A CHARACTERIZATION OF SYMMETRIC CONES
BY THE DEGREES OF
BASIC RELATIVE INVARIANTS

Takashi YAMASAKI and Takaaki NOMURA
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Abstract. In this paper, we characterize irreducible symmetric cones among homogeneous cones $\Omega$ of rank $r$ by the fact that the basic relative invariants for $\Omega$ and for $\Omega^*$ (the dual cone of $\Omega$) both have the degrees $1, 2, \ldots, r$, up to permutations.

1. Introduction

In linear algebra, we know that a real symmetric (or a complex Hermitian) matrix is positive-definite if and only if its (leading) principal minors are all positive. Thus the cone of positive-definite real symmetric matrices (or of positive-definite complex Hermitian matrices) is described as the positivity domain of the principal minors. Principal minors are generalized to Euclidean Jordan algebras in the book Faraut–Korányi [4], where Euclidean Jordan algebras are fundamental algebraic objects in studying symmetric cones. By Exercise 5 of [4, Ch. VI], an irreducible symmetric cone is still described as the positivity domain of the Jordan algebra principal minors. One of the notable features of the ordinary or the Jordan algebra version of the principal minors is that they are irreducible polynomials relatively invariant under the action of the simply transitive split solvable Lie group $H$. Thus each of them transforms according to a one-dimensional representation under the $H$-action. Basic relative invariants are further generalizations of principal minors introduced in Ishi [8] for homogeneous cones $\Omega$ with the same property of irreducibility and the relative invariance. Moreover, we have the description of $\Omega$ as the positivity domains of the basic relative invariants (Ishi [8, Proposition 2.3]). Homogeneous cones provide many non-reductive prehomogeneous vector spaces, and as such, basic relative invariants are of fundamental importance (see, for example, [11]) in studying homogeneous cones.

Now, it is obvious that the degrees of the principal minors of an irreducible symmetric cone of rank $r$ form an arithmetic progression $1, \ldots, r$ with the common difference one. One might wonder if this progression property characterizes symmetric cones. However, in [9, Section 3], an example is given of a rank-three homogeneous non-symmetric cone with this property of arithmetic progression of the degrees of basic relative invariants. Furthermore, Section 6 of the paper [13] (see also [14]) produces a series of non-symmetric cones of rank $r$ with $1, \ldots, r$ of the degrees of basic relative invariants for arbitrary $r \geq 3$ (see Theorem 6.3

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and Remark 6.5 (2) of [13]). Then arises a natural question: what happens if the basic relative invariants for the dual cone also have the same property of the degrees? This paper finds that the question actually gives a characterization of irreducible symmetric cones.

Let $\Omega$ be a homogeneous cone of rank $r$, and $V$ the corresponding clan. The symbol $\Omega^*$ will denote the dual cone of $\Omega$. We have a normal decomposition (2.1) of $V$ relative to a complete orthogonal system $c_1, \ldots, c_r$ of primitive idempotents. Corresponding to (2.1), we have the basic relative invariants $\Delta_1(x), \ldots, \Delta_r(x)$. Since the numbering of a complete orthogonal system of primitive idempotents on which the numbering of basic relative invariants depends is not necessarily unique (cf. Lemma 3.2 of this paper), but since they are the irreducible factors of $\det R(x)$ of right multiplication operators $R(x)$ $(x \in V)$ of $V$ by [9], we introduce the following terminology for the formulation of the main theorem. We say that the basic relative invariants $\Delta_1(x), \ldots, \Delta_r(x)$ are of the degree type $1, \ldots, r$, if there is a permutation $\sigma$ of $r$ letters $1, \ldots, r$ such that $\sigma(j) = \deg \Delta_j(x)$ $(j = 1, \ldots, r)$.

Then, our main theorem is stated as follows.

**Theorem 1.1.** The homogeneous cone $\Omega$ is irreducible and symmetric if and only if both of the basic relative invariants for $\Omega$ and for $\Omega^*$ are of the degree type $1, \ldots, r$.

The only if part of Theorem 1.1 follows from what we have written above about principal minors. The main objective of this paper is of course to prove the if part.

Let us describe the organization of this paper. In Section 2, we collect fundamental facts on which our discussion will be based. In particular, we list up the technical terms from graph theory which we borrow in this paper. These technical terms are very useful to clarify our argument similarly to the previous paper [18]. In Section 3, two important properties of basic relative invariants are presented. The first property established in the previous paper [18, Proposition 4.2] shows in particular that the actual variables in the basic relative invariant $\Delta_k(x)$ come only from the out-neighbor subclan $V^\text{out}[k]$. This observation will be used several times in this paper. The second one is about a behavior under $V$-admissible permutations (see Section 3 for the definition of $V$-admissible permutations). The $V$-admissible permutations give permutations of the basic relative invariants, and Lemma 3.2 describes the precise permutations. We would like to emphasize that Lemma 3.2 is well understood with the graph theory terminology.

Our actual study starts at Section 4 which is devoted to a detailed analysis of the degrees of basic relative invariants. Propositions 4.4 and 4.5 are key to the main theorem, and Corollary 4.6 plays a crucial role in determining the degrees. Section 4 ends with a small but an important lemma which states that the degree type $1, \ldots, r$ of basic relative invariants occurs only if the set of sinks of $\Gamma$ contains only one point. Section 5 deals with the special case where the set of sources of $\Gamma$ consists only of one element. This assumption yields that $\Gamma$ is connected (Lemma 5.1). In this case, Theorem 5.2 shows that basic relative invariants are of the degree type $1, \ldots, r$ only if $\Gamma$ is complete, that is, only if every pair of distinct vertices is joined by an arc. In the final section, Section 6, we prove the main theorem (Theorem 6.8). Proposition 6.2 connects degree analysis in the previous sections with dimension investigation of the subspaces $V_{kj}$ $(k > j)$ appearing in the normal decomposition (2.1) of $V$. Our main theorem is proved by showing that $\dim V_{kj}$ $(k > j)$ are all equal, which is the criterion given by Proposition 3 of Vinberg [16, p. 73] for $\Omega$ to be symmetric.
Our main theorem may suggest that homogeneous cones for which the corresponding basic relative invariants are of the degree type 1, . . . , r form an interesting class $C$, next to symmetric cones, for further researches. We note that $C$ contains the class of square cones introduced by Xu [17, Ch. 6] (see Remark 6.5 of the present paper), and in particular the classification given in Theorem 6.31 in [17] exhibits examples of cones in $C$. However, we have non-square cones belonging to the class $C$ (see Example 6.6 in this paper). Thus $C$ is strictly larger than the class of square cones. We also add that the cone $\Omega^0$ defined in [13] (see also [14]) is a dual square cone in the sense of Xu [17, Ch. 6] (see Remark 6.7 of this paper).

Finally it should be remarked here that our main theorem may be derived from the results of Nakashima [12]. However, Nakashima’s results depend on a delicate analytic argument as well as on the results about Gindikin–Riesz distributions in Graczyk and Ishi [6] and Ishi [7]. In contrast, our method is a direct algebraic one, and we consider the present paper worth publication from this viewpoint.

2. Preliminaries

2.1. Clans

We begin with the definition of a clan. Let $V$ be a finite-dimensional real vector space with a bilinear product $x \triangle y$. We will write $L(x)y = R(y)x = x \triangle y$ by introducing the left multiplication operators $L(x)$ and the right multiplication operators $R(y)$. For two linear operators $A, B$, we put $[A, B] := AB - BA$. Then, we say that the pair $(V, \triangle)$ (or simply $V$) is a clan if the following three conditions are satisfied:

(C1) $[L(x), L(y)] = L(x \triangle y - y \triangle x)$ for any $x, y \in V$;

(C2) there exists $s_0 \in V^*$ such that $s_0(x \triangle y)$ defines an inner product in $V$;

(C3) for each $x \in V$, the operator $L(x)$ has only real eigenvalues.

The linear form $s_0$ in (C2) is said to be admissible. We note that the associative law is not assumed. In this paper, we always assume that a clan has a unit element.

Now let $(V, \triangle)$ be a clan with unit element $e_0$, and fix an inner product $(x | y) := s_0(x \triangle y)$ by taking an admissible linear form $s_0$. We denote by $r$ the rank of $V$. This means that there is a complete system $c_1, \ldots, c_r$ of orthogonal primitive idempotents by which we have an orthogonal decomposition

$$V = \bigoplus_{1 \leq j \leq k \leq r} V_{kj},$$

(2.1)

where $V_{kk} = \mathbb{R}c_k$ ($k = 1, 2, \ldots, r$), and for $1 \leq j < k \leq r$ we have

$$V_{kj} := \{x \in V \mid L(c_i)x = 2^{-1}(\delta_{ji} + \delta_{ki})x, \ R(c_i)x = \delta_{ji}x \ (i = 1, \ldots, r)\}.$$  

(2.2)

The decomposition (2.1) is called the normal decomposition of $V$ relative to $c_1, \ldots, c_r$, and we fix it once and for all. We write every $x \in V$ as

$$x = \sum_{j=1}^{r} \lambda_j c_j + \sum_{1 \leq k < m \leq r} x_{mk} \quad (\lambda_j \in \mathbb{R}, \ x_{mk} \in V_{mk}).$$  

(2.3)

The rank of $V$ is sometimes written as rank$(V)$ in this paper.
The multiplication rules between $V_{kj}$ are given as follows:

\[ V_{ji} \triangle V_{lk} = \{0\} \quad (\text{if } i \neq k, l), \quad V_{kj} \triangle V_{ji} \subset V_{ki}, \]
\[ V_{ji} \triangle V_{ki} \subset V_{jk} \text{ or } V_{kj} \quad (\text{according to } j \geq k \text{ or } j \leq k). \]

(2.4)

By introducing the lexicographic order among $V_{kj}$ in (2.1), every left multiplication operator $L(x)$ of $V$ is simultaneously represented by a lower triangular matrix.

By (C1) and (C3), the space $\mathfrak{h} := \{L(x) \mid x \in V\}$ of left multiplication operators of $V$ forms a split solvable Lie algebra. Let $H := \exp \mathfrak{h}$ be the connected and simply connected Lie group corresponding to $\mathfrak{h}$. The $H$-orbit $\Omega := He_0$ through $e_0$ is an open convex cone in $V$, and every homogeneous cone arises in this way. We note that the $H$-action on $\Omega$ is simply transitive.

We write down two useful lemmas given by Vinberg [15] here (see also our paper [18, Section 2]).

**Lemma 2.1.** Suppose that $i < j < k$. If $x \in V_{kj}$, $y \in V_{ji}$ and $z \in V_{ki}$, then one has

\[ \langle x \triangle y \mid z \rangle = \langle x \mid y \triangle z \rangle = \langle x \mid z \triangle y \rangle. \]

Before stating the following lemma, we note that $s_0(c_j) = s_0(c_j \triangle c_j) = \|c_j\|^2 \neq 0$.

**Lemma 2.2.** Suppose that $i < j < k$. If $x \in V_{kj}$ and $y \in V_{ji}$, then one has

\[ \|x \triangle y\| = 2^{-1/2} s_0(c_j)^{-1/2}\|x\|\|y\|. \]

(2.5)

### 2.2. Vinberg’s polynomials and basic relative invariants

For every $j = 1, \ldots, r$, we put

\[ V^{[j]} := \bigoplus_{m \geq k \geq j} V_{mk}. \]

(2.6)

Multiplication rules (2.4) tell us that $V^{[j]}$ is indeed a subclan of $V$. The inductive construction below of elements

\[ x^{[j]} = \sum_{k=j}^{r} \lambda^{[j]}_k c_k + \sum_{m > k \geq j} x^{[j]}_{mk} \in V^{[j]} \quad (j = 1, 2, \ldots, r) \]

(2.7)

given by Vinberg [15, III, §3] (see also Ishi [8, (1.11)]) plays a fundamental role in our degree analysis of basic relative invariants. Given $x \in V$ with (2.3), we put $x^{[1]} := x \in V^{[1]} = V$. When $x^{[j]} \in V^{[j]}$ is defined, we define $x^{[j+1]} \in V^{[j+1]}$ by setting

\[ \lambda^{[j+1]}_k := \lambda^{[j]}_k + 2^{-1} s_0(c_k)^{-1}\|x^{[j]}_{kj}\|^2 \quad (k = j + 1, \ldots, r), \]
\[ x^{[j+1]}_{mk} := \lambda^{[j+1]}_m x^{[j]}_{mk} - x^{[j]}_{mj} \triangle x^{[j]}_{kj} \quad (j + 1 \leq k < m \leq r). \]

(2.8)

(2.9)

Vinberg’s polynomials $D_j(x)$ are defined as

\[ D_j(x) := \lambda^{[j]}_j \quad (j = 1, 2, \ldots, r). \]

(2.10)

See Vinberg [15, formula (25), p. 385] (see also Ishi [8]).
Let \( H \) be the split solvable Lie group acting simply transitively on \( \Omega \) introduced in Section 2.1. A function \( f \) on \( \Omega \) is said to be \textit{relatively \( H \)-invariant} if there is a one-dimensional representation \( \chi \) of \( H \) such that \( f(hx) = \chi(h)f(x) \) for all \( h \in H \) and \( x \in \Omega \). Note that since \( H \) contains positive scalar multiplications, a relatively \( H \)-invariant function \( f \) is necessarily homogeneous, that is, there is \( a \in \mathbb{R} \) such that \( f(tx) = ta f(x) \) for all \( t > 0 \) and \( x \in \Omega \). Vinberg’s polynomials \( D_j(x) \) are relatively \( H \)-invariant (see Gindikin [5, Section 2]), and we have the following description of \( \Omega \):

\[
\Omega = \{ x \in V \mid D_1(x) > 0, \ldots, D_r(x) > 0 \}.
\] (2.11)

Moreover, by Ishi [8, Theorem 2.2], there exist relatively \( H \)-invariant \textit{irreducible} polynomial functions \( \Delta_1, \ldots, \Delta_r \) on \( V \) such that every relatively \( H \)-invariant polynomial function \( P \) on \( V \) is written as

\[
P(x) = P(e_0) \cdot \Delta_1(x)^{a_1} \cdots \Delta_r(x)^{a_r} \quad (n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}).
\]

These polynomials \( \Delta_1(x), \ldots, \Delta_r(x) \) are extracted from Vinberg’s polynomials in the following manner. First we put \( \Delta_1(x) := D_1(x) \). For \( k \geq 2 \) let \( \Delta_k(x) \) be the polynomial on \( V \) determined by requiring that each \( D_k(x) \) is written as

\[
D_k(x) = \Delta_k(x)\Delta_1(x)^{a_1}D_2(x)^{a_2} \cdots \Delta_{k-1}(x)^{a_{k-1}} \quad (a_{k_1}, \ldots, a_{k_{k-1}} \in \mathbb{Z}_{\geq 0})
\] (2.12)

with the condition that \( \Delta_k(x) \) is not divisible by any of \( \Delta_1(x), \ldots, \Delta_{k-1}(x) \). These \( \Delta_k(x) \) are called the \textit{basic relative invariants} of \( \Omega \) (and of \( V \)). We have another description of \( \Omega \) (see Ishi [8, Proposition 2.3]):

\[
\Omega = \{ x \in V \mid \Delta_1(x) > 0, \ldots, \Delta_r(x) > 0 \}.
\] (2.13)

By [9, Section 5], we know that \( \Delta_1(x), \ldots, \Delta_r(x) \) are \textit{the} irreducible factors of \( \det R(x) \). This is an intrinsic characterization of the basic relative invariants.

Recalling that \( H \) is split solvable, every \( h \in H \) is expressed in the following way (cf. Ishi [7, Proposition 2.1 (ii)]):

\[
h = (\exp T_1)(\exp L_1)(\exp T_2) \cdots (\exp L_{r-1})(\exp T_r),
\] (2.14)

where \( T_j := (2 \log h_j)L(c_j) \) with \( h_j > 0 \) (\( j = 1, \ldots, r \)) and

\[
L_j := \sum_{k>j} L(v_{kj}) \quad \text{with } v_{kj} \in V_{kj} \quad (1 \leq j < k \leq r).
\]

Vinberg’s polynomials \( D_j(x) \) appear in the unique solution of \( he_0 = x \) for a given \( x \in \Omega \). Indeed, by Vinberg [15, Section III-3], the numbers \( h_j > 0 \) in (2.14) of the unique solution \( h \in H \) are given by

\[
h_1^2 = D_1(x), \quad h_j^2 = D_1(x)^{-1} \cdots D_{j-1}(x)^{-1} D_j(x) \quad (j = 2, \ldots, r).
\]

This explains the description (2.11) of \( \Omega \).
2.3. Terminology from graph theory

In the previous paper [18], we borrowed some terminology from graph theory. In this paper also, terminology from graph theory is helpful to clarify the discussion. We summarize here the technical terms which will be used later. Our references are the books [2] and [3].

First of all, by a directed graph (or digraph for brevity) \( \Gamma = (V, A) \), we mean a pair of a finite set \( V = V(\Gamma) \) of elements called vertices of \( \Gamma \) and a set \( A = A(\Gamma) \) of ordered pairs, called arcs, of distinct vertices of \( \Gamma \). The sets \( V \) and \( A \) are called the vertex set and the arc set of \( \Gamma \), respectively. The cardinality of \( V \) is called the order of \( \Gamma \). In this paper we write \( v \to u \) or \([v \to u]\) for the arc \((v, u)\) to express visually that the arc \(v \to u\) leaves \( v\) and enters \( u\).

Let \( \Gamma = (V, A) \) be a digraph. For \( v \in V \), we set

\[
N^\text{out}(v) := \{ u \in V \setminus \{v\} \mid [v \to u] \in A \},
\]

\[
N^\text{in}(v) := \{ w \in V \setminus \{v\} \mid [w \to v] \in A \}.
\]

Elements of \( N^\text{out}(v) \) are called out-neighbors of \( v \). Similarly, elements of \( N^\text{in}(v) \) are called in-neighbors of \( v \). A vertex \( v \) is called a source (respectively sink) if \( N^\text{in}(v) = \emptyset \) (respectively \( N^\text{out}(v) = \emptyset \)). The set of sources of \( \Gamma \) is denoted by \( S(\Gamma) \). Similarly \( K(\Gamma) \) signifies the set of sinks of \( \Gamma \).

A weighted digraph is a digraph \( \Gamma \) with a function \( c : A \to \mathbb{R} \), where \( c \) is called the capacity function of \( \Gamma \). The value \( c([v \to u]) \) is called the capacity of the arc \( v \to u \). Digraphs which concern us in this paper have the property that whenever we have \([v \to u] \in A\), it holds that \([u \to v] \notin A\). These digraphs are called oriented graphs.

3. Properties of basic relative invariants

Let \( V \) be a clan of rank \( r \) with unit element \( e_0 \). Fix a complete system \( c_1, \ldots, c_r \) of orthogonal primitive idempotents so that we have the normal decomposition (2.1). We continue to use the same notation introduced in Section 2. For \( 1 \leq i < j \leq r \) we put \( d_{ji} := \dim V_{ji} \). To \( V \) we assign a digraph \( \Gamma \) of order \( r \) by defining the vertex set \( V \) and the arc set \( A \) respectively as

\[
V := \{1, 2, \ldots, r\}, \quad A := \{(j \to i) \mid j > i \text{ and } d_{ji} > 0\}.
\]

Clearly \( \Gamma \) is an oriented graph. Moreover, by (2.5), we see that \( \Gamma \) is necessarily transitive, that is, if \([k \to j] \in A \) and \([j \to i] \in A \), then we have \([k \to i] \in A \). Defining the capacity function \( c \) by \( c([j \to i]) = d_{ji} \) for \([j \to i] \in A \), we thus obtain a transitive weighted oriented graph \( \Gamma \) from \( V \). For simplicity, we call this transitive \( \Gamma \) the weighted oriented graph of the clan \( V \).

For every \( k = 1, \ldots, r \), we put \( N^\text{out}[k] := \{k\} \cup N^\text{out}[k] \). Using the symbol \( \bigoplus^{[k]} \) for a direct sum formed only by indices belonging to \( N^\text{out}[k] \), we define†

\[
V^\text{out}[k] := \bigoplus^{[k]}_{i \leq j} V_{ji}, \quad E^\text{out}[k] := \bigoplus^{[k]}_{i} V_{ki}.
\]  

(3.1)

It should be noted here that none of the direct summands \( V_{k,i} \) appearing in the definition of \( E^\text{out}[k] \) reduce to \( 0 \). By multiplication rules (2.4), it is clear that \( V^\text{out}[k] \) is a subclan of \( V \),

†We have changed the notation. In [18], \( V^\text{out}[k] \) is denoted by \( V_{[k]} \), and similarly for \( E^\text{out}[k] \).
and that $E^{\text{out}}[k]$ is a two-sided ideal of $V^{\text{out}}[k]$. Moreover, $e_{[k]} := \sum_i [k] c_i$ is the unit element of $V^{\text{out}}[k]$, where $\Sigma_i [k]$ indicates that the summation variable $i$ runs only over $N^{\text{out}}[k]$. We call $V^{\text{out}}[k]$ the out-neighbor subclan of $V$ corresponding to the vertex $k$.

Let $\pi[k]$ denote the orthogonal projector $V \rightarrow V^{\text{out}}[k]$. Enumerating $N^{\text{out}}[k]$ as

$$N^{\text{out}}[k] : i_1 < \cdots < i_{t-1} < i_t = k,$$

we denote the basic relative invariants of $V^{\text{out}}[k]$ by

$$\Delta_1^{V^{\text{out}}[k]}(x), \ldots, \Delta_r^{V^{\text{out}}[k]}(x).$$

Then, we have the following proposition (see [18, Proposition 4.2]).

**Proposition 3.1.** One has $\Delta_i(x) = \Delta_j^{V^{\text{out}}[k]}(\pi[k](x)) (x \in V)$ for $j = 1, \ldots, t$.

Let $\mathfrak{S}_r$ denote the symmetric group over $r$ letters $1, \ldots, r$. We say that $\sigma \in \mathfrak{S}_r$ is $V$-admissible if $i < j$ and $\sigma(i) > \sigma(j)$ imply $V_{ji} = V_{\sigma(i)\sigma(j)} = 0$. As in Vinberg [15, Definition 4, p. 381], $V$-admissible permutations correspond to inessential changes in the grading. Let $\sigma \in \mathfrak{S}_r$ be $V$-admissible, and we set

$$c_{\sigma(i)}^i := c_i, \quad V_{\sigma(k)\sigma(j)}^\prime := V_{kj}.$$  

Then we have another normal decomposition of $V$ given by

$$V = \bigoplus_{i=1}^r \mathbb{R} c_i^i \bigoplus \left( \bigoplus_{k>j} V_{kj} \right).$$  

From (3.3), another set of basic relative invariants

$$\Delta_1^\prime(x), \ldots, \Delta_r^\prime(x)$$

arises through the procedure described in Section 2.2. Since the basic relative invariants are the irreducible factors of $\det R(x)$, the presentation (3.4) is just a permutation of the original basic relative invariants $\Delta_1(x), \ldots, \Delta_r(x).$ The precise description is given as follows.

**Lemma 3.2.** One has $\Delta_k(x) = \Delta_{\sigma(k)}^\prime(x) (x \in V)$.

**Proof.** This follows from the fact that $\sigma$ permutes the ‘components’ of the clan $V$ as noted in Remark 4.15 in [18]. To be more convinced, we add the prime symbol for those which arise from the normal decomposition (3.3). For example, $(N')^{\text{out}}[k]$ is the set of out-neighbors of a vertex $k$ in the oriented graph $\Gamma'$ of $V$ formed on the ground of the normal decomposition (3.3). Thus $(N')^{\text{out}}[k] := \{ k \} \cup (N')^{\text{out}}[k]$. Then, since $\sigma$ is $V$-admissible, the set $(N')^{\text{out}}[k]$ is described, with the enumeration of $N^{\text{out}}[k]$ as in (3.2), by

$$(N')^{\text{out}}[\sigma(k)] = \{ \sigma(i_1), \ldots, \sigma(i_t) \},$$

and $\sigma(i_t) = \sigma(k)$ is still the maximum in $(N')^{\text{out}}[\sigma(k)]$, although some pair $i_a, i_b \ (1 \leq a < b < t)$ might be inverted by $\sigma$, in which case we have $V_{\sigma(i_a)\sigma(i_b)} = V_{ib,ia} = 0$. As in (3.1), let $(V')^{\text{out}}[k]$ be the out-neighbor subclan of $V$ corresponding to the vertex $k$ of $\Gamma'$. Similarly, $\bigoplus_{p \leq q} [k]$ denotes the direct sum in which the summation variables $p \leq q$ run only over $(N')^{\text{out}}[k]$. Then, by (3.5),

$$\left( V^{\text{out}}[\sigma(k)] \right) = \bigoplus_{p \leq q} [\sigma(k)] V_{qp}^{\text{out}} = \bigoplus_{i \leq j} [k] V_{\sigma(j)\sigma(i)}^\prime = V^{\text{out}}[k].$$

Now Proposition 3.1 for $j = t$ gives the formula in the lemma.
4. Some degree analysis of the basic relative invariants

We keep to the same notation used in the previous sections. In particular, we express every element \( x \in V \) as in (2.3), and we emphasize that by a polynomial on \( V \) we mean a polynomial of the variables \( \lambda_j \) and \( x_{mk} \) in (2.3). The purpose of the present section is to study in detail the degrees in the \( \lambda_j \) variables of the basic relative invariants. In what follows, we denote, for numerical or vector-valued polynomials \( p(x) \) on \( V \), by \( \text{Deg} p(x) \) the degree of \( p(x) \) in the usual sense, and by \( \text{Deg}_{\lambda_j} p(x) \) the degree in the \( \lambda_j \)-variable of \( p(x) \).

Let \( \lambda^{[j]}_k \) \((k \geq j)\) and \( x^{[j]}_{mk} \) \((m > k \geq j)\) be the polynomial functions defined by the recursion formulae (2.8) and (2.9), respectively. We also recall the definition \( D_j(x) = \lambda^{[j]}_j \) of Vinberg’s polynomials \( D_j(x) \).

**Lemma 4.1.** Fix any integer \( j \) such that \( 1 \leq j \leq r \). Then, we have the following assertions (1)–(3).

1. For \( p \geq j \), it holds that
   \[
   \text{Deg}_{\lambda_{k}} \lambda^{[j]}_k = 0 \quad (p \neq k, \ k \geq j), \quad \text{Deg}_{\lambda_{k}} x^{[j]}_{mk} = 0 \quad (m > k \geq j).
   \]

2. If \( p > j \), then it holds that \( \text{Deg}_{\lambda_p} D_j(x) = 0 \).

3. Suppose \( p \geq j \). Then with a polynomial \( g^{[j]}_p(x) \) not containing the variable \( \lambda_p \), one has an expression
   \[
   \lambda^{[j]}_p = D_1(x)D_2(x) \cdots D_{j-1}(x)\lambda_p + g^{[j]}_p(x).
   \]
   In particular, it holds that \( \text{Deg}_{\lambda_p} \lambda^{[j]}_p = 1 \), and for \( p = j \) this means \( \text{Deg}_{\lambda_j} D_j(x) = 1 \).

**Remark 4.2.** We note here that an explicit formula for \( p = j \) in the above (3) is found in Ishi [8, Proposition 1.4].

**Proof.** (1) We prove the assertion by induction on \( j \). The case \( j = 1 \) is trivially true. We suppose that the assertion is true for \( j \), and proceed to \( j + 1 \). Let \( p \geq j + 1 \) and \( k \neq p \), then the induction hypothesis gives us

\[
\text{Deg}_{\lambda_{k}} \lambda^{[j+1]}_k = \text{Deg}_{\lambda_{k}} \lambda^{[j]}_k = \text{Deg}_{\lambda_{k}} x^{[j]}_{kj} = 0.
\]

Hence (2.8) yields \( \text{Deg}_{\lambda_{k}} \lambda^{[j+1]}_k = 0 \). Next suppose \( m > k \geq j + 1 \). Then, the induction hypothesis gives \( \text{Deg}_{\lambda_{p}} x^{[j]}_{mk} = \text{Deg}_{\lambda_{p}} x^{[j]}_{mj} = 0 \) in addition to (4.1). Thus by (2.9) we arrive at

\[
\text{Deg}_{\lambda_{p}} x^{[j+1]}_{mk} = 0.
\]

(2) Immediate from (1) and \( D_j(x) = \lambda^{[j]}_j \).

(3) The assertions are all evident for the case \( j = 1 \). Suppose that the formulae for \( \lambda^{[j]}_p \) \((p \geq j)\) are true for \( j \), and we consider the case \( j + 1 \). We suppose \( p \geq j + 1 \), and look at the formula (2.8) for \( k = p \). By (2), the polynomial \( \lambda^{[j]}_p = D_j(x) \) does not contain \( \lambda_p \), and the induction hypothesis says that the only term in \( \lambda^{[j]}_p \) that contains \( \lambda_p \) is \( D_1(x) \cdots D_{j-1}(x)\lambda_p \). Moreover, \( x^{[j]}_{kj} \) \((k > j)\) is independent of \( \lambda_p \) by (1). Hence (2.8) tells us that, in \( \lambda^{[j+1]}_p \), only the term \( D_j(x)D_1(x) \cdots D_{j-1}(x)\lambda_p \) contains \( \lambda_p \). This completes the induction, and we are done. \( \square \)
LEMMA 4.3. (1) For $i < j \leq k < m$, one has
\[ \text{Deg}_{\lambda_i} \lambda_k^{[j]} \leq 2^{j-i-1}, \quad \text{Deg}_{\lambda_i} x_{mk} \leq 2^{j-i-1}. \]

(2) $\text{Deg}_{\lambda_i} D_j (x) = 2^{j-i-1}$ for $i < j$, and $D_j (x)$ has the term
\[ \lambda_j \lambda_{j-1} \lambda_{j-2} \cdots \lambda_1^{2^{j-1}} \quad (j \geq 2), \quad \lambda_1 \quad (j = 1). \]

Proof. For $j = 1$, the only thing to be verified is the last assertion in (2), which is obvious, since $D_1 (x) = \lambda_1$. Hence we suppose $j \geq 2$ in the rest of the proof.

(1) We prove the assertion by induction on $j$. For $j = 2$, we have by (2.8),
\[ \lambda_k^{[2]} = \lambda_1 \lambda_k - \frac{1}{2} x_0 (c_k)^{-1} \| x_{k1} \|^2, \]
so that $\text{Deg}_{\lambda_k} \lambda_k^{[2]} = 1$. Furthermore, (2.9) tells us that
\[ x_{mk}^{[2]} = \lambda_1 x_{mk} - x_{m1} \Delta x_{k1}. \]

This also implies $\text{Deg}_{\lambda_i} x_{mk}^{[2]} = 1$.

Next we assume that the inequalities in (1) are true for $j$, and we proceed to $j + 1$. Suppose $i < j + 1 \leq k < m$. If $i < j$, then (2.8) shows that
\[ \text{Deg}_{\lambda_i} \lambda_k^{[j+1]} \leq \max (\text{Deg}_{\lambda_i} \lambda_k^{[j]} + \text{Deg}_{\lambda_i} x_{mk}^{[j]} \text{Deg}_{\lambda_i} \| x_{k1} \|^2) \leq 2^{j-i}. \]

For $i = j$, Lemma 4.1 tells us that
\[ \text{Deg}_{\lambda_j} \lambda_k^{[j]} = 1, \quad \text{Deg}_{\lambda_j} x_{kj}^{[j]} = 0, \]
which together with (2.8) give us $\text{Deg}_{\lambda_j} \lambda_k^{[j+1]} = 2^{j-j}$. For the second inequality, we have by (2.9),
\[ \text{Deg}_{\lambda_i} x_{mk}^{[j+1]} \leq \max (\text{Deg}_{\lambda_i} \lambda_k^{[j]} + \text{Deg}_{\lambda_i} x_{mk}^{[j]}, \text{Deg}_{\lambda_i} x_{mj}^{[j]} + \text{Deg}_{\lambda_i} x_{kj}^{[j]}). \tag{4.2} \]

If $i < j$, the induction hypothesis says that all of the four degrees appearing on the right-hand side of (4.2) are $\leq 2^{j-i-1}$, so that the right-hand side of (4.2) is $\leq 2^{j-i}$. If $i = j$, Lemma 4.1 shows that
\[ \text{Deg}_{\lambda_j} \lambda_k^{[j]} = 1, \quad \text{Deg}_{\lambda_j} x_{mk}^{[j]} = \text{Deg}_{\lambda_j} x_{mj}^{[j]} = \text{Deg}_{\lambda_j} x_{kj}^{[j]} = 0, \]
which give $\text{Deg}_{\lambda_j} x_{mk}^{[j+1]} = 2^{j-j}$ by (2.9). The induction is now complete.

(2) We also prove (2) by induction on $j$. For $j = 2$, we have by (2.8),
\[ D_2 (x) = \lambda_2^{[2]} = \lambda_1 \lambda_2 - \frac{1}{2} x_0 (c_k)^{-1} \| x_{21} \|^2. \]

This shows that $\text{Deg}_{\lambda_2} D_2 (x) = 1$, and that $D_2 (x)$ contains the term $\lambda_2 \lambda_1$. Next we assume the assertion for $\leq j$, and proceed to $j + 1$. Let $i < j + 1$. The formula in Lemma 4.1(3) shows that
\[ \text{Deg}_{\lambda_i} \lambda_{j+1}^{[j+1]} \geq \sum_{a=1}^{j} \text{Deg}_{\lambda_i} D_a (x) = \sum_{a=i}^{j} \text{Deg}_{\lambda_i} D_a (x) \] (by Lemma 4.1(2))
\[ = 1 + \sum_{a=i+1}^{j} 2^{a-i-1} \] (by the induction hypothesis)
\[ = 1 + (2^{j-i} - 1) = 2^{j-i}. \]
On the other hand, we have $\deg_{\lambda_i} \lambda_{j+1}^{[j]} \leq 2j-i$ by (1) of the present lemma. Hence (4.3) holds by equality, so that $\deg_{\lambda_i} D_{j+1}(x) = \deg_{\lambda_i} \lambda_{j+1}^{[j]} = 2j-i$. Moreover, by Lemma 4.1(3), we see that $\lambda_{j+1}^{[j]}$ contains the term

$$D_1(x)D_2(x) \cdots D_j(x)\lambda_{j+1} = (\lambda_1)(\lambda_2\lambda_1) \cdots (\lambda_j\lambda_{j-1}\lambda_{j-2}^2 \cdots \lambda_1^{2j-2})\lambda_{j+1}$$

$$= \lambda_{j+1}\lambda_j\lambda_{j-1}^2 \cdots \lambda_1^{2j-1}.$$ 

This finishes the proof. 

We now consider the oriented graph $\Gamma = \Gamma(V)$ of the clan $V$. Let $k$ be a vertex in $\Gamma$, and consider its out-neighbors $N^{\text{out}}(k)$. Let $\Gamma_{N^{\text{out}}(k)}$ be the oriented subgraph of $\Gamma$ given by $N^{\text{out}}(k)$, and we denote by $S^{\text{out}}(k)$ the sources of $\Gamma_{N^{\text{out}}(k)}$, that is, we put $S^{\text{out}}(k) := S(\Gamma_{N^{\text{out}}(k)})$.

**Proposition 4.4.** For any $k = 1, \ldots, r$, one has $\deg_{\lambda_k} \Delta_k(x) = 1$. Moreover, with polynomials $f_k(x)$ and $g_k(x)$ not containing the variable $\lambda_k$, the polynomial $\Delta_k(x)$ is expressed as

$$\Delta_k(x) = f_k(x)\left(\prod_{j \in S^{\text{out}}(k)} \Delta_j(x)\right)\lambda_k + g_k(x).$$

**Proof.** If the rank $r$ of $V$ equals one, the assertion is trivially true. Hence we suppose $r \geq 2$ in the rest of the proof. Moreover, Proposition 3.1 convinces us that to prove the proposition it is enough to discuss by supposing that the set $S$ of the sources of $\Gamma$ contains only one element, that is, $S = \{r\}$, and that $k = r$.

First of all, we note that $\deg_{\lambda_r} \Delta_r(x) = 1$ by Lemma 4.1(3), and that $\deg_{\lambda_j} \Delta_k(x) = 0$ for $k < r$ by Lemma 4.1(2). From the latter fact together with the requirement (2.12) for the definition of $\Delta_k(x)$, we see that $\Delta_k(x)$ for $k < r$ cannot contain the variable $\lambda_r$. A similar reasoning gives $\deg_{\lambda_r} \Delta_r(x) = 1$, which is actually the first assertion of the proposition.

Now, by what has been shown right now we express $\Delta_r(x)$ as

$$\Delta_r(x) = f(x)\lambda_r + g(x),$$

where both of the polynomials $f(x)$, $g(x)$ are independent of $\lambda_r$. Moreover, in the expression

$$D_r(x) = h(x)\Delta_r(x),$$

the polynomial $h(x)$ does not contain the variable $\lambda_r$. We have

$$D_r(x) = h(x)f(x)\lambda_r + h(x)g(x).$$

**Claim.** The polynomial $f(x)$ is divisible by $\Delta_j(x)$ for any $j \in S^{\text{out}}(r)$.

**Proof of the claim.** (1) Noting that $r - 1 \in S^{\text{out}}(r)$, we first prove the claim for $j = r - 1$. Fixing a vector $\xi \in V_{r-1}$ with $\|\xi\|^2 = 2s_0(c_r)$, we consider the following two-dimensional affine subspace $Y$ of $V$:

$$Y = c_1 + \cdots + c_{r-2} + \xi + (\mathbb{R}c_{r-1} \oplus \mathbb{R}c_r).$$

Every $y \in Y$ is written as

$$y = c_1 + \cdots + c_{r-2} + \xi + \lambda_r c_{r-1} + \lambda_r c_r \quad (\lambda_{r-1}, \lambda_r \in \mathbb{R}).$$
We restrict Vinberg’s polynomials $D_1(x), \ldots, D_r(x)$ to $Y$, and investigate the polynomials $D_1(y), \ldots, D_r(y)$ ($y \in Y$) in the variables $\lambda_{r-1}, \lambda_r$. A straightforward computation starting with $y$ using (2.8) and (2.9) yields elements $y^{[j]} \in V^{[j]}$ given by
\[
y^{[j]} = c_j + \cdots + c_{r-2} + \xi + \lambda_{r-1}c_{r-1} + \lambda_r c_r \quad (1 \leq j \leq r - 2),
\]
\[
y^{[r-1]} = \xi + \lambda_{r-1}c_{r-1} + \lambda_r c_r, \quad y^{[r]} = (\lambda_{r-1} - 1)c_r.
\]
Looking at the coefficient of $c_j$ in $y^{[j]}$, we obtain
\[
D_j(y) = 1 \quad (1 \leq j \leq r - 2), \quad D_{r-1}(y) = \lambda_{r-1}, \quad D_r(y) = \lambda_{r-1} - 1.
\] (4.8)
Since we have already shown that $\deg_{\lambda_{r-1}} \Delta_{r-1}(x) = 1$, and since $\Delta_{r-1}(y)$ is a factor of $D_{r-1}(y)$, we necessarily have $\deg_{\lambda_{r-1}} \Delta_{r-1}(y) = \lambda_{r-1}$. Then, (4.8) for $D_r(y)$ shows that $D_r(y)$ is not divisible by $\Delta_{r-1}(y)$. Hence the original Vinberg’s polynomial $D_r(x)$ on $V$ is not divisible by $\Delta_{r-1}(x)$ either. Thus, in (4.4), we see that $h(x)$ is not divisible by $\Delta_{r-1}(x)$.

Now we equate the expression of $D_r(x)$ in (4.5) with the expression in Lemma 4.1(3) $(p = j = r)$. Then, with a polynomial $g_r^{[r]}(x)$ not containing the variable $\lambda_r$, we obtain
\[
h(x)f(x)\lambda_r + h(x)g(x) = D_1(x)D_2(x) \cdots D_{r-1}(x)\lambda_r + g_r^{[r]}(x).
\]
By Lemma 4.1(2), this is an identity of polynomials in $\lambda_r$ written in descending order by degree, so that we obtain
\[
h(x)f(x) = D_1(x)D_2(x) \cdots D_{r-1}(x).
\]
Since $h(x)$ is not divisible by $\Delta_{r-1}(x)$ by the preceding paragraph, and since $D_{r-1}(x)$ is divisible by $\Delta_{r-1}(x)$, we conclude that $f(x)$ is divisible by $\Delta_{r-1}(x)$.

(2) Now we consider a general $j \in S^{\text{out}}(r)$ ($j < r - 1$). Let $\sigma = (r - 1, \ldots, j)$ be the cyclic permutation. In $\sigma$, the inversions occur only for the pairs $j, k$ ($k = j + 1, \ldots, r - 1$). Since $j \in S^{\text{out}}(r)$, we do not have an arc from any $k$ to $j$ ($k > j$) (note that we are assuming $S = \{r\}$, so that $N^{\text{out}}(r) = \{1, \ldots, r - 1\}$). This implies $V_{kj} = \{0\}$, so that the cyclic permutation $\sigma$ considered as an element of $S_r$ is $V$-admissible. Then, by Lemma 3.2, we have $\Delta_i(x) = \Delta_{\sigma(i)}(x)$ for any $i$ with the notation used therein applied to the current $\sigma$. In particular, we have $\Delta_j(x) = \Delta_{\sigma(j)}(x) = \Delta_{r-1}(x)$. Applying the discussion in (1) to $\Delta_{r-1}(x)$, and noting that $\sigma$ fixes $r$, we see that $f(x)$ is divisible by $\Delta_j(x)$. Therefore the claim is established.

From (1) and (2), and from the fact that the polynomials $\Delta_i(x)$ are irreducible and not divisible by each other, the proof of the proposition is complete.

PROPOSITION 4.5. Fix an integer $k$ ($1 \leq k \leq r$).

(1) It holds that $\deg_{\lambda_j} \Delta_k(x) \geq 1$ if and only if $j \in N^{\text{out}}[k]$.

(2) Let the enumeration of $N^{\text{out}}[k]$ be as in (3.2). Then, $\Delta_k(x)$ has the term $\lambda_{\beta_1}^{\beta_1} \cdots \lambda_{\beta_t}^{\beta_t} \lambda_k$ with $\beta_j := \deg_{\lambda_j} \Delta_k(x) \geq 1$ ($j = 1, \ldots, t - 1$).

Proof. (1) If $j \notin N^{\text{out}}[k]$, then Proposition 3.1 shows that the variable $\lambda_j$ does not appear in $\Delta_k(x)$. To study the case $j \in N^{\text{out}}[k]$, we suppose, as in the proof of Proposition 4.4, that $S = \{r\}$ and $k = r$. By the first assertion of Proposition 4.4, we know that $\deg_{\lambda_j} \Delta_r(x) = 1$.

CLAIM. For $j = 1, \ldots, r$ one has $\deg_{\lambda_j} \Delta_r(x) \geq 1$.
Proof of the claim. We prove the claim by induction on the rank $r$ of $V$. For $r = 1$, the claim is clear. We suppose that the claim is true for any clan of rank $\leq r - 1$. It suffices to consider the case $j \leq r - 1$. Let us set $N_{\text{in}}[j] := \{j\} \cup N_{\text{in}}(j)$. Let $k_0$ be the maximum of the elements in $N_{\text{in}}[j] \setminus \{r\}$. Then we have $k_0 \in S_{\text{out}}(r)$. In fact, if not, then there exists an integer $l$ with $r > l > k_0$ such that we have an arc $[l \to k_0] \in A$. Since our graph $\Gamma$ is transitive, we necessarily have $[l \to j] \in A$, so that $l \in N_{\text{in}}[j]$. Since $l > k_0$, this together with the definition of $k_0$ implies $l = r$, which contradicts $l < r$. Thus $k_0 \in S_{\text{out}}(r)$, and noting $j \in S_{\text{out}}[k_0]$, we have by Proposition 3.1 and the induction hypothesis

$$\deg_{\lambda_j} \Delta_{k_0}(x) = \deg_{\lambda_j} \Delta_{V_{\text{out}}[k_0]}(\pi_{[k_0]}(x)) \geq 1 \quad (t := \text{rank}(V_{\text{out}}[k_0]) < r).$$

Now Proposition 4.4 gives

$$\Delta_r(x) = f_r(x) \left( \prod_{j \in S_{\text{out}}(r)} \Delta_j(x) \right) \lambda_r + g_r(x). \quad (4.9)$$

Since $k_0 \in S_{\text{out}}(r)$, we arrive at $\deg_{\lambda_j} \Delta_r(x) \geq \deg_{\lambda_j} \Delta_{k_0}(x) \geq 1$. \hfill $\square$

(2) Here also we assume that $S = \{r\}$ and $k = r$. First of all, we express $D_r(x)$ as in (4.4). Then we have

$$\deg_{\lambda_j} \Delta_r(x) + \deg_{\lambda_j} h(x) = \deg_{\lambda_j} D_r(x). \quad (4.10)$$

On the other hand, the second assertion in Lemma 4.3(2) tells us that $D_r(x)$ has the term $\lambda_r \lambda_{r-1} \cdots \lambda_2^{r-2}$. Then, $\Delta_r(x)$ has the term of the form $\lambda_{j_1}^{\beta_{j_1}} \cdots \lambda_{j_r}^{\beta_{j_r}} (\beta_{j_1}, \ldots, \beta_r \in \mathbb{Z}_{\geq 0})$, and similarly for $h(x)$. Hence we get

$$\deg_{\lambda_j} \Delta_r(x) \geq \beta_j, \quad \deg_{\lambda_j} h(x) \geq \deg_{\lambda_j} D_r(x) - \beta_j, \quad (4.11)$$

where the second inequality follows from (4.4). In view of (4.10) two inequalities in (4.11) hold by equalities. In particular, we have $\beta_j = \deg_{\lambda_j} \Delta_r(x)$. \hfill $\square$

Proposition 4.5 has the following important corollary.

**COROLLARY 4.6.** One has $\deg \Delta_k(x) \geq \text{rank}(V_{\text{out}}[k])$, where the equality holds if and only if $\deg_{\lambda_j} \Delta_k(x) = 1$ holds for any $j \in N_{\text{out}}[k]$.

We say that the basic relative invariants $\Delta_1(x), \ldots, \Delta_r(x)$ are of the degree type $1, \ldots, r$ if there is a permutation $\sigma \in \mathfrak{S}_r$ such that $\deg \Delta_j(x) = \sigma(j)$ for any $j = 1, \ldots, r$. The following simple lemma is used in the subsequent sections.

**LEMMA 4.7.** If $\Delta_1(x), \ldots, \Delta_r(x)$ are of the degree type $1, \ldots, r$, then the set $\mathcal{K}$ of sinks of $\Gamma$ contains only one element, that is, $\mathcal{K} = \{1\}$.

**Proof.** Take any $i \in \mathcal{K}$. Then, since $V_{\text{out}}[i] = \mathbb{R}c_i$, we have by Proposition 3.1,

$$\deg \Delta_i(x) = \deg \Delta_1^{V_{\text{out}}[i]}(\pi_{[i]}(x)) = 1 = \deg \Delta_1(x).$$

Since $j \mapsto \deg \Delta_j(x)$ is injective by assumption, we obtain $i = 1$. \hfill $\square$
5. The case $\mathcal{S} = \{r\}$

We assume in this section that there is only one source in our oriented graph $\Gamma$, that is, $\mathcal{S} = \{r\}$. This implies that $N_{\text{out}}(r) = \{1, \ldots, r - 1\}$. We begin with the following simple lemma.

**Lemma 5.1.** Under the assumption that $\mathcal{S} = \{r\}$, our graph $\Gamma$ is connected.

*Proof.* Suppose that two vertices $i, j$ ($i < j$) are not joined by an arc. Since $\mathcal{S} = \{r\}$, they must have the common source $r$, so that there is a path connecting $i$ and $j$. In reality, the transitivity of $\Gamma$ says that actually we have the arcs $r \to i$ and $r \to j$. Anyway, $\Gamma$ is connected. \hfill $\Box$

**Theorem 5.2.** Suppose that $\Delta_r(x), \ldots, \Delta_r(x)$ are of the degree type $1, \ldots, r$. Then, one has the following.

1. The graph $\Gamma$ is complete. In other words, one has $[j \to i] \in A$ for any pair $i, j$ such that $i < j$.
2. $\deg \Delta_j(x) = j$ for any $j = 1, \ldots, r$.

*Proof.* By the assumption of the theorem, we take $\sigma \in \mathcal{S}_r$ such that $\sigma(j) = \deg \Delta_j(x)$ holds for any $j = 1, \ldots, r$.

(1) We prove the assertion by induction on the rank $r$ of $V$. For $r = 1$, we have nothing to prove. For $r = 2$, the assertion follows from Lemma 5.1. Thus we assume that $r \geq 3$, and suppose that the assertion is true for any clan of rank $r - 1$. The assumption of the theorem implies, in particular, that $\deg \Delta_r(x) \leq r$. Since Corollary 4.6 tells us the reverse inequality, we obtain $\deg \Delta_r(x) = r$. Hence $\sigma(r) = r$, so that we can consider $\sigma$ as an element of $\mathcal{S}_{r-1}$, and we have $\sigma(j) = \deg \Delta_j(x)$ for any $j = 1, \ldots, r - 1$. On the other hand, Lemma 4.7 says that $\mathcal{K} = \{1\}$. Then the transitivity of $\Gamma$ implies that $1 \in N_{\text{out}}[j]$ for any $j$. Hence Proposition 4.5(1) tells us that

$$\deg_{\mathcal{S}_r} \Delta_j(x) \geq 1 \quad \text{(for any $j$).} \quad (5.1)$$

We shall show that $\mathcal{S}_{\text{out}}(r)$ is a one-element set, that is, $\mathcal{S}_{\text{out}}(r) = \{r - 1\}$. Suppose contrarily that we have $l < r - 1$ such that $l \in \mathcal{S}_{\text{out}}(r)$. Then, (4.9) and (5.1) yield

$$\deg_{\mathcal{S}_r} \Delta_r(x) \geq \deg_{\mathcal{S}_r} \Delta_j(x) + \deg_{\mathcal{S}_r} \Delta_{r-1}(x) \geq 2.$$ 

Then the equality condition in Corollary 4.6 fails to hold, so that

$$\deg \Delta_r(x) \geq \text{rank}(V_{\text{out}}^*) = \text{rank}(V) = r,$$

which contradicts the already established fact that $\deg \Delta_r(x) = r$. Hence $\mathcal{S}_{\text{out}}(r) = \{r - 1\}$, and thus the transitivity of $\Gamma$ shows that $N_{\text{out}}[r - 1] = \{1, \ldots, r - 1\}$. Hence $r - 1$ is the unique source of the oriented graph $\Gamma(V_{\text{out}}[r - 1])$. By Proposition 3.1 we have $\Delta_j^{V_{\text{out}}[r - 1]}(x) = \Delta_j(x)$ ($x \in V_{\text{out}}[r - 1]$) for $j = 1, \ldots, r - 1$, and thus $\deg \Delta_j^{V_{\text{out}}[r - 1]}(x) = \sigma(j)$. Hence we apply the induction hypothesis to $V_{\text{out}}[r - 1]$ in order to obtain the completeness of the graph $\Gamma(V_{\text{out}}[r - 1])$. This together with $N_{\text{out}}(r) = \{1, \ldots, r - 1\}$ implies that the original graph $\Gamma$ is complete.

(2) By (1), we have $j - 1 \in N_{\text{out}}(j)$ for $j > 1$, so that $j - 1 \in \mathcal{S}_{\text{out}}(j)$. Then, Proposition 4.4 tells us that $\sigma(j) = \deg \Delta_j(x) \geq \deg \Delta_{j-1}(x) = \sigma(j - 1)$ for any $j = 2, \ldots, r$. This together with $\sigma \in \mathcal{S}_r$ forces that $\sigma(j) = j$ for any $j = 1, \ldots, r$. \hfill $\Box$
LEMMA 5.3. Under the same assumption as Theorem 5.2, suppose that \( r \geq 3 \). Then one has, for \( k = 3, \ldots, r \),

\[
D_k(x) = \Delta_1(x)^{2k-3} \Delta_2(x)^{2k-4} \cdots \Delta_{k-2}(x) \Delta_k(x) \\
= D_1(x)^{k-2} \cdots D_{k-2}(x) \Delta_k(x).
\]

Proof. The assertion Lemma 4.3(2) and (2.12) show that, for \( i<k \),

\[
2^{k-i-1} = \text{Deg}_{\lambda_i} D_k(x) = \sum_{j=1}^{k-1} a_{kj} \text{Deg}_{\lambda_i} \Delta_j(x) + \text{Deg}_{\lambda_i} \Delta_k(x).
\]  
(5.2)

Here we note first that \( \Delta_j(x) \) is independent of \( \lambda_i \) for \( i>j \) by Proposition 4.5, and that \( \text{rank}(V^{\text{out}}[j]) = j \) for any \( j = 1, \ldots, r \) by the completeness of \( \Gamma \) which is the conclusion Theorem 5.2(1). Then Theorem 5.2(2) shows that \( \text{Deg} \Delta_j(x) = j = \text{rank}(V^{\text{out}}[j]) \), so that Corollary 4.6 yields \( \text{Deg}_{\lambda_i} \Delta_j(x) = 1 \) for any \( i \leq j \). Hence (5.2) gives the equation

\[
A_i : \quad 2^{k-i-1} = 1 + \sum_{j=i+1}^{k-1} a_{kj} (i = 1, \ldots, k-1).
\]

We make \( A_i - A_{i+1} \) for \( i = 1, \ldots, k-2 \), and also we look at \( A_{k-1} \). Then we obtain

\[
2^{k-i-2} = a_{ki} (i = 1, \ldots, k-2), \quad a_{k,k-1} = 0.
\]

Thus we obtain the first equality.

Next we start with \( D_1(x) = \Delta_1(x) \), and an easy induction by using (2.12) gives

\[
D_k(x) = \Delta_k(x) D_1(x)^{\beta_{k1}} \cdots D_{k-1}(x)^{\beta_{k,k-1}} \quad (k = 1, 2, \ldots)
\]

with some integers \( \beta_{k1}, \ldots, \beta_{k,k-1} \). If \( i<k \), then Lemma 4.3(2) gives

\[
2^{k-i-1} = \text{Deg}_{\lambda_i} D_k(x) = \text{Deg}_{\lambda_i} \Delta_k(x) + \sum_{j=1}^{k-1} \beta_{kj} \text{Deg}_{\lambda_i} D_j(x).
\]  
(5.3)

Here we have \( \text{Deg}_{\lambda_i} D_j(x) = 2^{j-i-1} \) for \( j>i \) by Lemma 4.3(2), and

\[
\text{Deg}_{\lambda_i} D_j(x) = \delta_{ij} \quad \text{for } i \geq j \text{ by Lemma 4.1(2) and (3)},
\]

where \( \delta_{ij} \) is the Kronecker delta. Moreover, as shown in the previous discussion we have \( \text{Deg}_{\lambda_i} \Delta_k(x) = 1 \). Thus (5.3) yields

\[
B_i : \quad 2^{k-i-1} = 1 + \sum_{j=i+1}^{k-1} 2^{j-i-1} \beta_{kj} + \beta_{ki} \quad (i \leq k-1).
\]

For \( i=k-1 \), this gives \( \beta_{k,k-1} = 0 \). For \( i<k-1 \), we make \( B_i - 2B_{i+1} \), so that we obtain

\[
1 = \beta_{ki} - \beta_{k,i+1}, \text{ whence } \beta_{ki} = k - i - 1.
\]

\( \square \)
6. Main theorem

If the rank of a homogeneous cone is one or two, then it is a symmetric cone. Moreover, if a rank-two symmetric cone is the direct product $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ of two half-lines, then the corresponding basic relative invariants both have degree one. Thus, throughout this section, we assume that the rank $r$ of $V$ satisfies $r \geq 3$. We recall that $d_{ji} = \dim V_{ji}$ for $j > i$. The following lemma is an immediate consequence of Lemma 2.2.

**Lemma 6.1.** Suppose that $i < j < k$ and $d_{kj} > 0$, $d_{ji} > 0$. Then $\max(d_{kj}, d_{ji}) \leq d_{ki}$.

**Proposition 6.2.** The following (1) and (2) are equivalent.

(1) $\Gamma$ is complete, and $d_{r1} = \cdots = d_{r,r-1}$.

(2) $S = \{r\}$ and $\Delta_1(x), \ldots, \Delta_r(x)$ are of the degree type $1, \ldots, r$.

**Proof.** Step 1. We show that (1) implies (2). Since $\Gamma$ is complete, it is clear that $S = \{r\}$. Let us put $N := d_{r1} = \cdots = d_{r,r-1} > 0$. We now recall the representation $\varphi = \varphi_{[r]}$ of the clan $V = V^\text{out}[r]$ on the two-sided ideal $E^\text{out}[r]$ of $V$ given by $\varphi(x)\xi = \xi \triangle x$, where $x \in V$ and $\xi \in E^\text{out}[r]$ (see [18, Section 3]). Since $\det \varphi(x)$ is a relatively $H$-invariant polynomial taking the value one at $x = e_0$, we can find $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}_{\geq 0}$ such that

$$\det \varphi(x) = \Delta_1(x)^{\alpha_1} \cdots \Delta_{r-1}(x)^{\alpha_{r-1}} \Delta_r(x)^{\alpha_r}. \quad (6.1)$$

Let us put

$$V' := V^\text{out}(r) = \bigoplus_{p \leq q < r} V_{qp}, \quad E^\text{out}(r) := V_{r1} \oplus \cdots \oplus V_{r,r-1},$$

and express every $x \in V$ as $x = x' + \xi' + \lambda_r c_r$ ($x' \in V'$, $\xi' \in E^\text{out}(r)$, $\lambda_r \in \mathbb{R}$). By [18, (3.3)], the operator $\varphi(x)$ is of the form

$$\varphi(x) = \begin{pmatrix} \varphi'(x') & 2^{-1} \xi' \\ s_0(c_r)^{-1} \{\cdot | \xi'\} & \lambda_r \end{pmatrix},$$

where $\varphi'(x')\eta' = \eta' \triangle x'$ ($\eta' \in E^\text{out}(r)$). Therefore, we have $\deg_{\lambda_{r}} \det \varphi(x) = 1$ and $\deg_{\lambda_{j}} \det \varphi(x) = N$ for $i < r$ (see also [18, equation (3.11)]). On the other hand, we know by Propositions 4.4 and 4.5 that $\deg_{\lambda_{i}} \Delta_{j}(x) = \delta_{ij}$. Hence in (6.1) we have $\alpha_r = 1$. Moreover, since

$$\deg_{\lambda_{i}} \Delta_{j}(x) = \delta_{ij} (i \geq j), \quad \beta_{ji} := \deg_{\lambda_{i}} \Delta_{j}(x) > 0 (i < j)$$

again by Propositions 4.4 and 4.5, equation (6.1) yields

$$C_i : \quad N = \beta_{ri} + \sum_{j=i+1}^{r-1} \alpha_j \beta_{ji} + \alpha_i \quad (i = 1, \ldots, r-1).$$

In general, by Lemma 4.3(2) we have, for $k = 2, \ldots, r$,

$$0 \leq \beta_{k,k-1} = \deg_{\lambda_{k-1}} \Delta_{k}(x) \leq \deg_{\lambda_{k-1}} D_{k}(x) = 1,$$

so that $\beta_{k,k-1} = 1$. Hence, looking at $C_{r-1}$ above, we obtain $\alpha_{r-1} = N - 1$. Then by $C_i$ we have

$$N \geq \beta_{ri} + (N - 1)\beta_{r-1,i} \geq N \quad (i = 1, \ldots, r-2).$$
Therefore, this holds by equality, so that we obtain $\beta_{ri} = 1$ $(i = 1, \ldots, r - 2)$, in particular. Recalling that $\Delta_r(x)$ is a homogeneous polynomial, we see by Proposition 4.5(2) that

$$\text{Deg } \Delta_r(x) = \beta_{r1} + \beta_{r2} + \cdots + \beta_{r,r-1} + 1 = r.$$ 

This together with Proposition 4.4 shows that, since $i - 1 \in S^{\text{out}}(i)$ for $i > 1$ by the completeness of $\Gamma$, 

$$r = \text{Deg } \Delta_r(x) \geq \text{Deg } \Delta_{r-1}(x) \geq \cdots \geq \text{Deg } \Delta_1(x).$$ 

This forces $\text{Deg } \Delta_j(x) = j$ for any $j = 1, \ldots, r$.

Step 2. We next prove that (2) implies (1). First of all, we note that $\Gamma_1$ is complete by Theorem 5.2(1). This together with Lemma 6.1 gives 

$$d_{r1} \geq d_{r2} \geq \cdots \geq d_{r,r-1}. \quad (6.2)$$

By denying the conclusion, we have a strict inequality somewhere in (6.2), so that we assume $d_{r1} \geq d_{r,r-1}$. We take $\xi_{r-1} \in V_{r,r-1}$ and $\eta_1 \in V_{r-1,1}$ so that 

$$\|\xi_{r-1}\|^2 = 2s_0(c_r), \quad \|\eta_1\|^2 = 2s_0(c_{r-1}).$$

Since $V_{r,r-1} \triangle \eta_1 \subset V_{r1}$, and since $\dim(V_{r,r-1} \triangle \eta_1) \leq d_{r,r-1} < d_{r1}$ by Lemma 2.2, we take $\xi_1 \in V_{r1} \cap (V_{r,r-1} \triangle \eta_1)^\perp$ in such a way that $\|\xi_1\|^2 = 2s_0(c_r)$. Then we consider the following one-dimensional affine subspace $Y$ of $V$:

$$Y := \mathbb{R}c_1 + c_2 + \cdots + c_r + \xi_1 + \xi_{r-1} + \eta_1.$$ 

Every element $y$ of $Y$ is written as

$$y = \lambda c_1 + c_2 + \cdots + c_r + \xi_1 + \xi_{r-1} + \eta_1 \quad (\lambda \in \mathbb{R}).$$

Starting with $y$, we define elements

$$y^{[j]} = \sum_{k=j}^{r} \lambda_k^{[j]} c_k + \sum_{m>k} y_{mk}^{[j]} \in V^{[j]}$$

by (2.8) and (2.9). We have

$$\lambda_k^{[2]} = \begin{cases} \lambda & (k = 2, \ldots, r - 2), \\ \lambda - 1 & (k = r - 1, r). \end{cases}$$

Next, Lemma 2.1 shows, for any $\xi' \in V_{r,r-1}$, 

$$\langle \xi' \mid \xi_1 \triangle \eta_1 \rangle = \langle \xi' \triangle \eta_1 \mid \xi_1 \rangle = 0,$$

which implies $\xi_1 \triangle \eta_1 = 0$, so that

$$\lambda_{mk}^{[2]} = \begin{cases} \lambda \xi_{r-1} - \xi_1 \triangle \eta_1 = \lambda \xi_{r-1} & (m = r, \ k = r - 1), \\ 0 & (\text{otherwise}). \end{cases}$$
Continuing this process, we obtain, for $j = 3, \ldots, r - 1$,

$$
\lambda_k^{[j]} = \begin{cases} 
\lambda^{2j-2} & (k = j, \ldots, r - 2), \\
\lambda^{2j-2}(\lambda - 1) & (k = r - 1, r), 
\end{cases}
$$

and finally

$$
\lambda_{r}^{[r]} = \lambda^{2r-2}(\lambda - 1)^2 - \lambda^{2r-2} = \lambda^{2r-2}(1 - 2\lambda).
$$

Hence we arrive at

$$
D_1(y) = \lambda, \quad D_j(y) = \lambda^{2j-2} \quad (j = 2, \ldots, r - 2),
$$

$$
D_{r-1}(y) = \lambda^{2r-3}(\lambda - 1), \quad D_r(y) = \lambda^{2r-2}(1 - 2\lambda).
$$

From these formulae, we obtain

$$
D_1(y)y^{r-2}D_2(y)y^{r-3} \cdots D_{r-2}(y) = \lambda^{2r-2-1}. \quad (6.3)
$$

But the formula (6.3) says that $D_1(y)y^{r-2}D_2(y)y^{r-3} \cdots D_{r-2}(y)$ cannot divide $D_r(y) = \lambda^{2r-2}(1 - 2\lambda)$. Hence the original polynomial $D_1(x)y^{r-2}D_2(x)y^{r-3} \cdots D_{r-2}(x)$ on $V$ cannot divide $D_r(x)$ either, which contradicts Lemma 5.3 for $k = r$. This contradiction completes the proof.

**Lemma 6.3.** Suppose that $i < j < k < l$, and $d_{q,p} > 0$ for any $q > p$ such that $q, p \in \{i, j, k, l\}$. Then $d_{li} = d_{lj}$ implies that $d_{ki} = d_{kj}$.

**Proof.** We deny the conclusion. Then Lemma 6.1 shows that $d_{ki} > d_{kj}$. Take non-zero $x_{ji} \in V_{ji}$. Then, since $V_{kj} \triangle x_{ji} \subset V_{ki}$ and since $\dim(V_{kj} \triangle x_{ji}) = d_{kj} < d_{ki}$ by Lemma 2.2, we choose non-zero $x_{ki} \in V_{ki} \cap (V_{kj} \triangle x_{ji})^\perp$. Let us take non-zero $x_{lk} \in V_{lk}$ and put $y_{li} := x_{lk} \triangle x_{ki} \in V_{li}$. Lemma 2.2 guarantees that $y_{li} \neq 0$. We shall show that $y_{li} \in (V_{ij} \triangle x_{ji})^\perp$, which together with $\dim(V_{ij} \triangle x_{ji}) = d_{ij}$ implies

$$
d_{li} \geq \dim(V_{ij} \triangle x_{ji}) + \dim \mathbb{R} y_{li} \geq d_{lj},
$$

and the proof will be complete. Now we have, by (C1) and (2.4),

$$
y_{li} \triangle (V_{ij} \triangle x_{ji}) = V_{ij} \triangle (y_{li} \triangle x_{ji}) + (y_{li} \triangle V_{ij}) \triangle x_{ji} - (V_{ij} \triangle y_{li}) \triangle x_{ji}
$$

$$
= V_{ij} \triangle (y_{li} \triangle x_{ji}). \quad (6.4)
$$

Here it holds again by (C1) and (2.4) that

$$
y_{li} \triangle x_{ji} = (x_{lk} \triangle x_{ki}) \triangle x_{ji}
$$

$$
= (x_{ki} \triangle x_{lk}) \triangle x_{ji} + x_{lk} \triangle (x_{ki} \triangle x_{ji}) - x_{ki} \triangle (x_{lk} \triangle x_{ji})
$$

$$
= x_{lk} \triangle (x_{ki} \triangle x_{ji}). \quad (6.5)
$$

On the other hand, Lemma 2.1 and the definition of $x_{ki}$ give

$$
\langle V_{kj} \mid x_{ki} \triangle x_{ji} \rangle = \langle V_{kj} \triangle x_{ji} \mid x_{ki} \rangle = \{0\}.
$$

This implies that $x_{ki} \triangle x_{ji} = 0$, so that $y_{li} \triangle x_{ji} = 0$ by (6.5). Therefore, by (6.4), we arrive at $y_{li} \triangle (V_{ij} \triangle x_{ji}) = \{0\}$. Applying $s_0$, we obtain $\langle y_{li} \mid V_{ij} \triangle x_{ji} \rangle = \{0\}$. \qed
Remark 6.4. In the above proof, since \( x_{lk} \in V_{lk} \) is arbitrary, what we have actually shown is the implication

\[
(x_{ki} | V_{kj} \triangle x_{ji}) = 0 \implies (V_{lk} \triangle x_{ki} | V_{lj} \triangle x_{ji}) = 0.
\]

Although a proof of this implication is found in Vinberg [15, pp. 398–399], we have written it down for the readers’ convenience. This orthogonality relation is an important one, since it is required in the definition of \( N \)-algebras (see Vinberg [15, Definition 7, p. 395]).

Remark 6.5. Following Xu [17, Ch. 6], we call \( \Omega \) a square cone if there are positive integers \( N_2, \ldots, N_r \) such that \( d_{j1} = d_{j2} = \cdots = d_{j,j-1} = N_j \) for any \( j = 2, \ldots, r \). In view of Lemma 6.3, the condition Proposition 6.2(1) is nothing other than the requirement for \( \Omega \) to be a square cone. Hence the condition Proposition 6.2(2) tells us that the basic relative invariants of square cones are necessarily of the degree type 1, \ldots, r. In Example 6.6 below, we give non-square cones such that the corresponding basic relative invariants are of the degree type 1, \ldots, r. By Proposition 6.2, the oriented graphs for these non-square cones necessarily have the set \( \mathcal{S} \) of sources with an element \( s < r \).

Example 6.6. Here we present an example of a rank-four non-square cone such that the basic relative invariants are of the degree type 1, 2, 3, 4. The presentation follows our previous paper [18]. The oriented graph \( \Gamma \) which concerns us is of the type \((S^5_2)^*\) in the notation of Kaneyuki and Tsuji [10]. Thus, \( \Gamma \) is as in Figure 1, where \( d_{41} = m \geq 2 \), and we do not indicate \( \dim V_{kj} (k > j) \) if they are equal to one for simplicity. Moreover, we have \( V_{43} = \{0\} \).

![Figure 1. \( \Gamma \).](image)

Clearly we have the sources \( \mathcal{S} = \{3, 4\} \) of \( \Gamma \). Denoting by \( \text{Sym}(r, \mathbb{R}) \) the vector space of real symmetric matrices of size \( r \), we see that the source subclans, that is, the out-neighbor subclans \( V_{\text{out}}[3] \) and \( V_{\text{out}}[4] \) corresponding to the sources 3 and 4 respectively, are

\[
V_{\text{out}}[3] = \left\{ x_{[3]} := \begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{21} & x_{22} & x_{32} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \in \text{Sym}(3, \mathbb{R}) \right\},
\]

\[
V_{\text{out}}[4] = \left\{ x_{[4]} := \begin{pmatrix} x_{11} & I_m & x_{21} & x_{31} \\ x_{21} & e_1 & x_{22} & x_{32} \\ x_{31} & x_{41} & x_{42} & x_{44} \end{pmatrix} \in \text{Sym}(m + 2, \mathbb{R}) \right\},
\]

(6.6)

where \( I_m \) is the \( m \)th identity matrix, \( e_1 \) is the unit column vector in \( \mathbb{R}^m \) with the only non-zero first entry equal to one, and \( x_{41} \in \mathbb{R}^m \). We have \( V = [V_{\text{out}}[3], V_{\text{out}}[4]] \) with the shaded parts stapled as shown in (6.6). The corresponding source cones \( \Omega^0_{[3]} \) and \( \Omega^0_{[4]} \) are given by
the positive-definite matrices $x_{[3]}$ and $x_{[4]}$, respectively. Our homogeneous cone is the stapled one $\Omega^0 = [\Omega^0_{[3]}, \Omega^0_{[4]}]$. Since $m \geq 2$, the cone $\Omega^0$ is not square. However, the basic relative invariants $\Delta_j(x)$ for $\Omega^0$ are given as

$$\Delta_j(x) = \text{the leading principal minors of } x_{[3]} \quad (j = 1, 2, 3),$$

$$\Delta_4(x) = (x_{11}x_{44} - \|x_{41}\|^2)(x_{11}x_{22} - x_{21}^2) - (x_{11}x_{42} - x_{21}^1 e_1 x_{41})^2.$$

For $m = 2$, we have $\dim \Omega^0 = 10$. This $\Omega^0$ and the homogeneous cone given in [9, Section 3] are the only two homogeneous non-symmetric cones such that the basic relative invariants are of the type $1, \ldots, r$ among homogeneous cones of dimension $\leq 10$. We also note that the above $\Omega^0$ can be generalized to arbitrary rank $r + 1$ ($r \geq 3$) in such a way that $S = \{r, r + 1\}$, and that $V = [V_{\text{out}}[r], V_{\text{out}}[r + 1]]$ with

$$V_{\text{out}}[r] = \{x_{[r]} \in \text{Sym}(r, \mathbb{R})\},$$

$$V_{\text{out}}[r + 1] = \left\{ x_{[r+1]} := \begin{pmatrix} x_{11} I_{m} & x_{21} e_1 & \cdots & x_{r-1,1} e_1 & x_{r+1,1} \\ x_{21}^t e_1 & x_{22} & \cdots & x_{r-1,2} & x_{r+1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{r-1,1}^t e_1 & x_{r-1,2} & \cdots & x_{r-1,r-1} & \vdots \\ x_{r+1,1} & x_{r+1,2} & \cdots & \cdots & x_{r+1,r+1} \end{pmatrix} \right\}.$$

Note that $V_{\text{out}}[r + 1] \subset \text{Sym}(r + 1 - m, \mathbb{R})$, and that in the upper left $(r - 1) \times (r - 1)$ parts of $V_{\text{out}}[r]$ and $V_{\text{out}}[r + 1]$, we have the common coordinates from $\text{Sym}(r - 1, \mathbb{R})$. Then, for $j \leq r$, we have $\Delta_j(x)$ equal to the leading principal minors of $x_{[r]}$, and a closer inspection of $\det x_{[r+1]}$ using Proposition 4.5 (2) reveals that

$$\Delta_{[r+1]}(x) = x_{11} \det \tilde{x}_{[r+1]} - \|x_{r+1,1} - \pi(x_{r+1,1})\|^2 \Delta_{r-1}(x),$$

where $\pi$ is the orthogonal projector $\mathbb{R}^m \to \mathbb{R} e_1$, and $\tilde{x}_{[r+1]} \in \text{Sym}(r, \mathbb{R})$ is defined from $x_{[r+1]} \in V_{\text{out}}[r + 1]$ by

$$\tilde{x}_{[r+1]} := \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{r-1,1} & \pi(x_{r+1,1}) \\ x_{21} & x_{22} & \cdots & x_{r-1,2} & x_{r+1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{r-1,1} & x_{r-1,2} & \cdots & x_{r-1,r-1} & \vdots \\ \pi(x_{r+1,1}) & x_{r+1,2} & \cdots & \cdots & x_{r+1,r+1} \end{pmatrix}.$$

Details are omitted here.

Remark 6.7. Similarly to Remark 6.5, we call $\Omega$ a dual square cone, after Xu [17], if there are positive integers $M_1, \ldots, M_{r-1}$ such that $d_{j+1,j} = \cdots = d_{r-1,r-j} = M_j$ for any $j = 1, \ldots, r - 1$. The cone $\Omega^0$ treated in [13, Section 5] (see also [14]) is a dual square cone, and the basic relative invariants for its dual cone $(\Omega^0)^\ast$ are of the degree type $1, \ldots, r$.

Now we resume the general situation with which we have begun this section. Let $\Omega^\ast$ be the dual cone of $\Omega$ with respect to our inner product $\langle \cdot | \cdot \rangle$. The associated clan is obtained as follows. The underlying vector space is the same $V$, but the clan product is defined by the
transposes $^tL(x)$ of $L(x)$ ($x \in V$). The clan product obtained in this way is denoted by $x \triangledown y$, and we call the clan $(V, \triangledown)$ the dual clan of $V$. The complete system of primitive idempotents $c_1, \ldots, c_r$ of the original clan $(V, \triangle)$ is also a complete system of primitive idempotents of the new clan $(V, \triangledown)$, and $e_0$ is still its unit element. For details we refer the reader to the paper by Nakashima [12, Section 1]. The weighted oriented graph of $(V, \triangledown)$ is obtained by just reversing the orientation of each arc in the original weighted oriented graph $\Gamma$. However, in this case, the simply transitive group of $\Omega^*$ is the transpose $^tH$ of $H$ with respect to $\langle \cdot | \cdot \rangle$, and thus is represented by upper triangular matrices in the original ordering for $(V, \triangle)$ of the orthonormal basis of $V$. In order to stay within the framework that the simply transitive group is represented by lower triangular matrices, we reverse our ordering of the orthonormal basis of $V$. This gives rise to the renumbering $j \mapsto r - j + 1$ ($j = 1, \ldots, r$) of the vertices in the graph. The weighted oriented graph obtained in this way is denoted by $\Gamma^*$. To avoid any possible confusion, we write the corresponding normal decomposition as

$$V = \bigoplus_{1 \leq j \leq k \leq r} V^*_kj,$$

where $V^*_kj := V_{r-j+1,r-k+1}$. We put $d^*_kj := \dim V^*_kj$ ($k > j$). In reality we have $d^*_kj = d_{r-j+1,r-k+1}$, but it would be clearer to introduce this notation in applying the previous discussions to $\Gamma^*$. In particular, the numbering of the basic relative invariants of $\Omega^*$ is according to this normal decomposition. The set of sources and the set of sinks of $\Gamma^*$ are denoted by $S^*$ and $K^*$, respectively. To obtain $S^*$, we take $K$, and renumber the elements by $j \mapsto r - j + 1$. A similar procedure suffices to get $K^*$. Now we prove the main theorem.

**Theorem 6.8.** $\Omega$ is irreducible and symmetric if and only if both of the basic relative invariants $\Delta_1(x), \ldots, \Delta_r(x)$ of $\Omega$ and $\Delta^*_1(x), \ldots, \Delta^*_r(x)$ of $\Omega^*$ are of the degree type $1, \ldots, r$.

**Proof.** Clearly it is sufficient to prove the if part. First of all, Lemma 4.7 shows that $K = K^* = \{1\}$. This implies $S = S^* = \{r\}$. In particular, Lemma 5.1 says that $\Gamma$ is connected. Hence, by Asano [1], our cone $\Omega$ is irreducible. Now, Proposition 6.2 shows that $\Gamma$ is a complete graph, and that $d_{r-1} = \cdots = d_{r,r-1}$. Then, Lemma 6.3 yields $d_{k1} = \cdots = d_{k,k-1}$ for any $k = 1, \ldots, r$. Applying the same discussion to $\Gamma^*$, we obtain $d^*_{k1} = \cdots = d^*_{k,k-1}$ for any $k$, and this actually says that $d_{r,j} = \cdots = d_{j+1,j}$ for any $j$. Consequently we see that all $d_{kj}$ ($k > j$) are equal. Now Vinberg’s criterion given by Proposition 3 of [16, p. 73] tells us that $\Omega$ is symmetric.

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**References**


T. Yamasaki  
Faculty of Mathematics  
Kyushu University  
744 Motooka, Nishi-ku Fukuoka 819-0735  
Japan  
(E-mail: t-yamasaki@math.kyushu-u.ac.jp)

T. Nomura  
Faculty of Mathematics  
Kyushu University  
744 Motooka, Nishi-ku Fukuoka 819-0735  
Japan  
(E-mail: tnomura@math.kyushu-u.ac.jp)