IRRATIONALITY MEASURES FOR CONTINUED FRACTIONS WITH ASYMPTOTIC CONDITIONS

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Abstract. We are interested in lower and upper bounds of asymptotic irrationality measures for certain simple continued fractions. A general procedure for estimation of real numbers by rational numbers is described. It is applied for simple continued fractions with quotients satisfying some asymptotic density conditions.

1. Introduction

Standard definitions we use include the following.

Throughout the entire paper we denote by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ the set of all integers, positive integers, non-negative integers, real numbers and positive real numbers, respectively. Square brackets $a = [a_0; a_1, \dots]$ stand for the simple continued fraction expansion of the real number a , and the numerator and the denominator of the n th convergent $[a_0; a_1, \dots, a_n]$ are denoted by p_n and q_n , respectively. The notation $f(x) = O(g(x))$ and $f(x) = o(g(x))$ mean that

$$\limsup_{x \rightarrow \infty, (f(x), g(x)) \neq (0, 0)} \left| \frac{f(x)}{g(x)} \right| \neq \infty$$

and

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0,$$

respectively. By $\log x$ we mean the natural logarithm of the number x and we use the convention $0^0 = 1$. If x is a real number then $[x]$ is the integer part of the number x . The number of positive divisors of n is denoted by $d(n)$.

Following Euler [4], Perron [16], Mahler [14] and Bundschuh [3] we prove the following.

THEOREM 1.1. *Let $a = [a_0; a_1, \dots]$ be a continued fraction and let ε be a positive real number. Let $\{V_n\}_{n=0}^\infty$ be a sequence consisting of numbers 1 and 2 such that $\lim_{n \rightarrow \infty} (1/n)(\#\{j: V_j = 1, 1 \leq j \leq n\}) = \frac{1}{2}$. Set $a_n = n^{V_n}$ for all $n \in \mathbb{N}$. Then for every $M \in \mathbb{Z}$ and sufficiently large $N \in \mathbb{Z}^+$ we have*

$$\left(\frac{9}{4} - \varepsilon \right) \left(\frac{\log \log N}{N \log N} \right)^2 < \left| a - \frac{M}{N} \right|$$

and for infinitely many $(M, N) \in \mathbb{Z} \times \mathbb{Z}^+$ we have

$$\left(\frac{9}{4} + \varepsilon \right) \left(\frac{\log \log N}{N \log N} \right)^2 > \left| a - \frac{M}{N} \right|.$$

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This kind of problem has a long history which starts from Euler [4]. In 1998 Bundschuh [3] proved that

$$\left| a - \frac{M}{N} \right| \geq \frac{D \log \log N}{d N^2 \log N} \left(1 - \frac{\log \log n}{\log n} + O\left(\frac{1}{\log n}\right) \right),$$

where $N = q_n$ is the denominator of the n th convergent of the continued fraction

$$a = [a_0; a_1, \dots, a_s, \overline{c_1 + \lambda d_1, \dots, c_m + \lambda d_m}]_{\lambda=0}^{\infty} \quad (1)$$

satisfying the conditions

$$a_0, a_1, \dots, a_s, c_1, \dots, c_m \in \mathbb{Z}^+, \quad d_1, \dots, d_m \in \mathbb{N}, \quad d = \max(d_1, \dots, d_m) \neq 0,$$

and D is the number of the non-zero d_j . Later Tasoev [19] proved that if ε is a positive real number then for every $M \in \mathbb{Z}$ and sufficiently large $N \in \mathbb{Z}^+$ we have

$$\left(\frac{D}{d} - \varepsilon \right) \left(\frac{\log \log N}{N^2 \log N} \right) < \left| a - \frac{M}{N} \right|$$

and for infinitely many $(M, N) \in \mathbb{Z} \times \mathbb{Z}^+$ we have

$$\left(\frac{D}{d} + \varepsilon \right) \left(\frac{\log \log N}{N^2 \log N} \right) > \left| a - \frac{M}{N} \right|.$$

In the same paper, Tasoev also proved that if a_0, b and m are positive integers such that $b > 1$,

$$a = [a_0; \underbrace{b^\lambda, \dots, b^\lambda}_{m \text{ times}}]_{\lambda=1}^{\infty}$$

and ε is a positive real number, then for every $M \in \mathbb{Z}$ and sufficiently large $N \in \mathbb{Z}^+$ we have

$$\left(\frac{1}{\sqrt{b}} - \varepsilon \right) \left(\frac{1}{N^{2+\sqrt{2 \log b/(m \log N)}}} \right) < \left| a - \frac{M}{N} \right|$$

and for infinitely many $(M, N) \in \mathbb{Z} \times \mathbb{Z}^+$ we have

$$\left(\frac{1}{\sqrt{b}} + \varepsilon \right) \left(\frac{1}{N^{2+\sqrt{2 \log b/(m \log N)}}} \right) > \left| a - \frac{M}{N} \right|.$$

We improve this result in Theorem 3.7 by proving that there exists a positive real number H such that for every $M \in \mathbb{Z}$ and sufficiently large $N \in \mathbb{Z}^+$ we have

$$\frac{1 - Hb^{-2\sqrt{2 \log N/\log b}}}{N^{2b^{1/2} + \sqrt{1/4 + 2(\log N - C)/(m \log b)}}} < \left| a - \frac{M}{N} \right|$$

and for infinitely many $(M, N) \in \mathbb{Z} \times \mathbb{Z}^+$ we have

$$\frac{1 + Hb^{-2\sqrt{2 \log N/\log b}}}{N^{2b^{1/2} + \sqrt{1/4 + 2(\log N - C)/(m \log b)}}} > \left| a - \frac{M}{N} \right|,$$

where

$$e^C = b \prod_{j=2}^{\infty} \left(1 + \frac{q_{j-2}}{a_j q_{j-1}} \right)$$

is a constant and q_n is the denominator of the n th convergent of a .

Number e has a simple continued fraction expression like a has in (1). Many mathematicians used this fact to prove irrationality measures of e . We can find this in the papers of Bundschuh [2], Galochkin [6] and Komatsu [13]. See also Borwein and Borwein [1], Fel'dman and Nesterenko [5], and Shiokawa [17]. Better approximation of the number e we can find in Hančl *et al* [7] or in [9]. Also Sondow [18] obtained a new kind of irrationality measure for e . Hata [12] proved an interesting approximation of e^n . Mahler [14] approximated exponential and logarithmic functions.

Compared to the number a in (1), which has an almost periodic continued fraction expression, we work with asymptotical conditions only. This is connected to an earlier work of Hančl *et al* [10] using the concept of asymptotic density determining the Hausdorff dimension of special sets of continued fractions from the interval $(0, 1)$. Theorems 3.1–3.3 deal with continued fractions with quotients a_n growing somewhat linearly, as in (1). In Theorem 3.1, we only assume that a_n/n has finitely many accumulation points, and that they are positive and finite. The results of Theorems 3.2 and 3.3 are more exact but here we assume that we know the asymptotic densities or even the accumulation points. In the more general Theorems 3.4 and 3.5, we make some assumptions on the average behaviour of $\log a_n/\log n$. This extends the results in the modern mathematical theory of means when the function is irregular but the average of this function exists and can be simply expressed. One example of this is Theorem 3.6 where we study a continued fraction constructed using the divisor function $d(n)$, the average and maximal behaviour of which is known. More examples, including a refined version of Theorem 3.6, can be found in [8].

2. Notation

Some less-known notation we use is as follows.

For a real number a we define the function $J_a: \mathbb{Z}^+ \rightarrow \mathbb{R}_0^+$ as

$$J_a(N) = N^2 \min_{M \in \mathbb{Z}} \left| a - \frac{M}{N} \right|.$$

This is related to the concepts of the irrationality measure of a , which usually means any lower bound of $J_a(N)/N^2$, and the irrationality exponent of a , which is

$$\mu_a(N) = 2 - \liminf_{N \rightarrow \infty} \frac{\log J_a(N)}{\log N}.$$

For convenience of notation we say that $J_a(N) = f(N) + \overline{O}(g(N))$ if there exists $C \in \mathbb{R}^+$ such that:

- (1) $J_a(N) > f(N) - C |g(N)|$ for all $N \in \mathbb{Z}^+$ big enough;
- (2) $J_a(N) < f(N) + C |g(N)|$ for infinitely many $N \in \mathbb{Z}^+$.

Similarly we define that $J_a(N) = f(N) + \overline{o}(g(N))$ if for every $\varepsilon \in \mathbb{R}^+$:

- (1) $J_a(N) > f(N) - \varepsilon |g(N)|$ for all $N \in \mathbb{Z}^+$ big enough;
- (2) $J_a(N) < f(N) + \varepsilon |g(N)|$ holds for infinitely many $N \in \mathbb{Z}^+$.

In other words, we do not only provide irrationality measures but also show how sharp they are. Let a_0 be an integer, $a_1, \dots, a_s$ be positive integers and $F_j: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, $j = 1, \dots, m$

be functions. Then we denote

$$\begin{aligned} & [a_0; a_1, \dots, a_s, \overline{F_1(\lambda), \dots, F_m(\lambda)}]_{\lambda=1}^{\infty} \\ &= [a_0; a_1, \dots, a_s, F_1(1), \dots, F_m(1), F_1(2), \dots, F_m(2), \dots]. \end{aligned}$$

The reader will also need to know some basic facts about simple continued fractions. Let $a = [a_0; a_1, \dots]$. For the convergents we have the recurrence formulae

$$p_{n+2} = a_{n+2}p_{n+1} + p_n, \quad q_{n+2} = a_{n+2}q_{n+1} + q_n \quad (2)$$

for all $n \in \mathbb{N}$, with the initial values $p_0 = a_0$, $q_0 = 1$, $p_1 = a_1a_0 + 1$ and $q_1 = a_1$. It is a generally known fact that the best approximations of the number a are its convergents. In fact, p/q is a convergent of a whenever $|a - p/q| < 1/(2q^2)$, and for the convergents we have

$$\frac{1}{q_n^2(a_{n+1} + 2)} < \left| a - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2 a_{n+1}} \quad (3)$$

for all $n \in \mathbb{N}$. All of this can be found in Hardy and Wright [11].

We will also use some crude versions of Stirling's formula

$$n! = (n/e)^n e^{o(n)} \quad (4)$$

$$= (n/e)^n e^{O(\log n)} \quad (5)$$

and Abel's partial summation formula

$$\sum_{j=1}^n x_j y_j = y_n \sum_{j=1}^n x_j - \sum_{k=1}^{n-1} \left((y_{k+1} - y_k) \sum_{l=1}^k x_l \right). \quad (6)$$

3. Results

THEOREM 3.1. *Let $Z_1 < \dots < Z_K$ be positive real numbers. Suppose that $\{A_n\}_{n=0}^{\infty}$ is a sequence consisting of numbers $Z_1, \dots, Z_K$ such that every Z_j , $j = 1, \dots, K$, occurs infinitely many times. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive integers such that $a_n = A_n(1 + o(1))n$. Set*

$$x = \liminf_{n \rightarrow \infty} \frac{\#\{j: A_j = Z_K, 1 \leq j \leq n\}}{n},$$

$$y = \limsup_{n \rightarrow \infty} \frac{\#\{j: A_j = Z_1, 1 \leq j \leq n\}}{n},$$

$$X = Z_K^x Z_{K-1}^{1-x}$$

and

$$Y = Z_1^y Z_2^{1-y}.$$

Let ε be a positive real number. Then for all sufficiently large N we have

$$J_a(N) > \frac{(1 - \varepsilon) \log(Y \log N / (e \log \log N))}{Z_K \log N} \quad (7)$$

and for infinitely many N we have

$$J_a(N) < \frac{(1 + \varepsilon) \log(X \log N / (e \log \log N))}{Z_K \log N}. \quad (8)$$

Proof of Theorem 3.1. From the recurrence formulae of continued fractions (2) and the growth rate $a_n = A_n(1 + o(1))n$ we obtain

$$q_n = a_n q_{n-1} + q_{n-2} = a_n q_{n-1}(1 + o(1)) = \dots = e^{o(n)} \prod_{j=1}^n a_j = e^{o(n)} n! \prod_{j=1}^n A_j. \quad (9)$$

Now the proof of Theorem 3.1 falls into two cases.

First we prove upper bound (8). By the definition of x we have the bounds

$$Z_1^n \leq \prod_{j=1}^n A_j \leq Z_K^{n(x+o(1))} Z_{K-1}^{n(1-x+o(1))} = X^n e^{o(n)}.$$

Now applying Stirling's formula (4) to (9) gives

$$\left(\frac{Z_1}{e} (1 + o(1))n \right)^n \leq q_n \leq \left(\frac{X}{e} (1 + o(1))n \right)^n,$$

noting that $(1 + o(1))^n = e^{o(n)}$. This implies

$$\begin{aligned} n &\geq \frac{\log q_n}{\log\left(\frac{X}{e}(1 + o(1))n\right)} \geq \frac{\log q_n}{\log\left(\frac{X}{e}(1 + o(1)) \frac{\log q_n}{\log\left(\frac{Z_1}{e}(1 + o(1))n\right)}\right)} \\ &\geq \frac{\log q_n}{\log\left(\frac{X}{e}(1 + o(1)) \frac{\log q_n}{\log\left(\frac{Z_1}{e}(1 + o(1)) \frac{\log q_n}{\log\left(\frac{X}{e}(1 + o(1))n\right)}\right)}\right)} \geq \frac{\log q_n}{\log\left(\frac{X \log q_n}{e \log \log q_n}\right) + o(1)}. \end{aligned} \quad (10)$$

The last inequality is bounding $n = o(q_n)$ and simplifying. Consider only those values N that equal some q_n such that $A_{n+1} = Z_K$. By (3) we get

$$J_a(N) < \frac{1}{a_{n+1}} = \frac{1}{A_{n+1}(n+1)(1 + o(1))} = \frac{1 + o(1)}{Z_K n},$$

and so (8) follows from estimate (10).

Then we prove the lower bound (7). From (9) we obtain

$$\left(\frac{Y}{e} (1 + o(1))n \right)^n \leq q_n \leq \left(\frac{Z_K}{e} (1 + o(1))n \right)^n.$$

This yields

$$\begin{aligned} n &\leq \frac{\log q_n}{\log\left(\frac{Y}{e}(1 + o(1))n\right)} \leq \frac{\log q_n}{\log\left(\frac{Y}{e}(1 + o(1)) \frac{\log q_n}{\log\left(\frac{Z_K}{e}(1 + o(1))n\right)}\right)} \\ &\leq \frac{\log q_n}{\log\left(\frac{Y}{e}(1 + o(1)) \frac{\log q_n}{\log\left(\frac{Z_K}{e}(1 + o(1)) \frac{\log q_n}{\log\left(\frac{Y}{e}(1 + o(1))n\right)}\right)}\right)} \leq \frac{\log q_n}{\log\left(\frac{Y \log q_n}{e \log \log q_n}\right) + o(1)}. \end{aligned} \quad (11)$$

As all of the best approximations of a are the convergents (in fact, if N is not a convergent then $J_a(N) \geq 1/2$), we may assume that N equals q_n . As $A_{n+1} \leq Z_K$, we get

$$J_a(N) > \frac{1}{a_{n+1} + 2} = \frac{1}{A_{n+1}(n+1)(1 + o(1)) + 2} \geq \frac{1 + o(1)}{Z_K n},$$

by (3) and so (7) follows from estimate (11). $\square$

THEOREM 3.2. *Let $Z_1 < \dots < Z_K$ be positive real numbers and $z_1, \dots, z_K$ be non-negative real numbers such that $\sum_{j=1}^K z_j = 1$. Suppose that $\{A_n\}_{n=0}^\infty$ is a sequence consisting of the numbers $Z_1, \dots, Z_K$ such that every Z_j , $j = 1, \dots, K$, occurs infinitely many times and has asymptotic density z_j . Let $\{a_n\}_{n=0}^\infty$ be a sequence of positive integers. Set*

$$Z = \prod_{j=1}^K Z_j^{z_j}.$$

If $a_n = A_n(1 + o(1))n$, then we have

$$J_a(N) = \frac{(1 + \overline{o}(1)) \log(Z \log N / (e \log \log N))}{Z_K \log N}.$$

If there exists a positive real number C such that $a_n = A_n(1 + O(n^{-C}))n$, then we have

$$J_a(N) = \frac{\log(Z \log N / (e \log \log N)) + \overline{o}(1)}{Z_K \log N}.$$

Proof of Theorem 3.2. In both cases we have at least $a_n = A_n(1 + o(1))n$, so (9) is true. By Stirling's formula (4) and the fact that

$$\prod_{j=1}^n A_j = Z_1^{n(z_1+o(1))} \dots Z_K^{n(z_K+o(1))} = Z^n e^{o(n)}$$

we obtain

$$q_n = \left(\frac{Z}{e} (1 + o(1))n \right)^n.$$

Now

$$n = \frac{\log q_n}{\log(Z \log q_n / (e \log \log q_n)) + o(1)}$$

and in fact the whole first claim follows just as in the proof of Theorem 3.1.

Now suppose further that $a_n = A_n(1 + O(n^{-C}))n$ for some positive C . For the upper bound we again consider only those N that equal some q_n such that $A_{n+1} = Z_K$. Inequality (3) implies

$$J_a(N) < \frac{1}{a_{n+1}} = \frac{1}{A_{n+1}(1 + O(1/(n+1)^C))(n+1)} = \frac{1 + O(1/n^{\min\{C,1\}})}{Z_K n}$$

so everything is fine after substitution of n . For the lower bound we make the justified assumption that N equals some q_n , and the claim follows since (3) gives

$$J_a(N) > \frac{1}{a_{n+1} + 2} = \frac{1}{A_{n+1}(1 + O(1/(n+1)^C))(n+1) + 2} \geq \frac{1 + O(1/n^{\min\{C,1\}})}{Z_K n}. \quad \square$$

THEOREM 3.3. *Let $Z_1 < \dots < Z_K$ be positive real numbers and $z_1, \dots, z_K$ be non-negative real numbers such that $\sum_{j=1}^K z_j = 1$. Suppose that $\{A_n\}_{n=0}^\infty$ is a sequence consisting of the numbers $Z_1, \dots, Z_K$ such that for every Z_j , $j = 1, \dots, K$ occurs infinitely many times and*

$$\#\{k : A_k = Z_j, 1 \leq k \leq n\} = z_j n + O(n^{1-C})$$

for some $C \in (0, 1)$. Let $\{a_n\}_{n=0}^\infty$ be a sequence of positive integers such that $a_n = A_n(1 + O(n^{-C}))n$. Set

$$Z = \prod_{j=1}^K Z_j^{z_j}.$$

Then

$$J_a(N) = \frac{\log F_k(N) + \overline{O}(G(N))}{Z_K \log N},$$

where $F_k(N)$ and $G_k(N)$ are defined recursively as $F_0(N) = \log N$,

$$F_{k+1}(N) = \frac{Z \log N}{e \log F_k(N)},$$

$G_0(N) = 1$ and

$$G_{k+1}(N) = \frac{G_k(N)}{\log F_k(N)}.$$

Proof of Theorem 3.3. This time (9) will not be enough so we estimate

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2} = a_n q_{n-1} \left(1 + O\left(\frac{1}{n^2}\right)\right) = \dots = e^{O(1/n)} \prod_{j=1}^n a_j \\ &= e^{O(n^{1-C})} n! \prod_{j=1}^n A_j. \end{aligned}$$

This is because by integrating we see that

$$\log \prod_{j=1}^n \left(1 + O\left(\frac{1}{j^D}\right)\right) = \sum_{j=1}^n \log \left(1 + O\left(\frac{1}{j^D}\right)\right) = \sum_{j=1}^n O\left(\frac{1}{j^D}\right) = O(n^{1-D})$$

for any positive D (here $D = 2$ and $D = C$). Using Stirling's formula (5) and

$$\prod_{j=1}^n A_j = Z_1^{z_1 n + O(n^{1-C})} \dots Z_K^{z_K n + O(n^{1-C})} = Z^n e^{O(n^{1-C})}$$

gives

$$q_n = \left(\frac{Z}{e} n\right)^{ne^{O(n^{1-C})}}.$$

This yields

$$\begin{aligned} n &= \frac{\log q_n}{\log\left(\frac{Z}{e} n\right) + O\left(\frac{1}{n^C}\right)} = \frac{\log q_n}{\log\left(\frac{Z \log q_n}{e \log\left(\frac{Z}{e} n\right)}\right) + O\left(\frac{1}{n^C}\right)} \\ &= \frac{\log q_n}{\log\left(\frac{Z \log q_n}{e \log\left(\frac{Z \log q_n}{e \log\left(\frac{Z}{e} n\right)}\right)}\right) + O\left(\frac{1}{n^C}\right)} = \dots \end{aligned}$$

After k steps we make the estimate $n = O(\log q_n)$, to obtain

$$n = \frac{\log q_n}{\log(F_k(q_n)) + O(G_k(q_n))}$$

via mathematical induction. Note that the new error $O(G(\log q_n))$ is always bigger than the old $O(n^{-C})$. The proof is finished in the usual manner using (3) and only considering those N that equal some q_n with $A_{n+1} = Z_K$ for the upper bound. $\square$

THEOREM 3.4. *Let $\{V_n\}_{n=0}^\infty$ be a sequence of non-negative real numbers such that*

$$0 < \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V_j \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V_j < \infty.$$

Let $\{a_n\}_{n=0}^\infty$ be a sequence of positive integers such that $a_n = n^{V_n}$.

If $\limsup_{n \rightarrow \infty} V_n = \alpha < \infty$, then

$$J_a(N) = \left(\frac{\log \log N}{\log N} \right)^{\alpha + \bar{o}(1)}. \quad (12)$$

If $\limsup_{n \rightarrow \infty} V_n / \log n = \beta < \infty$, then

$$J_a(N) = \left(\frac{\log \log N}{\log N} \right)^{(\beta + \bar{o}(1)) \log(\log N / \log \log N)}. \quad (13)$$

Proof of Theorem 3.4. By Abel's partial summation formula (6)

$$\begin{aligned} \sum_{j=1}^n V_j \log j &= \log n \sum_{j=1}^n V_j - \sum_{k=1}^{n-1} \left((\log(k+1) - \log k) \sum_{l=1}^k V_l \right) \\ &= \log n \sum_{j=1}^n V_j + O(n). \end{aligned}$$

From this, the recurrence formulae of continued fractions (2) and the growth rate $a_n = n^{V_n}$, we obtain that

$$q_n = a_n q_{n-1} + q_{n-2} = a_n q_{n-1} O(1) = \dots = e^{O(n)} \prod_{j=1}^n a_j = n^{n \varepsilon_1(n)},$$

where $0 < a_1 \leq \varepsilon_1(n) \leq b_1 < \infty$. This yields

$$n = \frac{\log q_n}{\varepsilon_1(n) \log n} = \frac{\log q_n}{\varepsilon_1(n) \log(\log q_n / (\varepsilon_1(n) \log n))} = \frac{\log q_n}{\varepsilon_2(n) \log \log q_n},$$

where $0 < a_2 \leq \varepsilon_2(n) \leq b_2 < \infty$. This and (3) imply (12) and (13), if in the upper bounds we only consider those N equal to some q_n such that V_{n+1} is sufficiently close to α and $\beta \log n$, respectively. $\square$

THEOREM 3.5. *Let β and γ be positive real numbers. Let $\{V_n\}_{n=0}^\infty$ be a sequence of non-negative real numbers such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V_j = \gamma$$

and for infinitely many k we have $V_k = \beta = \max_{n \in \mathbb{N}} V_n$. Suppose that $\{A_n\}_{n=0}^\infty$ is a sequence of positive real numbers such that

$$0 < \alpha_1 \liminf_{n \rightarrow \infty} A_n \leq \limsup_{n \rightarrow \infty} A_n = \alpha_2 < \infty.$$

Let $\{a_n\}_{n=0}^\infty$ be a sequence of positive integers such that $a_n = A_n n^{V_n}$. Let ε be a positive real number. Then for all sufficiently large N we have

$$J_a(N) > \frac{(1 - \varepsilon)(\gamma \log \log N / \log N)^\beta}{\alpha_2} \quad (14)$$

and for infinitely many N we have

$$J_a(N) < \frac{(1 + \varepsilon)(\gamma \log \log N / \log N)^\beta}{\alpha_1}. \quad (15)$$

Proof of Theorem 3.5. We use Abel's summation formula (6) as in the proof of Theorem 3.5, taking into account that now

$$\sum_{j=1}^n V_j = \gamma n + o(n),$$

to see

$$q_n = n^{\gamma n + o(n)}$$

and

$$n = \frac{\log q_n}{\gamma(1 + o(1)) \log \log q_n}.$$

This and (3) imply (14) and (15) if we only consider those N equal to some q_n such that $V_{n+1} = \beta$ for the upper bound. $\square$

Proof of Theorem 1.1. Theorem 1.1 is a consequence of Theorem 3.5. $\square$

THEOREM 3.6. Set $a = [1; \overline{2^{d(\lambda)}}]_{\lambda=1}^\infty$. Then

$$J_a(N) = 2^{-2 \log \log N / (\log \log \log N - 1 + \overline{o(1)})}.$$

Proof of Theorem 3.6. From the fact that

$$\sum_{j=1}^n d(j) = n \log n + O(n)$$

(see Hardy and Wright [11, p. 347]) and the recurrence formulae (2) we get

$$q_n = a_n q_{n-1} + q_{n-2} = a_n q_{n-1} O(1) = \dots = e^{O(n)} \prod_{j=1}^n a_j = e^{\log(2)n \log n + O(n)}.$$

This yields

$$\begin{aligned} n &= \frac{\log q_n}{\log(2) \log n + O(1)} = \frac{\log q_n}{\log(2) \log(\log q_n / \log(2) \log n + O(1)) + O(1)} \\ &= \frac{\log q_n}{\log(2) \log \log q_n - \log(2) \log \log n + O(1)} \\ &= \frac{\log q_n}{\log(2) \log \log q_n - \log(2) \log \log \log q_n + O(1)}. \end{aligned} \quad (16)$$

From this, (3) and the fact that for every positive real number ε we have

$$d(n) \leq 2^{\log n / (\log \log n - 1 - \varepsilon)}$$

(see Nicolas [15]) we obtain the lower bound.

For the upper bound consider only those N that equal some q_n , where $n + 1$ is the product of k smallest prime numbers p_j . Now $d(n + 1) = 2^{k+1}$, so we need to express k in terms of n . By using the prime number theorem $p_n / (n \log n) = 1 + o(1)$ we see that

$$\log(n + 1) = \sum_{j=1}^k \log p_j = \sum_{j=1}^k \log j + \sum_{j=1}^k \log \log j + o(k).$$

Stirling's formula (4) gives

$$\sum_{j=1}^k \log j = k \log k - k + o(\log k) = k \log k - k + o(k)$$

and Abel's partial summation formula (6) gives

$$\sum_{j=1}^k \log \log j = k \log \log k + O(\text{li}(k)) = k \log \log k + o(k),$$

and after solving for k and simplifying we obtain

$$k = \frac{\log n}{\log \log n - 1 + o(1)}.$$

The claim now follows from (3) and (16). $\square$

THEOREM 3.7. *Let a_0 be an integer and $a_1, \dots, a_s, b$ be positive integers such that $b > 1$. Set $a = [a_0; a_1, \dots, a_s, \underbrace{b^\lambda, \dots, b^\lambda}_{m \text{ times}}]_{\lambda=1}^\infty$. Then*

$$J_a(N) = \frac{1 + O(b^{-2\sqrt{2 \log N / (m \log b)}})}{b^{1/2 + \sqrt{1/4 + 2(\log N - C)/(m \log b)}}}, \quad (17)$$

where

$$e^C = \left(\prod_{j=1}^s a_j \right) \left(\prod_{j=2}^\infty \left(1 + \frac{q_{n-2}}{a_n q_{n-1}} \right) \right).$$

Proof of Theorem 3.7. Since $a_{n+1} = b^{[(n-s)/m]+1}$, it is enough by (3) to only look at n such that $n \equiv s \pmod{m}$. Now recurrence formulae (2) yield

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2} = a_n q_{n-1} \left(1 + \frac{q_{n-2}}{a_n q_{n-1}} \right) = \dots = a_1 \prod_{j=2}^n a_j \left(1 + \frac{q_{j-2}}{a_j q_{j-1}} \right) \\ &= e^C \left(\prod_{j=n+1}^\infty \left(1 + \frac{q_{j-2}}{a_j q_{j-1}} \right) \right)^{-1} \prod_{j=s+1}^n a_j = e^{C+O(b^{-2n/m})} \prod_{j=s+1}^n a_j. \end{aligned}$$

Hence

$$\begin{aligned}\log q_n &= C + O(b^{-2n/m}) + m \log b \sum_{j=1}^{(n-s)/m} j \\ &= C + O(b^{-2n/m}) + \log b \frac{(n-s)(n-s+m)}{2m}.\end{aligned}$$

Therefore,

$$\begin{aligned}n &= s - \frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{2m}{\log b}(\log q_n - C + O(b^{-2n/m}))} \\ &= s - \frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{2m}{\log b}(\log q_n - C + O(b^{-2\sqrt{2\log q_n/(m\log b)}}))}.\end{aligned}$$

This and the fact that $a_{n+1} = b^{(n-s)/m+1}$ imply that for $n \equiv s \pmod{m}$ we have

$$\begin{aligned}J_a(N) &= \frac{1 + O(b^{-2n/m})}{a_{n+1}} = \frac{1 + O(b^{-2\sqrt{2\log q_n/(m\log b)}})}{b^{[(n-s)/m]+1}} \\ &= \frac{1 + O(b^{-2\sqrt{2\log q_n/(m\log b)}})}{b^{1/2 + \sqrt{1/4 + 2(\log q_n - C)/(m\log b)}}}\end{aligned}$$

and (17) follows. $\square$

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