SOLUTIONS IN TERMS OF INTEGRALS OF MULTIVALUED FUNCTIONS FOR THE CLASSICAL HYPERGEOMETRIC EQUATIONS AND THE HYPERGEOMETRIC SYSTEM ON THE CONFIGURATION SPACE

Katsuhisa MIMACHI and Masatoshi NOUMI

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Abstract. The systems of differential equations associated with the classical hypergeometric functions and the hypergeometric functions on the space of point configurations are investigated from the viewpoint of the twisted de Rham theory. In each case, it is proved that the integral of a certain multivalued function over an arbitrary twisted (or loaded) cycle satisfies the system of differential equations in question. The classical hypergeometric functions studied here include Appell’s hypergeometric functions F_1, F_2, F_3, F_4 , Lauricella’s hypergeometric functions F_A, F_B, F_C, F_D , and the generalized hypergeometric function ${}_{n+1}F_n$.

0. Introduction

In this paper, we investigate the systems of differential equations associated with the classical hypergeometric functions, as well as the hypergeometric functions on the space of point configurations, in the framework of the twisted de Rham theory. In each case, we prove that the integral of a certain multivalued function over an arbitrary twisted (or loaded) cycle satisfies the system of differential equations in question. The classical hypergeometric functions studied here include Appell’s hypergeometric functions F_1, F_2, F_3, F_4 , Lauricella’s hypergeometric functions F_A, F_B, F_C, F_D , and the generalized hypergeometric function ${}_{n+1}F_n$. The corresponding differential equations will be denoted by $E_1, E_2, E_3, E_4, E_A, E_B, E_C, E_D$, and ${}_{n+1}E_n$.

We briefly explain the point of our investigation, taking the example of integrals associated with ${}_{n+1}F_n$:

$$\int_D u(x; t) dt_1 \cdots dt_n, \quad u(x; t) = \prod_{i=1}^n t_i^{\alpha_{i+1} - \beta_i} \prod_{i=1}^{n+1} (t_{i-1} - t_i)^{\beta_i - \alpha_i - 1} \quad (0.1)$$

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where $t_0 = 1$, $t_{n+1} = x$, $\beta_{n+1} = 1$. It is shown in [M1, Proposition 2.1] that, when the variable x is real and $-\infty < x < 0$, the integrals (0.1) over the real domains

$$D_i^{(0)} = \{x < t_n < \cdots < t_i < 0, 1 < t_1 < \cdots < t_{i-1} < +\infty\}, \quad i = 1, \dots, n+1,$$

and

$$D_i^{(\infty)} = \{\infty < t_i < \cdots < t_n < x, 0 < t_{i-1} < \cdots < t_1 < 1\}, \quad i = 1, \dots, n+1,$$

are expressed in terms of ${}_{n+1}F_n$ in x and in $1/x$, respectively, under appropriate conditions on the exponents. It is also verified that they provide fundamental sets of solutions of the generalized hypergeometric equation ${}_{n+1}E_n$ around $x = 0$ and around $x = \infty$ (see Section 1.1 for the definitions of ${}_{n+1}F_n$ and ${}_{n+1}E_n$). Hence, from the viewpoint of twisted de Rham theory, it is expected that the integral (0.1) over an arbitrary twisted cycle D such as

$$D_i = \{x < t_n < \cdots < t_i < 0, t_i < t_{i-1} < \cdots < t_1 < 1\}, \quad i = 1, \dots, n+1,$$

satisfies the same differential equation ${}_{n+1}E_n$. It is indeed the case as we will prove in Section 6.1.

The contents of this paper are as follows. In Section 1, we recall the classical hypergeometric equations and the hypergeometric system $RE(k, n)$ on the configuration space of n points in the projective $(k-1)$ -space. Sections 2–7 are devoted to proving that the integral of a multivalued function for any twisted cycle satisfies the corresponding differential equation(s) in the cases of E_A , E_B , E_C , E_D , ${}_{n+1}E_n$, and $RE(k, n)$, respectively. As for the cases E_D and ${}_{n+1}E_n$, we treat two different types of integrals to help with future studies while in the case of ${}_{n+1}E_n$, each of which can be derived from the other by the change of integration variables.

We note that Yoshida showed in [Y1] that integrals associated with F_1 , F_2 , F_3 , and F_4 satisfy E_1 , E_2 , E_3 , and E_4 for any cycles, and Takano showed in [T] that an integral associated with F_C satisfies E_C for any cycle. Their arguments, based on the Euler integral transform, are different from ours.

Throughout this paper, we use the symbols

$$\theta_{x_i} = x_i \partial / \partial x_i, \quad \partial_x = \frac{\partial}{\partial x} \quad \text{and} \quad dt = dt_1 \cdots dt_n$$

for brevity.

1. Preliminaries

1.1. Classical hypergeometric functions

Lauricella's hypergeometric function F_A is the analytic continuation of Lauricella's hypergeometric series

$$\begin{aligned} & F_A(\alpha, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha)_{m_1+\dots+m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}, \quad |x_1| + \cdots + |x_n| < 1, \end{aligned}$$

where $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$, and it satisfies the system E_A of differential equations of rank 2^n :

$$(E_A) \quad \theta_{x_i}(\theta_{x_i} + \gamma_i - 1)F = x_i(\theta_{x_i} + \beta_i)(\theta_{x_1} + \cdots + \theta_{x_n} + \alpha)F, \quad i = 1, \dots, n.$$

Similarly, Lauricella's hypergeometric functions F_B , F_C , and F_D are the analytic continuations of the series

$$\begin{aligned} & F_B(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1 + \dots + m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}, \quad |x_1| < 1, \dots, |x_n| < 1, \\ & F_C(\alpha, \beta, \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha)_{m_1 + \dots + m_n} (\beta)_{m_1 + \dots + m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}, \quad \sqrt{|x_1|} + \cdots + \sqrt{|x_n|} < 1, \end{aligned}$$

and

$$\begin{aligned} & F_D(\alpha, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha)_{m_1 + \dots + m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1 + \dots + m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}, \quad |x_1| < 1, \dots, |x_n| < 1, \end{aligned}$$

and they satisfy the following systems E_B , E_C , and E_D of rank 2^n , 2^n , and $n + 1$, respectively:

$$\begin{aligned} (E_B) \quad & x_i(\theta_{x_i} + \alpha_i)(\theta_{x_i} + \beta_i)F = \theta_{x_i}(\theta_{x_1} + \cdots + \theta_{x_n} + \gamma - 1)F, \quad i = 1, \dots, n, \\ (E_C) \quad & \theta_{x_i}(\theta_{x_i} + \gamma_i - 1)F = x_i(\theta_{x_1} + \cdots + \theta_{x_n} + \alpha)(\theta_{x_1} + \cdots + \theta_{x_n} + \beta)F, \\ & i = 1, \dots, n, \end{aligned}$$

and

$$(E_D) \quad \begin{cases} \theta_{x_i}(\theta_{x_1} + \cdots + \theta_{x_n} + \gamma - 1)F = x_i(\theta_{x_i} + \beta_i)(\theta_{x_1} + \cdots + \theta_{x_n} + \alpha)F, \\ \quad \quad \quad i = 1, \dots, n, \\ x_i(\theta_{x_i} + \beta_i)\theta_{x_j}F = x_j(\theta_{x_j} + \beta_j)\theta_{x_i}F, \quad 1 \leq i \neq j \leq n. \end{cases}$$

Appell's hypergeometric functions F_2 , F_3 , F_4 , and F_1 are the special $n = 2$ cases of F_A , F_B , F_C , and F_D , respectively.

The generalized hypergeometric function ${}_{n+1}F_n$ is the analytic continuation of the generalized hypergeometric series

$${}_{n+1}F_n \left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{n+1} \\ \beta_1, \dots, \beta_n \end{matrix} ; x \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_{n+1})_k}{(\beta_1)_k \cdots (\beta_n)_k k!} x^k, \quad |x| < 1,$$

and it satisfies the following equation of rank $n + 1$:

$$({}_{n+1}E_n) \quad \left[\theta_x \prod_{i=1}^n (\theta_x + \beta_i - 1) - x \prod_{i=1}^{n+1} (\theta_x + \alpha_i) \right] F = 0.$$

1.2. Hypergeometric function on the configuration space

In this paper, by hypergeometric function $F(k, n)$, we mean the analytic continuation of the series

$$F(k, n; \lambda; x) = \sum_{\substack{m_{ip} \geq 0 \\ (2 \leq i \leq k, k+2 \leq p \leq n)}} \frac{\prod_{i=2}^k (\lambda_i + 1)_{\sum_{k+2 \leq p \leq n} m_{ip}} \prod_{p=k+2}^n (-\lambda_p)_{\sum_{2 \leq i \leq k} m_{ip}}}{(\sum_{2 \leq i \leq k+1} \lambda_i + k)_{\sum_{2 \leq i \leq k, k+2 \leq p \leq n} m_{ip}}} \prod_{\substack{2 \leq i \leq k \\ k+2 \leq p \leq n}} \frac{x_{ip}^{m_{ip}}}{m_{ip}!},$$

in the variables $x = (x_{ip})_{2 \leq i \leq k, k+2 \leq p \leq n}$. It satisfies the following system $RE(k, n)$ of differential equations of rank $\binom{n-2}{k-1}$:

$$\left\{ \begin{array}{l} \theta_{ip}(\theta + \lambda_0 - 1)F = x_{ip}(\theta_i + \lambda_i + 1)(\theta^p - \lambda_p)F, \quad i = 2, \dots, k, \\ \quad \quad \quad \text{and } p = k+2, \dots, n, \\ x_{iq}\theta_{ip}(\theta^q - \lambda_q)F = x_{ip}\theta_{iq}(\theta^p - \lambda_p)F, \quad i = 2, \dots, k, \\ \quad \quad \quad \text{and } p, q = k+2, \dots, n, \\ x_{jp}\theta_{ip}(\theta_j + \lambda_j + 1)F = x_{ip}\theta_{jp}(\theta_i + \lambda_i + 1)F, \quad i, j = 2, \dots, k, \\ \quad \quad \quad \text{and } p = k+2, \dots, n, \\ x_{iq}x_{jp}\theta_{ip}\theta_{jq}F = x_{ip}x_{jq}\theta_{iq}\theta_{jp}F, \quad i, j = 2, \dots, k, \text{ and } p, q = k+2, \dots, n, \end{array} \right.$$

where

$$\begin{aligned} \lambda_0 &= \lambda_2 + \dots + \lambda_{k+1} + k, \\ \theta_{jq} &= x_{jq} \frac{\partial}{\partial x_{jq}}, \\ \theta_j &= \theta_{j,k+2} + \theta_{j,k+3} + \dots + \theta_{j,n}, \\ \theta^q &= \theta_{2,q} + \theta_{3,q} + \dots + \theta_{k,q}, \\ \theta &= \theta^{k+2} + \dots + \theta^n = \theta_2 + \dots + \theta_k = \sum_{\substack{2 \leq i \leq k \\ k+2 \leq r \leq n}} \theta_{ir}. \end{aligned}$$

Let $Z_{k,n}$ denote the space of $k \times n$ complex matrices of which all the $k \times k$ minor determinants are non-zero, and let $X_{k,n} := \mathrm{GL}_k \backslash Z_{k,n} / H_n$ denote the configuration space of n points in the projective $(k-1)$ -space, where H_n is the group of diagonal $n \times n$ non-singular matrices acting on $Z_{k,n}$ by right multiplication. The dimension of $X_{k,n}$ is $(k-1) \times (n-k-1)$, and the variables $x = (x_{ip})_{i,p}$ correspond to a system of local coordinates of $X_{k,n}$.

The system of differential equations $RE(k, n)$ on $X_{k,n}$ is called the *hypergeometric system on the configuration space* of n points in $\mathbb{P}^{k-1}$. This system $RE(k, n)$ is formulated in [MS2] (see also [MSY, Theorem 1.5.1]) and the series $F(k, n)$ can be found, for example, in [MSY, Ki, NN]. We remark that the original system of differential equations $E(k, n)$ due to Gelfand [Ge] is defined on $Z_{k,n}$ and that $RE(k, n)$ is the reduction of $E(k, n)$ to $X_{k,n}$; $RE(k, n)$ is sometimes called the *reduced system of $E(k, n)$* . We refer the reader to [Ge, GKZ, Ho, MSY, MS2, Y2] for more details of these systems.

1.3. Twisted homology groups

Let $u(t) = \prod_i f_i(t)^{\lambda_i}$ be a multivalued function on $T \subset \mathbb{C}^m$, where $\lambda_i \in \mathbb{C}$ and T is the complement of the singular locus $\bigcup_i \{f_i(t) = 0\}$ in $\mathbb{C}^m$. Let $\mathcal{L}^\vee$ be the locally constant sheaf (the local system) defined by u : the sheaf consisting of the local solutions of $dL = L\omega$ for $\omega = du(t)/u(t)$.

Let $H_m(T, \mathcal{L}^\vee)$ be the m th homology group with coefficients in $\mathcal{L}^\vee$, and let $H_m^{\text{lf}}(T, \mathcal{L}^\vee)$ be the m th locally finite homology group with coefficients in $\mathcal{L}^\vee$. Elements of these twisted homology groups, called *twisted cycles* or *loaded cycles*, are represented by ∂ -closed twisted (finite or locally finite) chains

$$C = \sum_{\Delta} a_{\Delta} \Delta \otimes v_{\Delta} \quad (a_{\Delta} \in \mathbb{C}),$$

where each Δ is an m -simplex and v_{Δ} is a section of $\mathcal{L}^\vee$ on Δ .

Under some condition on the exponents λ_i , we have the isomorphism, said to be the *regularization*,

$$\text{reg} : H_m^{\text{lf}}(T, \mathcal{L}^\vee) \longrightarrow H_m(T, \mathcal{L}^\vee),$$

which is the inverse of the natural map $\iota : H_m(T, \mathcal{L}^\vee) \rightarrow H_m^{\text{lf}}(T, \mathcal{L}^\vee)$.

For example, in case $T = \mathbb{C} \setminus \{0, 1\}$ and $u(t) = t^\alpha (t-1)^\beta$, where $\alpha, \beta, \alpha + \beta \in \mathbb{R} \setminus \mathbb{Z}$, a regularization (regularized cycle) $\text{reg } C \in H_1(T, \mathcal{L}^\vee)$ of $C = (\overrightarrow{0, 1}) \in H_1^{\text{lf}}(T, \mathcal{L}^\vee)$ can be given by

$$\text{reg } C = \left\{ \frac{1}{d_\alpha} S(\epsilon; 0) + \overrightarrow{[\epsilon, 1-\epsilon]} - \frac{1}{d_\beta} S(1-\epsilon; 1) \right\} \otimes u(t),$$

where $d_\lambda = e^{2\pi\sqrt{-1}\lambda} - 1$, the symbol $S(a; z)$ stands for the positively oriented circle centered at the point z with starting and ending point a , ϵ is a small positive number and the argument of each factor of $u(t)$ on the oriented circle $S(\epsilon; 0)$ or $S(1-\epsilon; 1)$ is defined so that $\arg t$ takes values from 0 to 2π on $S(\epsilon; 0)$, and $\arg(1-t)$ from 0 to 2π .

We refer the reader to [Ki] for the construction of regularized cycles in higher-dimensional cases.

2. Lauricella's F_A

Let $x = (x_1, \dots, x_n)$ be a point of

$$X = \mathbb{C}^n \setminus \bigcup_{j=1}^n \{x_j = 0\} \cup \bigcup_{\{i_1, \dots, i_r\} \subset \{1, \dots, n\}} \left\{ \sum_{s=1}^r x_{i_s} = 1 \right\} \subset (\mathbb{P}^1)^n,$$

and let $u(t) = u(x; t)$ be a multivalued function

$$u(x; t) = (1 - x_1 t_1 - \dots - x_n t_n)^{-\alpha} \prod_{s=1}^n t_s^{\beta_s - 1} (1 - t_s)^{\gamma_s - \beta_s - 1}$$

on

$$T_x = \mathbb{C}^n \setminus \bigcup_{j=1}^n \{t_j = 0\} \cup \bigcup_{j=1}^n \{t_j = 1\} \cup \{1 = x_1 t_1 + \dots + x_n t_n\} \subset (\mathbb{P}^1)^n.$$

Let $\mathcal{L}_x^\vee$ be the locally constant sheaf (local system) determined by $u(t)$.

Then we get Theorem 2.1.

THEOREM 2.1. *We have*

$$\left[x_i \left(\sum_{s=1}^n \theta_{x_s} + \alpha \right) (\theta_{x_i} + \beta_i) - \theta_{x_i} (\theta_{x_i} + \gamma_i - 1) \right] \int_C u(x, t) dt_1 \cdots dt_n = 0,$$

$$i = 1, 2, \dots, n,$$

for any cycle $C \in H_n(T_x, \mathcal{L}_x^\vee)$.

Proof. Direct calculation shows that

$$\theta_{x_i} u(t) = \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t), \quad (2.1)$$

$$\theta_{t_i} u(t) = \left[\beta_i - 1 + \frac{(\gamma_i - \beta_i - 1)(-t_i)}{1 - t_i} + \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} \right] u(t). \quad (2.2)$$

It follows from (2.1) and (2.2) that

$$(\theta_{x_i} - \theta_{t_i}) u(t) = \left[1 - \beta_i - \frac{(\gamma_i - \beta_i - 1)(-t_i)}{1 - t_i} \right] u(t),$$

which is equivalent to

$$(\theta_{x_i} + \beta_i) u(t) = \partial_{t_i} \{t_i u(t)\} + \frac{(\gamma_i - \beta_i - 1)t_i}{1 - t_i} u(t), \quad (2.3)$$

since $\theta_{t_i} = -1 + \partial_{t_i} t_i$, as an operator. Since $t_i = 1 - (1 - t_i)$, (2.3) can be written as

$$(\theta_{x_i} + \gamma_i - 1) u(t) = \partial_{t_i} \{t_i u(t)\} + (\gamma_i - \beta_i - 1) \frac{1}{1 - t_i} u(t). \quad (2.4)$$

On the other hand, summing both sides of (2.1) over i , we have

$$\sum_{i=1}^n \theta_{x_i} u(t) = \frac{\alpha \sum_{i=1}^n t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t),$$

which can be written as

$$\left(\sum_{i=1}^n \theta_{x_i} + \alpha \right) u(t) = \frac{\alpha}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t), \quad (2.5)$$

since $\sum_i t_i x_i = 1 - (1 - \sum_i t_i x_i)$.

By (2.1), the action of θ_{x_i} on (2.4) implies

$$\begin{aligned} & \theta_{x_i} (\theta_{x_i} + \gamma_i - 1) u(t) \\ &= \partial_{t_i} \{t_i \theta_{x_i} u(t)\} + (\gamma_i - \beta_i - 1) \frac{1}{1 - t_i} \theta_{x_i} u(t) \\ &= \partial_{t_i} \left\{ \frac{\alpha t_i^2 x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t) \right\} + (\gamma_i - \beta_i - 1) \frac{1}{1 - t_i} \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t). \end{aligned} \quad (2.6)$$

By (2.5), the action of $(\sum_{j=1}^n \theta_{x_j} + \alpha)$ on (2.3) implies

$$\begin{aligned} & \left(\sum_{j=1}^n \theta_{x_j} + \alpha \right) (\theta_{x_i} + \beta_i) u(t) \\ &= \partial_{t_i} \left\{ t_i \left(\sum_{j=1}^n \theta_{x_j} + \alpha \right) u(t) \right\} + \frac{(\gamma_i - \beta_i - 1)t_i}{1 - t_i} \left(\sum_{j=1}^n \theta_{x_j} + \alpha \right) u(t) \\ &= \partial_{t_i} \left\{ \frac{\alpha t_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t) \right\} + (\gamma_i - \beta_i - 1) \frac{t_i}{1 - t_i} \frac{\alpha}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t). \end{aligned} \quad (2.7)$$

Combining (2.6) and (2.7), we obtain

$$\left[x_i \left(\sum_{s=1}^n \theta_{x_s} + \alpha \right) (\theta_{x_i} + \beta_i) - \theta_{x_i} (\theta_{x_i} + \gamma_i - 1) \right] u(t) = \partial_{t_i} \left\{ \frac{\alpha x_i t_i (1 - t_i)}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t) \right\}.$$

Hence, we reach the result. $\square$

Remark 2.2. If we assume the condition for the exponents

$$\beta_1, \dots, \beta_n, \gamma_1 - \beta_1, \dots, \gamma_n - \beta_n, -\alpha + \sum_{j \in J} \beta_j \notin \mathbb{Z}, \quad (2.8)$$

for all $J \subset \{1, \dots, n\}$, then the rank of the twisted homology group $H_n(T_x, \mathcal{L}_x^\vee)$ turns out to be 2^n (see [AK, C, KN]), and hence it is natural to expect that an appropriate set of 2^n cycles gives a fundamental set of solutions of E_A . In the case of $n = 2$, it is shown in [M2] by deriving a Wronskian formula that a certain set of four cycles gives a fundamental set of solutions of E_A , i.e. E_2 .

Remark 2.3. It is shown in [B] that E_A is irreducible if and only if (2.8) holds.

Remark 2.4. If $\beta_j, \gamma_j - \beta_j \notin \mathbb{Z}$, $j = 1, \dots, n$, then we have

$$\begin{aligned} & \int_{\text{reg}(0,1)^n} (1 - x_1 t_1 - \dots - x_n t_n)^{-\alpha} \prod_{s=1}^n t_s^{\beta_s-1} (1 - t_s)^{\gamma_s-\beta_s-1} dt_1 \dots dt_n \\ &= \prod_{j=1}^n \frac{\Gamma(\beta_j) \Gamma(\gamma_j - \beta_j)}{\Gamma(\gamma_j)} \times F_A(\alpha, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n; x_1, \dots, x_n), \end{aligned}$$

where the argument of each factor of the integrand is fixed to be zero when $0 < x_1, 0 < x_2, \dots, 0 < x_n$ and $x_1 + \dots + x_n < 1$; it is the analytic continuation of the integral representation given by Lauricella [L]. See also [A] and [AKdF].

3. Lauricella's F_B

Let $x = (x_1, \dots, x_n)$ be a point of

$$X = \mathbb{C}^n \setminus \bigcup_{j=1}^n \{x_j = 0\} \cup \bigcup_{\{i_1, \dots, i_r\} \subset \{1, \dots, n\}} \left\{ \sum_{s=1}^r x_{i_1} \cdots \widehat{x_{i_s}} \cdots x_{i_r} = x_{i_1} \cdots x_{i_r} \right\} \subset (\mathbb{P}^1)^n,$$

where the symbol $\widehat{A}$ means omitting A , and let $u(t) = u(x; t)$ be a multivalued function

$$u(x; t) = \left(1 - \sum_{s=1}^n t_s\right)^{\gamma-1-\sum_{1 \leq s \leq n} \alpha_s} \prod_{s=1}^n t_s^{\alpha_s-1} (1 - x_s t_s)^{-\beta_s}$$

on

$$T_x = \mathbb{C}^n \setminus \bigcup_{j=1}^n \{t_j = 0\} \cup \bigcup_{j=1}^n \{x_j t_j = 1\} \cup \{t_1 + \cdots + t_n = 1\} \subset (\mathbb{P}^1)^n.$$

Let $\mathcal{L}_x^\vee$ be the locally constant sheaf (local system) determined by $u(t)$.

Then we obtain the following.

THEOREM 3.1. *We have*

$$\left[\theta_{x_i} \left(\sum_{s=1}^n \theta_{x_s} + \gamma - 1 \right) - x_i (\theta_{x_i} + \alpha_i) (\theta_{x_i} + \beta_i) \right] \int_C u(t) dt_1 \cdots dt_n = 0, \quad i = 1, \dots, n,$$

for any cycle $C \in H_n(T_x, \mathcal{L}_x^\vee)$.

Proof. Direct calculation shows that, for $1 \leq i \leq n$,

$$\theta_{x_i} u(t) = \frac{\beta_i t_i x_i}{1 - x_i t_i} u(t), \quad (3.1)$$

$$\theta_{t_i} u(t) = \left[\alpha_i - 1 + \frac{\beta_i t_i x_i}{1 - x_i t_i} + \left(\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s \right) \frac{-t_i}{1 - \sum_{1 \leq s \leq n} t_s} \right] u(t). \quad (3.2)$$

Since $t_i x_i = 1 - (1 - t_i x_i)$, (3.1) can be written as

$$(\theta_{x_i} + \beta_i) u(t) = \frac{\beta_i}{1 - x_i t_i} u(t). \quad (3.3)$$

From (3.1) and (3.2), it follows that

$$(\theta_{x_i} - \theta_{t_i}) u(t) = \left[-\alpha_i + 1 + \left(\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s \right) \frac{t_i}{1 - \sum_{1 \leq s \leq n} t_s} \right] u(t),$$

which is equivalent to

$$(\theta_{x_i} + \alpha_i) u(t) = \partial_{t_i} \{t_i u(t)\} + \left(\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s \right) \frac{t_i}{1 - \sum_{1 \leq s \leq n} t_s} u(t), \quad (3.4)$$

since $\theta_{t_i} = -1 + \partial_{t_i} t_i$, as an operator. Summing both sides of (3.4) over i , we have

$$\left(\sum_{1 \leq s \leq n} \theta_{x_s} + \gamma - 1 \right) u(t) = \sum_{1 \leq s \leq n} \partial_{t_s} \{t_s u(t)\} + \left(\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s \right) \frac{1}{1 - \sum_{1 \leq s \leq n} t_s} u(t), \quad (3.5)$$

since $\sum_s t_s = 1 - (1 - \sum_s t_s)$.

By (3.1), the action of θ_{x_i} on (3.5) implies

$$\begin{aligned} & \theta_{x_i} \left(\sum_{1 \leq s \leq n} \theta_{x_s} + \gamma - 1 \right) u(t) \\ &= \sum_{1 \leq s \leq n} \partial_{t_s} \{ t_s \theta_{x_i} u(t) \} + \left(\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s \right) \frac{1}{1 - \sum_{1 \leq s \leq n} t_s} \theta_{x_i} u(t) \\ &= \sum_{1 \leq s \leq n} \partial_{t_s} \left\{ \frac{\beta_i t_s t_i x_i}{1 - x_i t_i} u(t) \right\} + \left(\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s \right) \frac{1}{1 - \sum_{1 \leq s \leq n} t_s} \frac{\beta_i t_i x_i}{1 - x_i t_i} u(t). \end{aligned} \quad (3.6)$$

By (3.3), the action of $(\theta_{x_i} + \beta_i)$ on (3.4) implies

$$\begin{aligned} & (\theta_{x_i} + \beta_i)(\theta_{x_i} + \alpha_i)u(t) \\ &= \partial_{t_i} \{ t_i (\theta_{x_i} + \beta_i)u(t) \} + \left(\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s \right) \frac{t_i}{1 - \sum_{1 \leq s \leq n} t_s} (\theta_{x_i} + \beta_i)u(t) \\ &= \partial_{t_i} \left\{ \frac{\beta_i t_i}{1 - x_i t_i} u(t) \right\} + \left(\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s \right) \frac{t_i}{1 - \sum_{1 \leq s \leq n} t_s} \frac{\beta_i}{1 - x_i t_i} u(t). \end{aligned} \quad (3.7)$$

Combining (3.6) and (3.7), we obtain

$$\begin{aligned} & \left[\theta_{x_i} \left(\sum_{s=1}^n \theta_{x_s} + \gamma - 1 \right) - x_i (\theta_{x_i} + \alpha_i)(\theta_{x_i} + \beta_i) \right] u(t) \\ &= \sum_{1 \leq s \leq n} \partial_{t_s} \left\{ \frac{\beta_i x_i t_i t_s}{1 - x_i t_i} u(t) \right\} - \partial_{t_i} \left\{ \frac{\beta_i x_i t_i}{1 - x_i t_i} u(t) \right\}. \end{aligned}$$

Therefore, we arrive at the result. $\square$

Remark 3.2. If we assume the condition for the exponents

$$\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma - \delta_1 - \dots - \delta_n \notin \mathbb{Z}, \quad (3.8)$$

for $\delta_i \in \{\alpha_i, \beta_i\}$, then the rank of $H_n(T_x, \mathcal{L}_x^\vee)$ is 2^n , and an appropriate set of 2^n cycles would give rise to a fundamental set of solutions of E_B . In the case of $n = 2$, it is shown in [M2] by deriving a Wronskian formula that a certain set of four cycles gives a fundamental set of solutions of E_B , i.e. E_3 .

Remark 3.3. It is shown in [B] that E_B is irreducible if and only if (3.8) holds.

Remark 3.4. If $\alpha_1, \dots, \alpha_n, \gamma - \alpha_1 - \dots - \alpha_n \notin \mathbb{Z}$, then we have

$$\begin{aligned} & \int_{\text{reg}(t_1 > 0, \dots, t_n > 0, 1 > t_1 + \dots + t_n)} \prod_{s=1}^n \{ t_s^{\alpha_s - 1} (1 - x_s t_s)^{-\beta_s} \} \left(1 - \sum_{s=1}^n t_s \right)^{\gamma - 1 - \sum_{1 \leq s \leq n} \alpha_s} dt_1 \cdots dt_n \\ &= \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n) \Gamma(\gamma - \alpha_1 - \dots - \alpha_n)}{\Gamma(\gamma)} \\ & \quad \times F_B(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n), \end{aligned}$$

where the argument of each factor of the integrand is fixed to be zero when $0 < x_i < 1$, $i = 1, \dots, n$; it is the analytic continuation of the integral representation given by Lauricella [L]. See also [AKdF].

4. Lauricella's F_C

Let $x = (x_1, \dots, x_n)$ be a point of

$$X = \mathbb{C}^n \setminus \bigcup_{s=1}^n \{x_s = 0\} \cup \bigcup_{\epsilon_i \in \{-1, 1\}} \{1 + \epsilon_1 \sqrt{x_1} + \dots + \epsilon_n \sqrt{x_n} = 0\} \subset (\mathbb{P}^1)^n,$$

and let $u(t) = u(x; t)$ be a multivalued function

$$u(x; t) = \left(1 - \sum_{s=1}^n t_s\right)^{-\alpha-n+\sum_{1 \leq s \leq n} \gamma_s} \left\{ \prod_{s=1}^n t_s^{-\gamma_s} \right\} \left(1 - \sum_{s=1}^n \frac{x_s}{t_s}\right)^{-\beta}$$

on

$$T_x = \mathbb{C}^n \setminus \bigcup_{s=1}^n \{t_s = 0\} \cup \left\{ \sum_{s=1}^n x_s t_1 \cdots \widehat{t_s} \cdots t_n = t_1 \cdots t_n \right\} \cup \left\{ \sum_{s=1}^n t_s = 1 \right\} \subset (\mathbb{P}^1)^n.$$

Let $\mathcal{L}_x^\vee$ be the locally constant sheaf (local system) determined by $u(t)$.

Then we obtain the following.

THEOREM 4.1. *We have*

$$\begin{aligned} & \left[\theta_{x_i} (\theta_{x_i} + \gamma_i - 1) - x_i \left(\sum_{1 \leq s \leq n} \theta_{x_s} + \beta \right) \left(\sum_{1 \leq s \leq n} \theta_{x_s} + \alpha \right) \right] \\ & \times \int_C u(t) dt_1 \cdots dt_n = 0, \quad i = 1, 2, \dots, n, \end{aligned}$$

for any cycle $C \in H_n(T_x, \mathcal{L}_x^\vee)$.

Proof. Direct calculation shows that

$$\theta_{x_i} u(t) = \frac{\beta(x_i/t_i)}{1 - \sum_{s=1}^n (x_s/t_s)} u(t), \quad (4.1)$$

$$\theta_{t_i} u(t) = \left[-\gamma_i + \frac{-\beta(x_i/t_i)}{1 - \sum_{s=1}^n (x_s/t_s)} + \left(\sum_{1 \leq s \leq n} \gamma_s - \alpha - n \right) \frac{-t_i}{1 - \sum_{1 \leq s \leq n} t_s} \right] u(t). \quad (4.2)$$

Summing both sides of (4.1) over i , we have

$$\left(\sum_{1 \leq s \leq n} \theta_{x_s} + \beta \right) u(t) = \frac{\beta}{1 - \sum_{s=1}^n (x_s/t_s)} u(t), \quad (4.3)$$

since $\sum_s (x_s/t_s) = 1 - (1 - \sum_s (x_s/t_s))$. From (4.1) and (4.2), it follows that

$$(\theta_{x_i} + \theta_{t_i} + \gamma_i) u(t) = \left(-\alpha - n + \sum_{1 \leq s \leq n} \gamma_s \right) \frac{-t_i}{1 - \sum_{1 \leq s \leq n} t_s} u(t),$$

which is equivalent to

$$(\theta_{x_i} - 1 + \gamma_i) u(t) = -\partial_{t_i} \{t_i u(t)\} + \left(\alpha + n - \sum_{1 \leq s \leq n} \gamma_s \right) \frac{t_i}{1 - \sum_{1 \leq s \leq n} t_s} u(t) \quad (4.4)$$

since $\theta_{t_i} = -1 + \partial_{t_i} t_i$, as an operator.

On the other hand, summing both sides of (4.4) over i , we have

$$\left(\sum_{1 \leq s \leq n} \theta_{x_s} + \alpha \right) u(t) = - \sum_{1 \leq s \leq n} \partial_{t_s} \{ t_s u(t) \} + \left(\alpha + n - \sum_{1 \leq s \leq n} \gamma_s \right) \frac{1}{1 - \sum_{1 \leq s \leq n} t_s} u(t), \quad (4.5)$$

since $-\sum_s t_s = -1 + (1 - \sum_s t_s)$.

By (4.1), the action of θ_{x_i} on (4.4) implies

$$\begin{aligned} \theta_{x_i} (\theta_{x_i} - 1 + \gamma_i) u(t) &= -\partial_{t_i} \{ t_i \theta_{x_i} u(t) \} + \left(\alpha + n - \sum_{1 \leq s \leq n} \gamma_s \right) \frac{t_i}{1 - \sum_{1 \leq s \leq n} t_s} \theta_{x_i} u(t) \\ &= -\partial_{t_i} \left\{ \frac{\beta x_i}{1 - \sum_{s=1}^n (x_s/t_s)} u(t) \right\} \\ &\quad + \left(\alpha + n - \sum_{1 \leq s \leq n} \gamma_s \right) \frac{1}{1 - \sum_{1 \leq s \leq n} t_s} \frac{\beta x_i}{1 - \sum_{s=1}^n (x_s/t_s)} u(t). \end{aligned} \quad (4.6)$$

By (4.3), the action of $(\sum_{1 \leq s \leq n} \theta_{x_s} + \beta)$ on (4.5) implies

$$\begin{aligned} &\left(\sum_{1 \leq s \leq n} \theta_{x_s} + \beta \right) \left(\sum_{1 \leq s \leq n} \theta_{x_s} + \alpha \right) u(t) \\ &= - \sum_{1 \leq s \leq n} \partial_{t_s} \left\{ t_s \left(\sum_{1 \leq s \leq n} \theta_{x_s} + \beta \right) u(t) \right\} \\ &\quad + \left(\alpha + n - \sum_{1 \leq s \leq n} \gamma_s \right) \frac{1}{1 - \sum_{1 \leq s \leq n} t_s} \left(\sum_{1 \leq s \leq n} \theta_{x_s} + \beta \right) u(t) \\ &= - \sum_{1 \leq s \leq n} \partial_{t_s} \left\{ \frac{\beta t_s}{1 - \sum_{s=1}^n (x_s/t_s)} u(t) \right\} + \left(\alpha + n - \sum_{1 \leq s \leq n} \gamma_s \right) \frac{\beta}{1 - \sum_{s=1}^n (x_s/t_s)} u(t). \end{aligned} \quad (4.7)$$

Combining (4.6) and (4.7), we obtain

$$\begin{aligned} &\left[\theta_{x_i} (\theta_{x_i} + \gamma_i - 1) - x_i \left(\sum_{1 \leq s \leq n} \theta_{x_s} + \beta \right) \left(\sum_{1 \leq s \leq n} \theta_{x_s} + \alpha \right) \right] u(t) \\ &= \sum_{1 \leq s \leq n} \partial_{t_s} \left\{ \frac{\beta x_i t_s}{1 - \sum_{s=1}^n (x_s/t_s)} u(t) \right\} - \partial_{t_i} \left\{ \frac{\beta x_i}{1 - \sum_{s=1}^n (x_s/t_s)} u(t) \right\}. \end{aligned}$$

Therefore, we arrive at the result. $\square$

Remark 4.2. If we assume the condition for the exponents

$$-\alpha + \sum_{i \in I} \gamma_i, -\beta + \sum_{i \in I} \gamma_i \notin \mathbb{Z}, \quad (4.8)$$

for all $I \subset \{1, \dots, n\}$, then the rank of $H_n(T_x, \mathcal{L}_x^\vee)$ is 2^n , and a set of 2^n cycles might give rise to a fundamental set of solutions of E_C .

Remark 4.3. It is shown in [B] that E_C is irreducible if and only if (4.8) holds.

Remark 4.4. If $\gamma_1, \dots, \gamma_n, \gamma_1 + \dots + \gamma_n - \alpha \notin \mathbb{Z}$, then we have

$$\begin{aligned} & \int_{\text{reg}(t_1 > 0, \dots, t_n > 0, 1 > t_1 + \dots + t_n)} \left\{ \prod_{s=1}^n t_s^{-\gamma_s} \right\} \\ & \quad \times \left(1 - \sum_{s=1}^n t_s \right)^{-\alpha - n + \sum_{1 \leq s \leq n} \gamma_s} \left(1 - \sum_{s=1}^n \frac{x_s}{t_s} \right)^{-\beta} dt_1 \cdots dt_n \\ & = \frac{\Gamma(1 - \gamma_1) \cdots \Gamma(1 - \gamma_n) \Gamma(\gamma_1 + \dots + \gamma_n + 1 - n - \alpha)}{\Gamma(1 - \alpha)} \\ & \quad \times F_C(\alpha, \beta, \gamma_1, \dots, \gamma_n; x_1, \dots, x_n), \end{aligned}$$

where the argument of each factor of the integrand is fixed to be zero when $x_1 > 0, \dots, x_n > 0$ are small. This formula was established by Aomoto and Kita [AK] and Kita [Ki]. See also Proposition 2 of [Ka], and [Pa]. In contrast to the other classical hypergeometric functions, the regularization of the cycle is essential in this integral representation. It is impossible to define the integral on the open set $\{t_1 > 0, \dots, t_n > 0, 1 > t_1 + \dots + t_n\}$ even in the case where the real part of the exponents are positive.

5. Lauricella's F_D

5.1. Integrals of one variable

Let $x = (x_1, \dots, x_n)$ be a point of

$$X = \mathbb{C}^n \setminus \bigcup_{j=1}^n \{x_j = 0\} \cup \bigcup_{j=1}^n \{x_j = 1\} \cup \bigcup_{1 \leq i < j \leq n} \{x_i = x_j\} \subset (\mathbb{P}^1)^n,$$

and let $u(t) = u(x; t)$ be a multivalued function

$$u(x; t) = t^{\lambda_0} \prod_{j=1}^{n+1} (t - x_j)^{\lambda_j} \quad \text{with } x_{n+1} = 1,$$

on

$$T_x = \mathbb{C} \setminus \{0, 1, x_1, \dots, x_n\} \subset \mathbb{P}^1,$$

where

$$\lambda_0 = \beta_1 + \dots + \beta_n - \gamma, \quad \lambda_j = -\beta_j (1 \leq j \leq n), \quad \lambda_{n+1} = \gamma - \alpha - 1. \quad (5.1)$$

Let $\mathcal{L}_x^\vee$ be the locally constant sheaf (local system) determined by $u(t)$.

Then we get Theorem 5.1.

THEOREM 5.1. *We have*

$$\begin{aligned} & \left[\theta_{x_i} \left(\sum_{s=1}^n \theta_{x_s} + \gamma - 1 \right) - x_i (\theta_{x_i} + \beta_i) \left(\sum_{s=1}^n \theta_{x_s} + \alpha \right) \right] \int_C u(t) dt = 0, \quad i = 1, \dots, n, \\ & [x_i (\theta_{x_i} + \beta_i) \theta_{x_j} - x_j (\theta_{x_j} + \beta_j) \theta_{x_i}] \int_C u(t) dt = 0, \quad 1 \leq i \neq j \leq n, \end{aligned}$$

for any cycle $C \in H_1(T_x, \mathcal{L}_x^\vee)$.

Proof. Direct calculation shows that

$$\theta_{x_i} u(t) = \frac{-\lambda_i x_i}{t - x_i} u(t), \quad (5.2)$$

$$\theta_t u(t) = \left[\lambda_0 + \sum_{1 \leq j \leq n} \frac{\lambda_j t}{t - x_j} + \frac{\lambda_{n+1} t}{t - 1} \right] u(t). \quad (5.3)$$

Since $-x_i = (t - x_i) - t$, (5.2) implies

$$(\theta_{x_i} - \lambda_i) u(t) = \frac{-\lambda_i t}{t - x_i} u(t). \quad (5.4)$$

Summing both sides of (5.2) over i and both sides of (5.3), we have

$$\left(\sum_{1 \leq s \leq n} \theta_{x_s} + \theta_t \right) u(t) = \left(\sum_{0 \leq j \leq n} \lambda_j + \frac{\lambda_{n+1} t}{t - 1} \right) u(t),$$

which is equivalent to

$$\left(\sum_{1 \leq s \leq n} \theta_{x_s} - 1 - \sum_{0 \leq j \leq n} \lambda_j \right) u(t) = -\partial_t \{t u(t)\} + \frac{\lambda_{n+1} t}{t - 1} u(t), \quad (5.5)$$

since $\theta_{t_i} = -1 + \partial_t t_i$, as an operator. Since $t = 1 + (t - 1)$, (5.5) can be written as

$$\left(\sum_{1 \leq s \leq n} \theta_{x_s} - 1 - \sum_{0 \leq j \leq n+1} \lambda_j \right) u(t) = -\partial_t \{t u(t)\} + \frac{\lambda_{n+1}}{t - 1} u(t). \quad (5.6)$$

By (5.2), the action of θ_{x_i} on (5.5) implies

$$\begin{aligned} \theta_{x_i} \left(\sum_{1 \leq s \leq n} \theta_{x_s} - 1 - \sum_{0 \leq j \leq n} \lambda_j \right) u(t) &= -\partial_t \{t \theta_{x_i} u(t)\} + \frac{\lambda_{n+1} t}{t - 1} \theta_{x_i} u(t) \\ &= -\partial_t \left\{ t \frac{-\lambda_i x_i}{t - x_i} u(t) \right\} + \frac{\lambda_{n+1} t}{t - 1} \frac{-\lambda_i x_i}{t - x_i} u(t). \end{aligned} \quad (5.7)$$

By (5.4), the action of $(\theta_{x_i} - \lambda_i)$ on (5.6) implies

$$\begin{aligned} (\theta_{x_i} - \lambda_i) \left(\sum_{1 \leq s \leq n} \theta_{x_s} - 1 - \sum_{0 \leq j \leq n+1} \lambda_j \right) u(t) &= -\partial_t \{t (\theta_{x_i} - \lambda_i) u(t)\} + \frac{\lambda_{n+1}}{t - 1} (\theta_{x_i} - \lambda_i) u(t) \\ &= -\partial_t \left\{ t \frac{-\lambda_i t}{t - x_i} u(t) \right\} + \frac{\lambda_{n+1}}{t - 1} \frac{-\lambda_i t}{t - x_i} u(t). \end{aligned} \quad (5.8)$$

Combining (5.7) and (5.8), we obtain

$$\begin{aligned} &\left[x_i (\theta_{x_i} - \lambda_i) \left(\sum_{1 \leq s \leq n} \theta_{x_s} - 1 - \sum_{0 \leq j \leq n+1} \lambda_j \right) - \theta_{x_i} \left(\sum_{1 \leq s \leq n} \theta_{x_s} - 1 - \sum_{0 \leq j \leq n} \lambda_j \right) \right] u(t) \\ &= \lambda_i x_i \partial_t \left\{ \frac{t(t - 1)}{t - x_i} u(t) \right\}, \end{aligned}$$

which leads to

$$\left[x_i (\theta_{x_i} - \lambda_i) \left(\sum_{1 \leq s \leq n} \theta_{x_s} - 1 - \sum_{0 \leq j \leq n+1} \lambda_j \right) - \theta_{x_i} \left(\sum_{1 \leq s \leq n} \theta_{x_s} - 1 - \sum_{0 \leq j \leq n} \lambda_j \right) \right] \int_C u(t) dt = 0, \quad i = 1, \dots, n,$$

for a cycle C .

On the other hand, combining (5.2) and (5.4), we have

$$\begin{aligned} x_i \theta_{x_j} (\theta_{x_i} - \lambda_i) u(t) &= x_i \theta_{x_j} \frac{-\lambda_i t}{t - x_i} u(t) \\ &= x_i \frac{-\lambda_i t}{t - x_i} \frac{-\lambda_j x_j}{t - x_j} u(t) = \frac{\lambda_i \lambda_j x_i x_j t}{(t - x_i)(t - x_j)} u(t), \end{aligned}$$

and

$$\begin{aligned} x_j \theta_{x_i} (\theta_{x_j} - \lambda_j) u(t) &= x_j \theta_{x_i} \frac{-\lambda_j t}{t - x_j} u(t) \\ &= x_j \frac{-\lambda_j t}{t - x_j} \frac{-\lambda_i x_i}{t - x_i} u(t) = \frac{\lambda_i \lambda_j x_i x_j t}{(t - x_i)(t - x_j)} u(t), \end{aligned}$$

which lead to

$$[x_i \theta_{x_j} (\theta_{x_i} - \lambda_i) - x_j \theta_{x_i} (\theta_{x_j} - \lambda_j)] u(t) = 0, \quad 1 \leq i \neq j \leq n,$$

and thus

$$[x_i \theta_{x_j} (\theta_{x_i} - \lambda_i) - x_j \theta_{x_i} (\theta_{x_j} - \lambda_j)] \int_C u(t) dt = 0, \quad 1 \leq i \neq j \leq n,$$

for a cycle C .

Substitution of (5.1) implies the result. $\square$

Remark 5.2. If we assume the condition for the exponents

$$\alpha, \beta_1, \dots, \beta_n, \gamma - \alpha, \gamma - \beta_1 - \dots - \beta_n \notin \mathbb{Z}, \quad (5.9)$$

then the rank of $H_1(T_x, \mathcal{L}_x^\vee)$ is $n + 1$, and it is proved in [MS1] that a set of $n + 1$ cycles give rise to a fundamental set of solutions of E_D .

Remark 5.3. It is shown in [B] and [MS1] that E_D is irreducible if and only if (5.9) holds.

Remark 5.4. If $\alpha, \gamma - \alpha \notin \mathbb{Z}$, then we have

$$\begin{aligned} &\int_{\text{reg}(1, \infty)} t^{\beta_1 + \dots + \beta_n - \gamma} (t - x_1)^{-\beta_1} \dots (t - x_n)^{-\beta_n} (t - 1)^{\gamma - \alpha - 1} dt \\ &= \frac{\Gamma(\alpha) \Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \times F_D(\alpha, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n), \end{aligned}$$

where the argument of each factor of the integrand is fixed to be zero when $x_1, \dots, x_n < 1$; this is a modification of the formula by Lauricella [L]. See also [MS1].

5.2. Integrals of many variables

Let $x = (x_1, \dots, x_n)$ be a point of

$$X = \mathbb{C}^n \setminus \bigcup_{j=1}^n \{x_j = 0\} \cup \bigcup_{j=1}^n \{x_j = 1\} \cup \bigcup_{1 \leq i < j \leq n} \{x_i = x_j\} \subset (\mathbb{P}^1)^n,$$

and let $u(t) = u(x; t)$ be a multivalued function

$$u(x; t) = \left(1 - \sum_{s=1}^n t_s\right)^{\lambda_0} \left(1 - \sum_{s=1}^n t_s x_s\right)^{-\alpha} \prod_{s=1}^n t_s^{\beta_s - 1},$$

on

$$T_x = \mathbb{C} \setminus \bigcup_{j=1}^n \{t_j = 0\} \cup \left\{ \sum_{s=1}^n t_s = 1 \right\} \cup \left\{ \sum_{s=1}^n t_s x_s = 1 \right\} \subset \mathbb{P}^n,$$

where

$$\lambda_0 = \gamma - 1 - \beta_1 - \dots - \beta_n. \quad (5.10)$$

Let $\mathcal{L}_x^\vee$ be the locally constant sheaf (local system) determined by $u(t)$.

Then we get the following.

THEOREM 5.5. *We have*

$$\begin{aligned} \left[\theta_{x_i} \left(\sum_{s=1}^n \theta_{x_s} + \gamma - 1 \right) - x_i (\theta_{x_i} + \beta_i) \left(\sum_{s=1}^n \theta_{x_s} + \alpha \right) \right] \int_C u(t) dt &= 0, \quad i = 1, \dots, n, \\ \left[x_i (\theta_{x_i} + \beta_i) \theta_{x_j} - x_j (\theta_{x_j} + \beta_j) \theta_{x_i} \right] \int_C u(t) dt &= 0, \quad 1 \leq i \neq j \leq n, \end{aligned}$$

for any cycle $C \in H_n(T_x, \mathcal{L}_x^\vee)$.

Proof. Direct calculation shows that

$$\theta_{x_i} u(t) = \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t), \quad (5.11)$$

$$\theta_{t_i} u(t) = \left[\beta_i - 1 + \frac{-\lambda_0 t_i}{1 - \sum_{1 \leq s \leq n} t_s} + \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} \right] u(t). \quad (5.12)$$

It follows from (5.11) and (5.12) that

$$(\theta_{x_i} - \theta_{t_i}) u(t) = \left[1 - \beta_i + \frac{\lambda_0 t_i}{1 - \sum_{1 \leq s \leq n} t_s} \right] u(t),$$

which is equivalent to

$$(\theta_{x_i} + \beta_i) u(t) = \partial_{t_i} \{t_i u(t)\} + \frac{\lambda_0 t_i}{1 - \sum_{1 \leq s \leq n} t_s} u(t), \quad (5.13)$$

since $\theta_{t_i} = -1 + \partial_{t_i} t_i$, as an operator. Moreover, summing both sides of (5.13) over i , we have

$$\left(\sum_{i=1}^n (\theta_{x_i} + \beta_i) \right) u(t) = \sum_{i=1}^n \partial_{t_i} \{t_i u(t)\} + \frac{\lambda_0 \sum_{i=1}^n t_i}{1 - \sum_{1 \leq s \leq n} t_s} u(t),$$

which can be written as

$$\left(\sum_{i=1}^n \theta_{x_i} + \gamma - 1\right)u(t) = \sum_{i=1}^n \partial_{t_i} \{t_i u(t)\} + \frac{\lambda_0}{1 - \sum_{1 \leq s \leq n} t_s} u(t), \quad (5.14)$$

since $\sum_{i=1}^n t_i = 1 - (1 - \sum_{i=1}^n t_i)$ and (5.10).

On the other hand, summing both sides of (5.11) over i , we have

$$\sum_{i=1}^n \theta_{x_i} u(t) = \frac{\alpha \sum_{i=1}^n t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t),$$

which can be written as

$$\left(\sum_{i=1}^n \theta_{x_i} + \alpha\right)u(t) = \frac{\alpha}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t), \quad (5.15)$$

since $\sum_i t_i x_i = 1 - (1 - \sum_i t_i x_i)$.

By (5.11), the action of θ_{x_i} on (5.14) implies

$$\begin{aligned} \theta_{x_i} \left(\sum_{s=1}^n \theta_{x_s} + \gamma - 1\right)u(t) &= \sum_{s=1}^n \partial_{t_s} \{t_s \theta_{x_i} u(t)\} + \frac{\lambda_0}{1 - \sum_{1 \leq s \leq n} t_s} \theta_{x_i} u(t) \\ &= \sum_{s=1}^n \partial_{t_s} \left\{ t_s \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t) \right\} \\ &\quad + \frac{\lambda_0}{1 - \sum_{1 \leq s \leq n} t_s} \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t). \end{aligned} \quad (5.16)$$

By (5.15), the action of $(\sum_{s=1}^n \theta_{x_s} + \alpha)$ on (5.13) implies

$$\begin{aligned} &\left(\sum_{s=1}^n \theta_{x_s} + \alpha\right)(\theta_{x_i} + \beta_i)u(t) \\ &= \partial_{t_i} \left\{ t_i \left(\sum_{s=1}^n \theta_{x_s} + \alpha\right)u(t) \right\} + \frac{\lambda_0 t_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} \left(\sum_{s=1}^n \theta_{x_s} + \alpha\right)u(t) \\ &= \partial_{t_i} \left\{ t_i \frac{\alpha}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t) \right\} + \frac{\lambda_0 t_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} \frac{\alpha}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t). \end{aligned} \quad (5.17)$$

Combining (5.16) and (5.17), we obtain

$$\begin{aligned} &\left[x_i \left(\sum_{s=1}^n \theta_{x_s} + \alpha\right)(\theta_{x_i} + \beta_i) - \theta_{x_i} \left(\sum_{s=1}^n \theta_{x_s} + \gamma - 1\right) \right] u(t) \\ &= x_i \partial_{t_i} \left\{ t_i \frac{\alpha}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t) \right\} - \sum_{s=1}^n \partial_{t_s} \left\{ t_s \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t) \right\}, \end{aligned}$$

which leads to

$$\left[x_i \left(\sum_{s=1}^n \theta_{x_s} + \alpha\right)(\theta_{x_i} + \beta_i) - \theta_{x_i} \left(\sum_{s=1}^n \theta_{x_s} + \gamma - 1\right) \right] \int_C u(t) dt = 0, \quad i = 1, \dots, n,$$

for any cycle C .

On the other hand, combining (5.11) and (5.13), we have

$$\begin{aligned} x_i \theta_{x_j} (\theta_{x_i} + \beta_i) u(t) &= x_i \partial_{t_i} \{t_i \theta_{x_j} u(t)\} + x_i \frac{\lambda_0 t_i}{1 - \sum_{1 \leq s \leq n} t_s} \theta_{x_j} u(t) \\ &= x_i \partial_{t_i} \{t_i \theta_{x_j} u(t)\} + x_i \frac{\lambda_0 t_i}{1 - \sum_{1 \leq s \leq n} t_s} \frac{\alpha t_j x_j}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t) \end{aligned}$$

and

$$\begin{aligned} x_j \theta_{x_i} (\theta_{x_j} + \beta_j) u(t) &= x_j \partial_{t_j} \{t_j \theta_{x_i} u(t)\} + x_j \frac{\lambda_0 t_j}{1 - \sum_{1 \leq s \leq n} t_s} \theta_{x_i} u(t) \\ &= x_j \partial_{t_j} \{t_j \theta_{x_i} u(t)\} + x_j \frac{\lambda_0 t_j}{1 - \sum_{1 \leq s \leq n} t_s} \frac{\alpha t_i x_i}{1 - \sum_{1 \leq s \leq n} x_s t_s} u(t), \end{aligned}$$

which lead to

$$\begin{aligned} [x_i \theta_{x_j} (\theta_{x_i} + \beta_i) - x_j \theta_{x_i} (\theta_{x_j} + \beta_j)] u(t) \\ = x_i \partial_{t_i} \{t_i \theta_{x_j} u(t)\} - x_j \partial_{t_j} \{t_j \theta_{x_i} u(t)\}, \quad 1 \leq i \neq j \leq n, \end{aligned}$$

and thus

$$[x_i \theta_{x_j} (\theta_{x_i} + \beta_i) - x_j \theta_{x_i} (\theta_{x_j} + \beta_j)] \int_C u(t) dt = 0, \quad 1 \leq i \neq j \leq n,$$

for a cycle C . This completes the proof. $\square$

Remark 5.6. If $\beta_1, \dots, \beta_n, \gamma - \beta_1 - \dots - \beta_n \notin \mathbb{Z}$, then we have

$$\begin{aligned} &\int_{\text{reg}(t_1 > 0, \dots, t_n > 0, 1 > t_1 + \dots + t_n)} \\ &\quad \times \prod_{s=1}^n t_s^{\beta_s-1} \left(1 - \sum_{s=1}^n t_s\right)^{\gamma-1-\sum_{1 \leq s \leq n} \beta_s} \left(1 - \sum_{s=1}^n t_s x_s\right)^{-\alpha} dt_1 \cdots dt_n \\ &= \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_n) \Gamma(\gamma - \beta_1 - \dots - \beta_n)}{\Gamma(\gamma)} \times F_D(\alpha, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n), \end{aligned}$$

where the argument of each factor of the integrand is fixed to be zero when $x_1, \dots, x_n$ are small positive numbers. This formula is given in Lauricella [L].

6. Generalized hypergeometric equation ${}_{n+1}E_n$

6.1. Integrals of difference products

Let x be a point of $\mathbb{C} \setminus \{0, 1\}$, and let $u(t) = u(x; t)$ be a multivalued function

$$u(t) = \prod_{i=1}^n t_i^{\lambda_i} \prod_{i=1}^{n+1} (t_{i-1} - t_i)^{\lambda_{i-1,i}} \quad \text{with } t_0 = 1, t_{n+1} = x,$$

on

$$T_x = \mathbb{C}^n \setminus \bigcup_{i=1}^n \{t_i = 0\} \cup \bigcup_{i=1}^{n+1} \{t_{i-1} - t_i = 0\},$$

where

$$\lambda_i = \alpha_{i+1} - \beta_i, \quad \lambda_{i-1,i} = \beta_i - \alpha_i - 1, \quad \text{and} \quad \beta_{n+1} = 1. \quad (6.1)$$

Let $\mathcal{L}_x^\vee$ be the locally constant sheaf (local system) determined by $u(t)$.

Then we obtain Theorem 6.1.

THEOREM 6.1. *We have*

$$\left[\theta_x \prod_{s=1}^n (\theta_x + \beta_s - 1) - x \prod_{s=1}^{n+1} (\theta_x + \alpha_s) \right] \int_C u(t) dt = 0$$

for any cycle $C \in H_n(T_x, \mathcal{L}_x^\vee)$.

Proof. Direct calculation shows that

$$\theta_x u(t) = \frac{-\lambda_{n,n+1}x}{t_n - x} u(t), \quad (6.2)$$

$$\begin{aligned} \theta_{t_j} u(t) &= t_j \left(\frac{\lambda_j}{t_j} + \frac{\lambda_{j,j+1}}{t_j - t_{j+1}} - \frac{\lambda_{j-1,j}}{t_{j-1} - t_j} \right) u(t) \\ &= \left(\lambda_j + \frac{\lambda_{j,j+1}t_j}{t_j - t_{j+1}} - \frac{\lambda_{j-1,j}t_j}{t_{j-1} - t_j} \right) u(t), \quad j = 1, \dots, n. \end{aligned} \quad (6.3)$$

Since $-x = (t_n - x) - t_n$, (6.2) can be written as

$$\theta_x u(t) = \left(\lambda_{n,n+1} + \frac{-\lambda_{n,n+1}t_n}{t_n - x} \right) u(t). \quad (6.4)$$

Since $t_j = (t_j - t_{j+1}) + t_{j+1}$ or $-t_j = (-t_j + t_{j-1}) - t_{j-1}$, (6.3) can be written as

$$\theta_{t_j} u(t) = \left(\lambda_j + \lambda_{j,j+1} + \frac{\lambda_{j,j+1}t_{j+1}}{t_j - t_{j+1}} - \frac{\lambda_{j-1,j}t_j}{t_{j-1} - t_j} \right) u(t), \quad j = 1, \dots, n, \quad (6.5)$$

or

$$\theta_{t_j} u(t) = \left(\lambda_j + \lambda_{j-1,j} + \frac{\lambda_{j,j+1}t_j}{t_j - t_{j+1}} - \frac{\lambda_{j-1,j}t_{j-1}}{t_{j-1} - t_j} \right) u(t), \quad j = 1, \dots, n. \quad (6.6)$$

Summing both sides of (6.2) and (6.5) for $s \leq j \leq n$, we get

$$\left(\theta_x + \sum_{j=s}^n \theta_{t_j} \right) u(t) = \left(\sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) - \frac{\lambda_{s-1,s}t_s}{t_{s-1} - t_s} \right) u(t), \quad s = 1, \dots, n. \quad (6.7)$$

Summing both sides of (6.4) and (6.6) for $s \leq j \leq n$, we get

$$\left(\theta_x + \sum_{j=s}^n \theta_{t_j} \right) u(t) = \left(\lambda_{n,n+1} + \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) - \frac{\lambda_{s-1,s}t_{s-1}}{t_{s-1} - t_s} \right) u(t), \quad s = 1, \dots, n. \quad (6.8)$$

Since $\theta_{t_j} = \partial_{t_j} t_j - 1$ as an operator, equalities (6.7) and (6.8) can be written as

$$\begin{aligned} & \left(\theta_x + \sum_{j=s}^n \partial_{t_j} t_j - (n-s+1) - \sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) \right) u(t) \\ &= -\frac{\lambda_{s-1,s} t_s}{t_{s-1} - t_s} u(t), \quad s = 1, \dots, n, \end{aligned} \quad (6.9)$$

and

$$\begin{aligned} & \left(\theta_x + \sum_{j=s}^n \partial_{t_j} t_j - (n-s+1) - \lambda_{n,n+1} - \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) \right) u(t) \\ &= -\frac{\lambda_{s-1,s} t_{s-1}}{t_{s-1} - t_s} u(t), \quad s = 1, 2, \dots, n. \end{aligned} \quad (6.10)$$

Considering the equality (6.9) multiplied by $\prod_{j=1}^{s-1} (t_j / (t_{j-1} - t_j))$ and the equality (6.10) multiplied by $\prod_{j=1}^{s-1} (t_{j-1} / (t_{j-1} - t_j))$, we have

$$\begin{aligned} & \left(\theta_x - (n-s+1) - \sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) \right) \int_C \prod_{j=1}^{s-1} \frac{t_j}{t_{j-1} - t_j} u(t) dt \\ &= -\lambda_{s-1,s} \int_C \prod_{j=1}^s \frac{t_j}{t_{j-1} - t_j} u(t) dt, \quad s = 1, \dots, n, \end{aligned} \quad (6.11)$$

and

$$\begin{aligned} & \left(\theta_x - (n-s+1) - \lambda_{n,n+1} - \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) \right) \int_C \prod_{j=1}^{s-1} \frac{t_{j-1}}{t_{j-1} - t_j} u(t) dt \\ &= -\lambda_{s-1,s} \int_C \prod_{j=1}^s \frac{t_{j-1}}{t_{j-1} - t_j} u(t) dt, \quad s = 1, \dots, n, \end{aligned} \quad (6.12)$$

for a cycle C . Successive application of (6.11) and (6.12) implies

$$\begin{aligned} & \prod_{s=1}^n \left(\theta_x - (n-s+1) - \sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) \right) \int_C u(t) dt \\ &= \prod_{s=2}^n \left(\theta_x - (n-s+1) - \sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) \right) (-\lambda_{0,1}) \int_C \frac{t_1}{t_0 - t_1} u(t) dt \\ &= \prod_{s=3}^n \left(\theta_x - (n-s+1) - \sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) \right) (-\lambda_{0,1}) (-\lambda_{1,2}) \int_C \prod_{j=1}^2 \frac{t_j}{t_{j-1} - t_j} u(t) dt \\ &= \dots \\ &= \prod_{s=1}^n (-\lambda_{s-1,s}) \int_C \prod_{j=1}^n \frac{t_j}{t_{j-1} - t_j} u(t) dt, \end{aligned} \quad (6.13)$$

and

$$\begin{aligned}
& \prod_{s=1}^n \left(\theta_x - (n-s+1) - \lambda_{n,n+1} - \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) \right) \int_C u(t) dt \\
&= \prod_{s=2}^n \left(\theta_x - (n-s+1) - \lambda_{n,n+1} - \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) \right) (-\lambda_{0,1}) \int_C \frac{t_0}{t_0 - t_1} u(t) dt \\
&= \prod_{s=3}^n \left(\theta_x - (n-s+1) - \lambda_{n,n+1} - \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) \right) (-\lambda_{0,1})(-\lambda_{1,2}) \\
&\quad \times \int_C \prod_{j=1}^2 \frac{t_{j-1}}{t_{j-1} - t_j} u(t) dt \\
&= \dots \\
&= \prod_{s=1}^n (-\lambda_{s-1,s}) \int_C \prod_{j=1}^n \frac{t_{j-1}}{t_{j-1} - t_j} u(t) dt. \tag{6.14}
\end{aligned}$$

On the other hand, equality (6.2) multiplied by $\prod_{j=1}^n (t_j/(t_{j-1} - t_j))$ leads to

$$\theta_x \int_C \prod_{j=1}^n \frac{t_j}{t_{j-1} - t_j} u(t) dt = (-\lambda_{n,n+1}) \int_C \prod_{j=1}^{n+1} \frac{t_j}{t_{j-1} - t_j} u(t) dt, \tag{6.15}$$

and equality (6.4) multiplied by $\prod_{j=1}^n (t_{j-1}/(t_{j-1} - t_j))$ leads to

$$(\theta_x - \lambda_{n,n+1}) \int_C \prod_{j=1}^n \frac{t_{j-1}}{t_{j-1} - t_j} u(t) dt = (-\lambda_{n,n+1}) \int_C \prod_{j=1}^{n+1} \frac{t_{j-1}}{t_{j-1} - t_j} u(t) dt. \tag{6.16}$$

Therefore, combination of (6.13) with (6.15) and combination of (6.14) with (6.16) imply

$$\begin{aligned}
& \theta_x \prod_{s=1}^n \left(\theta_x - (n-s+1) - \sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) \right) \int_C u(t) dt \\
&= \prod_{s=1}^{n+1} (-\lambda_{s-1,s}) \int_C \prod_{j=1}^{n+1} \frac{t_j}{t_{j-1} - t_j} u(t) dt,
\end{aligned}$$

and

$$\begin{aligned}
& (\theta_x - \lambda_{n,n+1}) \prod_{s=1}^n \left(\theta_x - (n-s+1) - \lambda_{n,n+1} - \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) \right) \int_C u(t) dt \\
&= \prod_{s=1}^{n+1} (-\lambda_{s-1,s}) \int_C \prod_{j=1}^{n+1} \frac{t_{j-1}}{t_{j-1} - t_j} u(t) dt,
\end{aligned}$$

which lead to

$$\left[\theta_x \prod_{s=1}^n \left(\theta_x - (n-s+1) - \sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) \right) - x(\theta_x - \lambda_{n,n+1}) \prod_{s=1}^n \left(\theta_x - (n-s+1) - \lambda_{n,n+1} - \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) \right) \right] \int_C u(t) dt.$$

Since

$$\sum_{j=s}^n (\lambda_j + \lambda_{j,j+1}) = \beta_{n+1} - \beta_s - (n-s+1),$$

$$\lambda_{n,n+1} + \sum_{j=s}^n (\lambda_j + \lambda_{j-1,j}) = -\alpha_s - (n-s+1),$$

which follow from (6.1), we reach the result. $\square$

Remark 6.2. If we assume the condition for the exponents

$$\alpha_i - \beta_j \notin \mathbb{Z}, \quad 1 \leq i, j \leq n+1, \quad \text{with } \beta_{n+1} = 1, \quad (6.17)$$

then the rank of $H_n(T_x, \mathcal{L}_x^\vee)$ is $n+1$, and a set of $n+1$ cycles gives rise to a fundamental set of solutions of ${}_{n+1}E_n$. Indeed, such a set of cycles is given in [M1] and [M3].

Remark 6.3. It is known that ${}_{n+1}E_n$ is irreducible if and only if (6.17) holds. See [BH] and [M3].

Remark 6.4. If $\alpha_j, \beta_j - \alpha_j \notin \mathbb{Z}, j = 1, \dots, n$, then we have

$$\int_{\text{reg}(1 < t_1 < \dots < t_n < \infty)} \prod_{s=1}^n \{ t_s^{\alpha_{s+1} - \beta_s} (t_s - t_{s-1})^{\beta_s - \alpha_s - 1} \} (t_n - x)^{-\alpha_{n+1}} dt_1 \dots dt_n$$

$$= \prod_{j=1}^n \frac{\Gamma(\alpha_j) \Gamma(\beta_j - \alpha_j)}{\Gamma(\beta_j)} \times {}_{n+1}F_n \left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{n+1} \\ \beta_1, \dots, \beta_n \end{matrix} ; x \right), \quad (6.18)$$

where $t_0 = 1$ and the argument of each factor of the integrand is fixed to be zero when $0 < x < 1$. This expression is given in (109) of [Po]. See also [A] and [M2].

6.2. Integrals including a polynomial of degree n

Let x be a point of $\mathbb{C} \setminus \{0, 1\}$, and let $u(t) = u(x; t)$ be a multivalued function

$$u(t) = (1 - t_1 t_2 \dots t_n x)^{-\alpha_{n+1}} \prod_{i=1}^n t_i^{\alpha_i - 1} (1 - t_i)^{\beta_i - \alpha_i - 1}$$

on

$$T_x = \mathbb{C}^n \setminus \bigcup_{i=1}^n \{t_i = 0\} \cup \bigcup_{i=1}^n \{t_i = 1\} \cup \{t_1 t_2 \dots t_n x = 1\}.$$

Let $\mathcal{L}_x^\vee$ be the locally constant sheaf (local system) determined by $u(t)$.

Then we obtain Theorem 6.5.

THEOREM 6.5. *We have*

$$\left[\theta_x \prod_{s=1}^n (\theta_x + \beta_s - 1) - x \prod_{s=1}^{n+1} (\theta_x + \alpha_s) \right] \int_C u(t) dt = 0$$

for any cycle $C \in H_n(T_x, \mathcal{L}_x^\vee)$.

Proof. Direct calculation shows that

$$\theta_x u(t) = \frac{\alpha_{n+1} t_1 t_2 \cdots t_n x}{1 - t_1 t_2 \cdots t_n x} u(t), \quad (6.19)$$

$$\theta_{t_i} u(t) = \left[\frac{\alpha_{n+1} t_1 t_2 \cdots t_n x}{1 - t_1 t_2 \cdots t_n x} + (\alpha_i - 1) + \frac{(\beta_i - \alpha_i - 1) t_i}{1 - t_i} \right] u(t), \quad i = 1, \dots, n. \quad (6.20)$$

Since $t_1 \cdots t_n x = 1 - (1 - t_1 \cdots t_n x)$, (6.19) can be written as

$$(\theta_x + \alpha_{n+1}) u(t) = \frac{\alpha_{n+1}}{1 - t_1 t_2 \cdots t_n x} u(t). \quad (6.21)$$

From (6.19) and (6.20), it follows that

$$(\theta_x - \theta_{t_i}) u(t) = \left[1 - \alpha_i + \frac{(\beta_i - \alpha_i - 1) t_i}{1 - t_i} \right] u(t),$$

which is equivalent to

$$(\theta_x + \alpha_i) u(t) = \partial_{t_i} \{t_i u(t)\} + \frac{(\beta_i - \alpha_i - 1) t_i}{1 - t_i} u(t), \quad (6.22)$$

since $\theta_{t_i} = -1 + \partial_{t_i} t_i$, as an operator. Moreover, (6.22) can be written as

$$(\theta_x + \beta_i - 1) u(t) = \partial_{t_i} \{t_i u(t)\} + \frac{(\beta_i - \alpha_i - 1)}{1 - t_i} u(t), \quad (6.23)$$

since $t_i = 1 - (1 - t_i)$. Successive application of (6.22) implies

$$\prod_{i=1}^n (\theta_x + \alpha_i) \int_C u(t) dt = \prod_{i=1}^n (\beta_i - \alpha_i - 1) \int_C \prod_{i=1}^n \frac{t_i}{1 - t_i} u(t) dt,$$

and application of $(\theta_x - \alpha_{n+1})$ by (6.21) induces

$$\prod_{i=1}^{n+1} (\theta_x + \alpha_i) \int_C u(t) dt = \prod_{i=1}^n (\beta_i - \alpha_i - 1) \int_C \frac{\alpha_{n+1}}{1 - t_1 t_2 \cdots t_n x} \prod_{i=1}^n \frac{t_i}{1 - t_i} u(t) dt \quad (6.24)$$

for any cycle $C \in H_n(T_x, \mathcal{L}_x^\vee)$. Similarly, successive application of (6.23) and application of θ_x by (6.19) imply

$$\theta_x \prod_{i=1}^n (\theta_x - \beta_i - 1) \int_C u(t) dt = \prod_{i=1}^n (\beta_i - \alpha_i - 1) \int_C \frac{\alpha_{n+1} t_1 \cdots t_n x}{1 - t_1 t_2 \cdots t_n x} \prod_{i=1}^n \frac{1}{1 - t_i} u(t) dt. \quad (6.25)$$

Combining (6.24) and (6.25), we get

$$\left[x \prod_{i=1}^{n+1} (\theta_x + \alpha_i) - \theta_x \prod_{i=1}^n (\theta_x - \beta_i - 1) \right] \int_C u(t) dt = 0.$$

This completes the proof. $\square$

Remark 6.6. If $\alpha_j, \beta_j - \alpha_j \notin \mathbb{Z}$, $j = 1, \dots, n$, then we have

$$\begin{aligned} & \int_{\text{reg}(0,1)^n} (1 - t_1 t_2 \cdots t_n x)^{-\alpha_{n+1}} \prod_{i=1}^n t_i^{\alpha_i - 1} (1 - t_i)^{\beta_i - \alpha_i - 1} dt_1 \cdots dt_n \\ &= \prod_{j=1}^n \frac{\Gamma(\alpha_j) \Gamma(\beta_j - \alpha_j)}{\Gamma(\beta_j)} \times {}_{n+1}F_n \left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{n+1} \\ \beta_1, \dots, \beta_n \end{matrix} ; x \right), \end{aligned} \quad (6.26)$$

where the argument of each factor of the integrand is fixed to be zero when $0 < x < 1$. This expression is given on [Go, p. 281], on [Po, p. 78], and is derived from [W, equations (38) and (38a)]. See also [A].

Remark 6.7. By the change of integration variables such that $t_1 \cdots t_j = 1/u_j$, $j = 1, \dots, n$, we obtain the left-hand side of (6.18) from the left-hand side of (6.26).

7. Hypergeometric system $RE(k, n)$ on the configuration space

Let $u(t) = u(x; t)$ be a multivalued function

$$u(x; t) = \left\{ \prod_{s=2}^k t_s^{\lambda_s} \right\} \left(1 - \sum_{s=2}^k t_s \right)^{\lambda_{k+1}} \left\{ \prod_{p=k+2}^n \left(1 - \sum_{s=2}^k t_s x_{sp} \right)^{\lambda_p} \right\}$$

on

$$T_x = \mathbb{C}^{k-1} \setminus \bigcup_{s=2}^k \{t_s = 0\} \cup \left\{ \sum_{s=2}^k t_s = 1 \right\} \cup \bigcup_{p=k+2}^n \left\{ \sum_{s=2}^k t_s x_{sp} = 1 \right\} \subset (\mathbb{P}^1)^{k-1}.$$

Let $\mathcal{L}_x^\vee$ be the locally constant sheaf (local system) determined by $u(t)$.

Then we get Theorem 7.1.

THEOREM 7.1. *We have*

$$\begin{aligned} & [x_{ip}(\theta_i + \lambda_i + 1)(\theta^p - \lambda_p) - \theta_{ip}(\theta + \lambda_0 - 1)] \int_C u(t) dt = 0, \\ & \quad 2 \leq i \leq k \text{ and } k+2 \leq p \leq n, \\ & [x_{iq}\theta_{ip}(\theta^q - \lambda_q) - x_{ip}\theta_{iq}(\theta^p - \lambda_p)] \int_C u(t) dt = 0, \quad 2 \leq i \leq k \text{ and } k+2 \leq p, q \leq n, \\ & [x_{ip}\theta_{jp}(\theta_i + \lambda_i + 1) - x_{jp}\theta_{ip}(\theta_j + \lambda_j + 1)] \int_C u(t) dt = 0, \\ & \quad 2 \leq i, j \leq k \text{ and } k+2 \leq p \leq n, \\ & [x_{iq}x_{jp}\theta_{ip}\theta_{jq} - x_{ip}x_{jq}\theta_{iq}\theta_{jp}] \int_C u(t) dt = 0, \quad 2 \leq i, j \leq k \text{ and } k+2 \leq p, q \leq n, \end{aligned}$$

for any cycle $C \in H_{k-1}(T_x, \mathcal{L}_x^\vee)$, where

$$\begin{aligned}\lambda_0 &= \lambda_2 + \cdots + \lambda_{k+1} + k, \\ \theta_{jq} &= x_{jq} \frac{\partial}{\partial x_{jq}}, \\ \theta_j &= \theta_{j,k+2} + \theta_{j,k+3} + \cdots + \theta_{j,n}, \\ \theta^q &= \theta_{2,q} + \theta_{3,q} + \cdots + \theta_{k,q}, \\ \theta &= \theta^{k+2} + \cdots + \theta^n = \theta_2 + \cdots + \theta_k = \sum_{\substack{2 \leq i \leq k \\ k+2 \leq r \leq n}} \theta_{ir}.\end{aligned}\tag{7.1}$$

Proof. Direct calculation shows that

$$\theta_{x_{ip}} u(t) = \frac{-\lambda_p t_i x_{ip}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t),\tag{7.2}$$

$$\theta_{t_i} u(t) = \left[\lambda_i + \frac{-\lambda_{k+1} t_i}{1 - \sum_{2 \leq s \leq k} t_s} + \sum_{p=k+2}^n \frac{-\lambda_p t_i x_{ip}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} \right] u(t)\tag{7.3}$$

for $2 \leq i \leq k$, $k+2 \leq p \leq n$. Summing both sides of (7.2) over i , we have

$$\left(\sum_{2 \leq i \leq k} \theta_{x_{ip}} - \lambda_p \right) u(t) = \frac{-\lambda_p}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t),\tag{7.4}$$

since $\sum_s t_s x_{sp} = 1 - (1 - \sum_s t_s x_{sp})$.

It follows from (7.2) and (7.3) that

$$\left(\sum_{k+2 \leq p \leq n} \theta_{x_{ip}} - \theta_{t_i} \right) u(t) = \left[-\lambda_i + \frac{\lambda_{k+1} t_i}{1 - \sum_{2 \leq s \leq k} t_s} \right] u(t),$$

which is equivalent to

$$\left(\sum_{k+2 \leq p \leq n} \theta_{x_{ip}} + \lambda_i + 1 \right) u(t) = \partial_{t_i} \{t_i u(t)\} + \frac{\lambda_{k+1} t_i}{1 - \sum_{2 \leq s \leq k} t_s} u(t),\tag{7.5}$$

since $\theta_{t_i} = -1 + \partial_{t_i} t_i$, as an operator. Moreover, summing both sides of (7.5) over i , we have

$$\left(\sum_{2 \leq i \leq k} \sum_{k+2 \leq p \leq n} \theta_{x_{ip}} + \sum_{2 \leq i \leq k} \lambda_i + (k-1) \right) u(t) = \sum_{2 \leq i \leq k} \partial_{t_i} \{t_i u(t)\} + \frac{\lambda_{k+1} \sum_{2 \leq i \leq k} t_i}{1 - \sum_{2 \leq s \leq k} t_s} u(t),$$

which is equivalent to

$$\left(\sum_{2 \leq i \leq k} \sum_{k+2 \leq p \leq n} \theta_{x_{ip}} + \sum_{2 \leq i \leq k+1} \lambda_i + (k-1) \right) u(t) = \sum_{2 \leq i \leq k} \partial_{t_i} \{t_i u(t)\} + \frac{\lambda_{k+1}}{1 - \sum_{2 \leq s \leq k} t_s} u(t),\tag{7.6}$$

since $\sum_{2 \leq i \leq k} t_i = 1 - (1 - \sum_{2 \leq i \leq k} t_i)$.

By (7.4), the action of $(\sum_{2 \leq i \leq k} \theta_{x_{ip}} - \lambda_p)$ on (7.5) implies

$$\begin{aligned} & \left(\sum_{2 \leq t \leq k} \theta_{x_{tp}} - \lambda_p \right) \left(\sum_{k+2 \leq p' \leq n} \theta_{x_{ip'}} + \lambda_i + 1 \right) u(t) \\ &= \partial_{t_i} \left\{ t_i \left(\sum_{2 \leq t \leq k} \theta_{x_{tp}} - \lambda_p \right) u(t) \right\} + \frac{\lambda_{k+1} t_i}{1 - \sum_{2 \leq s \leq k} t_s} \left(\sum_{2 \leq t \leq k} \theta_{x_{tp}} - \lambda_p \right) u(t) \\ &= \partial_{t_i} \left\{ t_i \frac{-\lambda_p}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t) \right\} + \frac{\lambda_{k+1} t_i}{1 - \sum_{2 \leq s \leq k} t_s} \frac{-\lambda_p}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t). \end{aligned} \quad (7.7)$$

By (7.2), the action of $\theta_{x_{ip}}$ on (7.6) implies

$$\begin{aligned} & \theta_{x_{ip}} \left(\sum_{2 \leq i' \leq k} \sum_{k+2 \leq p' \leq n} \theta_{x_{i'p'}} + \sum_{2 \leq i' \leq k+1} \lambda_{i'} + (k-1) \right) u(t) \\ &= \sum_{2 \leq i' \leq k} \partial_{t_{i'}} \{ t_{i'} \theta_{x_{ip}} u(t) \} + \frac{\lambda_{k+1}}{1 - \sum_{2 \leq s \leq k} t_s} \theta_{x_{ip}} u(t) \\ &= \sum_{2 \leq i' \leq k} \partial_{t_{i'}} \left\{ t_{i'} \frac{-\lambda_p t_i x_{ip}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t) \right\} + \frac{\lambda_{k+1}}{1 - \sum_{2 \leq s \leq k} t_s} \frac{-\lambda_p t_i x_{ip}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t). \end{aligned} \quad (7.8)$$

Combining (7.7) and (7.8), we obtain

$$\begin{aligned} & \left[x_{ip} \left(\sum_{2 \leq t \leq k} \theta_{x_{tp}} - \lambda_p \right) \left(\sum_{k+2 \leq p' \leq n} \theta_{x_{ip'}} + \lambda_i + 1 \right) \right. \\ & \quad \left. - \theta_{x_{ip}} \left(\sum_{2 \leq i' \leq k} \sum_{k+2 \leq p' \leq n} \theta_{x_{i'p'}} + \sum_{2 \leq i' \leq k+1} \lambda_{i'} + (k-1) \right) \right] u(t) \\ &= x_{ip} \partial_{t_i} \left\{ t_i \frac{-\lambda_p}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t) \right\} - \sum_{2 \leq i' \leq k} \partial_{t_{i'}} \left\{ t_{i'} \frac{-\lambda_p t_i x_{ip}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t) \right\}, \end{aligned}$$

which implies

$$\begin{aligned} & \left[x_{ip} \left(\sum_{2 \leq t \leq k} \theta_{x_{tp}} - \lambda_p \right) \left(\sum_{k+2 \leq p' \leq n} \theta_{x_{ip'}} + \lambda_i + 1 \right) \right. \\ & \quad \left. - \theta_{x_{ip}} \left(\sum_{2 \leq i' \leq k} \sum_{k+2 \leq p' \leq n} \theta_{x_{i'p'}} + \lambda_0 - 1 \right) \right] \int_C u(t) dt = 0. \end{aligned}$$

Next, the action of $\theta_{x_{ip}}$ on (7.4) by (7.2) implies

$$\theta_{x_{ip}} \left(\sum_{2 \leq i' \leq k} \theta_{x_{i'q}} - \lambda_q \right) u(t) = \frac{-\lambda_q}{1 - \sum_{2 \leq s \leq k} t_s x_{sq}} \frac{-\lambda_p t_i x_{ip}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t)$$

for $k+2 \leq p \neq q \leq n$, which leads to

$$\left[x_{iq} \theta_{x_{ip}} \left(\sum_{2 \leq i' \leq k} \theta_{x_{i'q}} - \lambda_q \right) - x_{ip} \theta_{x_{iq}} \left(\sum_{2 \leq i' \leq k} \theta_{x_{i'p}} - \lambda_p \right) \right] u(t) = 0,$$

and

$$\left[x_{iq} \theta_{x_{ip}} \left(\sum_{2 \leq i' \leq k} \theta_{x_{i'q}} - \lambda_q \right) - x_{ip} \theta_{x_{iq}} \left(\sum_{2 \leq i' \leq k} \theta_{x_{i'p}} - \lambda_p \right) \right] \int_C u(t) dt = 0.$$

On the other hand, the action of $\theta_{x_{jp}}$ on (7.5) by (7.2) implies

$$\begin{aligned} \theta_{x_{jp}} \left(\sum_{k+2 \leq p' \leq n} \theta_{x_{ip'}} + \lambda_i + 1 \right) u(t) &= \partial_{t_i} \{ t_i \theta_{x_{jp}} u(t) \} + \frac{\lambda_{k+1} t_i}{1 - \sum_{2 \leq s \leq k} t_s} \theta_{x_{jp}} u(t) \\ &= \partial_{t_i} \left\{ t_i \frac{-\lambda_p t_j x_{jp}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t) \right\} \\ &\quad + \frac{\lambda_{k+1} t_i}{1 - \sum_{2 \leq s \leq k} t_s} \frac{-\lambda_p t_i x_{jp}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t), \end{aligned}$$

which leads to

$$\begin{aligned} &\left[x_{ip} \theta_{x_{jp}} \left(\sum_{k+2 \leq p' \leq n} \theta_{x_{ip'}} + \lambda_i + 1 \right) - x_{jp} \theta_{x_{ip}} \left(\sum_{k+2 \leq p' \leq n} \theta_{x_{jp'}} + \lambda_j + 1 \right) \right] u(t) \\ &= x_{ip} \partial_{t_i} \left\{ t_i \frac{-\lambda_p t_j x_{jp}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t) \right\} - x_{jp} \partial_{t_j} \left\{ t_j \frac{-\lambda_p t_i x_{ip}}{1 - \sum_{2 \leq s \leq k} t_s x_{sp}} u(t) \right\}, \end{aligned}$$

and

$$\left[x_{ip} \theta_{x_{jp}} \left(\sum_{k+2 \leq p' \leq n} \theta_{x_{ip'}} + \lambda_i + 1 \right) - x_{jp} \theta_{x_{ip}} \left(\sum_{k+2 \leq p' \leq n} \theta_{x_{jp'}} + \lambda_j + 1 \right) \right] \int_C u(t) dt = 0.$$

Finally, using (7.2) twice, we have, for $p \neq q$,

$$\begin{aligned} \theta_{x_{ip}} \theta_{x_{jq}} u(t) &= \theta_{x_{ip}} \frac{-\lambda_q t_j x_{jq}}{1 - \sum_{2 \leq s \leq k} t_s x_{sq}} u(t) = \frac{-\lambda_q t_i x_{jq}}{1 - \sum_{2 \leq s \leq k} t_s x_{sq}} \theta_{x_{ip}} u(t) \\ &= \frac{-\lambda_q t_j x_{jq}}{1 - \sum_{2 \leq s \leq k} t_s x_{sq}} \frac{-\lambda_p t_i x_{ip}}{1 - \sum_{2 \leq s \leq k} t_s x_{sq}} u(t), \end{aligned}$$

which leads to

$$[x_{jp} x_{iq} \theta_{x_{ip}} \theta_{x_{jq}} - x_{jq} x_{ip} \theta_{x_{iq}} \theta_{x_{jp}}] u(t) = 0,$$

and

$$[x_{jp} x_{iq} \theta_{x_{ip}} \theta_{x_{jq}} - x_{jq} x_{ip} \theta_{x_{iq}} \theta_{x_{jp}}] \int_C u(t) dt = 0.$$

This completes the proof. $\square$

Remark 7.2. If $\lambda_1, \dots, \lambda_{k+1} \notin \mathbb{Z}$, then we have

$$\begin{aligned} &\int_{\text{reg}(t_2 > 0, \dots, t_k > 0, 1 > t_2 + \dots + t_k)} \left\{ \prod_{s=2}^k t_s^{\lambda_s} \right\} \left(1 - \sum_{s=2}^k t_s \right)^{\lambda_{k+1}} \prod_{p=k+2}^n \left(1 - \sum_{s=2}^k t_s x_{sp} \right)^{\lambda_p} dt_2 \cdots dt_k \\ &= \frac{\Gamma(\lambda_2 + 1) \cdots \Gamma(\lambda_{k+1} + 1)}{\Gamma(\lambda_2 + \dots + \lambda_{k+1} + k)} \times F(k, n; \lambda; x), \end{aligned}$$

where the argument of each factor of the integrand is fixed to be zero when all x_{sp} ($2 \leq s \leq k$ and $k+2 \leq p \leq n$) are positive and small enough. See also [Ki, equation (8.5)].

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Katsuhisa Mimachi
Department of Pure and Applied Mathematics
Osaka University
Suita, Osaka, 565-0871
Japan
(E-mail: mimachi@math.sci.osaka-u.ac.jp)

Masatoshi Noumi
Department of Mathematics
Kobe University
Rokko, Kobe, 657-8501
Japan
(E-mail: noumi@math.kobe-u.ac.jp)