A NOTE ON FINITE REAL
MULTIPLE ZETA VALUES

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(Received 14 October 2015)

Abstract. We prove three theorems on finite real multiple zeta values: the symmetric formula, the sum formula and the height-one duality theorem. These are analogues of their counterparts on finite multiple zeta values.

1. Main theorems

For positive integers \( k_1, k_2, \ldots, k_n \) with \( k_1 \geq 2 \), the multiple zeta value (MZV) and the multiple zeta star value (MZSV) are defined by

\[
\zeta(k_1, k_2, \ldots, k_n) := \sum_{m_1 > m_2 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},
\]

\[
\zeta^*(k_1, k_2, \ldots, k_n) := \sum_{m_1 \geq m_2 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.
\]

The finite real multiple zeta values (or symmetric multiple zeta values), which were first introduced by Kaneko and Zagier [9], are defined for any positive integers \( k_1, k_2, \ldots, k_n \) as follows:

\[
\zeta^\ast_F(k_1, k_2, \ldots, k_n) := \sum_{i=0}^{n} (-1)^{k_1+k_2+\cdots+k_i} \zeta^\ast(k_i, k_{i-1}, \ldots, k_1) \zeta^\ast(k_{i+1}, k_{i+2}, \ldots, k_n),
\]

\[
\zeta^\ast_X(k_1, k_2, \ldots, k_n) := \sum_{i=0}^{n} (-1)^{k_1+k_2+\cdots+k_i} \zeta^\ast(k_i, k_{i-1}, \ldots, k_1) \zeta^\ast_X(k_{i+1}, k_{i+2}, \ldots, k_n).
\]

Here, the symbols \( \zeta^\ast \) and \( \zeta^\ast_X \) on the right-hand sides stand for the regularized values coming from harmonic and shuffle regularizations respectively, i.e., real values obtained by taking constant terms of harmonic and shuffle regularizations as explained in [6]. In the sums, we understand \( \zeta^\ast(\emptyset) = \zeta^\ast_X(\emptyset) = 1 \).

Let \( \mathbb{Z} \) be the \( \mathbb{Q} \)-vector subspace of \( \mathbb{R} \) spanned by the MZVs. It is known that this is a \( \mathbb{Q} \)-algebra. In [9], Kaneko and Zagier proved that the difference \( \zeta^\ast_F(k_1, k_2, \ldots, k_n) - \zeta^\ast_X(k_1, k_2, \ldots, k_n) \) is in the principal ideal of \( \mathbb{Z} \) generated by \( \zeta(2) \) (or \( \pi^2 \)), in other words, that the congruence

\[
\zeta^\ast_F(k_1, k_2, \ldots, k_n) \equiv \zeta^\ast_X(k_1, k_2, \ldots, k_n) \pmod{\zeta(2)}
\]

2010 Mathematics Subject Classification: Primary 11M32; Secondary 05A19.

Keywords: finite real multiple zeta values.
holds in \( Z \). They then defined the finite real multiple zeta value (FRMZV) \( \zeta_F(k_1, k_2, \ldots, k_n) \) as an element in the quotient ring \( Z/\zeta(2) \) by
\[
\zeta_F(k_1, k_2, \ldots, k_n) := \zeta^*_F(k_1, k_2, \ldots, k_n) \pmod{\zeta(2)}.
\]
We also refer to the values \( \zeta^*_F(k_1, k_2, \ldots, k_n) \) and \( \zeta^{sh}_F(k_1, k_2, \ldots, k_n) \) as (harmonic and shuffle versions of) FRMZVs.

In this paper, we prove the following theorems.

**Theorem 1.1.** (Symmetric formula) Let \( (k_1, k_2, \ldots, k_n) \) be any index set \( (k_i \in \mathbb{N}) \) and let \( S_n \) be the symmetric group of degree \( n \). Then, we have
\[
\sum_{\sigma \in S_n} \zeta_F(k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(n)}) = 0 \quad \text{(in } Z/\zeta(2) \text{)}.
\]

**Theorem 1.2.** (Sum formula) Let \( (k_1, k_2, \ldots, k_n) \) be any index set \( (k_i \in \mathbb{N}) \). For positive integers \( k, n \) and \( i \) with \( 1 \leq i \leq n \leq k-1 \), we have
\[
\sum_{k_1+k_2+\cdots+k_n=k, k_i \geq 2} \zeta^*_F(k_1, k_2, \ldots, k_n) \equiv (-1)^{i-1} \left( \binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} \right) \zeta(k) \pmod{\zeta(2)},
\]
where the congruences are mod \( \zeta(2) \) in the \( \mathbb{Q} \)-algebra \( Z \).

**Theorem 1.3.** (Height-one duality theorem) For positive integers \( k \) and \( n \), we have the equality
\[
\zeta_F(k, 1, \ldots, 1) = \zeta_F(n, 1, \ldots, 1)
\]
in \( Z/\zeta(2) \).

2. **Proofs**

2.1. **Proof of Theorem 1.1**

Let \( k_1, k_2 \) and \( k \) be any index sets. We note that the FRMZVs \( \zeta^*_F(k_1, k_2, \ldots, k_n) \) satisfy the harmonic product rule:
\[
\zeta^*_F(k_1) \zeta^*_F(k_2) = \zeta^*_F(k_1 * k_2),
\]
where the right-hand side is a linear combination of \( \zeta^*_F(k) \) coming from the harmonic product in [4], e.g., \( \zeta^*_F(2 * 2) = 2\zeta^*_F(2, 2) + \zeta^*_F(4) \).

Hoffman’s theorem [5, Theorem 4.1] states that any symmetric sum
\[
\sum_{\sigma \in S_n} \zeta^*_F(k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(n)})
\]
is a polynomial in the Riemann zeta values \( \zeta(k) \). His proof only uses the harmonic product rule of MZVs, and hence applies to our \( \zeta^*_F(k) \). Therefore, we conclude in a similar manner as in [3, 5] that the symmetric sum above is a sum of products of \( \zeta^*_F(k) = (1 + (-1)^k)\zeta(k) \), which is 0 when \( k \) is odd and a multiple of \( \zeta(2) \) when \( k \) is even.
Remark. One can also prove Theorem 1.1 directly by using the definition. For example, we compute

\[
\sum_{\sigma \in S_3} \xi_{F}^{*}(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)})
\]

\[
= (1 + (-1)^{k_1})(1 + (-1)^{k_2})(1 + (-1)^{k_3}) \sum_{\sigma \in S_3} \xi_{F}^{*}(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)})
\]

\[
+ ((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3})(\xi(k_1 + k_2, k_3) + \xi^{*}(k, k_1 k_3))
\]

\[
+ ((-1)^{k_1} + (-1)^{k_3})(1 + (-1)^{k_2})(\xi(k_1 + k_3, k_2) + \xi^{*}(k, k_1 + k_3))
\]

\[
+ ((-1)^{k_2} + (-1)^{k_3})(1 + (-1)^{k_1})(\xi(k_2 + k_3, k_1) + \xi^{*}(k, k_1 + k_2 + k_3)).
\]

When the weight (= sum of the indices) \( k \) is odd, the coefficients \( (1 + (-1)^{k_1})(1 + (-1)^{k_2})(1 + (-1)^{k_3}) \) becomes 0 if at least one \( k_i \) is odd. When all \( k_i \) are even, then \( \sum_{\sigma \in S_3} \xi^{*}(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}) = \xi^{*}(k_1)\xi^{*}(k_2)\xi^{*}(k_3) - \xi(k_1 + k_2)\xi^{*}(k_3) - \xi(k_1 + k_3)\xi^{*}(k_2) - \xi(k_2 + k_3)\xi^{*}(k_1) + 2\xi(k_1 + k_2 + k_3) \) is 0 modulo \( \xi(2) \). As for the term \( ((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3})(\xi(k_1 + k_2, k_3) + \xi^{*}(k, k_1 k_3)) \), etc., if we write this as \( ((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3})(\xi(k_1 + k_2)\xi^{*}(k_3) - \xi(k_1 + k_2 + k_3)) \), we see that either \( ((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3}) = 0 \) or \( (\xi(k_1 + k_2)\xi^{*}(k_3) - \xi(k_1 + k_2 + k_3)) \) is a multiple of \( \xi(2) \).

2.2. Proof of Theorem 1.2

We can prove Theorem 1.2 in the same manner as in [11]. Set

\[
S_{k,n,i} := \sum_{k_1 + k_2 + \cdots + k_n = k \atop k_i \geq 2} \xi_{F}^{*}(k_1, k_2, \ldots, k_n).
\]

We notice that the harmonic version of the FRMZVs satisfy the harmonic product rule. Thus, \( S_{k,n,i} \) enjoy the recursion relation in the following lemma, which can be proved in the same way as in [11, Proposition 2.2].

**Lemma 2.1.** For positive integers \( k, n \) and \( i \) with \( 2 \leq i + 1 \leq n \leq k - 1 \), we have

\[
(n - i)S_{k,n,i} + iS_{k,n,i+1} + (k - n)S_{k,n-1,i} = 0.
\]

We prove Theorem 1.2 by backward induction on \( n \). To do this, we need the initial value.

**Lemma 2.2.** For positive integers \( k \) and \( i \) with \( 1 \leq i \leq k - 1 \), we have

\[
S_{k,k-1,i} \equiv (-1)^{i-1} \binom{k}{i} \frac{\zeta(k)}{2} \pmod{\zeta(2)}.
\]
Proof. Since \( S_{k,k-1,i} = \xi^g_X(1, \ldots, 1, 2, 1, \ldots, 1) \), we compute \( \xi^g_F(1, \ldots, 1, 2, 1, \ldots, 1) \) instead. Because of the fact that \( \xi^g_F(1, \ldots, 1) = 0 \), we have by definition that

\[
S_{k,k-1,i} \equiv \xi^g_F(1, \ldots, 1, 2, 1, \ldots, 1) \pmod{\xi(2)}
\]

\[
= \xi^g(1, \ldots, 1, 2, 1, \ldots, 1) + (-1)^k \xi^g(1, \ldots, 1, 2, 1, \ldots, 1).
\]

By using [6, equation (5.2)] for \( w_0 = x y^j \), we have \( \xi^g(1, \ldots, 1, 2, 1, \ldots, 1) = (-1)^m \binom{m+l}{l} \xi(2, 1, \ldots, 1) \). Thus,

\[
S_{k,k-1,i} \equiv (-1)^{i-1} \left( \binom{k-1}{i-1} + \binom{k-1}{i} \right) \xi(2, 1, \ldots, 1) \pmod{\xi(2)}.
\]

Let us consider the case \( n = k-1 \) of Theorem 1.2. If \( k \) is even, the identity holds from Lemma 2.2. If \( k \) is odd, then \( n \) is even and the identity again follows because

\[
\binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} = \binom{k-1}{i-1} + \binom{k-1}{i} = \binom{k}{i}.
\]

We assume the identity holds for \( n \). By Lemma 2.1,

\[
(n-k)S_{k,n-1,i} = (n-i)S_{k,n,i} + iS_{k,n,i+1}
\]

\[
= (n-i)(-1)^{i-1} \left( \binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} \right) \xi(k)
\]

\[
+ i(-1)^i \left( \binom{k-1}{i} + (-1)^n \binom{k-1}{n-i-1} \right) \xi(k)
\]

\[
= (-1)^{i-1} \binom{k-1}{i-1} + (k-n+i)(-1)^n \binom{k-1}{n-i-1} \xi(k)
\]

\[
+ (-1)^i \binom{k-1}{i-1} + i(-1)^n \binom{k-1}{n-i-1} \xi(k)
\]

\[
= (n-k)(-1)^{i-1} \binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i-1} \xi(k).
\]

Thus, the identity holds for \( n - 1 \).

Remark. We mention an analogy of Theorem 1.2 on FRMZSVs. For positive integers \( k_1, k_2, \ldots, k_n \), let us define \( \xi^{\circ,\ast}_F \) by

\[
\xi^{\circ,\ast}_F(k_1, k_2, \ldots, k_n) := \sum_{\circ \text{ is either a comma "," or a plus "+"}} \xi_F(k_1 \circ k_2 \circ \cdots \circ k_n).
\]
Set \( S_{k,n,i}^{*} := \sum_{k_1 + \ldots + k_n = k} \sum_{k_i \geq 2} \zeta_{k_1,k_2,\ldots,k_n}^{*,*}(k_1,k_2,\ldots,k_n). \) Since these \( \zeta_{k_1,k_2,\ldots,k_n}^{*,*}(k_1,k_2,\ldots,k_n) \) satisfy the same harmonic product rule as \( \zeta^{*}(k_1,k_2,\ldots,k_n) \), \( S_{k,n,i}^{*} \) enjoy the same recursion relation as \([11, Proposition 2.2]\), that is, \((n-i)S_{k,n,i}^{*} + iS_{k,n,i+1}^{*} - (k-n)S_{k,n-1,i}^{*} = 0.\)

Writing \( k_i \sqcup k_j \) for juxtaposition of index sets \( k_i \) and \( k_j \), we see from \([5, Theorem 3.1]\) that

\[
\zeta_{k_1,k_2,\ldots,k_n}^{*,*}(k_1,k_2,\ldots,k_n) = (-1)^n \sum_{k_1 \sqcup \ldots \sqcup k_n = (k_1,k_2,\ldots,k_n)} (-1)^i \zeta_{i}^{*}(k_1) \cdots \zeta_{i}^{*}(k_i).
\]

Consider the case \((k_1,k_2,\ldots,k_n) = (1,\ldots,1,2,1,\ldots,1)\) in this equality. Since \( \zeta_{k_1,k_2,\ldots,k_n}^{*}(1,\ldots,1,1) \equiv \zeta_{k_1,k_2,\ldots,k_n}^{*}(1,\ldots,1,0) \pmod{\zeta(2)} \), the right-hand side is equal modulo \( \zeta(2) \) to \( \zeta_{k_1,k_2,\ldots,k_n}^{*}(1,\ldots,1,2,1,\ldots,1) \). Thus, we find \( S_{k,k-1,i}^{*} \equiv S_{k,k-1,i} \equiv (-1)^{i-1} \zeta(k) \pmod{\zeta(2)} \). In a similar way as for the proof of Theorem 1.2 (i.e., by backward induction on \( n \)), we obtain

\[
\sum_{k_1 + k_2 + \ldots + k_n = k} \zeta_{k_1,k_2,\ldots,k_n}^{*,*}(k_1,k_2,\ldots,k_n)
\equiv (-1)^i(-1)^{i-1}\left((-1)^i\binom{k-1}{i} + \binom{k-1}{n-i}\right) \zeta(k) \pmod{\zeta(2)}.
\]

2.3. **Proof of Theorem 1.3**

For a given index \( k \), we call the number of its elements greater than 1 the height. With this terminology, we shall call

\[
\zeta_{k}(1,\ldots,1)^{n-1}
\]

height-one FRMZVs. In this section, we prove Theorem 1.3. To this end, we state the following key lemma.

**Lemma 2.3.** *For positive integers \( k \) and \( n \) with \( k \geq 2 \), we have*

\[
\zeta^{*}(1,\ldots,1,\underbrace{k}_{n-1}) = (-1)^{n-1} \zeta^{*}(1,\ldots,1,\underbrace{k}_{n-1}).
\]

**Proof.** We note that the MZVs \( \zeta^{*}(k_1,k_2,\ldots,k_n) \) satisfy the shuffle product rule (for a precise definition, see \([4]\)) coming from the iterated integral expressions of the MZVs: \( \zeta^{*}(k_1)\zeta^{*}(k_2) = \zeta^{*}(k_1 \sqcup k_2) \), e.g., \( \zeta^{*}(1,1,1)^{(2)} = 3\zeta^{*}(2,1,1) + 2\zeta^{*}(1,2,1) + \zeta^{*}(1,1,2) \). Here, the notation \( k_1 \sqcup k_2 \) is a \( \mathbb{Z} \)-linear combination of indices and we extend \( \zeta^{*} \) linearly. To make the notation easier, let \( \zeta^{*}(1 \oplus (k_1,k_2,\ldots,k_n)) = \zeta^{*}(k_1+1,k_2,\ldots,k_n) \) and \( \zeta^{*}(1,\ldots,1,1,m,\ldots) = \zeta^{*}(\ldots,1,m-1,\ldots). \)
By the regularization formula \([6, \text{equation (5.2)}]\), we have (extending \(1 \oplus (\cdot)\) also linearly)

\[
\zeta^{\text{III}}(1, \ldots, 1, k) \\
\quad = (-1)^{n-1} \zeta^{\text{III}}(1 \oplus ((1, \ldots, 1)_{n-1}(k-1))) \\
\quad = (-1)^{n-1} \sum_{\substack{a_1 + \cdots + a_k = n-1 \geq 0 \atop a_1, \ldots, a_k \geq 0}} \zeta(2, 1, \ldots, 1)_{a_1-1} \ldots \zeta(2, 1, \ldots, 1)_{a_k-1} \\
\quad = (-1)^{n-1} \sum_{\substack{a_1 + \cdots + a_k = n-1 \geq 0 \atop a_1, \ldots, a_k \geq 0}} (a_{k-1} + 1) \zeta(2, 1, \ldots, 1)_{a_1-1} \ldots \zeta(2, 1, \ldots, 1)_{a_k-1} \\
\quad = (-1)^{n-1} \sum_{\substack{a_1 + \cdots + a_k = n-1 \geq 0 \atop a_1, \ldots, a_k \geq 0}} (a_{k-1} + 1) \zeta(a_{k-1} + 2, a_{k-2} + 1, \ldots, a_1 + 1).
\]

For the last equality, we used the duality formula of MZVs. That the last sum equals \(\zeta^*(n, 1, \ldots, 1)\) is due to Ohno \([10, \text{Proof of Theorem 2}]\), see also \([7, \text{Section 3}]\). Thus

\[
\zeta^{\text{III}}(1, \ldots, 1, k) = (-1)^{n-1} \zeta^*(k, 1, \ldots, 1).
\]

Now, we prove Theorem 1.3. When either \(k \) or \(n = 1\), the theorem clearly holds. We consider the case when \(k, n \geq 2\). From the above Lemma 2.3, we have

\[
\zeta^{\text{II}}(k, 1, \ldots, 1) - \zeta^{\text{II}}(n, 1, \ldots, 1) \\
\quad = \zeta(k, 1, \ldots, 1)_{n-1} + (-1)^k \zeta^*(k, 1, \ldots, 1)_{n-1} - (\zeta(n, 1, \ldots, 1)_{k-1} + (-1)^n \zeta^*(n, 1, \ldots, 1)_{k-1}).
\]

Let \(\psi(X) = \Gamma'(X)/\Gamma(X)\). By using the well-known generating series

\[
1 - \sum_{k,n \geq 1} \zeta(k, 1, \ldots, 1)_{n-1} X^k Y^n = \exp \left( \sum_{n \geq 2} \zeta(n) X^n + Y^n - (X + Y)^n \right) \\
\quad = \frac{\Gamma(1 - X)\Gamma(1 - Y)}{\Gamma(1 - X - Y)}
\]

(cf. Aomoto \([1]\) and Drinfel’d \([2]\)) and \(\psi(1 - X) = -\sum_{k \geq 2} \zeta(k) X^{k-1} - \gamma\) \((\gamma\) is Euler’s constant), we have

\[
\sum_{k,n \geq 2} \left( \zeta(k, 1, \ldots, 1)_{n-1} - \zeta(n, 1, \ldots, 1)_{k-1} \right) X^k Y^{n-1} \\
\quad = \left( \frac{1}{Y} - \frac{1}{X} \right) \left( 1 - \frac{\Gamma(1 - X)\Gamma(1 - Y)}{\Gamma(1 - X - Y)} \right) + \psi(1 - X) - \psi(1 - Y).
\]
On the other hand, from Kaneko and Ohno [8, Theorem 2],
\[
\sum_{k,n \geq 2} \left( (-1)^k \zeta^*(k, 1, \ldots, 1) - (-1)^n \zeta^*(n, 1, \ldots, 1) \right) X^{k-1} Y^{n-1} \\
= -\psi(X) + \psi(Y) - \pi (\cot(\pi X) - \cot(\pi Y)) \frac{\Gamma(1-X) \Gamma(1-Y)}{\Gamma(1-X-Y)}.
\]
From these, and by the well-known equalities
\[
\pi \cot(\pi X) = \frac{1}{X} + \psi(1-X) - \psi(1+X),
\]
\[
\psi(X) = \psi(1+X) - \frac{1}{X},
\]
we have
\[
\sum_{k,n \geq 2} \left( \zeta^\mu_F(k, 1, \ldots, 1) - \zeta^\mu_F(n, 1, \ldots, 1) \right) X^{k-1} Y^{n-1} \\
= \left( \frac{1}{Y} - \frac{1}{X} \right) \left( 1 - \frac{\Gamma(1-X) \Gamma(1-Y)}{\Gamma(1-X-Y)} \right) \psi(1-X) - \psi(1-Y) \\
- \psi(X) + \psi(Y) - \pi (\cot(\pi X) - \cot(\pi Y)) \frac{\Gamma(1-X) \Gamma(1-Y)}{\Gamma(1-X-Y)} \\
= \left( 1 - \frac{\Gamma(1-X) \Gamma(1-Y)}{\Gamma(1-X-Y)} \right) \left( \psi(1-X) - \psi(1+X) - \psi(1-Y) + \psi(1+Y) \right) \\
= -2 \left( 1 - \frac{\Gamma(1-X) \Gamma(1-Y)}{\Gamma(1-X-Y)} \right) \sum_{l \geq 1} \zeta(2l)(X^{2l-1} - Y^{2l-1}).
\]
Since the coefficients of $\Gamma(1-X) \Gamma(1-Y)/\Gamma(1-X-Y)$ belong to the $\mathbb{Q}$-algebra $\mathcal{Z}$, we have
\[
\zeta^\mu_F(k, 1, \ldots, 1) \equiv \zeta^\mu_F(n, 1, \ldots, 1) \pmod{\zeta(2)}.
\]
This proves Theorem 1.3.

Acknowledgement. The author would like to thank Professor Masanobu Kaneko for valuable comments and suggestions.

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