NON-HOMEOMORPHIC TOPOLOGICAL RANK AND EXPANSIVENESS

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Abstract. Downarowicz and Maass (2008) have shown that every Cantor minimal homeomorphism with finite topological rank $K > 1$ is expansive. Bezuglyi *et al* (2009) extended the result to non-minimal cases. On the other hand, Gambaudo and Martens (2006) had expressed all Cantor minimal continuous surjections as the inverse limit of graph coverings. In this paper, we define a topological rank for every Cantor minimal continuous surjection, and show that every Cantor minimal continuous surjection of finite topological rank has the natural extension that is expansive.

1. Introduction

By a *zero-dimensional system*, we mean a pair (X, f) of a compact zero-dimensional metrizable space X , and a continuous surjective map $f : X \rightarrow X$. In [6], we showed that every zero-dimensional system is expressed as an inverse limit of a sequence of covers of finite directed graphs. In this paper, instead of the term a ‘sequence of graph covers’, we use the term a ‘graph covering’ or just ‘covering’ for short. On the other hand, for Cantor minimal homeomorphisms, properly ordered Bratteli diagrams have been used for deep investigations. In [2], Downarowicz and Maass presented a remarkable theorem that states that every Cantor minimal system of finite topological rank $K > 1$ is expansive. They used the properly ordered Bratteli diagrams with astonishing technique. In [1], Bezuglyi *et al* extended the result to non-minimal cases. In this paper, we want to extend the result of Downarowicz and Maass to the non-homeomorphic case. As for general Cantor minimal continuous surjections, Gambaudo and Martens in [3], presented a way of expressing every Cantor minimal continuous surjection by a kind of graph covering. Hereafter, we refer to the coverings that Gambaudo and Martens gave in [3] as *GM-coverings* (see Definition 5.1 for details). Applying the proof of [2] and GM-coverings, we show an analogous theorem for Cantor minimal continuous surjections. We can define the topological rank for a Cantor minimal continuous surjection using GM-coverings. Here, because we have extended the kind of coverings, logically, the topological rank may decrease from that of Downarowicz and Maass [2]. Nevertheless, a simple observation shows that the topological rank is the same for Cantor minimal homeomorphisms (see Corollary 6.6). Our main result is as follows: for a Cantor minimal continuous surjection that has topological rank $K > 1$, the natural extension is expansive.

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2. Preliminaries

Let $\mathbb{Z}$ denote the set of all integers, and $\mathbb{N}$ the set of all non-negative integers. In this section, we repeat the construction of general graph coverings for general zero-dimensional systems originally given in [6, Section 3]. We describe some notation for later use. For $m \geq n$, we denote $[n, m] := \{n, n+1, \dots, m\}$. A pair $G = (V, E)$ consisting of a finite set V and a relation $E \subseteq V \times V$ on V can be considered as a directed graph with vertices V and an edge from u to v when $(u, v) \in E$. Unlike Bratteli diagrams that are well known and defined in Section 3, multiple edges from a vertex u to v are not permitted. We note here that the expression (V, E) is also used to mean a Bratteli diagram in this paper. If we write ‘graph G ’, ‘graph $G = (V, E)$ ’ or ‘surjective directed graph $G = (V, E)$ ’, we mean a finite directed graph. When the expression (V, E) mean a Bratteli diagram, we explicitly write this as ‘Bratteli diagram (V, E) ’.

Notation 2.1. In this paper, we assume that a finite directed graph G is a surjective relation, i.e., for every vertex $v \in V$ there exist edges (u_1, v) , $(v, u_2) \in E$.

For directed graphs $G_i = (V_i, E_i)$ with $i = 1, 2$, a map $\varphi : V_1 \rightarrow V_2$ is said to be a *graph homomorphism* if for every edge $(u, v) \in E_1$, it follows that $(\varphi(u), \varphi(v)) \in E_2$. In this case, we write as $\varphi : G_1 \rightarrow G_2$. For a graph homomorphism $\varphi : G_1 \rightarrow G_2$, we say that φ is *edge-surjective* if $\varphi(E_1) = E_2$.

Definition 2.2. Suppose that a graph homomorphism $\varphi : G_1 \rightarrow G_2$ satisfies the following condition:

$$(u, v), (u, v') \in E_1 \text{ implies that } \varphi(v) = \varphi(v').$$

In this case, φ is said to be *+directional*. Suppose that a graph homomorphism φ satisfies both of the following conditions:

$$(u, v), (u, v') \in E_1 \text{ implies that } \varphi(v) = \varphi(v')$$

and

$$(u, v), (u', v) \in E_1 \text{ implies that } \varphi(u) = \varphi(u').$$

Then, φ is said to be *bidirectional*.

Definition 2.3. A graph homomorphism $\varphi : G_1 \rightarrow G_2$ is called a *cover* if it is a +directional edge-surjective graph homomorphism.

For a sequence $G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$ of graph homomorphisms and $m > n$, we write $\varphi_{m,n} := \varphi_n \circ \varphi_{n+1} \circ \dots \circ \varphi_{m-1}$. Then, $\varphi_{m,n}$ is a graph homomorphism. If all φ_i ($i \in \mathbb{N} \setminus \{0\}$) are edge surjective, then every $\varphi_{m,n}$ is edge surjective. If all φ_i ($i \in \mathbb{N} \setminus \{0\}$) are covers, every $\varphi_{m,n}$ is a cover. Let $G_0 := (\{v_0\}, \{(v_0, v_0)\})$ be a singleton graph. For a sequence of graph covers $G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$, we attach the singleton graph G_0 at the head. We call a sequence of graph covers $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$ a *graph covering* or just a *covering*. Let us write the directed graphs as $G_i = (V_i, E_i)$ for $i \in \mathbb{N}$. Define

$$V_{\mathcal{G}} := \left\{ (x_0, x_1, x_2, \dots) \in \prod_{i=0}^{\infty} V_i \mid x_i = \varphi_i(x_{i+1}) \text{ for all } i \in \mathbb{N} \right\}$$

and

$$E_{\mathcal{G}} := \{(x, y) \in V_{\mathcal{G}} \times V_{\mathcal{G}} \mid (x_i, y_i) \in E_i \text{ for all } i \in \mathbb{N}\},$$

each equipped with the product topology.

Notation 2.4. For each $n \in \mathbb{N}$, the projection from $V_{\mathcal{G}}$ to V_n is denoted by $\varphi_{\infty, n}$. For $v \in V_n$, we denote a clopen set $U(v) := \varphi_{\infty}^{-1}(v)$. For a subset $V \subset V_n$, we denote a clopen set $U(V) := \bigcup_{v \in V} U(v)$.

Notation 2.5. Let X be a compact metrizable zero-dimensional space. A finite partition of X by non-empty clopen sets is called a *decomposition*. The set of all decompositions of X is denoted by $\mathcal{D}(X)$. Each $\mathcal{U} \in \mathcal{D}(X)$ is endowed with the discrete topology.

We recall that a zero-dimensional system is a pair (X, f) of a zero-dimensional compact metrizable space X and a continuous surjection $f : X \rightarrow X$. The +directionality and bidirectionality is defined in Definition 2.2. We can state the following result.

THEOREM 2.6. [6, Theorem 3.9 and Lemma 3.5] *Let $\mathcal{G}$ be a covering $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$. Then, $E_{\mathcal{G}}$ is a continuous surjective mapping and $(V_{\mathcal{G}}, E_{\mathcal{G}})$ is a zero-dimensional system. Conversely, every zero-dimensional system can be written in this way. Furthermore, if all φ_n are bidirectional, then this zero-dimensional system is a homeomorphism and every compact zero-dimensional homeomorphism is written in this way.*

We write $(V_{\mathcal{G}}, E_{\mathcal{G}})$ as G_{∞} . Take a subsequence $n_0 = 0 < n_1 < n_2 < \dots$. Then, we can obtain an essentially same covering

$$G_0 \xleftarrow{\varphi_{n_1, 0}} G_{n_1} \xleftarrow{\varphi_{n_2, n_1}} G_{n_2} \xleftarrow{\varphi_{n_3, n_2}} \dots$$

It is evident that the new covering produces a naturally topologically conjugate zero-dimensional system. Following the naming in the theory of Bratteli–Vershik systems, we call this procedure *telescoping*.

Notation 2.7. Let $G = (V, E)$ be a surjective directed graph. A sequence of vertices $w = (v_0, v_1, \dots, v_l)$ of G is said to be a *walk* of length l if $(v_i, v_{i+1}) \in E$ for all $0 \leq i < l$. We denote as $l(w) := l$. We say that a walk $w = (v_0, v_1, \dots, v_l)$ is a *path* if v_i ($0 \leq i \leq l$) are mutually distinct. A walk $c = (v_0, v_1, \dots, v_l)$ is said to be a *cycle* of period l if $v_0 = v_l$, and a cycle $c = (v_0, v_1, \dots, v_l)$ is a *circuit* of period l if the v_i ($0 \leq i < l$) are mutually distinct. A circuit c and a path p are also considered to be subgraphs of G with period $l(c)$ and length $l(p)$ respectively. For a walk $w = (v_0, v_1, \dots, v_l)$, we define as $V(w) := \{v_i \mid 0 \leq i \leq l\}$ and $E(w) := \{(v_i, v_{i+1}) \mid 0 \leq i < l\}$. For a subgraph G' of G , we also define $V(G')$ and $E(G')$ in the same manner, especially, $V(G) = V$ and $E(G) = E$.

Notation 2.8. Let $w_1 = (u_0, u_1, \dots, u_l)$ and $w_2 = (v_0, v_1, \dots, v_{l'})$ be walks such that $u_l = v_0$. Then, we denote $w_1 w_2 := (u_0, u_1, \dots, u_l, v_1, v_2, \dots, v_{l'})$. Evidently, we obtain $l(w_1 w_2) = l + l'$. If c is a cycle of length l , then for any positive integer n , a cycle c^n of length ln is well defined. When a walk $w c w'$ can be constructed, a walk $w c^0 w'$ means $w w'$.

3. Bratteli–Vershik systems

We follow Gjerde and Johansen [4, Section 1] to describe the Bratteli–Vershik representation for Cantor minimal homeomorphisms.

Definition 3.1. A *Bratteli diagram* is an infinite directed graph (V, E) , where V is the vertex set and E is the edge set. These sets are partitioned into non-empty disjoint finite sets $V = V_0 \cup V_1 \cup V_2 \cup \dots$ and $E = E_1 \cup E_2 \cup \dots$, where $V_0 = \{v_0\}$ is a one-point set. Each E_n is a set of edges from V_{n-1} to V_n . Therefore, there exist two maps $r, s : E \rightarrow V$ such that $r : E_n \rightarrow V_n$ and $s : E_n \rightarrow V_{n-1}$ for $n \geq 1$, the range map and the source map respectively. Moreover, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$. We say that $u \in V_{n-1}$ is connected to $v \in V_n$ if there exists an edge $e \in E_n$ such that $s(e) = u$ and $r(e) = v$. Unlike graph coverings, multiple edges between u and v are permitted. A Bratteli diagram has *rank* $K \geq 1$ if $\#V_n \leq K$ for all $n \geq 0$ and K is the least such number.

Let (V, E) be a Bratteli diagram and let $m > n$ be non-negative integers. We define

$$E_{n,m} := \{p \mid p \text{ is a path from a } u \in V_n \text{ to } v \in V_m\}.$$

Then, we can construct a new Bratteli diagram (V', E') as follows:

$$\begin{aligned} V' &:= V_0 \cup V_1 \cup \dots \cup V_n \cup V_m \cup V_{m+1} \cup \dots \\ E' &:= E_1 \cup E_2 \cup \dots \cup E_n \cup E_{n,m} \cup E_{m+1} \cup \dots \end{aligned}$$

The source map and the range map are also defined naturally. This procedure is called *telescoping*.

Definition 3.2. A Bratteli diagram is called *simple* if we make (at most, countably many times of) telescopings, all pairs of vertices $u \in V_n$ and $v \in V_{n+1}$ are joined by at least one edge for all $n \geq 0$.

Definition 3.3. Let (V, E) be a Bratteli diagram such that $V = V_0 \cup V_1 \cup V_2 \cup \dots$ and $E = E_1 \cup E_2 \cup \dots$ are the partitions, where $V_0 = \{v_0\}$ is a one-point set. Let $r, s : E \rightarrow V$ be the range map and the source map respectively. We say that $(V, E, \leq)$ is an *ordered* Bratteli diagram if the partial order $\leq$ is defined on E such that $e, e' \in E$ is comparable if and only if $r(e) = r(e')$. That is, we have a linear order on each set $r^{-1}(v)$ with $v \in V \setminus V_0$. The edges $r^{-1}(v)$ are numbered from 1 to $\#(r^{-1}(v))$.

Let $n > 0$ and let $e = (e_n, e_{n+1}, e_{n+2}, \dots)$, $e' = (e'_n, e'_{n+1}, e'_{n+2}, \dots)$ be cofinal paths from vertices of V_{n-1} , which might be different. We obtain the lexicographic order $e < e'$ as follows:

if $k \geq n$ is the largest number such that $e_k \neq e'_k$, then $e_k < e'_k$.

Definition 3.4. Let $(V, E, \leq)$ be an ordered Bratteli diagram. Let $E_{\max}$ and $E_{\min}$ denote the set of maximal and minimal edges, respectively. An infinite path is maximal (minimal) if all the edges making up the path are elements of $E_{\max}$ ($E_{\min}$).

Definition 3.5. An ordered Bratteli diagram is properly ordered if it is simple and if it has a unique maximal and a unique minimal path, denoted respectively by $x_{\max}$ and $x_{\min}$.

Definition 3.6. (Vershik map) Let $(V, E, \leq)$ be a properly ordered Bratteli diagram. Let

$$E_{0,\infty} := \{(e_1, e_2, \dots) \mid r(e_i) = s(e_{i+1}) \text{ for all } i \geq 1\},$$

with the subspace topology of the product space $\prod_{i=1}^{\infty} E_i$. We can define the *Vershik map* $\phi : E_{0,\infty} \rightarrow E_{0,\infty}$ as follows.

If $e = (e_1, e_2, \dots) \neq x_{\max}$, then there exists the least $n \geq 1$ such that e_n is not maximal in $r^{-1}(r(e_n))$. Then, we can select the least $f_n > e_n$ in $r^{-1}(r(e_n))$. Let $v_{n-1} = s(f_n)$. Then, it is easy to obtain the unique least path $(f_1, f_2, \dots, f_{n-1})$ from v_0 to v_{n-1} . We define

$$\phi(e) := (f_1, f_2, \dots, f_{n-1}, f_n, e_{n+1}, e_{n+2}, \dots).$$

We define $\phi(x_{\max}) = x_{\min}$. The map $\phi : E_{0,\infty} \rightarrow E_{0,\infty}$ is called the *Vershik map*.

In [5, Theorem 4.7], Herman *et al* gave a correspondence that for a properly ordered Bratteli diagram, the Vershik map is a minimal homeomorphism from a compact metrizable zero-dimensional space to itself; conversely, a minimal homeomorphism from a compact metrizable zero-dimensional space to itself is represented as the Vershik map of a properly ordered Bratteli diagram. In [2], Downarowicz and Maass introduced the topological rank for a Cantor minimal homeomorphism.

Definition 3.7. Let (X, f) be a Cantor minimal homeomorphism. Then, the topological rank of (X, f) is $1 \leq K \leq \infty$, if it has a Bratteli–Vershik representation with Bratteli diagram with rank K , and K is the least such number.

4. Covering of Kakutani–Rohlin type

Before we proceed to the coverings for general minimal continuous surjections that are described in Section 5, we will introduce a type of coverings that is closely related to Bratteli diagrams. We shall construct a covering $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$ that can express a certain set of Cantor minimal continuous surjections that contains all Cantor minimal homeomorphisms but not all minimal continuous surjections. Let G_0 be the singleton graph that corresponds to the top vertex of the Bratteli diagram. We shall construct graphs G_n ($n \geq 1$) such that for each $n \geq 1$, G_n is a generalized figure of 8. A generalized figure of 8 is a finite directed graph that has a unique central vertex, and distinct circuits that start and end at the central vertex. We denote the central vertex as $v_{n,0}$, and distinct circuits as $\{c_{n,1}, c_{n,2}, \dots, c_{n,l_n}\}$. We write the period of each circuit $c_{n,i}$ with $1 \leq i \leq l_n$ as $l(n, i) \geq 1$. Thus, we write

$$c_{n,i} = (v_{n,i,0} = v_{n,0}, v_{n,i,1}, v_{n,i,2}, \dots, v_{n,i,l(n,i)} = v_{n,0})$$

for $n \geq 1$ and $1 \leq i \leq l_n$. As we are constructing each G_n as a generalized figure of 8, we assume that $V(c_{n,i}) \cap V(c_{n,j}) = \{v_{n,0}\}$ for all $n \geq 1$ and $i \neq j$. In [3], Gambaudo and Martens expressed every Cantor minimal continuous surjection by certain graph coverings (see Section 5). We make use of their coverings to express all Cantor minimal homeomorphisms, i.e., all Bratteli–Vershik systems with properly ordered Bratteli diagrams.

Definition 4.1. We say that a covering $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ is of *Kakutani–Rohlin type* if G_n is a generalized figure of 8 and $\varphi_n(v_{n+1,0}) = v_{n,0}$ for every $n \geq 0$. We call a covering of Kakutani–Rohlin type a *KR-covering* for short. A KR-covering has *rank* $1 \leq K \leq \infty$ if $l_n \leq K$ for all $n \geq 0$ and K is the least such number.

Note that we can write for each $n > 0$ and i with $1 \leq i \leq l_n$,

$$\varphi_{n-1}(c_{n,i}) = c_{n-1,a(n,i,1)}c_{n-1,a(n,i,2)} \cdots c_{n-1,a(n,i,k(n,i))}.$$

Furthermore, because of the +directionality of a graph cover, we obtain

$$a(n, 1, 1) = a(n, 2, 1) = \cdots = a(n, l_n, 1).$$

For an $n \geq 0$, φ_n is bidirectional if and only if

$$a(n, 1, k(n, 1)) = a(n, 2, k(n, 2)) = \cdots = a(n, l_n, k(n, l_n)).$$

Remark 4.2. For a KR-covering $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$, we write $G_\infty = (X, f)$. Although, f may not be a homeomorphism, we are able to consider a refining sequence of Kakutani–Rohlin partitions. For each $n \geq 0$ and $c_{n,i}$ with $1 \leq i \leq l_n$, let $c_{n,i} = (v_0, v_1, \dots, v_{l(n,i)} = v_0)$. We write $B_i = f^{-1}(U(v_1)) \subset U(v_0)$. Then, $f^k(B_i) = U(v_k)$ for $1 \leq k < l(n, i)$. We write $\xi_i = \{f^k(B_i) \mid 0 \leq k < l(n, i)\}$. The zero-dimensional system (X, f) is partitioned into Kakutani–Rohlin partition $\Xi_n := \bigcup_{1 \leq i \leq l_n} \xi_i$. It is clear that Ξ_{n+1} refines Ξ_n for all $n \geq 0$.

Remark 4.3. A KR-covering is linked with an ordered Bratteli diagram (that may not be properly ordered nor simple). To see this, let $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ be a KR-covering. The ordered Bratteli diagram (V, E) is constructed as follows: if G_n consists of circuits $c_{n,i}$ ($1 \leq i \leq l_n$), then we define $V_n = \{c_{n,i} \mid 1 \leq i \leq l_n\}$ for each $n \geq 0$; if $\varphi_{n-1}(c_{n,i}) = c_{n-1,a(n,i,1)}c_{n-1,a(n,i,2)} \cdots c_{n-1,a(n,i,k(n,i))}$, an edge that belongs to E_n is made from $c_{n,i}$ to each $c_{n-1,a(n,i,j)}$ for $1 \leq j \leq k(n, i)$ that is numbered by j . Conversely, a properly ordered Bratteli–Vershik system has Kakutani–Rohlin partitions. Therefore, we can obtain a KR-covering with respect to these partitions.

THEOREM 4.4. *Let $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ be a KR-covering of rank $1 \leq K \leq \infty$, and let (X, f) be its inverse limit. Then, there exists an $x_0 \in X$ such that for all $x \neq x_0$, it follows that $\#f^{-1}(x) = 1$ and $\#f^{-1}(x_0) \leq K$.*

Proof. Let $v_{n,0}$ be the central vertex of G_n for each $n \geq 0$. We define as $x_0 := (v_0, v_{1,0}, v_{2,0}, \dots)$. Let $x \neq x_0$ and $x = (v_0, u_1, u_2, \dots)$. Because $x \neq x_0$, there exists an $N > 0$ such that $u_n \neq v_{n,0}$ for all $n \geq N$. Take a $y \in X$ such that $f(y) = x$. We write $y = (v_0, v_1, v_2, \dots)$. It follows that (v_n, u_n) is an edge of a circuit of G_n . This circuit cannot be $(v_{n,0}, v_{n,0})$ for $n \geq N$. Thus, v_n is uniquely determined for all $n \geq 0$. Therefore, y is uniquely determined, i.e., $\#f^{-1}(x) = 1$. For the number $\#f^{-1}(x_0)$, it is sufficient to consider the case in which K is finite. Suppose that there exist mutually distinct $x_1, x_2, \dots, x_{K+1}$ such that $f(x_i) = x_0$ for all $1 \leq i \leq K+1$. For each $1 \leq i \leq K+1$, let $x_i = (v_0, v_{1,i}, v_{2,i}, \dots)$. Then, there exists a pair $i \neq i'$ with $1 \leq i, i' \leq K+1$ such that for infinitely many n , $v_{n,i} = v_{n,i'}$. This implies that $x_i = x_{i'}$, a contradiction. $\square$

Here, we introduce a proposition that describes a condition of minimality of a graph covering.

PROPOSITION 4.5. *Let $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$ be a covering. Then, the resulting zero-dimensional system G_∞ is minimal if and only if for all $n \geq 0$, there exists an $m > n$ such that $V(\varphi_{m,n}(c_{m,i})) = V(G_n)$.*

Proof. See [7, (a), (d), (e) of Theorem 3.5]. $\square$

With this proposition, when we consider a minimal zero-dimensional system, telescoping if necessary, we can assume that $V(\varphi_{n-1}(c_{n,i})) = V(G_{n-1})$ and $E(\varphi_{n-1}(c_{n,i})) = E(G_{n-1})$ for all $n > 0$ and $1 \leq i \leq l_n$, i.e., all circuits $c_{n-1,j}$ ($1 \leq j \leq l_{n-1}$) appear in every $\varphi_{n-1}(c_{n,i})$ with $1 \leq i \leq l_n$.

Definition 4.6. A KR-covering $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$ is said to be *simple* if for every $n \geq 0$, there exists an $m > n$ such that $V(\varphi_{m,n}(c_{m,i})) = V(G_n)$.

Definition 4.7. A Cantor minimal system that is an inverse limit of some KR-covering of finite rank has *topological rank K* if it is represented by some KR-covering of rank K and K is the least such number.

The notion of KR-covering is indispensable for stating a partial answer to a problem that appears in a later section.

5. Covering of Gambaudo–Martens type

In [3, Theorem 2.5], Gambaudo and Martens showed that every Cantor minimal system is an inverse limit of a special kind of graph covering. In our context, their construction of a graph covering is as follows: let $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$ be a graph covering. As usual, we assume that G_0 is the singleton graph $(\{v_0\}, \{(v_0, v_0)\})$. We shall construct graphs G_n with an $n \geq 1$ such that there exist a unique vertex $v_{n,0}$ and a finite number of circuits $c_{n,i}$ ($1 \leq i \leq l_n$) that starts and ends at $v_{n,0}$; and, roughly, if two circuits meet at a vertex, then the rest of the two circuits merge until they reach the end.

Definition 5.1. We say that a covering $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \dots$ is of *Gambaudo–Martens type* if for each $n > 0$, there exist a vertex $v_{n,0}$, a finite number of circuits $c_{n,i}$ ($1 \leq i \leq l_n$) and a covering map φ_n such that:

- (a) $c_{n,i}$ can be written as $(v_{n,0} = v_{n,i,0}, v_{n,i,1}, v_{n,i,2}, \dots, v_{n,i,l(n,i)} = v_{n,0})$ with $l(n, i) \geq 1$;
- (b) $\bigcup_{i=1}^{l_n} E(c_{n,i}) = E(G_n)$;
- (c) if $v_{n,i,j} = v_{n,i',j'}$ with $j, j' \geq 1$, then $v_{n,i,j+k} = v_{n,i',j'+k}$ for $k = 0, 1, 2, \dots$, until $j+k = l(n, i)$ and $j'+k = l(n, i')$ at the same time;
- (d) $\varphi_n(v_{n+1,0}) = v_{n,0}$; and
- (e) $\varphi_n(v_{n+1,i,1}) = v_{n,1,1}$ for all $1 \leq i \leq l_n$.

We say that a covering of this type is a *GM-covering* for short. A GM-covering is said to be *simple* if for all $n > 0$, there exists an $m > n$ such that for each $1 \leq i \leq l_m$, $E(\varphi_{m,n}(c_{m,i})) = E(G_n)$. As listed in Proposition 4.5, this condition makes the resulting zero-dimensional system minimal. If we want to avoid the case in which the resulting zero-dimensional system has an isolated point, we have to add the following condition: for every $n \geq 1$ and every vertex v of G_n , there exist an $m > n$ and distinct vertices u_1, u_2 of G_m such that

$\varphi_{m,n}(u_1) = \varphi_{m,n}(u_2) = v$. A GM-covering has *rank* $K \geq 1$ if $l_n \leq K$ for all $n \geq 0$ and K is the least such number.

Remark 5.2. For $n \geq 0$ and $1 \leq i \leq l_{n+1}$, we can write

$$\varphi_n(c_{n+1,i}) = c_{n,a(n,i,1)}c_{n,a(n,i,2)} \cdots c_{n,a(n,i,k(n,i))},$$

such that $a(n, i, 1) = 1$ for all i with $1 \leq i \leq l_{n+1}$.

Notation 5.3. By telescoping, we can make a simple GM-covering have the following condition: for every $n > 0$ and every i ($1 \leq i \leq l_n$), $E(\varphi_n(c_{n+1,i})) = E(G_n)$. Henceforth, if we say that a GM-covering is simple, we assume that this condition is satisfied.

THEOREM 5.4. (Gambaudo and Martens [3]) *A zero-dimensional system is minimal (not necessarily homeomorphic) if and only if it is represented as the inverse limit of a simple GM-covering.*

Proof. See the proof of Theorem 2.5 of [3]. □

Remark 5.5. Unlike KR-covering, GM-covering may not have Kakutani–Rohlin decomposition. Therefore, it may not be possible to link with the Bratteli diagrams.

As an analogue of topological rank for Cantor minimal homeomorphisms, we say that a minimal zero-dimensional system has a *topological rank* K , if there exists a simple GM-covering of rank K and K is the least such integer (see [2]). In [2], they showed that a Cantor minimal homeomorphism whose topological rank K is finite and $K > 1$ is expansive, i.e., topologically conjugate to a minimal two-sided subshift. We show an analogous theorem for (not necessarily homeomorphic) Cantor minimal systems. Suppose that a simple GM-covering $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ produces a minimal zero-dimensional system G_∞ . We write $G_\infty = (X, f)$ and we assume that (X, f) is not a single periodic orbit. Then, because of minimality, (X, f) is a Cantor system and has no periodic orbits. Therefore, the minimal length of the circuits of G_n grows to infinity, i.e., we obtain $l(n, i) \rightarrow \infty$ uniformly as $n \rightarrow \infty$.

Notation 5.6. For (X, f) , we construct the natural extension $(\hat{X}_f, \sigma)$ as follows:

- $\hat{X}_f := \{(\dots, x_{-1}, x_0, x_1, x_2, \dots) \in X^{\mathbb{Z}} \mid f(x_i) = x_{i+1} \text{ for all } i \in \mathbb{Z}\};$
- for $\hat{x} = (\dots, x_{-1}, x_0, x_1, x_2, \dots) \in \hat{X}_f$, σ shifts $\hat{x}$ to the left, i.e., $(\sigma(\hat{x}))_i = x_{i+1}$ for all $i \in \mathbb{Z}$.

It is easy to check that if (X, f) is minimal, then $(\hat{X}_f, \sigma)$ is minimal. For an $\hat{x} \in \hat{X}_f$ and an $i \in \mathbb{Z}$, we denote $\hat{x}(i) = x_i$. Then, $(\sigma(\hat{x}))(i) = x_{i+1}$ for all $i \in \mathbb{Z}$. We use a lot of the notation and ideas that are found in [2]. For every $\hat{x} \in \hat{X}_f$ and $i \in \mathbb{Z}$, there exists a unique $u_{n,i} \in V(G_n)$ such that $x_i \in U(u_{n,i})$. Therefore, a unique sequence $\hat{x}|_n := (\dots, u_{n,-2}, u_{n,-1}, u_{n,0}, u_{n,1}, \dots)$ of vertices of G_n is defined such that $x_i \in U(\hat{x}|_n(i))$. Although the vertex $u_{n,i}$ is uniquely determined for each $\hat{x}$, $n \geq 0$ and $i \in \mathbb{Z}$, the circuit $c_{n,t}$ with $u_{n,i} \in V(c_{n,t})$ may not be unique. Nevertheless, if $x_i \in U(v_{n,0})$ for some $i \in \mathbb{Z}$, then there exists a unique t ($1 \leq t \leq l_n$) such that $x_{i+1} \in U(v_{n,t,1})$; and therefore, $x_{i+j} \in U(v_{n,t,j})$ for all $0 \leq j \leq l(n, t)$.

Notation 5.7. We write this t as $t(\hat{x}, n, i)$, and $c_{n,t}$ as $c(\hat{x}, n, i)$, for all $n \geq 0$ and $i \in \mathbb{Z}$.

Let $k(0) \in \mathbb{Z}$ be such that $x_{k(0)} \in U(v_{n,0})$, and let $k(1) > k(0)$ be the least $k > k(0)$ such that $x_k \in U(v_{n,0})$. Then, we have combined the interval $u_{n,k(0)}, u_{n,k(0)+1}, \dots, u_{n,k(1)-1}$ with the unique circuit $c(\hat{x}, n, i)$ with $k(0) \leq i < k(1)$. Thus, we have obtained a sequence of $c_{n,i}$, and we denote it as $\hat{x}[n]$. We include, in $\hat{x}[n]$, the information of the beginnings (and endings) of all circuits. This may be done by changing the first appearance of each circuit c by $\check{c}$. We write $\hat{x}[n](i) = c(\hat{x}, n, i)$ for all $n \geq 0$ and all $i \in \mathbb{Z}$. Note that we can recover the sequence of vertices of G_n from $\hat{x}[n]$. A $\hat{x}[0]$ becomes just a sequence of v_0 . For an interval $[n, m]$ with $m > n$, the combination of rows $\hat{x}|_{n'}$ with $n \leq n' \leq m$ is denoted as $\hat{x}|_{[n,m]}$, and the combination of rows $\hat{x}[n']$ with $n \leq n' \leq m$ is denoted as $\hat{x}[n, m]$. The *array system* of $\hat{x}$ is the infinite combination $\hat{x}|_{[0,\infty)}$ of all rows $\hat{x}|_n$, $0 \leq n < \infty$. The *linked array system* of $\hat{x}$ is the infinite combination $\hat{x}[0, \infty)$ of all rows $\hat{x}[n]$, $0 \leq n < \infty$. Note that from the information of $\hat{x}[0, \infty)$, we can recover $\hat{x}|_{[0,\infty)}$ and can also identify $\hat{x}$ itself. If each circuit of G_n is considered to be just an alphabet, then for $n \geq 0$ and $I < J$, we can consider a finite sequence of circuits of G_n :

$$\hat{x}[n](I), \hat{x}[n](I+1), \dots, \hat{x}[n](J)$$

even if the completion of the circuits are cut off at the right or the left end in the above sequence.

Definition 5.8. Let $\check{\hat{X}}_f := \{\hat{x}|_{[0,\infty)} \mid \hat{x} \in \hat{X}_f\}$ be a set of the sequences of symbols that are vertices of G_n ($0 \leq n < \infty$). The topology is generated by *cylinders* such that for $\hat{x} \in \hat{X}_f$ and $N, I > 0$,

$$\check{C}(\hat{x}, N, I) := \{\hat{y}|_{[0,\infty)} \mid \hat{y} \in \hat{X}_f, \hat{y}|_{[0,N]}(i) = \hat{x}|_{[0,N]}(i) \text{ for all } i \text{ with } -I \leq i \leq I\}.$$

The shift map $\sigma : \check{\hat{X}}_f \rightarrow \check{\hat{X}}_f$ is defined as above. Then, $(\check{\hat{X}}_f, \sigma)$ is a zero-dimensional system and we call it as an *array system* of $(\hat{X}_f, \sigma)$. Let $\bar{\hat{X}}_f := \{\hat{x}[0, \infty) \mid \hat{x} \in \hat{X}_f\}$ be a set of the sequences of symbols that are circuits of G_n ($0 \leq n < \infty$). The topology is generated by *cylinders* such that for $\hat{x} \in \hat{X}_f$ and $N, I > 0$,

$$\bar{C}(\hat{x}, N, I) := \{\hat{y}[0, \infty) \mid \hat{y} \in \hat{X}_f, \hat{y}[0, N](i) = \hat{x}[0, N](i) \text{ for all } i \text{ with } -I \leq i \leq I\}.$$

The shift map $\sigma : \bar{\hat{X}}_f \rightarrow \bar{\hat{X}}_f$ is defined as above. Then, $(\bar{\hat{X}}_f, \sigma)$ is a zero-dimensional system and we call it a *linked array system* of $(\hat{X}_f, \sigma)$.

Remark 5.9. Clearly, $(\check{\hat{X}}_f, \sigma)$ is topologically conjugate to $(\hat{X}_f, \sigma)$. Moreover, because $(\check{\hat{X}}_f, \sigma)$ has a continuous factor map to $(\bar{\hat{X}}_f, \sigma)$ and it is bijective as described above, $(\bar{\hat{X}}_f, \sigma)$ is also topologically conjugate to $(\hat{X}_f, \sigma)$.

For each sequence $\hat{x}[n]$ of circuits of G_n ($n > 0$), we make an n -cut at the position $i \in \mathbb{Z}$ when $\hat{x}|_n(i) = v_{n,0}$, i.e., if there exists an n -cut at position $i \in \mathbb{Z}$, then $c(\hat{x}, n, i-1)$ and $c(\hat{x}, n, i)$ are separated by the cut. The row $\hat{x}[n]$ is separated into circuits exactly by the cuts (see Figures 1 and 2). Note that for $m > n$, if there exists an m -cut at position i , then there exists an n -cut at position i . For each circuit $c_{n,i}$, we can determine a series of circuits by $\varphi_{n-1}(c_{n,i}) = c_{n-1,1}c_{n-1,a(n,i,2)} \cdots c_{n-1,a(n,i,k(n,i))}$. Furthermore, each $c_{n-1,a(n,i,j)}$ determines a series of circuits by the map φ_{n-2} . In this way we can determine a set of circuits arranged in a square form as in Figure 3. Following [2], this form is said

$c_{n,1}$	$c_{n,3}$	$c_{n,1}$	$c_{n,3}$	$c_{n,2}$	$c_{n,1}$
$c_{n+1,5}$		$c_{n+1,1}$			$c_{n+1,3}$

FIGURE 1. The n and $(n + 1)$ th rows of a linked array system with cuts.

v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0
$c_{1,3}$	$c_{1,3}$			$c_{1,1}$			$c_{1,3}$			$c_{1,2}$		$c_{1,1}$		
$c_{2,3}$				$c_{2,1}$								$c_{2,3}$		
$c_{3,3}$				$c_{3,1}$										
$\vdots$														

FIGURE 2. The first four rows of an array system.

v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0
$c_{1,1}$			$c_{1,3}$		$c_{1,2}$		
$c_{2,1}$							

FIGURE 3. The 2-symbol corresponding to the circuit $c_{2,1}$ of Figure 2.

to be the n -symbol and denoted also by $c_{n,i}$. For $n' < n$, the projection $c_{n,i}[n']$ that is a finite sequence of circuits of $G_{n'}$ is also defined. We have to make it clear that the author has borrowed the main ideas of the proofs from [2]. Our work was just to check that the proofs in [2] are valid in our case of Cantor minimal continuous surjections. It is clear that $\hat{x}[n] = \hat{x}'[n]$ implies $\hat{x}[0, n] = \hat{x}'[0, n]$. If $\hat{x} \neq \hat{x}'$ ($\hat{x}, \hat{x}' \in \hat{X}_f$), then there exists an $n > 0$ with $\hat{x}[n] \neq \hat{x}'[n]$. For $\hat{x}, \hat{x}' \in \hat{X}_f$, we say that the pair $(\hat{x}, \hat{x}')$ is n -compatible if $\hat{x}[n] = \hat{x}'[n]$. If $\hat{x}[n] \neq \hat{x}'[n]$, then we say that $\hat{x}$ and $\hat{x}'$ are n -separated. We recall that if there exists an n -cut at position k , then there exists an n' -cut at position k for all $0 \leq n' \leq n$. Let $\hat{x} \neq \hat{x}'$. If a pair $(\hat{x}, \hat{x}')$ is n -compatible and $(n + 1)$ -separated, then, we say the *depth of compatibility* of $\hat{x}$ and $\hat{x}'$ is n , or the pair $(\hat{x}, \hat{x}')$ has *depth* n . If $(\hat{x}, \hat{x}')$ is a pair of depth n and $(\hat{x}, \hat{x}'')$ is a pair of depth $m > n$, then the pair $(\hat{x}', \hat{x}'')$ has depth n (and hence are never equal). An n -separated pair $(\hat{x}, \hat{x}')$ is said to *have a common n -cut* if both $\hat{x}$ and $\hat{x}'$ have an n -cut at the same position. If a pair has a common n -cut then it also has a common m -cut for each $0 \leq m \leq n$. The set $X_n := \{\hat{x}[n] \mid \hat{x} \in \hat{X}_f\}$ is a two-sided subshift of the finite set $\{c_{n,1}, c_{n,2}, \dots, c_{n,l_n}\}$. The factoring map is denoted by $\pi_n : \hat{X}_f \rightarrow X_n$, and the shift map is denoted by $\sigma_n : X_n \rightarrow X_n$. We write $\sigma = \sigma_n$ for all n , if there is no confusion.

6. Main theorem

In this section, we state our main result, and prove the theorem. Again, we have to make it clear that the author has borrowed the main ideas of the proofs from [2]. Our work from now on is just to check that the proofs in [2] are valid in our case of Cantor minimal continuous surjections.

THEOREM 6.1. (Main result) *Let (X, f) be a minimal zero-dimensional system whose topological rank is K ($1 < K < \infty$). Then, its natural extension $(\hat{X}_f, \sigma)$ is expansive.*

Proof. As in the proof of [2], we show by contradiction. Suppose that the claim fails. Then, for all $L > 0$, there exists a pair $(\hat{x}, \hat{x}')$ with distinct elements of $\hat{X}_f$ that is L -compatible. Because $\hat{x} \neq \hat{x}'$, for some $m > L$, $(\hat{x}, \hat{x}')$ is m -separated. Therefore, $(\hat{x}, \hat{x}')$ has depth n with $L \leq n < m$. Therefore, for infinitely many n , there exists a pair $(\hat{x}_n, \hat{x}'_n)$ of depth n . By telescoping, we can assume that every $n > 0$ has a pair $(\hat{x}_n, \hat{x}'_n)$ of depth n . Note that even after additional telescoping, this quality still holds. As in the proof of [2], we prove two separate cases:

- (1) there exists an N such that for all $n > N$ and every $m > n$, there exists a pair $(\hat{x}_n, \hat{x}'_n)$ of depth n with a common m -cut;
- (2) for infinitely many n , and then every sufficiently large $m > n$, any pair of depth n has no common m -cut.

Proof in case (1). As in the proof of [2], we prove that such a case never occurs, even for $K = 1$. Fix some $m > N + K$ and for an integer $n \in [m - K, m - 1]$ let $(\hat{x}_n, \hat{x}'_n)$ be a pair of depth n with a common m -cut. For $n = m - 1$, we have a $(m - 1)$ -compatible m -separated pair $(\hat{x}_{m-1}, \hat{x}'_{m-1})$ with a common m -cut. Suppose that all of the m -cuts of $\hat{x}_{m-1}$ and $\hat{x}'_{m-1}$ are the same. Because the pair $(\hat{x}_{m-1}, \hat{x}'_{m-1})$ is m -separated, at least two distinct m -symbols are in the same place of $\hat{x}_{m-1}$ and $\hat{x}'_{m-1}$. These m -symbols have the same rows from 0 to $m - 1$. Therefore, at least two distinct m -symbols have the same rows from 0 to $m - 1$. If a set of m -symbols whose rows from 0 to $m - 1$ are the same, we factor these to the same alphabet, i.e., we make a new GM-covering identifying such circuits of G_m . The new directed graph is also a GM-covering, the projection mapping to G_{m-1} from the new G_m is well defined, and the projection mapping to the new G_m is also well defined. It is clear that the new covering is $+$ -directional. Furthermore, the inverse limit of the new covering and the X_{m-1} of the new covering is the same as the old one. Because the pair $(\hat{x}_{m-2}, \hat{x}'_{m-2})$ was $(m - 1)$ -separated, and this factorization does not affect in the $(m - 1)$ th row, the pair $(\hat{x}_{m-2}, \hat{x}'_{m-2})$ is still m -separated. Note that the number of circuits of the m th row has been decreased by at least one. Next, we consider the case in which after a common m -cut of the pair $(\hat{x}_{m-1}, \hat{x}'_{m-1})$, the coincidence of the positions of m -cuts do not continue to the right or left end. Suppose that after a common m -cut at position k_0 , the continuation of common m -cuts does not continue to the right end. Let $k_1 > k_0$ be the position of the first common m -cut such that the right m -symbols c and c' of $\hat{x}_{m-1}$ and $\hat{x}'_{m-1}$ have different lengths. We assume without loss of generality that $l(c') < l(c)$. Let $k_2 = k_1 + l(c')$, then, k_2 is the position of the next m -cut of $\hat{x}'_{m-1}$. Because $\hat{x}_{m-1}[m - 1] = \hat{x}'_{m-1}[m - 1]$, the m -symbol c itself has an $(m - 1)$ -cut at $l(c')$. Let $c = (v_{m,0}, v_1, v_2, \dots, v_{l(c)})$. Then, c is separated into two parts $(v_{m,0}, v_1, v_2, \dots, v_{l(c')})$ and $(v_{l(c')}, v_{l(c')+1}, \dots, v_{l(c)})$ which have the vertex $v_{l(c')}$ in common. Let $c = c_1, c_2, \dots, c_s$ be the circuits of G_m that pass $v_{l(c')}$. We make a new graph identifying $v_{l(c')}$ with $v_{m,0}$. Note that $(c[m - 1])(l(c'))$ is the ending position of c' . Therefore, the first circuit that begins from $(c[m - 1])(l(c'))$ is $c_{m-1,1}$. Because of this, the new cover map satisfies $+$ -directionality. Each circuit c_i $i = 1, 2, \dots, s$ is decomposed into two circuits $\bar{c}_i$ that are distinct for each i and the common c'' , i.e., each original circuit c_i is now a cycle and $c_i = \bar{c}_i c''$. Unlike the proof in [2], we have to consider two cases:

FIGURE 4. Change of an m -symbol. There is a common cut at the left end.

- (a) the circuit c'' has already existed;
- (b) the circuit c'' has not existed.

Let us consider case (a). Then, in the new graph G_m , the number of circuits is not changed. The projection from G_m is well defined, and the projection onto G_m is also well defined. In this way, we have obtained a new covering. Because $c' \neq \bar{c}$, it follows that the pair $(\hat{x}, \hat{x}')$ is m -separated.

Next, let us consider case (b). Then, the number of circuits has now been increased by one. For two circuits $\bar{c}_1$ (or $\bar{c}$) and c' , we obtain $\bar{c}_1[m-1] = c'[m-1]$. We merge these two circuits and construct a new graph and the covering. The projection from G_m is well defined, and the projection onto G_m is also well defined. For the last covering, the new G_m has the same number of circuits compared to the original one (see Figure 4 under the condition that $\bar{c} = c'$). Because c'' is a new circuit, the circuits of $\hat{x}[m]$ and $\hat{x}'[m]$ that start from the position k_1 are different. Therefore, the pair $(\hat{x}, \hat{x}')$ is m -separated. In both cases, we obtain new graph coverings that have the same inverse limits and the same X_{m-1} . After this modification no cut that existed is removed. The number of m -symbols has not increased. The coincidence of the m -cut from k_0 to the right might be shortened. Nevertheless, the same modification is possible, and finally we come to the point that we have the same m -cuts from k_0 to the right end. The same argument is valid for the left direction from k_0 . Now, the pair $(\hat{x}_{m-1}, \hat{x}'_{m-1})$ is m -separated and have the same m -cuts throughout the sequences. Then, we can apply the previous argument. In this way, we obtain a factor in the m th row that decreases the number of circuits of G_m , and yet $(\hat{x}_{m-2}, \hat{x}'_{m-2})$ is $(m-1)$ -separated. Therefore, $(\hat{x}_{m-2}, \hat{x}'_{m-2})$ is still m -separated. We can now delete the $(m-1)$ th row, and continue this process. Finally, the m th row is represented by only one circuit, and still the pair $(\hat{x}_{m-K}, \hat{x}'_{m-K})$ is m -separated and has a common m -cut, a contradiction.

Proof in case (2). As in the proof of [2], we show that $(\hat{X}_f, \sigma)$ becomes an odometer. For arbitrarily large n , there exists an $m(n) > n$ such that every pair $(\hat{x}_n, \hat{x}'_n)$ of depth n has no common $m(n)$ -cut. As described briefly in [2], by telescoping, we wish to show that the condition of (2) holds for every n . Fix n . Take an $n' > m(n)$ such that for every pair $(\hat{x}_{n'}, \hat{x}'_{n'})$ of depth n' has no common $m(n')$ -cut. Then, because $n' > m(n)$, every pair $(\hat{x}_n, \hat{x}'_n)$ of depth n has no common n' -cut. Thus, by telescoping (from n' to n), every pair $(\hat{x}_n, \hat{x}'_n)$ of depth n has no common $(n+1)$ -cut. Thus, by consecutive application of this telescoping, we obtain a covering such that for every n , every pair $(\hat{x}_n, \hat{x}'_n)$ of depth n has no common $(n+1)$ -cut.

LEMMA 6.2. *Let $(\hat{x}_n, \hat{x}'_n)$ be a pair of depth n that has no common $(n+1)$ -cut. Let $\hat{y} \in \hat{X}_f$. Then, there exists a $\hat{y}' \in \hat{X}_f$ such that $\hat{y}[n] = \hat{y}'[n]$ and the pair $(\hat{y}, \hat{y}')$ has no common $(n+1)$ -cut. Especially, $(\hat{y}, \hat{y}')$ has depth n .*

Proof. By minimality of $(\hat{X}_f, \sigma)$, there exists a sequence $n_1 < n_2 < \dots$ such that $\lim_{k \rightarrow \infty} \sigma^{n_k}(\hat{x}_n) = \hat{y}$. Taking a subsequence if necessary, we also obtain $\lim_{k \rightarrow \infty} \sigma^{n_k}(\hat{x}'_n) = \hat{y}'$ for some $\hat{y}' \in \hat{X}_f$. Take an $N > n$ arbitrarily. We obtain $\pi_N(\sigma^{n_k}(\hat{x}_n)) = \sigma^{n_k}((\hat{x}_n[N])) \rightarrow \hat{y}[N]$ and $\pi_N(\sigma^{n_k}(\hat{x}'_n)) = \sigma^{n_k}((\hat{x}'_n[N])) \rightarrow \hat{y}'[N]$ as $k \rightarrow \infty$. Because $\sigma^k(\hat{x})[n] = \sigma^k(\hat{x}')[n]$ for all $k \in \mathbb{Z}$, we obtain $\hat{y}[n] = \hat{y}'[n]$. Note that this does not directly imply that the depth of the pair $(\hat{y}, \hat{y}')$ is n . Nevertheless, by the same reasoning, it follows that $(\hat{y}, \hat{y}')$ has no common $(n+1)$ -cut. Therefore, $(\hat{y}, \hat{y}')$ is $(n+1)$ -separated, and $(\hat{y}, \hat{y}')$ has depth n . $\square$

Fix some $n_0 \geq 1$ and let $m = n_0 + K^{K+1}$. For $n \in [n_0, m-1]$, fix a pair $(\hat{x}_n, \hat{x}'_n)$ of depth n . Fix $\hat{y}_0 \in X$. By Lemma 6.2, we can take pairs $(\hat{y}_0, \hat{y}_n)$ with $n \in [n_0, m-1]$ such that $\hat{y}_0[n] = \hat{y}_n[n]$ and has no common $(n+1)$ -cut. Then, for any $n, n' \in [i_0, m-1]$, $n' > n$, the pair $(\hat{y}_n, \hat{y}_{n'})$ has depth n and has no common $(n+1)$ -cut. In this way, we have found $K^{K+1} + 1$ number of n_0 -compatible and pairwise m -separated elements $\hat{y}_0, \hat{y}_{i_0}, \hat{y}_{i_0+1}, \dots, \hat{y}_{m-1}$ with no common m -cuts.

The next lemma is said to be the infection lemma in [2], with which we can show that X_{n_0} is periodic. Then, because n_0 is arbitrary, we can conclude that $(\hat{X}_f, \sigma)$ is an odometer system.

LEMMA 6.3. (Downarowicz and Maass [2]) *If there exist at least $K^{K+1} + 1$ n -compatible points $\hat{y}_k$ ($k \in [1, K^{K+1} + 1]$), which, for some $m > n$, are pairwise m -separated with no common m -cuts, then X_n is periodic.*

Proof. As in the proof of [2], denote by $\hat{y}$ the common image of the points $\hat{y}_k$ in X_n . Think of the sequences $\hat{y}$ and $\hat{y}_k[m]$. Let us fix a position k_0 . Then, for at least $K^K + 1$ times, the lines $\hat{y}_k[m]$ overlap with the same m -symbol. This m -symbol is denoted by c . Because, c overlaps at least twice, we can find a periodicity in $c[n]$, i.e., there exists an $l > 0$ such that $(c[n])(i+l) = (c[n])(i)$ for $0 \leq i < l(c) - l$. Let l_c be the least such number. Thus, we have the periodicity in $\hat{y}$ around k_0 of length at least $l(c)$. Suppose that this periodicity continues to the right end. Then, $\hat{y}$ is eventually periodic, and by minimality, X_n turns out to be a periodic orbit as required. Therefore, suppose that this periodicity is finitely limited and broken for the first time at k_1 . We have chosen at least $K^K + 1$ lines already, and we get rid of the rest. These overlap with the same m -symbol c at the position k_0 . At the position k_1 , one can obtain at least $K^{K-1} + 1$ lines $y_k[m]$ with the same m -symbol c' overlapping at k_1 . Suppose that $c' = c$. Then, because at least two lines overlap, at least one line has slid left more than l_c from the position k_1 . This shows that $\hat{y}(k_1 - l_c) = \hat{y}(k_1)$, contradicting the limited periodicity. Thus, suppose that $c' \neq c$. We have the two cases: (a) $l_{c'} \geq l_c$, and (b) $l_{c'} < l_c$.

Suppose that case (a) occurs. The lines overlapping at k_1 have the same symbol c over k_0 . Thus, the left end of c' that occurs over k_1 , does not exceed k_0 . Now, because we have at least three lines overlapping at k_1 , we have two c' that have slid left from k_1 . Therefore, the interval $[k_1 - 2l_{c'} - l_c, k_1 - 1]$ has periodicity l_c . Furthermore, for $0 < s \leq l_c$, we obtain $\hat{y}(k_1 - s) = \hat{y}(k_1 - s - l_{c'})$. If $l_{c'} = l_c$, then the overlapping of c' on k_1 implies the periodicity $\hat{y}(k_1 - l_c) = \hat{y}(k_1)$. This contradicts the limited periodicity. Therefore, we suppose that $l_{c'} > l_c$. Then, $\hat{y}(k_1 - l_{c'} + al_c)$ with $a = 0, 1, 2, \dots$ have the same symbol with $\hat{y}(k_1)$

while $-l_{c'} + al_c < 0$. Because $l_{c'} > l_c$, there exists an $0 < s_1 \leq l_c$ and an $a_1 > 0$ such that $k_1 - s_1 = k_1 - l_{c'} + a_1 l_c$, and we obtain $\hat{y}(k_1 - s_1) = \hat{y}(k_1 - l_{c'} + a_1 l_c) = \hat{y}(k_1)$. It follows that $k_1 - s_1 - l_{c'} \in [k_1 - 2l_{c'} - l_c, k_1 - 1]$. Then, we obtain $\hat{y}(k_1 - s_1) = \hat{y}(k_1 - s_1 - l_{c'}) = \hat{y}(k_1 - 2l_{c'} + a_1 l_c)$. Again, we obtain $\hat{y}(k_1 - s_2) = \hat{y}(k_1 - s_1 - l_{c'}) = \hat{y}(k_1 - 2l_{c'} + a_2 l_c) = \hat{y}(k_1)$ for some $0 < s_2 \leq l_c$ and $a_2 > 0$. In this way, we obtain a sequence $0 < s_i \leq l_c$ and $0 < a_i$ with $1 \leq i < \infty$ such that $\hat{y}(k_1 - s_i) = \hat{y}(k_1 - il_{c'} + a_i l_c) = \hat{y}(k_1)$ for all $1 \leq i < \infty$. Because $\gcd(l_c, l_{c'}) = s_i$ for some $i > 0$, we also obtain $l_c = s_j$ for some $j > 0$. Therefore, we have $\hat{y}(k_1 - l_c) = \hat{y}(k_1)$, contradicting the limited periodicity. Therefore, case (a) implies a contradiction.

Suppose that case (b) occurs. Then, we restart the argument from the position k_1 , and continue in the same way with $K^{K-1} + 1$ lines. If the periodicity is right limited, we shall encounter only case (b). Further, because the period of the periodicity decreases by condition (b), the right end is covered with an m -symbol that has not appeared previously. At the last step, there exists only one symbol exists, and there remains at least $K + 1$ lines. Therefore, case (a) must happen, the final contradiction.

We have proved that the right end is not limited. This concludes the proof. $\square$

Because the infection lemma is shown, then Theorem 6.1 is proved. $\square$

Remark 6.4. If the natural extension of a Cantor minimal system of finite topological rank has finite topological rank, then our main result becomes a waste of effort. Nevertheless, we could not answer this question.

In the rest of this section, we shall show a theorem that gives a partial affirmative answer.

THEOREM 6.5. *Let (X, f) be a Cantor minimal system with topological rank $1 \leq K \leq \infty$. The natural extension of (X, f) is denoted by $(\hat{X}_f, \sigma)$. Then, there exists a Cantor minimal system of Kakutani–Rohlin type (X', f') with rank $\leq K$, such that the natural extension $(\hat{X}'_{f'}, \sigma)$ is naturally topologically conjugate to $(\hat{X}_f, \sigma)$. Specifically, as a Cantor minimal system of Kakutani–Rohlin type, (X', f') is one-to-one except one point $x_0 \in X'$ with which $\#f'^{-1}(x_0) \leq K$.*

Proof. Let $\mathcal{G} : G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ be a simple GM-covering of rank $1 \leq K \leq \infty$ such that the inverse limit is topologically conjugate to (X, f) . As a GM-covering, every G_n ($n > 0$) consists of circuits $\{c_{n,1}, c_{n,2}, \dots, c_{n,l_n}\}$ such that the central vertex is $v_{n,0}$. Then, as in Notation 5.6, we can define its natural extension $(\hat{X}_f, \sigma)$, the array system and the linked array system. For each $n > 0$, we shall construct another directed graph G'_n and cover maps $\varphi'_n : G'_{n+1} \rightarrow G'_n$. Let $c_{n,i}$ ($1 \leq i \leq l_n$) be circuits of G_n and let $v_{n,0}$ be its central vertex. Then, as in the remark after the definition of GM-covering, we can write

$$\varphi_n(c_{n+1,i}) = c_{n,a(n,i,1)} c_{n,a(n,i,2)} \cdots c_{n,a(n,i,k(n,i))}$$

for each $n \geq 0$ and each $1 \leq i \leq l_{n+1}$, where $a(n, i, 1) = 1$ for all $1 \leq i \leq l_{n+1}$. Then, we can write $c_{n,i} = (v_{n,0} = v_{n,i,0}, v_{n,i,1}, v_{n,i,2}, \dots, v_{n,i,l(n,i)} = v_{n,0})$. It may happen that $v_{n,i,j} = v_{n,i',j'}$ for some $i \neq i'$, $1 \leq j < l(n, i)$ and $1 \leq j' < l(n, i')$. Nevertheless, for the construction of G'_n , we make distinct vertices $v'_{n,i,j}$ and $v'_{n,i',j'}$. Therefore, we make use of $1 + \sum_{1 \leq i \leq l_n} (l(n, i) - 1)$ vertices

$$\{v'_{n,0}\} \cup \{v'_{n,i,j} \mid 1 \leq i \leq l_n, 1 \leq j < l(n, i)\}.$$

As in G_n , we assume that $v'_{n,0} = v'_{n,i,0} = v'_{n,i,l(n,i)}$. The set of edges is defined as

$$\{(v'_{n,i,j}, v'_{n,i,j+1}) \mid 1 \leq i \leq l_n \text{ and } 0 \leq j < l(n, i)\}.$$

Thus, we have constructed a new directed graph G'_n that consists of circuits $c'_{n,i}$ corresponding to $c_{n,i}$ for each $1 \leq i \leq l_n$. The graph homomorphisms $\varphi'_n : G'_{n+1} \rightarrow G'_n$ are defined as:

- $\varphi'_n(v'_{n+1,0}) = v'_{n,0}$;
- $\varphi'_n(c'_{n+1,i}) = c'_{n,a(n,i,1)} c'_{n,a(n,i,2)} \cdots c'_{n,a(n,i,k(n,i))}$ for each $1 \leq i \leq l_{n+1}$.

Because $a(n, i, 1) = 1$ for all $1 \leq i \leq l_n$, it is evident that the sequence $\mathcal{G}' : G'_0 \xleftarrow{\varphi'_0} G'_1 \xleftarrow{\varphi'_1} G'_2 \xleftarrow{\varphi'_2} \cdots$ is a graph covering. From the construction, this covering is a KR-covering. We can obtain its inverse limit $G'_\infty = (X', f')$. It is evident that (X', f') has rank $\leq K$. It is also obvious that (X', f') has a natural factor map onto (X, f) . Then, as in Notation 5.6, we can define its natural extension $(\hat{X}'_{f'}, \sigma)$, the array system and the linked array system. Note that the linked array systems of $\mathcal{G}$ and $\mathcal{G}'$ coincide. As has been shown, the linked array systems are topologically conjugate to the natural extensions. Thus, it follows that $(\hat{X}_f, \sigma)$ and $(\hat{X}'_{f'}, \sigma)$ are naturally topologically conjugate. $\square$

COROLLARY 6.6. *For a Cantor minimal homeomorphism, the topological rank that is defined by Bratteli diagrams and the topological rank that is defined by GM-coverings coincide.*

Proof. Let (X, f) be a Cantor minimal homeomorphism such that the topological rank with regard to the Bratteli diagram representation is $1 \leq K \leq \infty$. Then, there exists a properly ordered Bratteli diagram with rank K . Then, as described in Remark 4.3, (X, f) is an inverse limit of a KR-covering with rank K . It is evident that a KR-covering of rank K is a GM-covering of rank K . Therefore, (X, f) has a topological rank $\leq K$ with regard to GM-coverings. On the other hand, suppose that there exists a GM-covering with rank $1 \leq K' \leq \infty$ such that the inverse limit is topologically conjugate to (X, f) . Then, there exists a KR-covering with rank K' such that the inverse limit is topological conjugate to the natural extension of (X, f) , i.e., (X, f) itself. By telescoping, this KR-covering becomes bidirectional. Therefore, the corresponding ordered Bratteli diagram has a unique maximal infinite path. The uniqueness of minimal infinite path is evident. The simplicity of the Bratteli diagram comes from the minimality of the Cantor system. Therefore, this Bratteli diagram is properly ordered and has rank $\leq K'$. This concludes the coincidence of two definitions of the topological rank. $\square$

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