simon Sobolev inequality and an iteration lemma due to De Giorgi, they attained the desired estimate. In the hyperbolic space case, Gerhardt [5] proved that the flows with star-shaped initial hypersurface exist for all time. In the rotationally symmetric space case, Ding [2] defined the support function in terms of the geodesic polar coordinate. He proved that the IMCF has a unique solution for all time and certain rescaled flows converge to a uniquely determinate (geodesic) sphere in terms of the evolution equation for the support function.

We shall consider the case where the ambient space is a rank one symmetric space of non-compact type. Let $N = G/K$ be a rank one symmetric space of non-compact type and let $(\mathfrak{g}, \theta)$ be the orthogonal symmetric Lie algebra associated with $G/K$. Put $p := \{X \in \mathfrak{g} | \theta(X) = -X\}$, which is identified with the $TeK(G/K)$ ($e$: the identity element of $G$). Without loss of generality, we may assume that $eK$ belongs to the interior region of the initial data $F_0(M)$. For simplicity, we put $p_0 := eK$. Denote by $\bar{g}$ the Riemannian metric of $N$, by $\bar{\nabla}$ the Levi-Civita connection of $\bar{g}$ and by $\exp_{p_0}$ the exponential map of $N$ at $p_0$. We define the support function $w_t$ of $M_t$ by

$$w_t(p) := \bar{g}_{F(p, t)}(c'_{v(p, t)}(1), v(p, t)) \quad (p \in M),$$

where $v(p, t) := \exp^{-1}_{p_0}(F(p, t))$ and $c_{v(p, t)}$ is the maximal geodesic of the direction $v(p, t) \in T_{p_0}(G/K)$. Define a function $w$ on $M \times [0, T)$ by $w(p, t) := w_t(p) \ ((p, t) \in M \times [0, T))$. If $w > 0$ on $M \times [0, T)$, we call $F$ a star-shaped solution (with respect to $p_0$).

First, we derive the evolution equation for the support function $w$. By differentiating $w$ with respect to $t$ and using Lemma 2.3, we have

$$\frac{\partial w}{\partial t}(p, t) = \bar{g}_{F(p, t)}((\bar{\nabla}_{\partial_t} c'_{v(p, t)}(1))(p, t), v(p, t))$$

$$+ \frac{1}{H(p, t)^2} \bar{g}_{F(p, t)}(c'_{v(p, t)}(1), dF(p, t)(\text{grad } H)). \quad (3.1)$$

We investigate the first term of (3.1). We define a geodesic variation $V$ by

$$V(s, t) := c_{v(p, t)}(s) = \exp_{p_0}(sv(p, t)).$$

Put

$$Y_t(s) := dV_{(s,t)} \left( \frac{\partial}{\partial t} \right).$$
Then $Y_t$ is the Jacobi field along the geodesic $c_{v(p,t)}$, and
\[
(\tilde{\nabla}^F_{\partial_t}c'_{v}(1))_{(p,t)} = \tilde{\nabla}^V_{\partial_t} \left( \frac{\partial}{\partial s} V(s, t) \right)_{s=1} = \tilde{\nabla}^V_{\partial_t} \left( dV(s,t) \left( \frac{\partial}{\partial s} \right) \right)_{s=1} = \tilde{\nabla}^V_{\partial_t} Y_t(s)_{s=1} = \frac{DY_t}{ds}(1).
\]

The Jacobi field $Y_t$ satisfies the following initial condition:
\[
Y_t(0) = 0 \quad \text{and} \quad \frac{DY_t}{ds}(0) = \frac{1}{H(p, t)}(d \exp^{-1}_{p_0})F(p,t)(v(p,t)). \quad (3.2)
\]

The simultaneous eigenspace decomposition of $p$ with respect to the operator $\bar{R}(\cdot, v(p, t))v(p, t) = -\text{ad}(v(p, t))^2$ coincides with the root space decomposition of $p$ with respect to the maximal abelian subspace $\text{Span}\{v(p, t)\}$ of $p$, where $\text{ad}$ is the adjoint representation of $g$. Let $c < 0$ be the maximal sectional curvature of $G/K$, where we note that the minimal sectional curvature of $G/K$ is equal to $4c$. Let $\triangle(p,t)$ be the root system with respect to $\text{Span}\{v(p, t)\}$. Then $\triangle(p,t)$ is given by
\[
\triangle(p,t) = \left\{ \pm \sqrt{-c} \|v(p,t)\| \tilde{g}F(p,t)(v(p,t), \cdot), \pm 2\sqrt{-c} \|v(p,t)\| \tilde{g}F(p,t)(v(p,t), \cdot) \right\}.
\]

For simplicity,
\[
a^0_{(p,t)} := 0, \quad a^1_{(p,t)} := \sqrt{-c} \|v(p,t)\| \tilde{g}F(p,t)(v(p,t), \cdot)
\]
and
\[
a^2_{(p,t)} := \frac{2}{\|v(p,t)\|} \tilde{g}F(p,t)(v(p,t), \cdot).
\]

Denote by $l$ the dimension of the root space for $a^2_{(p,t)}$. The explicit description of a Jacobi field $J$ along the geodesic $c_{v(p,t)}$ in symmetric spaces is given by Terng–Thorbergsson [12]. (See also [9].) According to the description, a Jacobi field $J$ along $c_{v(p,t)}$ is described as
\[
J(s) = P_{c_{v(p,t)}|[0,s]} \left( D_{sv(p,t)}^{co}(J(0)) + s D_{sv(p,t)}^{ij}(\frac{DJ}{ds}(0)) \right), \quad (3.3)
\]
where $P_{c_{v(p,t)}|[0,s]}$ is the parallel translation along $c_{v(p,t)}|[0,s]$, and $D_{sv(p,t)}^{co}$ and $D_{sv(p,t)}^{ij}$ are defined by
\[
D_{sv(p,t)}^{co} := \cosh (s \cdot \text{ad}(v(p,t))), \quad D_{sv(p,t)}^{ij} := \frac{\sinh(s \cdot \text{ad}(v(p,t)))}{s \cdot \text{ad}(v(p,t))}.
\]

For simplicity, we put $X := (d \exp^{-1}_{p_0})v(p,t)$. From (3.2) and (3.3), we obtain
\[
\frac{DY_t}{ds}(s) = \frac{1}{H} P_{c_{v(p,t)}|[0,s]}(D_{sv(p,t)}^{co}(X)),
\]
and hence
\[
(\tilde{\nabla}^F_{\partial_t}c'_{v}(1))_{(p,t)} = \frac{DY_t}{ds}(1) = \frac{1}{H} P_{c_{v(p,t)}|[0,1]}(D_{v(p,t)}^{co}(X)). \quad (3.4)
\]

Define geodesic variations $\tilde{V}$ and $\hat{V}$ by
\[
\tilde{V}(s, \hat{t}) := s(v(p, t) + \hat{t}X) \quad \text{and} \quad \hat{V}(s, \hat{t}) := \exp_{p_0}(\tilde{V}(s, \hat{t})).
\]
respectively. Since \( \hat{Y}(s) := d\hat{V}_{(s,0)}(\partial/\partial \hat{t}) \) is the Jacobi field with the initial conditions

\[
\hat{Y}(0) = 0 \quad \text{and} \quad \frac{d\hat{Y}}{ds}(0) = X,
\]

it is described as

\[
\hat{Y}(s) = P_{c_v(p,t)\|\partial_i}(sD_{\nu(s\nu(p,t))}^{\partial_i}(X)).
\]

Hence we have \( \nu(p, t) = \hat{Y}(1) = P_{c_v(p,t)\|\partial_i}(D_{\nu(s\nu(p,t))}^{\partial_i}(X)) \). From (3.4) and this relation, we have

\[
\bar{F}(p, t) = \nabla F^\partial_i c_v(p,t) (\nu(p, t)) = \frac{1}{H} \bar{F}_{c_v(p,t)}(D_{\nu(s\nu(p,t))}^{\partial_i}(X), D_{\nu(s\nu(p,t))}^{\partial_i}(X)). \tag{3.5}
\]

On the other hand, the Laplacian \( \triangle w \) of \( w \) is calculated as follows. Fix \( (p, t) \in M \times [0, T) \). Let \( (x^1, \ldots, x^n) \) be the normal coordinate of \( M \) with respect to an orthonormal base \( (e_1, \ldots, e_n) \) of \( T_pM \) with respect to \( (gt)_p \). Set \( \tilde{\partial}_i := \partial/\partial x^i \) \((i = 1, \ldots, n)\). At \( (p, t) \), \( \triangle w \) is calculated as

\[
(\triangle w)(p, t) = \sum_{i=1}^n \bar{F}_{c_v(p,t)}((\nabla F^\partial_i c_v(p,t))(\nu(p, t))) + \sum_{i=1}^n \bar{F}_{c_v(p,t)}(c_v'(p,t)(1), (\nabla F^\partial_i c_v(p,t))(\nu(p, t))) + 2 \sum_{i=1}^n \bar{F}_{c_v(p,t)}((\nabla F^\partial_i c_v(p,t))(\nu(p, t))). \tag{3.6}
\]

The second term of the right-hand side of (3.6) is easily calculated as

\[
\sum_{i=1}^n \bar{F}_{c_v(p,t)}(c_v'(p,t)(1), (\nabla F^\partial_i c_v(p,t))(\nu(p, t))) = \bar{F}_{c_v(p,t)}(c_v'(p,t)(1), \nabla H) - \|A(p, t)\|^2 w(p, t). \tag{3.7}
\]

To calculate the third term of the right-hand side of (3.6), we define a geodesic variation \( V_i \) by

\[
V_i(s, \tilde{t}) := c_v(y_i(\tilde{t}), s) = \exp_{p_0}(sv(y_i(\tilde{t}), t)),
\]

where \( y_i \) is a curve in \( M \) such that \( y_i(0) = p \) and \( y_i' = \partial_i \). Put

\[
Y_i^\partial_i(s) := (dV_i(s, \tilde{t}))(\frac{\partial}{\partial \tilde{t}}) \quad \text{and} \quad Y_i(s) := Y_i^\partial_i(0).
\]

Then \( Y_i \) is the Jacobi field along the geodesic \( c_v(p,t) \) and we have

\[
\frac{dY_i}{ds}(1) = \nabla Y_i^\partial_i dV_i \left( \frac{\partial}{\partial \tilde{t}} \right) \bigg|_{s=1, \tilde{t}=0} = \nabla Y_i^\partial_i dV_i \left( \frac{\partial}{\partial \tilde{t}} \right) \bigg|_{s=1, \tilde{t}=0} = (\nabla F^\partial_i c_v'(1))(p, t).
\]

The Jacobi field \( Y_i \) satisfies the following initial condition:

\[
Y_i(0) = 0 \quad \text{and} \quad \frac{dY_i}{ds}(0) =\left( d \exp_{p_0}^{-1}(dF_i)_p(\partial_i) \right).
\]

For simplicity, we put \( X_i := d \exp_{p_0}^{-1}((dF_i)_p(\partial_i)) \). According to (3.3), we have

\[
Y_i(s) = P_{c_v(p,t)\|\partial_i}(sD_{\nu(s\nu(p,t))}^{\partial_i}(X_i))
\]
and

\[
\frac{DY_i}{ds}(s) = P_{c_v(p,t)[0,s]}(D^{co}_{v(p,t)}(X_i)).
\]

Hence we have

\[
(\tilde{\nabla}_{\tilde{t}} F p'(1))_{(p,t)} = P_{c_v(p,t)[0,1]}(D^{co}_{v(p,t)}(X_i)).
\]

From \(Y_i(1) = (d F_t)_p(\partial_t)\) and the commutativity of \(D^{co}_{v(p,t)}\) and \(D^{si}_{v(p,t)}\), we obtain

\[
\sum_{i=1}^{n} \tilde{g}((\tilde{\nabla}_{\tilde{t}} F p'(1))_{(p,t)}, (\tilde{\nabla}_{\tilde{t}} v)(p,t))
= \sum_{i=1}^{n} \tilde{g}(P_{c_v(p,t)[0,1]}(D^{co}_{v(p,t)}(X_i)), (d F_t)_p((A_t)_p(\partial_t)))
= \sum_{i=1}^{n} \tilde{g}((d F_t)^{-1}_p \circ P_{c_v(p,t)[0,1]} \circ D^{co}_{v(p,t)}(X_i), (d F_t)_p((A_t)_p(\partial_t)))
= \sum_{i=1}^{n} g\left((d F_t)^{-1}_p \circ \frac{\partial}{\partial t} dF_t(T M) \circ D^{co}_{v(p,t)}(X_i), (d F_t)_p((A_t)_p(\partial_t))\right)
= \text{Tr}\left((d F_t)^{-1}_p \circ \frac{\partial}{\partial t} dF_t(T M) \circ D^{co}_{v(p,t)}(X_i), (d F_t)_p((A_t)_p(\partial_t))\right)
= (3.8)
\]

where \(\text{pr}_{dF_t(T M)}\) is the orthogonal projection from \(F^*(TN)\) to \(dF_t(TM)\).

Next we shall calculate the first term of the right-hand side of (3.6). Since \(Y_i^{\tilde{t}}(0) = 0\) and \(DY_i^{\tilde{t}}(0) = (d \exp_{p_0})^{-1}_{F_t(\gamma, \tilde{t})}((d F_t)_\gamma(\tilde{t}))\), we have

\[
\frac{DY_i^{\tilde{t}}}{ds}(1) = P_{c_v(\gamma(\tilde{t}), t)[0,1]}(D^{co}_{v(\gamma(\tilde{t}), t)}(X_i)),
\]

where \(X_i := (d \exp_{p_0})^{-1}_{F_t(\gamma, \tilde{t})}((d F_t)_\gamma(\tilde{t}))\). Put \(B_t := D^{co}_{v(\gamma(t), t)} \circ (D^{si}_{v(\gamma(t), t)})^{-1}\). Define \(Q_{v(p,t), \bullet}^{co}\) and \(Q_{v(p,t), \bullet}^{si}\) by

\[
Q_{v(p,t), \bullet}^{co} := \sum_{k=0}^{\infty} \frac{1}{(2k)!} \sum_{l=1}^{2k} \text{ad}(v(p,t))^l \circ \text{ad}(\bullet) \circ \text{ad}(v(p,t))^{2k-l}
\]

and

\[
Q_{v(p,t), \bullet}^{si} := \sum_{k=0}^{\infty} \frac{1}{(2k + 1)!} \sum_{l=1}^{2k} \text{ad}(v(p,t))^l \circ \text{ad}(\bullet) \circ \text{ad}(v(p,t))^{2k+1-l},
\]
respectively. Then we can show that

\[
\nabla_{\partial \bar{t}} \left( \sum_{k=0}^{\infty} \frac{1}{(2k)!} \text{ad}(v(\gamma_t(i), t))^{2k} \right) \circ (D_{v(p,t)}^{ii})^{-1} - B \circ (D_{v(p,t)}^{ii})^{-1} = \frac{1}{(2k+1)!} \text{ad}(v(\gamma_t(i), t))^{2k} \circ (D_{v(p,t)}^{ii})^{-1}.
\]

(3.10)

From (3.9) and (3.10), we have

\[
\sum_{i=1}^{n} \bar{g}_F(p,t)((\nabla F^i_{\partial i}((\nabla F^j_{\partial j}((1))))(p,t), v(p,t))
\]

\[
= \sum_{i=1}^{n} g((\nabla V^i_{\partial i}((1)), v(p,t)))
\]

\[
= \sum_{i=1}^{n} \bar{g}_F(p,t)(Q^{co}_{v(p,t),\gamma_i}(X_i), D_{v(p,t)}^{ii}(X))
\]

\[
- \sum_{i=1}^{n} \bar{g}_F(p,t)((B \circ (D_{v(p,t)}^{ii})^{-1}, X_i), D_{v(p,t)}^{ii}(X))
\]

\[
- H \bar{g}_F(p,t)(D_{v(p,t)}^{co}(X), D_{v(p,t)}^{ii}(X)).
\]

(3.11)

Let \( E \) be a vector bundle over \( M \). For \( S \in \Gamma(\pi^*_M(T^{(0,1)}M \otimes E)) \), we define \( \text{Tr}^\#_g S(\cdots, \bullet, \cdots, \bullet, \cdots) \) by

\[
(\text{Tr}^\#_g S(\cdots, \bullet, \cdots, \bullet, \cdots))(p,t)
\]

\[
:= \sum_{i=1}^{n} S_{(p,t)}(\cdots, \bullet^i, \cdots, \bullet^k, \cdots) \quad ((p,t) \in M \times [0, T]),
\]

where \((e_1, \ldots, e_n)\) is an orthonormal base of \( T_p M \) with respect to \((g_t)_p\), \( S(\cdots, \bullet, \cdots, \bullet, \cdots) \) means that \( \bullet \) is entered into the \( j \)th component and the \( k \)th component of \( S \) and \( S_{(p,t)}(\cdots, \bullet^j, \cdots, \bullet^k, \cdots) \) means that \( e_i \) is entered into the \( j \)th component and the \( k \)th component of \( S_{(p,t)} \). Since \((D_{v(p,t)}^{ii}(X_1), \ldots, D_{v(p,t)}^{ii}(X_n))\) is an orthonormal base with respect to \((g_t)_p\), we have

\[
\sum_{i=1}^{n} \bar{g}_F(p,t)(Q^{co}_{v(p,t),\gamma_i}(X_i), D_{v(p,t)}^{ii}(X))
\]

\[
= \bar{g}_F(p,t)(\text{Tr}^\#_g((Q^{co}_{v(p,t),\gamma_i}(D_{v(p,t)}^{ii})^{-1}(\bullet)) \circ (D_{v(p,t)}^{ii})^{-1}(\bullet)), D_{v(p,t)}^{ii}(X))
\]
and
\[ \sum_{i=1}^{n} \bar{g}_{F(p,t)}((B_{0} \circ Q_{v(p,t),X_{i}}^{i}(X_{i}), D_{v(p,t)}^{i}(X)) = \bar{g}_{F(p,t)}(Tr_{g}^{*}((B_{0} \circ Q_{v(p,t),D_{v(p,t)}^{i}^{-1}(\bullet)}^{i} \circ (D_{v(p,t)}^{i}^{-1}(\bullet)), D_{v(p,t)}^{i}(X)).\]

From (3.6)–(3.8) and (3.11), these relations and the arbitrariness of \((p, t)\), we have
\[ \Delta w = \bar{g}(Tr_{g}^{*}((Q_{v(D_{v})}^{co})^{-1}(\bullet) \circ (D_{v}^{i})^{-1}(\bullet)), D_{v}^{i}(X)) \]
\[ - \bar{g}(Tr_{g}^{*}((D_{v}^{co} \circ (D_{v}^{i}^{-1} \circ (Q_{v(D_{v})}^{i})^{-1}(\bullet))(\bullet)), D_{v}^{i}(X)) \]
\[ - H \bar{g}(D_{v}^{co}(X), D_{v}^{i}(X)) + \bar{g}(c_{v}(1), d F (\text{grad } H)) - \| A \|^{2} w + 2 \text{Tr}((d F_{t})^{-1} \circ pr_{d F_{t}(TM)} \circ P_{cv[0,1]} \circ D_{v}^{co} \circ (D_{v}^{i})^{-1} \circ P_{cv[0,1]}^{-1} \circ d F_{t} \circ A). (3.12) \]

Thus we obtain the evolution equation for the support function of \(w\).

**Lemma 3.1.** The support function \(w\) satisfies the following evolution equation:
\[ \frac{\partial w}{\partial t} = \frac{1}{H^{2}} \Delta w + \frac{2}{H} \bar{g}(D_{v}^{co}(X), D_{v}^{i}(X)) + \frac{\| A \|^{2}}{H^{2}} w \]
\[ - \frac{1}{H^{2}} \bar{g}(Tr_{g}^{*}((Q_{v(D_{v})}^{co})^{-1}(\bullet) \circ (D_{v}^{i})^{-1}(\bullet)), D_{v}^{i}(X)) \]
\[ + \frac{1}{H^{2}} \bar{g}(Tr_{g}^{*}((D_{v}^{co} \circ (D_{v}^{i}^{-1} \circ (Q_{v(D_{v})}^{i})^{-1}(\bullet))(\bullet)), D_{v}^{i}(X)) \]
\[ - \frac{2}{H^{2}} \text{Tr}((d F_{t})^{-1} \circ pr_{d F_{t}(TM)} \circ P_{cv[0,1]} \circ D_{v}^{co} \circ (D_{v}^{i})^{-1} \circ P_{cv[0,1]}^{-1} \circ d F_{t} \circ A). \]

As the next step, we shall show that the star-shapedness is preserved along the IMCF in terms of this lemma.

**Lemma 3.2.** Assume that \(M\) is a closed hypersurface in \(G/K\) and \(F : M \times [0, T) \to G/K\) is a smooth solution of (IMCF) such that the initial data \(F_{0} : M \to G/K\) satisfies the estimate
\[ 0 < R_{1} \leq w_{0} \leq R_{2}, \]
for some positive constants \(R_{1}\) and \(R_{2}\). Then the inequality
\[ \frac{1}{2 \sqrt{-c}} \arcsinh(e^{t/n} \sinh(2 \sqrt{-c} R_{1})) \leq w \leq \| v \| \leq \frac{1}{\sqrt{-c}} \arcsinh(e^{t/n} \sinh(\sqrt{-c} R_{2})) \]
holds everywhere on \(M \times [0, T)\).

**Proof.** Denote by \(A^{r}\) and \(H^{t}\) the shape operator and the mean curvature of the geodesic sphere of radius \(r\) in \(G/K\). By a simple calculation, we obtain the following estimates:
\[ \frac{\| A \|^{2}}{H^{2}} w - \frac{1}{H^{2}} \bar{g}(Tr_{g}^{*}((Q_{v(D_{v})}^{co})^{-1}(\bullet) \circ (D_{v}^{i})^{-1}(\bullet)), D_{v}^{i}(X)) \]
\[ - \frac{2}{H^{2}} \text{Tr}((d F_{t})^{-1} \circ pr_{d F_{t}(TM)} \circ P_{cv[0,1]} \circ D_{v}^{co} \circ (D_{v}^{i})^{-1} \circ P_{cv[0,1]}^{-1} \circ d F_{t} \circ A) \]
\[ \geq \frac{w}{H^{2}} \| A - A \|^{2} \geq 0 \]
and
\[
\int H^2 \overline{g}(T_{g}^{*}((D_v^o \circ (D_v^l)^{-1} \circ Q_{v, (D_v^l)^{-1}}(\bullet) \circ (D_v^l)^{-1})(\bullet)), D_v^l(X))
\]
\[
+ \frac{2}{H} \overline{g}(D_v^o(X), D_v^l(X)) \geq \frac{2}{H} - \frac{wH\|v\|}{\|v\|H^2}.
\]

Let \( \bar{p}_t \) be a minimal point of \( w_t \). From Lemma 3.1 and these estimates, we can derive the following estimate:
\[
(\partial w/\partial t)_{(\bar{p}_t, t)} - \frac{1}{H^2(\bar{p}_t, t)}(\triangle w_t)_{\bar{p}_t} \geq \frac{1}{(H_t)_{\bar{p}_t}} \geq \frac{\tanh(2\sqrt{-c}(w_t)_{\bar{p}_t})}{2\sqrt{-c} n}.
\]

Since
\[
\psi(t) := \frac{1}{2\sqrt{-c}} \arcsinh(e^{t/n} \sinh(2\sqrt{-c} w_0(\bar{p}_0)))
\]
satisfies
\[
\frac{d\psi}{dt} = \frac{\tanh(2\sqrt{-c} \psi)}{2n\sqrt{-c}},
\]
we obtain
\[
w \geq \frac{1}{2\sqrt{-c}} \arcsinh(e^{t/n} \sinh(2\sqrt{-c} w_0(\bar{p}_0))) \geq \frac{1}{2\sqrt{-c}} \arcsinh(e^{t/n} \sinh(2\sqrt{-c} R_1)).
\]
The second inequality is clear. Let \( S_t \) be the geodesic sphere with the radius \( r_t := \max_M \|v_t\| \) centered at \( p_0 \), \( H_t \) the mean curvature of \( S_t \) and \( \hat{p}_t \) a maximum point of \( \|v_t\| \), where we note that \( H_t \) is given by
\[
H_t = \frac{\sqrt{-c}}{\tanh(\sqrt{-c} r_t)}(n - 1) + \frac{2\sqrt{-c}}{\tanh(2\sqrt{-c} r_t)}l.
\]
(See [9].) Then we have
\[
(H_t)_{\hat{p}_t} \geq H_t \geq \frac{\sqrt{-c} n}{\tanh(\sqrt{-c} r_t)},
\]
and by using the Gauss lemma, we obtain
\[
\frac{d\|v_t(\hat{p}_t)\|}{dt} \leq \frac{1}{(H_t)_{\hat{p}_t}} \leq \frac{\tanh(\sqrt{-c}\|v_t(\hat{p}_t)\|)}{\sqrt{-c} n}.
\]

Since
\[
\phi(t) := \frac{1}{\sqrt{-c}} \arcsinh(e^{t/n} \sinh(\sqrt{-c}\|v_0(\hat{p}_0)\|))
\]
satisfies
\[
\frac{d\phi}{dt} = \frac{\tanh(\sqrt{-c} \phi)}{\sqrt{-c} n},
\]
we obtain
\[
\|v\| \leq \frac{1}{\sqrt{-c}} \arcsinh(e^{t/n} \sinh(\sqrt{-c}\|v_0(\hat{p}_0)\|)) \leq \frac{1}{\sqrt{-c}} \arcsinh(e^{t/n} \sinh(\sqrt{-c} R_2)).
\]
We define the modified speed function $u_t$ by

$$u_t(p) := \frac{1}{H_t(p) w_t(p)} \quad (p \in M)$$

and the function $u$ on $M \times [0, T]$ by $u(p, t) := u_t(p)$ ($((p, t) \in M \times [0, T]$). We shall investigate the evolution of $u_t$ to attain our purpose.

**Lemma 3.3.** The modified speed function $u$ satisfies the following evolution equation:

$$\frac{\partial u}{\partial t} - \frac{1}{H^2} \Delta u = -\frac{u}{w} \left( \frac{\partial w}{\partial t} - \frac{1}{H^2} \Delta w \right) + \frac{u}{H^2} (\|A\|^2 + \text{Ric}_{00}) - \frac{2}{H^2} \langle \text{grad} u, \text{grad} H \rangle - \frac{2}{H^2 u} \|\text{grad} u\|^2$$

Proof. We can show this lemma in terms of the evolution equations in Section 2 and Lemma 3.1.

4. **Proof of Theorem 1.1**

In this section, we prove Theorem 1.1. Let $c (< 0)$ be the maximal sectional curvature of $G/K$, where we note that the minimum sectional curvature of $G/K$ is equal to $4c$. We shall determine a lower bound of the mean curvature. The strategy of the proof aims for the De Giorgi iteration lemma to estimate an upper bound of $u$. The first step is to calculate an $L^p$-estimate for the modified speed function $u := 1/Hw$ for each $p > 2$.

To proceed further, we need the following Sobolev inequality for Riemannian submanifolds given by Hoffman and Spruck [6, Theorem 2.1].

**Theorem 4.1.** Let $M$ be an $n$-dimensional manifold, $(N, \bar{g})$ an $m$-dimensional Riemannian manifold and $f : M \hookrightarrow (N, \bar{g})$ an immersion, where $M$ may have the boundary. Denote by $d\mu_M$ the volume element of $f^* \bar{g}$, $H$ the mean curvature vector field of $f$, $\bar{K}$ the maximal sectional curvature of $N$, $\bar{R}(M)$ the injective radius of $N$ restricted to $M$ and by $\omega_m$ the volume of the unit ball in the Euclidean space $\mathbb{R}^m$. Let $b$ be a positive real number or a purely imaginary one satisfying $\bar{K} \leq b^2$ and $\psi$ a non-negative $C^1$ function on $M$ vanishing on $\partial M.$ Then

$$\left( \int_M \psi^{n/(n-1)} d\mu_M \right)^{(n-1)/n} \leq \tilde{C}(n) \int_M (\|\nabla \psi\| + \psi \|H\|) d\mu_M$$

(4.1)
provided
\[ b^2(1 - \alpha)^{-2/n} (\omega_n^{-1} \cdot \text{vol}(\text{supp} \ \psi))^2/n \leq 1 \quad \text{and} \quad 2\rho_0 \leq \bar{R}(M), \]  
(4.2)
where
\[ \rho_0 := \begin{cases} 
\bar{b}^{-1} \sin^{-1} b \cdot (1 - \alpha)^{-1/n} (\omega_n^{-1} \cdot \text{vol}(\text{supp} \ \psi))^{1/n} & \text{(for } b \text{ real)}, \\
(1 - \alpha)^{-1/n} (\omega_n^{-1} \cdot \text{vol}(\text{supp} \ \psi))^{1/n} & \text{(for } b \text{ purely imaginary)}.
\end{cases} \]

Here \( \alpha \) is a free parameter with \( 0 < \alpha < 1 \), and
\[ \tilde{C}(n) = \tilde{C}(n, \alpha) := \frac{\pi}{2} \cdot 2^{n-2} (1 - \alpha)^{-1/n} \left( \frac{n}{n-1} \right) \omega_n^{-1/n}. \]

Remark 4.1. Since the maximal sectional curvature of \( G/K \) is equal to \( c \), we have \( b = \sqrt{-c} i \). Also, we have \( \bar{R}(M) = \infty \). Hence the condition (4.2) holds for any \( C^1 \) function \( \psi \).

**Theorem 4.2.** Let \( F: M \times [0, T) \to G/K \) be as in Theorem 1.1. Put \( n := \dim M \). We use the notation from the previous section. For any \( p > 2 \), there are positive constants \( C(n, p) \) depending on only \( n \) and \( p \) such that the \( L^p \)-norm \( \| u_t \|_{L^p(M)} \) of \( u_t = 1/H_t \omega_t \) with respect to \( g_t \) is estimated from above as
\[ \| u_t \|_{L^p(M)} \leq C(n, p)(\text{vol}(M_0))^{(n+p)/np} e^{(2/p + C_{A_0, R_3})t} \left( \Psi_p(t) \right)^{1/2} \left(0 \leq t < T\right) \]
where
\[ C_{A_0, R_3} := \max_M \frac{\| A_0 \|^2}{R_3^2} \]
and
\[ \Psi_p(t) := \int_0^t \left( \frac{\text{arcsinh}(e^{\tau/n} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 e^{(2C_{A_0, R_3} + (n-p)/np)\tau} d\tau. \]

**Proof.** Take \( p \in (2, \infty) \) and \( t \in (0, T) \). We shall calculate a \( L^p \)-estimate for a modified speed function \( u \). From Lemma 2.2, we have
\[ \frac{d}{dt} \int_M u_t^p \ d\mu_t = \int_M \frac{\partial}{\partial t}(u_t^p) \ d\mu_t + \int_M u_t^p \ d\mu_t. \]
(4.3)
From Lemma 3.3, we have the following estimate:
\[ \int_M \frac{\partial}{\partial t}(u_t^p) \ d\mu_t = \int_M p u_t^{p-1} \frac{\partial u_t}{\partial t} \ d\mu_t \]
\[ = p \int_M \frac{u_t^{p-1}}{H_t^2} \Delta u_t \ d\mu_t - p \int_M \frac{u_t^p}{w_t} \left( \frac{\partial w_t}{\partial t} - \frac{1}{H_t^2} \Delta w_t \right) \ d\mu_t + p \int_M \frac{u_t^p}{H_t^2} \left( \| A_t \|^2 + \text{Ric}_{00} \right) \]
\[ - 2p \int_M u_t^3 w_t u_t^{p+2} \left( \text{grad } u_t, \text{grad } H_t \right) \ d\mu_t - 2p \int_M u_t^2 w_t^2 \| \text{grad } u_t \|^2 \ d\mu_t \]
\[ = -p(p+1) \int_M u_t^2 \| \text{grad } u_t \|^2 \ d\mu_t \]
\[ + p \int_M u_t^p \left( -\frac{1}{w_t} \left( \frac{\partial w_t}{\partial t} - \frac{1}{H_t^2} \Delta w_t \right) + \frac{1}{H_t^2} \left( \| A_t \|^2 + \text{Ric}_{00} \right) \right) \ d\mu_t \]
\[ \leq -p(p+1) \int_M u_t^2 \| \text{grad } u_t \|^2 \ d\mu_t + p \int_M u_t^p \| A_t \|^2 \left( \frac{R_3^2}{R_3} \right) \ d\mu_t, \]
(4.4)
where we use
\[
\int_M \left(p u_t^{p-1} \frac{\Delta u_t}{H_t^2} - \frac{2p}{H_t^2} u_t^{p-1} \langle \text{grad } u_t, \text{grad } H_t \rangle \right) d\mu_t = -\int_M p(p-1) \frac{u_t^{p-2}}{H_t^2} \|\text{grad } u_t\| d\mu_t
\]
(this relation follows from the divergence theorem).

By applying the maximum principle to the evolution equation for the tensor $HA$ (which is derived from Lemmas 2.1, 2.5 and 2.6), we can show that
\[
\|A_t\|^2 \leq \max_M \|A_0\|^2. \tag{4.5}
\]

Put $\varphi := u^{p/2+1}$. From Lemma 3.2, we have
\[
-p(p+1) \int_M u_t^2 u_t^p \|\text{grad } u_t\|^2 d\mu_t \\
\leq -\left(\frac{\text{arcsinh}(e^t/n \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}}\right)^2 \int_M \|\text{grad } \varphi_t\|^2 d\mu_t. \tag{4.6}
\]

From (4.3)–(4.6), we have
\[
\frac{d}{dt} \int_M u_t^p d\mu_t \leq -\left(\frac{\text{arcsinh}(e^t/n \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}}\right)^2 \int_M \|\text{grad } \varphi_t\|^2 d\mu_t \\
+ \left(1 + \frac{p \max_M \|A_0\|^2}{R_3^2}\right) \int_M u_t^p d\mu_t. \tag{4.7}
\]

We distinguish the two cases of $n = 2$ and $n \geq 3$. First we consider the case of $n = 2$. For any $p > 2$, put $q := 2p/(p+2)$. By using Theorem 4.1 as $\psi := |\varphi|^q$ and Remark 4.1, the Hölder inequality and the Minkowski inequality, we can show
\[
\left(\int_M \varphi_t^{2q} d\mu_t\right)^{1/q} \leq C(2) \cdot q^2 \text{vol}(M_t)^{1/q} \int_M (\|\text{grad } \varphi_t\|^2 + H_t^2 \varphi_t^2) d\mu_t, \tag{4.8}
\]
where $C(2) := 2\tilde{C}(2)^2$. By using the Hölder inequality, we obtain
\[
\text{vol}(M_t)^{-1/q} \left(\int_M \varphi_t^q d\mu_t\right)^{2/q} \leq \left(\int_M \varphi_t^{2q} d\mu_t\right)^{1/q}. \tag{4.9}
\]
Combining (4.7)–(4.9) and Lemma 3.2, we have

\[
\frac{d}{dt} \int_M \phi_t^q \, d\mu_t = \frac{d}{dt} \int_M u_t^p \, d\mu_t \\
\leq -\left( \frac{\arcsinh\left( e^{t/2} \sinh\left( 2\sqrt{-c} R_1 \right) \right)}{2\sqrt{-c}} \right)^2 \\
\times \int_M \| \nabla \phi_t \|^2 \, d\mu_t + (1 + pC_{A_0, R_3}) \int_M u_t^p \, d\mu_t \\
\leq -\left( \frac{\arcsinh\left( e^{t/2} \sinh\left( 2\sqrt{-c} R_1 \right) \right)}{2\sqrt{-c}} \right)^2 \\
\times \left( \frac{\text{vol}(M_t)^{-1/q}}{C(2)q^2} \left( \int_M \phi_t^{2q} \, d\mu_t \right)^{1/q} \right)^2 \\
+ (1 + pC_{A_0, R_3}) \int_M \phi_t^{q} \, d\mu_t \\
\leq -\frac{\text{vol}(M_t)^{-2/q}}{C(2)q^2} \left( \frac{\arcsinh\left( e^{t/2} \sinh\left( 2\sqrt{-c} R_1 \right) \right)}{2\sqrt{-c}} \right)^2 \left( \int_M \phi_t^{q} \, d\mu_t \right)^{2/q} \\
+ \int_M u_t^2 H_t^2 \phi_t^2 \, d\mu_t + (1 + pC_{A_0, R_3}) \int_M \phi_t^{q} \, d\mu_t \\
= -\frac{\text{vol}(M_0)^{-2/q}}{C(2)q^2} \left( \frac{\arcsinh\left( e^{t/2} \sinh\left( 2\sqrt{-c} R_1 \right) \right)}{2\sqrt{-c}} \right)^2 e^{-2t/q} \left( \int_M \phi_t^{q} \, d\mu_t \right)^{2/q} \\
+ (2 + pC_{A_0, R_3}) \int_M \phi_t^{q} \, d\mu_t,
\]

where we use the fact that \( \text{vol}(M_t) = e^t \cdot \text{vol}(M_0) \) for the last equality. Put

\[
\omega(t) := e^{-(2+pC_{A_0, R_3})t} \int_M \phi_t^{q} \, d\mu_t.
\]

Then the above differential inequality is written as follows:

\[
\frac{d\omega}{dt}(t) \leq -\frac{\text{vol}(M_0)^{-2/q}}{C(2)q^2} \left( \frac{\arcsinh\left( e^{t/2} \sinh\left( 2\sqrt{-c} R_1 \right) \right)}{2\sqrt{-c}} \right)^2 e^{(2C_{A_0, R_3}+2/p-1)t} \omega(t)^{2/q}.
\]

From this differential inequality, we obtain

\[
P \cdot (\omega(t))^{-2/p} \geq \frac{P}{2} (\omega(0))^{-2/p} + \frac{\text{vol}(M_0)^{-2/q}}{C(2)q^2} \Psi_p(t) \geq \frac{\text{vol}(M_0)^{-2/q}}{C(2)q^2} \Psi_p(t).
\]

Hence we derive the desired \( L^p \)-estimate for \( u_t \):

\[
\|u_t\|_{L^p(M)} \leq C(2, p) \text{vol}(M_0)^{(p+2)/2} e^{(2/p+C_{A_0, R_3})t} \frac{1}{\sqrt{\Psi_p(t)}},
\]

where \( C(2, p) := \sqrt{C(2)pq^2/2} \) is depending on \( n = 2 \) and \( p > 2 \).
Next we consider the case of \( n \geq 3 \). Put \( \psi := |\varphi|^{2(n-1)/(n-2)} \). From Theorem 4.1, the Hölder inequality and the Minkowski inequality, we have

\[
\left( \int_M |\varphi_t|^{2n/(n-2)} \, d\mu_t \right)^{(n-2)/n} \leq C(n) \int_M (\|\text{grad} \varphi_t\|^2 + H_t^2 \varphi_t^2) \, d\mu_t,
\]

where \( C(n) := (4(n - 1)/(n - 2))\tilde{C}(n)^2 \). For any \( p > 2 \), set \( q := 2p/(p + 2) \). By using the Hölder inequality, we have

\[
\left( \int_M \varphi_t^q \, d\mu_t \right)^{2/q} \leq \left( \int_M \varphi_t^{2n/(n-2)} \, d\mu_t \right)^{(n-2)/n} \text{vol}(M_t)^{(2(n+p)/np)}.
\]

Combining these inequalities and Lemma 3.2 again, we obtain

\[
\frac{d}{dt} \int_M \varphi_t^q \, d\mu_t = \frac{d}{dt} \int_M u_t^p \, d\mu_t \leq -\left( \frac{1}{2\sqrt{-c}} \operatorname{arcsinh}(e^{t/n} \sinh(2\sqrt{-c} R_1)) \right)^2 \times \int_M \|\text{grad} \varphi_t\|^2 \, d\mu_t + (1 + pC_{A_0, R_3}) \int_M u_t^p \, d\mu_t \leq \frac{1}{C(n)} \left( \frac{\operatorname{arcsinh}(e^{t/n} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 \left( \int_M \varphi_t^{2n/(n-2)} \, d\mu_t \right)^{(n-2)/n} + \int_M u_t^2 H_t^2 \varphi_t^2 \, d\mu_t + (1 + pC_{A_0, R_3}) \int_M \varphi_t^q \, d\mu_t \leq -\frac{\text{vol}(M_t)^{-2(n+p)/np}}{C(n)} \left( \frac{\operatorname{arcsinh}(e^{t/n} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 \left( \int_M \varphi_t^q \, d\mu_t \right)^{2/q} + (2 + pC_{A_0, R_3}) \int_M \varphi_t^q \, d\mu_t \leq -\frac{\text{vol}(M_0)^{-2(n+p)/np}}{C(n)} \left( \frac{\operatorname{arcsinh}(e^{t/n} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 e^{-2(n+p)t/np} \times \left( \int_M \varphi_t^q \, d\mu_t \right)^{2/q} + (2 + pC_{A_0, R_3}) \int_M \varphi_t^q \, d\mu_t ,
\]

where we use the fact that \( \text{vol}(M_t) = e^t \cdot \text{vol}(M_0) \) for the last equality. As in the case of \( n = 2 \), put

\[
\omega(t) := e^{-(2+pC_{A_0, R_3})t} \int_M \varphi_t^q \, d\mu_t.
\]

Using a similar argument in the case of \( n = 2 \), we obtain

\[
\|u_t\|_{L^p(M)} \leq C(n, p) \text{vol}(M_0)^{(n+p)/np} e^{(2/p+C_{A_0, R_3})t} \frac{1}{\sqrt{\Psi_p(t)}},
\]

where \( C(n, p) := \sqrt{C(n)p/2} \). This completes the proof. \( \square \)

**Proof of Theorem 1.1.** Fix \( t_0 \in (0, T) \). To obtain the supremum estimate for \( u \) from the \( L^p \)-estimate, we define the function \( z \) on \( M \times [t_0, T) \) by

\[
z(p, t) := (t - t_0)^{\beta} \cdot u(p, t) \quad ((p, t) \in M \times [t_0, T)),
\]

where \( \beta := 

where $\beta \in (0, 1)$ will be suitably chosen later. Clearly we have
\[
\frac{\partial z}{\partial t} - \frac{1}{H^2} \Delta z = \frac{\beta}{t - t_0} z + (t - t_0)^\beta \left( \frac{\partial u}{\partial t} - \frac{1}{H^2} \Delta u \right), \tag{4.10}
\]
Fix $k \in [0, \infty)$. Put $z_k := \max \{z - k, 0\}$, $A_t(k) := \{p \in M; z(p, t) > k\}$ and $A(k) := \bigcup_{0 < t < T} A_t(k)$. For $t \in (t_0, T)$, from Lemma 3.3 and (4.10), we have
\[
d \int_M (z_k)_t^2 d\mu_t = \int_M (z_k)_t^2 \frac{d}{dt} d\mu_t + \int_M (z_k)_t \frac{\partial z_t}{\partial t} d\mu_t
\]
\[
= \int_M (z_k)_t^2 d\mu_t + 2 \int_{A_t(k)} \frac{1}{H_t^2} (z_k)_t \Delta (z_t) d\mu_t + \frac{2\beta}{t - t_0} \int_M (z_k)_t z_t d\mu_t
\]
\[
- 4 \int_M \frac{(z_k)_t}{H_t^2} \|\text{grad } z_t\|^2 d\mu_t - 4 \int_{A_t(k)} \frac{(z_k)_t}{H_t^3} (\text{grad } z_t, \text{grad } H_t) d\mu_t
\]
\[
+ 2 \int_M (z_k)_t z_t \left(- \frac{1}{\nu_t} \langle \partial w_t, \frac{\partial z_t}{\partial t} - \frac{1}{H_t^2} \Delta w_t \rangle + \frac{1}{H_t^2} \|A_t\|^2 + \frac{\text{Ric}_{00}}{q} \right) d\mu_t.
\]
By using Green’s theorem and $(z_k)_t \leq z_t$ (on $A_t(k)$), we obtain
\[
d \int_M (z_k)_t^2 d\mu_t \leq \int_M (z_k)_t^2 d\mu_t + \frac{2\beta}{t - t_0} \int_{A_t(k)} z_t^2 d\mu_t + 2C_{A_0, R_3} \int_M (z_k)_t z_t d\mu_t
\]
\[
- \frac{2}{(t - t_0)^2\beta} \int_{A_t(k)} \|\text{grad } z_t\|^2 d\mu_t.
\]
From $z_t > k$ on $A_t(k)$ and Lemma 3.2, we have
\[
d \int_M (z_k)_t^2 d\mu_t
\]
\[
+ 2k^2 \left( \frac{\text{arcsinh}(e^{t_0} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 (t - t_0)^{-2\beta} \int_M (\|z_k\|^2 + H_t^2 (z_k)_t^2) d\mu_t
\]
\[
\leq \left( \frac{2\beta}{t - t_0} + 2C_{A_0, R_3} \right) \int_{A_t(k)} z_t^2 d\mu_t + 3 \int_M (z_k)_t^2 d\mu_t. \tag{4.11}
\]
We distinguish the cases of $n = 2$ and $n \geq 3$ as in Theorem 4.2. First we consider the case of $n = 2$. Fix $q \in (1, \infty)$. The inequality (4.8) holds for $(z_k)_t$ (instead of $\varphi_t$). By using (4.8) for $(z_k)_t$ and (4.11), we have
\[
d \int_M (z_k)_t^2 d\mu_t
\]
\[
+ \frac{2k^2}{C(2)q^2 (t - t_0)^{2\beta \text{vol}(M_t)^{1/q}}} \left( \frac{\text{arcsinh}(e^{t_0} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 \left( \int_M (z_k)_t^{2q} d\mu_t \right)^{1/q}
\]
\[
\leq \left( \frac{2\beta}{t - t_0} + 2C_{A_0, R_3} \right) \int_{A_t(k)} z_t^2 d\mu_t + 3 \int_M (z_k)_t^2 d\mu_t. \tag{4.12}
\]
According to the Hölder inequality, we have
\[
\int_M (z_k)_t^2 d\mu_t \leq \left( \int_M (z_k)_t^{2q} d\mu_t \right)^{1/q} \text{vol}(M_t)^{(q - 1)/q}.
\]
If 

\[ \text{vol}(M_t) \leq \frac{1}{3} C(2)^{-1} k^2 q^{-2} \left( \frac{\arcsinh(e^{t/2} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 (t - t_0)^{-2\beta}. \]

then the second term of the right-hand side of (4.12) is absorbed by the second term of the left-hand side. Fix \( t_1 \in (t_0, T) \) and put

\[ k_0 := \sqrt{3C(2)}(t_1 - t_0)^{\beta} q^{1/2} \text{vol}(M_0)^{1/2} \frac{2\sqrt{-c}}{\arcsinh(e^{t_0/2} \sinh(2\sqrt{-c} R_1))}. \] (4.13)

From \( \text{vol}(M_t) = e^t \cdot \text{vol}(M_0) \), this inequality holds for any \( t \in [t_0, t_1] \) and any \( k \geq k_0 \). Fix \( k \in [k_0, \infty) \). Hence we obtain

\[
\frac{d}{dt} \int_M (z_k)^2 d\mu_t \\
+ \frac{k^2}{C(2)q^2(t - t_0)^{2\beta} \text{vol}(M_t)^{1/q}} \left( \frac{\arcsinh(e^{t/2} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 \left( \int_M (z_k)^{2q} d\mu_t \right)^{1/q} \\
\leq \left( \frac{2\beta}{t - t_0} + 2C_{A_0, R_3} \right) \int_{A((k)} z_t^2 d\mu_t. \] (4.14)

Next we consider the case of \( n \geq 3 \). Put \( q := n/(n - 2) \). As in (4.12), we have

\[
\frac{d}{dt} \int_M (z_k)^2 d\mu_t + \frac{2k^2}{C(n)} \left( \frac{\arcsinh(e^{t/2n} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 (t - t_0)^{-2\beta} \left( \int_M (z_k)^{2q} d\mu_t \right)^{1/q} \\
\leq \left( \frac{2\beta}{t - t_0} + 2C_{A_0, R_3} \right) \int_{A((k)} z_t^2 d\mu_t + 3 \int_M (z_k)^2 d\mu_t. \] (4.15)

According to the Hölder inequality, we have

\[
\int_M (z_k)^2 d\mu_t \leq \left( \int_M (z_k)^{2q} d\mu_t \right)^{1/q} \text{vol}(M_t)^{2/n}. \]

If

\[
\text{vol}(M_t)^{2/n} \leq \frac{1}{3} \frac{k^2}{C(n)} \left( \frac{\arcsinh(e^{t/2n} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 (t - t_0)^{-2\beta},
\]

then the second term of the right-hand side of (4.15) is absorbed by the second term of the left-hand side. Fix \( t_1 \in (t_0, T) \) and put

\[ k_0 := \sqrt{3C(n)(t_1 - t_0)^{\beta} e^{t_1/n} \text{vol}(M_0)^{1/n} \frac{2\sqrt{-c}}{\arcsinh(e^{t_0/n} \sinh(2\sqrt{-c} R_1))}}. \] (4.16)

From \( \text{vol}(M_t) = e^t \cdot \text{vol}(M_0) \), this inequality holds for any \( t \in [t_0, t_1] \) and any \( k \geq k_0 \). Fix \( k \in [k_0, \infty) \). We obtain

\[
\frac{d}{dt} \int_M (z_k)^2 d\mu_t + \frac{k^2}{C(n)} \left( \frac{\arcsinh(e^{t/2n} \sinh(2\sqrt{-c} R_1))}{2\sqrt{-c}} \right)^2 (t - t_0)^{-2\beta} \left( \int_M (z_k)^{2q} d\mu_t \right)^{1/q} \\
\leq \left( \frac{2\beta}{t - t_0} + 2C_{A_0, R_3} \right) \int_{A((k)} z_t^2 d\mu_t. \] (4.17)
Now we define a constant \( B(n, k) \) by

\[
B(n, k) := \begin{cases} 
\frac{k^2}{C(2)q^2} \left( \frac{\arcsinh(e^{t_0/2} \sinh(2\sqrt{-c} R_1)))}{2\sqrt{-c}} \right)^2 \text{vol}(M_t) \ , & (n = 2), \\
\frac{k^2}{C(n)} \left( \frac{\arcsinh(e^{t_0/n} \sinh(2\sqrt{-c} R_1)))}{2\sqrt{-c}} \right)^2 \text{vol}(M_t) \ , & (n \geq 3).
\end{cases}
\]

Then both (4.14) and (4.17) are described as

\[
\frac{d}{dt} \int_M (z_k)_t^2 \, d\mu_t + B(n, k)(t - t_0) - 2\beta \left( \int_M (z_k)_t^{2q} \, d\mu_t \right)^{1/q} \leq \left( \frac{2\beta}{t - t_0} + 2C_{A_0, R_3} \right) \int_{A_t(k)} z_t^2 \, d\mu_t.
\]

Fix \( k \in [k_0, \infty) \), where \( k_0 \) is the constant as in (4.13) (in the case of \( n = 2 \)) and (4.16) (in the case of \( n \geq 3 \)). Integrating from \( t_0 \) to any \( t \in [t_0, t_1] \) and noticing \( (z_k)_{t_0} = 0 \), we obtain

\[
\sup_{t \in [t_0, t_1]} \int_M (z_k)_t^2 \, d\mu_t + B(n, k) \int_{t_0}^{t_1} (t - t_0)^{-2\beta} \left( \int_M (z_k)_t^{2q} \, d\mu_t \right)^{1/q} \, dt \\
\leq 4\beta \int_{t_0}^{t_1} \frac{1}{t - t_0} \left( \int_{A_t(k)} z_t^2 \, d\mu_t \right) \, dt + 4C_{A_0, R_3} \int_{t_0}^{t_1} \left( \int_{A_t(k)} z_t^2 \, d\mu_t \right) \, dt.
\]

(4.18)

To proceed further, we define an interpolating exponent \( q_0 \in (1, q) \) by

\[
\frac{1}{q_0} = \frac{a}{q} + (1 - a) \cdot \left( a := \frac{1}{q_0} \right).
\]

With this choice we have \( 1 - a = (1/q_0)(1 - 1/q) \) and

\[
\begin{align*}
1 < q < \infty, \quad q_0 &= 2 - \frac{1}{q} \quad (n = 2), \\
q &= \frac{n}{n - 2}, \quad q_0 = 2 - \frac{1}{q} = \frac{n + 2}{n} \quad (n \geq 3).
\end{align*}
\]

(4.19)

By using the interpolation inequality and Young’s inequality, we obtain

\[
\left( \int_{t_0}^{t_1} B(n, k)(t - t_0)^{-2\beta} \left( \int_{A_t(k)} (z_k)_t^{2q_0} \, d\mu_t \right) \, dt \right)^{1/q_0} \\
\leq \left( \int_{t_0}^{t_1} B(n, k)(t - t_0)^{-2\beta} \left( \int_{A_t(k)} (z_k)_t^2 \, d\mu_t \right)^{q_0 \cdot (1 - a)} \left( \int_{A_t(k)} (z_k)_t^{2q} \, d\mu_t \right)^{a q_0/q} \, dt \right)^{1/q_0} \\
\leq \left( \sup_{t \in [t_0, t_1]} \int_{A_t(k)} (z_k)_t^2 \, d\mu_t \right)^{1 - a} \left( \int_{t_0}^{t_1} B(n, k)(t - t_0)^{-2\beta} \left( \int_{A_t(k)} (z_k)_t^{2q} \, d\mu_t \right)^{1/q} \, dt \right)^a \\
\leq (1 - a) \left( \sup_{t \in [t_0, t_1]} \int_{A_t(k)} (z_k)_t^2 \, d\mu_t \right) \\
+ a \left( \int_{t_0}^{t_1} B(n, k)(t - t_0)^{-2\beta} \left( \int_{A_t(k)} (z_k)_t^{2q} \, d\mu_t \right)^{1/q} \, dt \right) \\
\leq \tilde{C}(n) \sup_{t \in [t_0, t_1]} \int_{A_t(k)} (z_k)_t^2 \, d\mu_t + \int_{t_0}^{t_1} B(n, k)(t - t_0)^{-2\beta} \left( \int_{A_t(k)} (z_k)_t^{2q} \, d\mu_t \right)^{1/q} \, dt,
\]

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where \( \bar{C}(n) := \max \{1 - a, a\}(< 1) \). Combining this estimate with (4.18), we have
\[
\left( \int_{t_0}^{t_1} B(n, k)(t - t_0)^{-2\beta} \left( \int_{A_t(k)} (z_k, t)^{2q_0} d\mu_t \right) dt \right)^{1/q_0} \\
\leq 4\beta \bar{C}(n) \int_{t_0}^{t_1} \frac{1}{t - t_0} \left( \int_{A_t(k)} z_t^2 d\mu_t \right) dt + 4\bar{C}(n) C_{A_0, R_3} \int_{t_0}^{t_1} \left( \int_{A_t(k)} z_t^2 d\mu_t \right) dt.
\]
(4.20)

Put \( d\sigma_t := (t - t_0)^{-\beta} d\mu_t dt \) and \( \|A(k)\| := \int_{t_0}^{t_1} \int_{A_t(k)} d\sigma_t \). Fix a positive constant \( r \) satisfying
\[
r > \begin{cases} 
\frac{2q - 1}{q - 1} & (n = 2), \\
\frac{n + 2}{2} & (n \geq 3).
\end{cases}
\]

By using (4.20) and Hölder’s inequality, we obtain
\[
\int_{t_0}^{t_1} \int_{A_t(k)} (z_k, t)^2 d\sigma_t \\
\leq 4\bar{C}(n) B(n, k)^{-1/q_0} \|A(k)\|^{1-1/q_0} \int_{t_0}^{t_1} \left( \frac{\beta}{t - t_0} + C_{A_0, R_3} \right) \operatorname{vol}(A_t(k))^{1-1/r} \|z_t\|^2_{L^{2r}(M)} dt.
\]

To properly match the powers of \( t - t_0 \) in both sides of this inequality, a good choice for \( \beta \) is \( \beta = \frac{1}{4} \). From the \( L^p \)-estimate in Theorem 4.2 for \( u_t \), we have
\[
\|z_t\|^2_{L^{2r}(M)} = (t - t_0)^{1/2} \|u_t\|^2_{L^{2r}(M)} \\
\leq (t - t_0)^{1/2} C(n, 2r)^2 \operatorname{vol}(M_0)^{(n+2r)/nr} e^{2(1/r+C_{A_0, R_3})t} \Psi_{2r}(t).
\]

Thus we have
\[
\int_{t_0}^{t_1} \int_{A_t(k)} (z_k, t)^2 d\sigma_t \\
\leq 4\bar{C}(n) C(n, 2r)^2 \operatorname{vol}(M_0)^{(n+2r)/nr} B(n, k_0)^{-1/q_0} \|A(k)\|^{1-1/q_0} e^{2(1/r+C_{A_0, R_3})t_1} \frac{1}{\Psi_{2r}(t_0)} \\
\times \left( \frac{1}{4} \int_{t_0}^{t_1} (t - t_0)^{-1/2} \|A_t(k)\|^{(r-1)/r} dt + C_{A_0, R_3} \int_{t_0}^{t_1} (t - t_0)^{1/2} \|A_t(k)\|^{(r-1)/r} dt \right).
\]

By using Hölder’s inequality, the integrals on the right-hand side of this inequality can be estimated as
\[
\int_{t_0}^{t_1} (t - t_0)^{-1/2} \|A_t(k)\|^{(r-1)/r} dt \leq \|A(k)\|^{1-1/r} 2^{1/r} (t_1 - t_0)^{1/2r}
\]
and
\[
\int_{t_0}^{t_1} (t - t_0)^{1/2} \|A_t(k)\|^{(r-1)/r} dt \leq \|A(k)\|^{1-1/r} \left( \frac{2}{2r + 1} \right)^{1/r} (t_1 - t_0)^{1+1/2r}.
\]
So we obtain

\[
\int_{t_0}^{t_1} \int_{A_t(k)} (z_k)^2 \, d\sigma_t \leq 4\tilde{C}(n)C(n, 2r)^2 \text{vol}(M_0)^{(n+2r)/nr} B(n, k_0)^{-1/q_0} e^{2(1/r+C_{A_0,R_3})t_0} \Psi_{2r}(t_0) (t_1 - t_0)^{1/2r} \\
\times 2^{1/r} \left( \frac{1}{4} + \frac{C_{A_0,R_3} (t_1 - t_0)}{(2r + 1)^{1/r}} \right) \|A(k)\|^{2-1/q_0-1/r}.
\]

Put \( \gamma := 2 - 1/q_0 - 1/r \). Clearly we have \( \gamma > 1 \). Thus we obtain the following inequality for \( h > k \geq k_0 (> 0) \):

\[
|h - k|^2 \|A(h)\| = \int_{t_0}^{t_1} \int_{A_t(h)} |h - k|^2 (t - t_0)^{-1/2} \, d\mu_t \, dt \\
\leq \int_{t_0}^{t_1} \int_{A_t(h)} (z_t - k)^2 (t - t_0)^{-1/2} \, d\mu_t \, dt \\
\leq \int_{t_0}^{t_1} \int_{A_t(k)} (z_k)^2 \, d\sigma_t \, dt \\
\leq 4\tilde{C}(n)C(n, 2r)^2 \text{vol}(M_0)^{(n+2r)/nr} B(n, k_0)^{-1/q_0} e^{2(1/r+C_{A_0,R_3})t_0} \Psi_{2r}(t_0) (t_1 - t_0)^{1/2r} \\
\times 2^{1/r} \left( \frac{1}{4} + \frac{C_{A_0,R_3} (t_1 - t_0)}{(2r + 1)^{1/r}} \right) \|A(k)\|^{\gamma}.
\]

De Giorgi’s iteration lemma (see [1]) yields

\[
\|A(k_0 + d)\| = 0,
\]

where

\[
d^2 := 4\tilde{C}(n)C(n, 2r)^2 \text{vol}(M_0)^{(n+2r)/nr} B(n, k_0)^{-1/q_0} e^{2(1/r+C_{A_0,R_3})t_0} \Psi_{2r}(t_0) (t_1 - t_0)^{1/2r} \\
\times 2^{1/r} \left( \frac{1}{4} + \frac{C_{A_0,R_3} (t_1 - t_0)}{(2r + 1)^{1/r}} \right) \|A(k_0)\|^{\gamma - 1} 2^{\gamma/(\gamma - 1)}.
\]

From \( \|A(k_0 + d)\| = 0 \), we obtain \( A_{t_1}(k_0 + d) = \emptyset \), that is,

\[
z_{t_1} \leq k_0 + d. \tag{4.21}
\]

From

\[
\|A(k_0)\| = \int_{t_0}^{t_1} (t - t_0)^{-1/2} \int_{A_t(k_0)} d\mu_t \, dt \\
\leq \int_{t_0}^{t_1} (t - t_0)^{-1/2} \left( \int_M d\mu_t \right) \, dt \leq 2e^{t_1} \text{vol}(M_0)(t_1 - t_0)^{1/2},
\]

we have

\[
\|A(k_0)\| \leq 2e^{t_1} \text{vol}(M_0)(t_1 - t_0)^{1/2}.
\]
we obtain
\[ d^2 \leq \tilde{C}(n, 2r) B(n, k_0)^{-1/q_0} \text{vol}(M_0)^{1+2/n-1/q_0} \]
\[ \times \left( \frac{1}{4} + \frac{C_{A_0, R_3} t_1 - t_0}{(2r + 1)^{1/r}} \right) (t_1 - t_0)^{1/2} (1 - 1/q_0 + 1/r + 2C_{A_0, R_3} t_1), \]
where \( \tilde{C}(n, 2r) := 4 \tilde{C}(n, 2r) 2^{3/(3^r - 2)} 2^{2/y - 12^1/r} \).

Here we distinguish two cases of \( t_0 < T/2 \) and \( 1 \leq t_0 < T - 1 \). First we consider the case of \( t_0 < T/2 \). Let \( t_1 = 2t_0 \). By using the relation between \( q, q_0 \) and \( n \) as in (4.19) we note that
\[ 1 - \frac{2}{q_0} + \frac{1}{q_0} = 0 \quad (n = 2), \quad 1 - \frac{1}{q_0} - 2 \frac{1}{nq_0} = 0 \quad (n \geq 3). \]
In the case of \( n = 2 \), we obtain
\[ B(2, k_0) = 3t_0^{1/2} e^{(1-1/q)t_1} \text{vol}(M_0)^{1-1/q} \]
and
\[ d^2 \leq 3^{-1/q_0} \tilde{C}(2, 2r) \left( \frac{1}{4} + \frac{C_{A_0, R_3} t_0}{(2r + 1)^{1/r}} \right) \text{vol}(M_0)^{1-2/q_0} \frac{e^{(1/r + 2C_{A_0, R_3} t_1)}}{\Psi_2 r (t_0)}. \]
In the case of \( n \geq 3 \), we obtain
\[ B(n, k_0) = 3t_0^{1/2} e^{2t_1/n} \text{vol}(M_0)^{2/n} \]
and
\[ d^2 \leq 3^{-1/q_0} \tilde{C}(n, 2r) \left( \frac{1}{4} + \frac{C_{A_0, R_3} t_0}{(2r + 1)^{1/r}} \right) \text{vol}(M_0)^{2/n} \frac{e^{(1/r + 2C_{A_0, R_3} t_1)}}{\Psi_2 r (t_0)}. \]
Hence we have
\[ d^2 \leq \tilde{C}(n, 2r) \left( \frac{1}{4} + \frac{C_{A_0, R_3} t_0}{(2r + 1)^{1/r}} \right) \text{vol}(M_0)^{2/n} \frac{e^{(1/r + 2C_{A_0, R_3} t_1)}}{\Psi_2 r (t_0)}. \]
(4.22)
where \( \tilde{C}(n, 2r) := 3^{-1/q_0} \tilde{C}(n, 2r) \).

Next we consider the case of \( 1 \leq t_0 < T - 1 \). Let \( t_1 = t_0 + 1 \). In the case of \( n = 2 \), we obtain
\[ B(2, k_0) = 3e^{(1-1/q)t_1} \text{vol}(M_0)^{1-1/q} \]
and
\[ d^2 \leq 3^{-1/q_0} \tilde{C}(2, 2r) \left( \frac{1}{4} + \frac{C_{A_0, R_3}}{(2r + 1)^{1/r}} \right) \text{vol}(M_0) \frac{e^{(1/r + 2C_{A_0, R_3} t_1)}}{\Psi_2 r (t_0)}. \]
In the case of \( n \geq 3 \), we obtain
\[ B(n, k_0) = 3e^{2/t_1} \text{vol}(M_0)^{2/n} \]
and
\[ d^2 \leq 3^{-1/q_0} \tilde{C}(n, 2r) \left( \frac{1}{4} + \frac{C_{A_0, R_3}}{(2r + 1)^{1/r}} \right) \text{vol}(M_0)^{2/n} \frac{e^{(1/r + 2C_{A_0, R_3} t_1)}}{\Psi_2 r (t_0)}. \]
Hence we have
\[ d^2 \leq \tilde{C}(n, 2r) \left( \frac{1}{4} + \frac{C_{A_0, R_3}}{(2r + 1)^{1/r}} \right) \text{vol}(M_0)^{2/n} \frac{e^{(1/r + 2C_{A_0, R_3} t_1)}}{\Psi_2 r (t_0)}. \]
(4.23)
where \( \tilde{C}(n, 2r) := 3^{-1/q_0} \tilde{C}(n, 2r) \).
According to our choice of \( t_1 \), we have \( t_1 = \min \{ 2t_0, t_0 + 1 \} \). Hence, it follows from (4.13), (4.16), (4.21), (4.22) and (4.23) that in both cases, the following inequality holds:

\[
  z_t \leq k_0 + d \\
  \leq C'(n) \min \left\{ \left( \frac{t_1}{2} \right)^{1/4}, 1 \right\} \frac{2 \sqrt{-c}}{\arcsinh(e^{\max\{t_1/2n, (t_1-1)/n\}/n}) \sinh(2 \sqrt{-c} R_1))} \epsilon(t_1/n) \text{vol}(M_0)^{1/n} \\
  + \sqrt{\hat{C}(n, 2r) \left( \frac{1}{4} + \frac{C_{A_0, R_3}}{(2r + 1)^{1/r}} \right) \max \left\{ \left( \frac{t_1}{2} \right)^{-1/4}, 1 \right\}} \text{vol}(M_0)^{1/n} \\
  \times \frac{e^{(1/2r + C_{A_0, R_3})t}}{\sqrt{\Psi_{2r}^2 \max\{t_1/2, t_1 - 1\}}},
\]

where \( C'(n) \) is a positive constant depending only on \( n \). In the case of \( T > 2, t_0 \) moves over \([0, T-1)\), \( t_1 \) moves over \([0, T)\), and, in the case of \( T \leq 2, t_0 \) moves over \([0, T/2)\), \( t_1 \) moves over \([0, T)\). Hence this inequality holds for any \( t_1 \in [0, T) \). Since \( z_t = (t - t_0)^{1/4} \epsilon_t \), we have

\[
  \frac{1}{H_t u_t} = u_t \leq C'(n) \frac{2 \sqrt{-c}}{\arcsinh(e^{\max\{t/2n, (t-1)/n\}/n}) \sinh(2 \sqrt{-c} R_1))} \epsilon(t_1/n) \text{vol}(M_0)^{1/n} \\
  + \sqrt{\hat{C}(n, 2r) \left( \frac{1}{4} + \frac{C_{A_0, R_3}}{(2r + 1)^{1/r}} \right) \max \left\{ \left( \frac{t_1}{2} \right)^{-1/2}, 1 \right\}} \text{vol}(M_0)^{1/n} \\
  \times \frac{e^{(1/2r + C_{A_0, R_3})t}}{\sqrt{\Psi_{2r}^2 \max\{t_1/2, t_1 - 1\}}},
\]

for any \( t \in [0, T) \). Therefore, from Lemma 3.2, we obtain the desired estimates for \( H_t \) and \( \int_M H_t \text{d}\mu_t \).

\[
\square
\]

REFERENCES

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