SUBMAJORIZATION OF THE ARAKI–LIEB–THIRRING INEQUALITY

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Abstract. We proved the Araki–Lieb–Thirring-type inequality for positive $\tau$-measurable operators associated with finite von Neumann algebra $(\mathcal{M}, \tau)$.

1. Introduction

Let $\mathbb{M}_n$ be a von Neumann algebra of $n \times n$ complex matrices. The Lieb–Thirring inequality [11] states that for positive matrices $A$ and $B$ in $\mathbb{M}_n$, and $p \geq 1$,

$$\text{Tr}((B^{1/2}AB^{1/2})^p) \leq \text{Tr}(A^pB^p).$$

A notable generalization of this inequality is the Araki–Lieb–Thirring inequality [1]. It can be rephrased as a norm inequality (see [5, Theorem IX.2.10]): for positive matrices $A$ and $B$ in $\mathbb{M}_n$, and any unitarily invariant norm $\| \cdot \|$, $\| (B A B)^p \| \leq \| B^p A^p B^p \|$, $\forall p \in [1, \infty)$

(1)

and

$\| B^p A^p B^p \| \leq \| (B A B)^p \|$, $\forall p \in (0, 1]$.

(2)

One can find some related results to this topics in [2, 3, 10].

We will use the notion of a generalized singular value studied by Fack and Kosaki [8], and the main result of [10], to generalize the Araki–Lieb–Thirring inequality ((1) and (2)) for positive $\tau$-measurable operators associated with a finite von Neumann algebra $\mathcal{M}$ and for norms of symmetric quasi-Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathcal{M}$ a finite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a normal faithful finite normalized trace $\tau$. The closed densely defined linear operator $T$ in $\mathcal{H}$ with domain $D(T)$ is said to be affiliated with $\mathcal{M}$ if and only if $U^*TU = T$ for all unitary $U$ which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If $T$ is affiliated with $\mathcal{M}$, then $T$ is said to be $\tau$-measurable if for every $\varepsilon > 0$ there exists a projection $P \in \mathcal{M}$ such that $P(\mathcal{H}) \subseteq D(T)$ and $\tau(P^\perp) < \varepsilon$ (where for any projection $P$ we let $P^\perp = 1 - P$). The set of all $\tau$-measurable operators will be denoted by $\overline{\mathcal{M}}$. The set $\overline{\mathcal{M}}$ is a $*$-algebra with sum and intersection.

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product being the respective closure of the algebraic sum and product. The sets
\[ N(\varepsilon, \delta) = \{ T \in \mathcal{M} : \exists \text{ projection } P \in \mathcal{M} \text{ such that } \| TP \| < \varepsilon \text{ and } \tau(P^\perp) < \delta \} \]
\((\varepsilon, \delta > 0)\) form a base at 0 for an metrizable Hausdorff topology in \(\mathcal{M}\) called the measure topology in \(\mathcal{M}\) and equipped with the measure topology; \(\mathcal{M}\) is a complete topological \(*\)-algebra [13]. We denote the set of all \(\tau\)-measurable positive operators by \(\mathcal{M}^+\).

Let \(T \in \mathcal{M}\). We can write \(T = U |T|\), where as usual \(|T| = (T^*T)^{1/2}\) is the absolute value of \(T\), but where \(U\) is a unitary operator. This is because all partial isometries can be extended to unitaries in a finite von Neumann algebra (see [14, Proposition 1.38, p. 304]).

For a positive self-adjoint operator \(T \in \mathcal{M}\), we can write \(T = U |T|\), where as usual \(|T| = (T^*T)^{1/2}\) is the absolute value of \(T\), but where \(U\) is a unitary operator. This is because all partial isometries can be extended to unitaries in a finite von Neumann algebra (see [14, Proposition 1.38, p. 304]).

For \(0 < p < \infty\), \(L^p(\mathcal{M}; \tau)\) is defined as the set of all \(\tau\)-measurable operators \(T\) affiliated with \(\mathcal{M}\) such that \(\|T\|_p = \tau(|T|^p)^{1/p} < \infty\).

In addition, we put \(L^\infty(\mathcal{M}; \tau) = \mathcal{M}\) and denote by \(\|\cdot\|_\infty\) (\(= \|\cdot\|\)) the usual operator norm.

It is well known that \(L^p(\mathcal{M}; \tau)\) is a Banach space under \(\|\cdot\|_p\) \((1 \leq p \leq \infty)\) satisfying all the expected properties such as duality.

**Definition 2.1.** Let \(T\) be a \(\tau\)-measure operator and \(t > 0\). The ‘\(t\)th singular number of \(T\’
\[ \mu_t(T) = \inf \{ \|TP\| : P \text{ is a projection in } \mathcal{M} \text{ with } \tau(P^\perp) \leq t \} \]
It is clear that \(\mu_t(T) = 0\), for all \(t \geq \tau(I)\). We will denote simply by \(\mu(T)\) the function \(t \mapsto \mu_t(T)\). It is easy to check that \(\mu(T)\) is decreasing and continuous from the right on \((0, 1)\). For further information we refer the reader to [8].

If \(T, S \in \mathcal{M}\) then we say that \(S\) is submajorized by \(T\) and write \(S \preccurlyeq T\) if and only if
\[ \int_0^a \mu_t(S) \, dt \leq \int_0^a \mu_t(T) \, dt, \quad \forall a \geq 0. \]

We remark that if \(T \in \mathcal{M}\), then for all \(s > 0\)
\[ \mu_s(T) = \mu_s(T^*) = \mu_s(|T|) \] (3)
and
\[ f(\mu_s(T)) = \mu_s(f(|T|)), \] (4)
where \(f\) is a continuous increasing function on \([0, \infty)\) such that \(f(0) = 0\) [8].

Now let \(E\) be a quasi-Banach lattice. Let \(0 < \alpha < \infty\). \(E\) is said to be \(\alpha\)-convex (respectively \(\alpha\)-concave) if there exists a constant \(C > 0\) such that for all finite sequence \((x_n)\) in \(E\),
\[ \left\| \left( \sum |x_n|^\alpha \right)^{1/\alpha} \right\|_E \leq C \left( \sum \|x_n\|^\alpha \right)^{1/\alpha}_E, \]
\[ \left( \text{respectively } \left\| \left( \sum |x_n|^\alpha \right)^{1/\alpha} \right\|_E \geq C^{-1} \left( \sum \|x_n\|^\alpha \right)^{1/\alpha}_E \right). \]
The least such constant $C$ is called the $\alpha$-convexity (respectively $\alpha$-concavity) constant of $E$ and is denoted by $M^{(\alpha)}(E)$ (respectively $M^{(\alpha\alpha)}(E)$). For $0 < p < \infty$, $E^{(p)}$ will denote the quasi-Banach lattice defined by

$$E^{(p)} = \{x : |x|^p \in E\},$$

equipped with the quasi-norm

$$\|x\|_{E^{(p)}} = \| |x|^p \|_E^{1/p}.$$  

It is easy to check that if $E$ is $\alpha$-convex and $\beta$-concave then $E^{(p)}$ is $p\alpha$-convex and $p\beta$-concave with $M^{(p\alpha)}(E^{(p)}) = M^{(\alpha)}(E)$ and $M^{(p\beta)}(E^{(p)}) = M^{(\beta)}(E)$. Therefore, if $E$ is $\alpha$-convex then $E^{(1/\alpha)}$ is 1-convex, so it can be renormed as a Banach lattice (cf. [12, p. 54]).

By a symmetric quasi-Banach space on $[0, 1]$ we mean a quasi-Banach lattice $E$ of measurable functions on $[0, 1]$ satisfying the following properties: (i) $E$ contains all simple functions; (ii) if $x \in E$ and $y$ is a measurable function such that $|y|$ is equi-distributed with $|x|$, then $y \in E$ and $\|x\|_E = \|y\|_E$. For convenience we shall always assume $E$ additionally satisfies

$$0 \leq x_n \uparrow x, \quad x_n, x \in E \Rightarrow \|x_n\|_E \uparrow \|x\|_E.$$  

For example, $E$ satisfies (5) if it is $\sigma$-order continuous, i.e. for every sequence $(x_n)_{n \geq 0}$ in $E$, $x_n \downarrow 0$ implies $\|x_n\|_E \downarrow 0$; a fortiori $E$ satisfies (5) if it fails to contain $c_0$.

Let $E$ be a symmetric quasi-Banach space on $[0, 1]$. We define

$$L_E(M) = \{x \in \overline{M} : \mu.(x) \in E\},$$

$$\|x\|_{L_E(M)} = \\|\mu.(x)\|_E, \quad x \in L_E(M).$$

Then $(L_E(M), \|\|_{L_E(M)})$ is a quasi-Banach space (cf. [6, 15]).

We state for easy reference the following facts that will be applied below.

**Theorem 2.2.** [10] Let $r \geq 1$ and let $f$ be a continuous increasing function on $[0, \infty)$ such that $f(0) = 0$ and $t \rightarrow f(\epsilon t)$ is convex. Then

$$f(|ST|^r) \preceq f(|S'T'|), \quad \forall S, T \in \overline{M}^+.$$  

**3. Main results**

**Lemma 3.1.** Let $S, T \in \overline{M}^+$ and let $f$ be a continuous increasing function on $[0, \infty)$ such that $f(0) = 0$ and $t \rightarrow f(\epsilon t)$ is convex.

(i) If $1 \leq p < \infty$, then $f((ST)^p) \preceq f(S^pT^pS^p)$.

(ii) If $0 < p \leq 1$, then $f(S^pT^pS^p) \preceq f((ST)^p)$.
Proof. (i) Since \( g(t) = f(t^2) \) is a continuous increasing function on \([0, \infty)\) such that \( f(0) = 0 \) and \( t \to g(e^t) = f(e^{2t}) \) is convex, by (3), (4) and Theorem 2.2, we have that

\[
\int_0^t \mu_s(f((ST S)^p)) \, ds = \int_0^t f(\mu_s(|T^{1/2} S|^{2p})) \, ds \\
= \int_0^t f((\mu_s(|T^{1/2} S|^{p}))^2) \, ds \\
\leq \int_0^t f((\mu_s(|T^{p/2} S|^{p}))^2) \, ds \\
= \int_0^t \mu_s(f(S^p T^p S^p)) \, ds.
\]

(ii) Since \( g(t) = f(t^{2p}) \) is a continuous increasing function on \([0, \infty)\) such that \( f(0) = 0 \) and \( t \to g(e^t) = f(e^{2pt}) \) is convex, using (4) and Theorem 2.2 we obtain

\[
\int_0^t \mu_s(f(S^p T^p S^p)) \, ds = \int_0^t f((\mu_s(|T^{1/2} S|^{2p}))^2) \, ds \\
= \int_0^t f((\mu_s(|T^{p/2} S|^{p}))^2) \, ds \\
= \int_0^t \mu_s(f(S^p T^p S^p)) \, ds.
\]

\[\square\]

**Theorem 3.2.** Let \( f : [0, +\infty) \to [0, +\infty) \) be a continuous increasing function such that \( f(0) = 0 \) and \( t \to f(e^t) \) is convex. Let \( T, S \in \overline{\mathcal{M}} \) and \( T \geq 0 \).

(i) If \( 1 \leq p < \infty \), then \( f((STS^*)^p) \preceq f(|S|^p T^p |S|)^p \).

(ii) If \( 0 < p \leq 1 \), then \( f(|S|^p T^p |S|)^p \preceq f((STS^*)^p) \).

Proof. Let the polar decomposition of \( S \) be \( S = U|S| \), where \( U \) is unitary. Then \( STS^* = U|S|^T S^* U^* \) and \( f((STS^*)^p) = U f(|S|^T |S|)^p U^* \). Applying Lemma 3.1 we obtain the desired result. \(\square\)

Now we can extend the inequalities (1) and (2) as follows.

**Theorem 3.3.** Let \( E \) be a symmetric quasi-Banach function space on \([0, 1]\) with \( M^{(\infty)}(E) = 1 \) for some \( 0 < \alpha < \infty \). Let \( T, S \in \overline{\mathcal{M}} \) and \( T \geq 0 \).

(i) If \( 1 \leq p < \infty \), then \( \|STS^*\|^p \|L_E(\mathcal{M}) \| \leq \|S|^p T^p |S|^p \|L_E(\mathcal{M}) \| \).

(ii) If \( 0 < p \leq 1 \), then \( \|S|^p T^p |S|^p \|L_E(\mathcal{M}) \| \leq \|(STS^*)^p\|L_E(\mathcal{M}) \| \).

Proof. Let \( A \in \overline{\mathcal{M}} \), then

\[
\|A\|^p_{L_E(\mathcal{M})} = \sup_{v \in V} \int_0^\infty \int_0^t (\mu_s(A))^\alpha \, ds \, dv(t).
\]

Here, \( V \) is a certain family of positive measures on \((0, \infty)\) explained in [15, p. 550]. Using Theorem 3.2 we obtain the desired result. \(\square\)

**Corollary 3.4.** Let \( 0 < r < \infty \) and let \( T, S \in \overline{\mathcal{M}} \) and \( T \geq 0 \). Then

(i) \( \|(ST S^*)^p\|_r \leq \|S|^p T^p |S|^p \|_r, \quad \forall p \in [1, \infty) \).

(ii) \( \|S|^p T^p |S|^p \|_r \leq \|(STS^*)^p\|_r, \quad \forall p \in (0, 1] \).

The following lemma is a special case of [9, Theorem 10 (see especially pp. 141–142)].
Lemma 3.5. Let \( f : [0, +\infty) \to [0, +\infty) \) be a continuous increasing function such that \( f(0) = 0 \) and \( t \to f(e^t) \) is convex, and let \( S, T \) be \( \tau \)-measurable normal operators. Then
\[
f(|S + T|) \leq f(|S| + |T|).
\]

Theorem 3.6. Let \( f : [0, +\infty) \to [0, +\infty) \) be a continuous increasing function such that \( f(0) = 0 \) and \( t \to f(e^t) \) is convex. Let \( T, S \in \mathcal{M} \) and \( T \) be self-adjoint. Then
\[
f(|T S^*|^p) \leq f(|S|^p |T|^p |S|^p), \quad \forall p \in [1, \infty).
\]

Proof. Let the Jordan decomposition of \( T \) be \( T = T_+ - T_- \), where \( T_+ \) (respectively \( T_- \)) is the positive part (respectively the negative part) of \( T \). Then \( ST S^* = ST_+ S^* - ST_- S^* \). Applying Lemma 3.5 and Theorem 3.2 we obtain that
\[
f(|ST S^*|^p) = f(|ST_+ S^* - ST_- S^*|^p) \\
\leq f((ST_+ S^* + ST_- S^*)^p) \\
= f((S|T|S^*)^p) \\
\leq f(|S|^p |T|^p |S|^p).
\]

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