REDUCIBILITY OF THE SYSTEMS OF DIFFERENTIAL EQUATIONS SATISFIED BY APPELL'S F_2 , F_3 AND F_4

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Abstract. By using the contiguity operators, we give a simple and elementary derivation of sufficient conditions that Appell's hypergeometric systems for F_2 , F_3 and F_4 are reducible.

0. Introduction

We denote by E_2 , E_3 and E_4 the systems of differential equations satisfied by Appell's hypergeometric functions F_2 , F_3 and F_4 , respectively. A necessary and sufficient condition for each of the systems E_2 , E_3 and E_4 being irreducible is obtained by Bod [Bo] and Kato [Ka1, Ka2]. To obtain the conditions, Bod applies the result by Beukers for the A -hypergeometric systems [Be], and Kato uses his connection relations among several kinds of solutions of E_2 and E_4 .

In this paper we give a simple and elementary derivation of the conditions required for the systems E_2 , E_3 and E_4 to be reducible. Maps given by the contiguity operators play crucial roles. The idea comes from the work by Beukers and Heckman [BH] studying the generalized hypergeometric function ${}_{n+1}F_n$ and from the work by Sasaki [Sa] studying the contiguity operators related with F_3 . The present argument might be a clue to the study of differential equations and their solutions in reducible cases, which have not been understood well. See also our work [MS] studying the reducibility of the system $E(k, n)$ by using the contiguity operators.

1. Appell's hypergeometric functions F_2 , F_3 and F_4

Appell's hypergeometric functions F_2 , F_3 and F_4 are the analytic continuations of Appell's hypergeometric series

$$\begin{aligned} F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) &= \sum_{m,n \geq 0} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_m(\gamma')_n} x^m y^n, \quad |x| + |y| < 1, \\ F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) &= \sum_{m,n \geq 0} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}} x^m y^n, \quad |x| < 1, |y| < 1, \\ F_4(\alpha, \beta, \gamma, \gamma'; x, y) &= \sum_{m,n \geq 0} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n m!n!} x^m y^n, \quad \sqrt{|x|} + \sqrt{|y|} < 1, \end{aligned}$$

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where $(\alpha)_n = a(a+1) \cdots (a+n-1)$. The functions F_2 , F_3 and F_4 satisfy the following systems E_2 , E_3 and E_4 , respectively:

$$\begin{aligned} (E_2) \quad & \begin{cases} [(\theta_x + \gamma - 1)\theta_x - x(\theta_x + \beta)(\theta_x + \theta_y + \alpha)]F = 0, \\ [(\theta_y + \gamma' - 1)\theta_y - y(\theta_y + \beta')(\theta_x + \theta_y + \alpha)]F = 0, \end{cases} \\ (E_3) \quad & \begin{cases} [x(\theta_x + \alpha)(\theta_x + \beta) - \theta_x(\theta_x + \theta_y + \gamma - 1)]F = 0, \\ [y(\theta_y + \alpha')(\theta_y + \beta') - \theta_y(\theta_x + \theta_y + \gamma - 1)]F = 0, \end{cases} \\ (E_4) \quad & \begin{cases} [\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta)]F = 0, \\ [\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta)]F = 0, \end{cases} \end{aligned}$$

where $\theta_x = x\partial/\partial x$ and $\theta_y = y\partial/\partial y$. All of these systems have rank 4. On the other hand, the necessary and sufficient conditions for the systems E_2 , E_3 and E_4 being irreducible are the following [Bo, Ka1, Ka2, Mi]:

$$(E_2) \quad \alpha, \beta, \beta', \gamma - \beta, \gamma' - \beta', \gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha \notin \mathbb{Z}, \quad (1.1)$$

$$(E_3) \quad \alpha, \alpha', \beta, \beta', \gamma - \beta - \beta', \gamma - \alpha - \beta', \gamma - \alpha' - \beta, \gamma - \alpha - \alpha' \notin \mathbb{Z}, \quad (1.2)$$

$$(E_4) \quad \alpha, \beta, \gamma - \alpha, \gamma' - \alpha, \gamma - \beta, \gamma' - \beta, \gamma + \gamma' - \alpha, \gamma + \gamma' - \beta \notin \mathbb{Z}. \quad (1.3)$$

We write the systems E_2 , E_3 and E_4 as $E_2(\alpha, \beta, \beta', \gamma, \gamma')$, $E_3(\alpha, \alpha', \beta, \beta', \gamma)$ and $E_4(\alpha, \beta, \gamma, \gamma')$, if we need to express the parameters explicitly. We also denote by $S_2(\alpha, \beta, \beta', \gamma, \gamma')$, $S_3(\alpha, \alpha', \beta, \beta', \gamma)$ and $S_4(\alpha, \beta, \gamma, \gamma')$ the local solution spaces of $E_2(\alpha, \beta, \beta', \gamma, \gamma')$, $E_3(\alpha, \alpha', \beta, \beta', \gamma)$ and $E_4(\alpha, \beta, \gamma, \gamma')$ around a generic point.

2. Reducibility of E_2 and E_3

We give a proof of the following.

THEOREM 2.1. (1) *If at least one of*

$$\alpha, \beta, \beta', \gamma - \beta, \gamma' - \beta', \gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha$$

is an integer, then the monodromy representation of the system E_2 is reducible.

(2) *If at least one of*

$$\alpha, \alpha', \beta, \beta', \gamma - \beta - \beta', \gamma - \alpha - \beta', \gamma - \alpha' - \beta, \gamma - \alpha - \alpha'$$

is an integer, then the monodromy representation of the system E_3 is reducible.

The following relation between E_2 and E_3 is classically known. We refer to [AKdF, p. 51].

LEMMA 2.2. *The function $u(x, y)$ is a solution of $E_3(\alpha, \alpha', \beta, \beta', \gamma)$, if and only if the function $v(x, y) = x^{-\beta}y^{-\beta'}u(1/x, 1/y)$ is a solution of the system $E_2(\beta + \beta' + 1 - \gamma, \beta, \beta', \beta + 1 - \alpha, \beta' + 1 - \alpha')$.*

Lemma 2.2 implies the equivalence between the conditions (1.1) and (1.2). Hence in what follows we consider only the E_2 case.

We now introduce two operators:

$$K(\alpha) = \alpha + \theta_x + \theta_y, \quad L(\beta) = \beta + \theta_x.$$

As operators, since $\theta_x x^\lambda = x^\lambda(\theta_x + \lambda)$, we have

$$\begin{aligned} K(\alpha)\{\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha)(\theta_x + \beta)\} \\ = \{\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha + 1)(\theta_x + \beta)\}K(\alpha), \\ K(\alpha)\{\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha)(\theta_y + \beta')\} \\ = \{\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha + 1)(\theta_y + \beta')\}K(\alpha), \end{aligned}$$

and

$$\begin{aligned} L(\beta)\{\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha)(\theta_x + \beta)\} \\ = \{\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha)(\theta_x + \beta + 1)\}L(\beta), \\ L(\beta)\{\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha)(\theta_y + \beta')\} \\ = \{\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha)(\theta_y + \beta')\}L(\beta), \end{aligned}$$

which imply

$$K(\alpha) : S_2(\alpha, \beta, \beta', \gamma, \gamma') \longrightarrow S_2(\alpha + 1, \beta, \beta', \gamma, \gamma')$$

and

$$L(\beta) : S_2(\alpha, \beta, \beta', \gamma, \gamma') \longrightarrow S_2(\alpha, \beta + 1, \beta', \gamma, \gamma').$$

We also note that

$$\begin{aligned} K(\alpha)F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) &= \alpha F_2(\alpha + 1, \beta, \beta', \gamma, \gamma'; x, y), \\ L(\beta)F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) &= \beta F_2(\alpha, \beta + 1, \beta', \gamma, \gamma'; x, y). \end{aligned}$$

The differential operators $K(\alpha)$ and $L(\beta)$ map the function F_2 to the contiguous functions up to constant multiples. Hence it is adequate to call $K(\alpha)$ and $L(\beta)$ *contiguity operators*.

We have the following lemma.

LEMMA 2.3. *The kernel of the map $K(\alpha)$ is a non-trivial proper subspace if and only if $\alpha = 0$ or $\alpha = \gamma - 1$ or $\alpha = \gamma' - 1$ or $\alpha = \gamma + \gamma' - 2$. In these cases, the dimension of $\ker K(\alpha)$ is one or two.*

Proof. If $K(\alpha)u = 0$, the system E_2 reduces to

$$\begin{cases} \theta_x(\theta_x + \gamma - 1)u = 0, \\ \theta_y(\theta_y + \gamma' - 1)u = 0. \end{cases} \quad (2.1)$$

We first give a proof of sufficiency by dividing it into several cases.

(a) *Case $\gamma - 1 \neq 0$, $\gamma' - 1 \neq 0$.* It is seen that

$$u = c_1 + c_2 x^{1-\gamma} + c_3 y^{1-\gamma'} + c_4 x^{1-\gamma} y^{1-\gamma'},$$

where c_i are constants, is a general solution of the system and that

$$\begin{aligned} K(\alpha)u &= c_1 \alpha + c_2(\alpha + 1 - \gamma)x^{1-\gamma} + c_3(\alpha + 1 - \gamma')y^{1-\gamma'} \\ &\quad + c_4(\alpha + 2 - \gamma - \gamma')x^{1-\gamma}y^{1-\gamma'}. \end{aligned}$$

(a-i) When $\alpha = 0$, we have

$$K(\alpha)u = c_2(1 - \gamma)x^{1-\gamma} + c_3(1 - \gamma')y^{1-\gamma'} + c_4(2 - \gamma - \gamma')x^{1-\gamma}y^{1-\gamma'}.$$

Hence $K(\alpha)u = 0$ if and only if $c_2 = c_3 = c_4(2 - \gamma - \gamma') = 0$. It means that the dimension of $\ker K(\alpha)$ is one or two.

(a-ii) When $\alpha = \gamma - 1$, we have

$$K(\alpha)u = c_1(\gamma - 1) + c_3(\gamma - \gamma')y^{1-\gamma'} + c_4(1 - \gamma')x^{1-\gamma}y^{1-\gamma'}.$$

Hence $K(\alpha)u = 0$ if and only if $c_1 = c_4 = c_3(\gamma - \gamma') = 0$. It means that the dimension of $\ker K(\alpha)$ is one or two.

(a-iii) When $\alpha = \gamma' - 1$, by the change of the role of the parameters in the previous subcase (a-ii), it is seen that $K(\alpha)u = 0$ if and only if $c_1 = c_4 = c_2(\gamma' - \gamma) = 0$. It means that the dimension of $\ker K(\alpha)$ is one or two.

(a-iv) When $\alpha = \gamma + \gamma' - 2$, we have

$$K(\alpha)u = c_1(\gamma + \gamma' - 2) + c_2(\gamma' - 1)x^{1-\gamma} + c_3(\gamma - 1)y^{1-\gamma'}.$$

Hence $K(\alpha)u = 0$ if and only if $c_1(\gamma + \gamma' - 2) = c_2 = c_3 = 0$. It means that the dimension of $\ker K(\alpha)$ is one or two.

(b) Case $\gamma - 1 = 0$, $\gamma' - 1 \neq 0$. The general solution is written as

$$u = c_1 + c_2 \log x + c_3 y^{1-\gamma'} + c_4 (\log x) y^{1-\gamma'},$$

and we have

$$\begin{aligned} K(\alpha)u &= (c_1\alpha + c_2) + c_2\alpha \log x + (c_3(\alpha + 1 - \gamma') + c_4)y^{1-\gamma'} \\ &\quad + c_4(\alpha + 1 - \gamma')(\log x)y^{1-\gamma'}. \end{aligned}$$

(b-i) When $\alpha = 0$, we have

$$K(\alpha)u = c_2 + (c_3(1 - \gamma') + c_4)y^{1-\gamma'} + c_4(1 - \gamma')(\log x)y^{1-\gamma'}.$$

Hence $K(\alpha)u = 0$ if and only if $c_2 = c_3 = c_4 = 0$. It means that the dimension of $\ker K(\alpha)$ is one.

(b-ii) When $\alpha = \gamma - 1$, we have $\alpha = 0$ since $\gamma - 1 = 0$. This case is reduced to the previous subcase (b-i). Hence the dimension of $\ker K(\alpha)$ is one.

(b-iii) When $\alpha = \gamma' - 1$, we have

$$K(\alpha)u = (c_1(\gamma' - 1) + c_2) + c_2(\gamma' - 1) \log x + c_4 y^{1-\gamma'}.$$

Hence $K(\alpha)u = 0$ if and only if $c_1 = c_2 = c_4 = 0$. It means that the dimension of $\ker K(\alpha)$ is one.

(b-iv) When $\alpha = \gamma + \gamma' - 2$, we have $\alpha = \gamma' - 1$ since $\gamma = 1$. This subcase reduces to the previous subcase (b-iii).

(c) Case $\gamma - 1 \neq 0$, $\gamma' - 1 = 0$. The assertion in this case is obtained from the previous case (b) by the change of the roles of γ and γ' .

(d) Case $\gamma - 1 = 0$, $\gamma' - 1 = 0$. The general solution is written as

$$u = c_1 + c_2 \log x + c_3 \log y + c_4(\log x)(\log y),$$

and we have

$$K(\alpha)u = (c_1\alpha + c_2 + c_3) + (c_2\alpha + c_4) \log x + (c_3\alpha + c_4) \log y + c_4\alpha(\log x)(\log y).$$

(d-i) When $\alpha = 0$, we have

$$K(\alpha)u = (c_2 + c_3) + c_4 \log x + c_4 \log y.$$

Hence $K(\alpha)u = 0$ if and only if $c_2 + c_3 = c_4 = 0$. It means that the dimension of $\ker K(\alpha)$ is two.

(d-ii) When $\alpha = \gamma - 1$ or $\gamma' - 1$ or $\gamma + \gamma' - 2$, the subcase reduces to the previous subcase (d-i).

This completes the proof of the sufficiency. We next give a proof of the necessity.

Using the equality $K(\alpha)u = (\theta_x + \theta_y + \alpha)u = 0$, we rewrite the first equation of (2.1) as

$$(\theta_y + \alpha)(\theta_y + \alpha + 1 - \gamma)u = 0. \quad (2.2)$$

Subtraction of the second equation of (2.1) from (2.2) leads to

$$[(2\alpha + 2 - \gamma - \gamma')\theta_y + \alpha(\alpha + 1 - \gamma)]u = 0. \quad (2.3)$$

We use equation (2.3) to express the second equation of (2.1) as

$$\alpha(\alpha + 1 - \gamma)(\alpha + 1 - \gamma')(\alpha + 2 - \gamma - \gamma')u = 0.$$

Hence we need

$$\alpha(\alpha + 1 - \gamma)(\alpha + 1 - \gamma')(\alpha + 2 - \gamma - \gamma') = 0$$

to have a non-zero solution u . This completes the proof. $\square$

LEMMA 2.4. *The kernel of the map $L(\beta)$ is a non-trivial proper subspace if and only if $\beta = 0$ or $\beta = \gamma - 1$. In these cases, the dimension of $\ker L(\beta)$ is two.*

Proof. The equality $L(\beta)u(x, y) = 0$ implies that $u(x, y)$ must be $x^{-\beta}f(y)$ up to a constant multiple, and that the system E_2 reduces to

$$\begin{cases} -\beta(-\beta + \gamma - 1)f(y) = 0, \\ [\theta_y(\theta_y + \gamma' - 1) - y(\theta_y - \beta + \alpha)(\theta_y + \beta')]f(y) = 0. \end{cases} \quad (2.4)$$

Note that the second equation is the differential equation satisfied by the Gauss hypergeometric function ${}_2F_1(\alpha - \beta, \beta', \gamma'; y)$. From the first equation of (2.4), we need $\beta(-\beta + \gamma - 1) = 0$ for a non-trivial function $f(y)$. On the other hand, if $\beta = 0$ or $\beta = \gamma - 1$, the second equation turns out to be $[\theta_y(\theta_y + \gamma' - 1) - y(\theta_y + \alpha)(\theta_y + \beta')]f(y) = 0$ or $[\theta_y(\theta_y + \gamma' - 1) - y(\theta_y + 1 - \gamma + \alpha)(\theta_y + \beta')]f(y) = 0$, which is the equation satisfied by ${}_2F_1(\alpha, \beta', \gamma'; y)$ or ${}_2F_1(\alpha + 1 - \gamma, \beta', \gamma'; y)$. This completes the proof. $\square$

Proof of Theorem 2.1. We first prove the case when α is an integer. For brevity, we denote $S_2(\alpha, \beta, \beta', \gamma, \gamma')$ by $S(\alpha, *)$.

If $\alpha = 0$, then we have a proper non-trivial subspace $V_0 = \ker(K(0))$ by Lemma 2.3.

If $\alpha > 0$, then we consider the sequence

$$S(0, *) \xrightarrow{K(0)} S(1, *) \xrightarrow{K(1)} \cdots \xrightarrow{K(\alpha-1)} S(\alpha, *).$$

Note that $K(0)$ has a proper non-trivial kernel. Choose $j \in \{0, 1, \dots, \alpha - 1\}$ maximal such that $K(j)$ has a proper non-trivial kernel. Then the image of the map $K(\alpha - 1) \cdots K(j)$ is a proper non-trivial invariant subspace in $S(\alpha, *)$.

If $\alpha < 0$, then we consider the sequence

$$S(\alpha, *) \xrightarrow{K(\alpha)} S(\alpha + 1, *) \xrightarrow{K(\alpha+1)} \cdots \xrightarrow{K(-1)} S(0, *) \xrightarrow{K(0)} S(1, *).$$

Note that $K(0)$ has a proper non-trivial kernel. Choose $j \in \{\alpha, \alpha + 1, \dots, -1, 0\}$ minimal such that $K(j)$ has a proper non-trivial kernel. Then the kernel of the map $K(j) \cdots K(\alpha)$ is a proper non-trivial invariant subspace in $S(\alpha, *)$. This completes the proof of the case when α is an integer.

The case when $\alpha - \gamma$ is an integer, the case when $\alpha - \gamma'$ is an integer, and the case when $\alpha - \gamma - \gamma'$ is an integer follow from Lemma 2.3 and the same argument as above. Moreover, the case when β is an integer and the case when $\beta - \gamma$ is an integer follow from Lemma 2.4 and the same argument for $L(\beta)$ as the argument for $K(\alpha)$ with α being an integer. At the final step, we note that a change of variables and parameters such as $(\beta, \beta', \gamma, \gamma', x, y) \leftrightarrow (\beta', \beta, \gamma', \gamma, y, x)$ leaves E_2 invariant. It implies that the case when β' is an integer and the case when $\beta' - \gamma'$ is an integer follow from the case when β is an integer and the case when $\beta - \gamma$ is an integer. This completes the proof of Theorem 2.1. $\square$

3. Reducibility of E_4

In this section, we give a proof of the following theorem.

THEOREM 3.1. *If at least one of*

$$\alpha, \beta, \gamma - \alpha, \gamma' - \alpha, \gamma - \beta, \gamma' - \beta, \gamma + \gamma' - \alpha, \gamma + \gamma' - \beta$$

is an integer, then the monodromy representation of the system E_4 is reducible.

Proof. We consider the contiguity operator

$$K(\alpha) = \alpha + \theta_x + \theta_y,$$

as in the previous section. As operators, we have

$$\begin{aligned} & K(\alpha)\{\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta)\} \\ &= \{\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha + 1)(\theta_x + \beta)\}K(\alpha), \\ & K(\alpha)\{\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta)\} \\ &= \{\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha + 1)(\theta_x + \theta_y + \beta)\}K(\alpha), \end{aligned}$$

and

$$\begin{aligned} & K(\beta)\{\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta)\} \\ &= \{\theta_x(\theta_x + \gamma - 1) - x(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta + 1)\}K(\beta), \\ & K(\beta)\{\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta)\} \\ &= \{\theta_y(\theta_y + \gamma' - 1) - y(\theta_x + \theta_y + \alpha)(\theta_x + \theta_y + \beta)\}K(\beta), \end{aligned}$$

which imply

$$K(\alpha) : S_4(\alpha, \beta, \gamma, \gamma') \longrightarrow S_4(\alpha + 1, \beta, \gamma, \gamma')$$

and

$$K(\beta) : S_4(\alpha, \beta, \gamma, \gamma') \longrightarrow S_4(\alpha, \beta + 1, \gamma, \gamma').$$

We also note that

$$\begin{aligned} K(\alpha)F_4(\alpha, \beta, \gamma, \gamma'; x, y) &= \alpha F_4(\alpha + 1, \beta, \gamma, \gamma'; x, y), \\ K(\beta)F_4(\alpha, \beta, \gamma, \gamma'; x, y) &= \beta F_4(\alpha, \beta + 1, \gamma, \gamma'; x, y). \end{aligned}$$

Let u satisfy the equation $K(\alpha)u = 0$ or $K(\beta)u = 0$. Then the system E_4 reduces to (2.1) in the proof of Lemma 2.3. Hence, Theorem 3.1 holds. This completes the proof. $\square$

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