ON A FUNDAMENTAL SET OF SOLUTIONS TO THE GENERALIZED HYPERGEOMETRIC EQUATION $n+1E_n$ PRESENTED BY INTEGRALS OF EULER TYPE

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Abstract. We give a fundamental set of solutions to the generalized hypergeometric equation $n+1E_n$ in terms of integrals of Euler type and explicitly determine the matrix elements of the circuit matrices with respect to this set of solutions.

0. Introduction

The generalized hypergeometric series $n+1F_n$ is defined to be

$$n+1F_n\left(\frac{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}}{\beta_1, \ldots, \beta_n}; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_{n+1})_k}{(\beta_1)_k \cdots (\beta_n)_k} \frac{z^k}{k!}, \quad |z| < 1,$$

where $(a)_k = a(a+1) \cdots (a+k-1)$. It satisfies the generalized hypergeometric differential equation $n+1E_n$, which has regular singular points $z = 0, 1, \infty$ and whose rank is $n+1$:

$$\left\{ \prod_{1 \leq i \leq n+1} (\theta z + \beta_i - 1) - z \left\{ \prod_{1 \leq i \leq n+1} (\theta z + \alpha_i) \right\} \right\} F = 0,$$

where $\theta z = zd/dz$ and $\beta_{n+1} = 1$.

The purpose of the present paper is to give a fundamental set of solutions of $n+1E_n$ in terms of integrals of Euler type, and to express explicitly the matrix elements of the circuit matrices with respect to this set of solutions.

We introduce some notation.

When $z$ is real and $0 < z < 1$, let $D_j, j = 1, \ldots, n+1$, be the domain

$$D_j = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 < t_j < \cdots < t_n < z, \quad t_j < t_{j-1} < \cdots < t_1 < 1\}$$

of the real manifold $(T_z)_{\mathbb{R}}$, which is the real locus of

$$T_z = \mathbb{C}^n \setminus \bigcup_{i=1}^{n} \{t_i = 0\} \cup \bigcup_{i=1}^{n+1} \{t_{i-1} - t_i = 0\}.$$

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Let $I_j(z)$ be

$$I_j(z) = \int_{D_j} u_{D_j}(z; t) \, dt_1 \cdots dt_n, \quad j = 1, \ldots, n + 1,$$

where

$$u_{D_j}(z; t) = \prod_{i=1}^{n} t_i^{\alpha_{i+1} - \beta_i} \prod_{i=1}^{n+1} (\epsilon'_i(t_{i-1} - t_i))^{\beta_i - \alpha_i - 1}$$

with $t_0 = 1, \ t_{n+1} = z, \ \beta_{n+1} = 1,$

and, for each $i = 1, \ldots, n + 1, \ \epsilon'_i = \pm$ is so determined that $\epsilon'_i(t_{i-1} - t_i)$ is positive on $D_j$ and the argument of $\epsilon'_i(t_{i-1} - t_i)$ is assigned to be zero on $D_j.$

In Section 1, we show that each $I_j(z)$ satisfies

$$1 + \alpha_j - \beta_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i < j \leq n + 1, \quad \text{and} \quad \beta_i - \alpha_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i \leq n + 1, \quad (0.1)$$

and that the integrals $I_j(z), \ j = 1, \ldots, n + 1,$ give a fundamental set of solutions of $n + 1 E_n$ if the conditions (0.1) and

$$\beta_j - \alpha_i \notin \mathbb{Z}_{< 0}, \quad 1 \leq i < j \leq n + 1 \quad (0.2)$$

hold. In Section 2, we express explicitly the matrix elements of $\rho(\gamma_0)$ and $\rho(\gamma_1)$ with respect to the basis $(I_1(z), \ldots, I_{n+1}(z));$ in other words, the matrix elements of the circuit matrices $M(\gamma_0)$ and $M(\gamma_1),$ where $M(\gamma_k) = (m^{(k)}_{ij})_{1 \leq i, j \leq n+1}, \ k = 0, 1,$ are defined by

$$\rho(\gamma_k)I_j(z) = \sum_{1 \leq i \leq n+1} I_i(z)m^{(k)}_{ij}.$$}

Here, $\gamma_0$ and $\gamma_1$ are generators of $\pi_1(C \setminus [0, 1], z^0),$ which correspond to the paths encircling the points 0 and 1, respectively, and $\rho(\gamma_k)I_j(z)$ denotes the analytic continuation of $I_j(z)$ along $\gamma_k.$ To derive the matrix elements, we deform, cut and paste the loaded chains, as in [M1].

Regarding $n + 1 E_n,$ Beukers and Heckman [BH] showed that the monodromy representations associated with $n + 1 E_n$ are irreducible if and only if $\alpha_i - \beta_j \notin \mathbb{Z}$ for any $i, \ j = 1, \ldots, n + 1.$ The expressions in Theorem 2.8 are valid even if the representation is not irreducible, and thus include some examples of reducible representations. We refer the reader to [M2] for the importance of the reducible cases of $n + 1 E_n.$ In [M2], we used another fundamental set of solutions to investigate examples of subrepresentations in the reducible cases, each of which is a generalization of what some physicists considered before.

As for the expression of the circuit matrices in the cases of Gauss’ $E_1,$ Lauricella’s $E_D,$ Jordan–Pochhammer’s $E_{JP}$ and Appell’s $E_1, \ E_2, \ E_3,$ we refer the reader to [MS1, MS2, MS3, M3].

In this paper, the symbols

$$e(A) = \exp(\pi \sqrt{-1} A), \quad \langle A \rangle = A - A^{-1}$$

and

$$\epsilon : \text{a sufficiently small positive number}$$

are frequently used.
1. A fundamental set of solutions

1.1. Generalized hypergeometric function \( n+1 F_n \)

The generalized hypergeometric function \( n+1 F_n \) is the analytic continuation of the generalized hypergeometric series defined by

\[
n+1 F_n \left( \alpha_1, \alpha_2, \ldots, \alpha_{n+1} ; \beta_1, \ldots, \beta_n ; z \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_{n+1})_k}{(\beta_1)_k \cdots (\beta_n)_k} k! z^k, \quad |z| < 1,
\]

where \((a)_k = a(a+1) \cdots (a+k-1)\). It satisfies the generalized hypergeometric differential equation \( n+1 E_n \) with regular singular points \( z = 0, 1, \infty \):

\[
\left\{ \theta z \left( \prod_{1 \leq i \leq n} (\theta z + \beta_i - 1) \right) - z \left( \prod_{1 \leq i \leq n+1} (\theta z + \alpha_i) \right) \right\} F = 0, \quad (1.1)
\]

where \(\theta z = zd/dz\). The rank of (1.1) is \(n+1\). The exponents of (1.1) are

\[
1 - \beta_1, 1 - \beta_2, \ldots, 1 - \beta_n, 0 \quad \text{at} \quad z = 0,
\]

\[0, 1, \ldots, n - 1, \sum_{i=1}^{n+1} \beta_i - \sum_{i=1}^{n+1} \alpha_i \quad \text{at} \quad z = 1,
\]

\[\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \quad \text{at} \quad z = \infty.
\]

The function \( n+1 F_n \) has an integral representation given by

\[
n+1 F_n \left( \alpha_1, \alpha_2, \ldots, \alpha_{n+1} ; \beta_1, \beta_2, \ldots, \beta_n ; z \right) = \prod_{s=1}^{n} \frac{\Gamma(\beta_s)}{\Gamma(\alpha_s) \Gamma(\beta_s - \alpha_s)} \times \int_{1 < t_1 < t_2 < \cdots < t_n < \infty} \prod_{i=1}^{n} t_i^{-\alpha_{i+1}} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\beta_i - \alpha_i - 1} dt_1 \cdots dt_n,
\]

where \(t_0 = 1, t_{n+1} = z, \beta_{n+1} = 1\) and \(\text{Re} \alpha_i, \text{Re} (\beta_i - \alpha_i) > 0 \ (1 \leq i \leq n + 1)\), and the branch of each factor is fixed to be zero on the integration domain.

1.2. Integral representation of the solutions

Let \(z\) be a point of \(\mathbb{C} \setminus \{0, 1\}\), and let \(u(t) = u(z; t)\) be a multivalued function

\[
u(t) = \prod_{i=1}^{n} t_i^{\lambda_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\lambda_i - 1}
\]
on
\[T_z = \mathbb{C}^n \setminus \bigcup_{i=1}^{n} \{t_i = 0\} \cup \bigcup_{i=1}^{n+1} \{t_i - t_{i-1} = 0\},
\]
where
\[ \lambda_i = \alpha_{i+1} - \beta_i, \quad \lambda_{i-1,i} = \beta_i - \alpha_i - 1, \quad t_0 = 1, \quad t_{n+1} = z, \quad \text{and} \quad \beta_{n+1} = 1. \]

Let \( \mathcal{L}_z \) be the locally constant sheaf (the local system) defined by \( u \): the sheaf consisting of the local solutions of \( dL = L\omega \) for \( \omega = du(t)/u(t) \). Let \( H_m(T_z, \mathcal{L}_z) \) be the \( m \)th homology group with coefficients in \( \mathcal{L}_z \), and let \( H_{lf}^m(T_z, \mathcal{L}_z) \) be the \( m \)th locally finite homology group with coefficients in \( \mathcal{L}_z \). Elements of these twisted homology groups, called \textit{twisted cycles} or \textit{loaded cycles}, are represented by \( \partial \)-closed twisted or loaded (finite or locally finite) chains

\[ C = \sum_{\Delta} a_\Delta \Delta \otimes v_\Delta \quad (a_\Delta \in \mathbb{C}), \]

where each \( \Delta \) is an \( m \)-simplex and \( v_\Delta \) is a section of \( \mathcal{L}_z \) on \( \Delta \).

For \( 1 \leq i \leq n + 1 \) and \( 1 \leq p \leq q \leq n \), let
\[ \lambda_{t_{i-1}=t_i}, \quad \lambda_{t_0=t_p=t_{p+1}=\cdots=t_q}, \quad \lambda_{t_p=t_{p+1}=\cdots=t_q=\infty} \quad \text{and} \quad \mu_i \]
be defined to be

\[ \lambda_{t_{i-1}=t_i} := \lambda_{i-1,i}, \]
\[ \lambda_{t_0=t_p=t_{p+1}=\cdots=t_q} := \sum_{r=p}^{q} \lambda_r + \sum_{r=p+1}^{q} \lambda_{r-1,r}, \]
\[ \lambda_{t_p=t_{p+1}=\cdots=t_q=\infty} := -\sum_{r=p}^{q} \lambda_r - \sum_{r=p}^{q+1} \lambda_{r-1,r}, \]

and
\[ \mu_i := \lambda_i + \lambda_{i,i+1}. \]

Then we have
\[ \lambda_{t_{i-1}=t_i} = -1 + \beta_i - \alpha_i, \]
\[ \lambda_{t_0=t_p=t_{p+1}=\cdots=t_q} = -(q - p) + \alpha_{q+1} - \beta_p, \]
\[ \lambda_{t_p=t_{p+1}=\cdots=t_q=\infty} = 2 + q - p + \alpha_p - \beta_{q+1}, \]
\[ \lambda_{p-1,p} + \lambda_{t_p=t_{p+1}=\cdots=t_q=\infty} = -\sum_{r=p}^{q} (\lambda_r + \lambda_{r,r+1}), \]
\[ \lambda_{t_0=t_{p+1}=\cdots=t_q} + \lambda_{q,q+1} = \sum_{r=p}^{q} (\lambda_r + \lambda_{r,r+1}), \]
\[ \lambda_{p-1,p} + \lambda_{t_p=t_{p+1}=\cdots=t_q=\infty} = 1 + q - p + \beta_p - \beta_{q+1}, \]
\[ \lambda_{t_0=t_{p+1}=\cdots=t_q} + \lambda_{q,q+1} = -q + p - 1 + \beta_{q+1} - \beta_p \]

and
\[ \mu_i = -1 + \beta_{i+1} - \beta_i. \]
The conditions
\[ \lambda_{t_{i-1}} = t_i, \quad \lambda_0 = t_p = \cdots = t_q, \quad \lambda_{t_p} = t_{p+1} = \cdots = t_q = \infty \notin \mathbb{Z}, \quad 1 \leq i \leq n + 1, \quad 1 \leq p \leq q \leq n \]  
(1.2)
is equivalent to the condition
\[ \alpha_i - \beta_j \notin \mathbb{Z}, \quad 1 \leq i, j \leq n + 1 \text{ with } \beta_{n+1} = 1. \]  
(1.3)

If (1.3) holds, it follows that \( \text{rank } H_j(T_z, \mathcal{L}_z) = \text{rank } H^{\text{lf}}_j(T_z, \mathcal{L}_z) = 0 \) for \( j \neq n \), \( \text{rank } H_n(T_z, \mathcal{L}_z) = \text{rank } H^{\text{lf}}_n(T_z, \mathcal{L}_z) = n + 1 \), and that the natural map \( \iota : H_n(T_z, \mathcal{L}_z) \to H^{\text{lf}}_n(T_z, \mathcal{L}_z) \) is an isomorphism. The inverse of the natural map is said to be the \textit{regularization},
\[ \text{reg} : H^{\text{lf}}_m(T, \mathcal{L}) \to H_m(T, \mathcal{L}) \]
(see [A1, A2, C, KN]).

When a complex variable \( z \) is real, for a domain \( D \) of the real manifold \( (T_z)_{\mathbb{R}} \) (the real locus of \( T_z \)), we load \( D \) with a section
\[ u_D(z; t) = \prod_{i=1}^{n} (\epsilon_i t_i)^{\lambda_j} \prod_{i=1}^{n+1} (\epsilon'_i (t_{i-1} - t_i))^{\lambda_{i-1,i}} \]
of \( \mathcal{L}_z \), and consider a loaded chain \( \mathcal{D} \otimes u_D(t) \), where \( \epsilon_i, \epsilon'_i = \pm 1 \) is determined so that each of \( \epsilon_i t_i \) and \( \epsilon'_i (t_{i-1} - t_i) \) is positive on \( D \) and the argument of each of \( \epsilon_i t_i \) and \( \epsilon'_i (t_{i-1} - t_i) \) is assigned to be zero on \( D \) (we also use an appropriate subset of \( D \) to assign the argument of each factor).

This choice of a section is said to be \textit{standard over} \( D \). The function \( u_p(t) \) for \( p \in (T_z)_{\mathbb{R}} \) is also defined in the same way: each of \( \epsilon_i t_i \) and \( \epsilon'_i (t_{i-1} - t_i) \) is positive at \( p \) and the argument of each of \( \epsilon_i t_i \) and \( \epsilon'_i (t_{i-1} - t_i) \) is assigned to be zero at \( p \). This choice of a section is said to be \textit{standard over the point} \( p \).

When the complex variable \( z \) is real and \( 0 < z < 1 \), we assign the name \( D_j \) for \( 1 \leq j \leq n + 1 \) to each domain of the real manifold \( (T_z)_{\mathbb{R}} \),
\[ D_j = \{(t_1, \ldots, t_n) \mid 0 < t_j < \cdots < t_n < z, \ t_j < t_{j-1} < \cdots < t_1 < 1\}, \]
where the orientation of \( D_j \) is fixed to be the natural one induced from \( (T_z)_{\mathbb{R}} \). Frequently, we write only \( D_j \) to express the loaded chain
\[ D_j \otimes u_D(t) = D_j \otimes u_{\{0 < t_j < \cdots < t_n < \epsilon, \ 1-\epsilon < t_{j-1} < \cdots < t_1 < 1\}(t)} \]
= \[ D_j \otimes u_{\{1-\epsilon^{j-1}, \ 1-\epsilon^{j-2}, \ldots, 1-\epsilon, \ \epsilon^{n-j+1}, \ldots, \ \epsilon^2, \ \epsilon\}}(t) \]
for simplicity. We note that \( \{0 < t_j < \cdots < t_n < \epsilon, \ 1-\epsilon < t_{j-1} < \cdots < t_1 < 1\} \) is a small part of \( D_j \) and the point \( (1-\epsilon^{j-1}, 1-\epsilon^{j-2}, \ldots, 1-\epsilon, \ \epsilon^{n-j+1}, \ldots, \ \epsilon^2, \ \epsilon) \) is located at \( D_j \). We recall that \( \epsilon \) is a sufficiently small number.

Next, we consider the integrals:
\[ I_j(z) = \int_{D_j} u_D(z; t) \, dt_1 \cdots dt_n, \quad j = 1, \ldots, n + 1. \]
The conditions for the existence of the integrals are imposed as follows. The conditions for the existence of \( I_1(z) \) are \( \text{Re}(q + \lambda_{0=t_1=\cdots=t_q}) > 0 \) for \( q = 1, \ldots, n \) and \( \text{Re}(1 + \lambda_{p+1}) > 0 \)
for $p = 1, \ldots, n$; in other words, $\Re(1 + \alpha_{q+1} + \beta_1) > 0$ for $q = 1, \ldots, n$ and $\Re(\beta_{p+1} - \alpha_{p+1}) > 0$ for $p = 1, \ldots, n$. The conditions for the existence of each $I_j(z)$ for $2 \leq j \leq n$ are $\Re(q - p + 1 + \lambda_0 = t_p = \ldots = t_q) > 0$ for $1 \leq p \leq j \leq n$ and $\Re(1 + \lambda_p, p+1) > 0$ for $p = 0, 1, \ldots, n$; in other words, $\Re(1 + \alpha_{q+1} + \beta_p) > 0$ for $1 \leq p \leq j \leq q \leq n$ and $\Re(\beta_{p+1} - \alpha_{p+1}) > 0$ for $p = 0, 1, \ldots, n$. The conditions for the existence of $I_{n+1}(z)$ are $\Re(1 + \lambda_p, p+1) > 0$ for $p = 0, 1, \ldots, n$; in other words, $\Re(\beta_{p+1} - \alpha_{p+1}) > 0$ for $p = 0, 1, \ldots, n$. Therefore, the conditions $\Re(q - p + 1 + \lambda_0 = t_p = \ldots = t_q) > 0$ for $1 \leq p \leq q \leq n$ and $\Re(1 + \lambda_p, p+1) > 0$ for $p = 0, 1, \ldots, n$, guarantee the existence of $I_1(z), \ldots, I_{n+1}(z)$. Moreover, analytic continuation relaxes the conditions into

$$q - p + 1 + \lambda_0 = t_p = \ldots = t_q \notin \mathbb{Z} \leq 0, \quad 1 \leq p \leq q \leq n \quad \text{and} \quad 1 + \lambda_p, p+1 \notin \mathbb{Z} \leq 0, \quad 0 \leq p \leq n,$$

in other words,

$$1 + \alpha_j - \beta_i \notin \mathbb{Z} \leq 0, \quad 1 \leq i < j \leq n+1, \quad \text{and} \quad \beta_i - \alpha_i \notin \mathbb{Z} \leq 0, \quad 1 \leq i \leq n+1, \quad (1.5)$$

where $\beta_{n+1} = 1$.

As for the relation with the differential equation $n+1 E_n$, we have Proposition 1.1.

**Proposition 1.1.** Suppose that

$$1 + \alpha_j - \beta_i \notin \mathbb{Z} \leq 0, \quad 1 \leq i < j \leq n+1, \quad \text{and} \quad \beta_i - \alpha_i \notin \mathbb{Z} \leq 0, \quad 1 \leq i \leq n+1,$$

where $\beta_{n+1} = 1$. Then each of $I_j(z), \quad j = 1, \ldots, n+1$, satisfies the differential equation $n+1 E_n$.

**Proof.** Theorem 6.1 in [MN] shows that

$$\int_C u(z; t) dt_1 \cdots dt_n$$

satisfies $n+1 E_n$ for any cycle $C \in H_m(T_z, \mathcal{L}_z)$. Hence, it implies that each of

$$\int_{\operatorname{reg} D_j} u_{D_j}(z; t) dt_1 \cdots dt_n, \quad j = 1, \ldots, n+1,$$

satisfies $n+1 E_n$ when $\alpha_i - \beta_j \notin \mathbb{Z}, \quad 1 \leq i \neq j \leq n+1$, with $\beta_{n+1} = 1$. Moreover, even if some $\alpha_i - \beta_j$ might be an integer, analytic continuation implies that $I_j(z)$ satisfies $n+1 E_n$ when the conditions (1.5) hold. This completes the proof. $\square$

Next, we give a Wronskian formula for $I_j(z), \quad 1 \leq j \leq n+1$, to clarify the condition for the set $\{I_j(z) \mid 1 \leq j \leq n+1\}$ being linearly independent. The formula is stated as follows.

**Proposition 1.2.** Fix $z$ to be $0 < z < 1$. Suppose that

$$1 + \alpha_j - \beta_i \notin \mathbb{Z} \leq 0, \quad 1 \leq i < j \leq n+1 \quad \text{and} \quad \beta_i - \alpha_i \notin \mathbb{Z} \leq 0, \quad i = 1, \ldots, n+1,$$

where $\beta_{n+1} = 1$. Then we have

$$\det(\partial^{i-1} I_j(z))_{1 \leq i, j \leq n+1}$$

$$= (-1)^{\frac{1}{2}n(n+1)} \prod_{i=1}^{n+1} \Gamma(\beta_i - \alpha_i)^n \prod_{1 \leq i \neq j \leq n+1} \frac{\Gamma(1 + \alpha_j - \beta_i)}{\Gamma(\beta_j - \alpha_i)}$$

$$\times z^{-\sum_{i=1}^{n+1} \beta_i - \frac{1}{2}n(n+1)} (1 - z)^{\sum_{i=1}^{n} \beta_i - \sum_{j=1}^{n+1} \alpha_i - n}, \quad (1.6)$$
where \( \partial = d/dz \) and each argument of \( z \) and \( 1 - z \) of the right-hand side is assigned to be zero.

**Proof.** We rewrite equation (1.1) as

\[
\partial^{n+1} F + p_1(z) \partial^n F + \cdots = 0,
\]

where

\[
p_1(z) = \frac{\sum_{s=1}^{n+1} \alpha_s - \sum_{s=1}^{n} \beta_s + n}{z - 1} + \frac{\sum_{s=1}^{n} \beta_s + \frac{1}{2} n(n-1)}{z}.
\]

It is seen that the determinant \( W(z) = \det(\partial_i - 1 I_j (z))_{1 \leq i, j \leq n+1} \) satisfies the differential equation \( \partial W(z) = - p_1(z) W(z) \). Hence, \( W(z) \) is a constant multiple of \( z^{-\sum_{s=1}^{n} \beta_s - \frac{1}{2} n(n-1)} (1 - z)^{-\sum_{s=1}^{n+1} -n} \). Therefore, what remains is to determine the multiplicative constant. For this purpose, we proceed as follows.

The change of variables from \( t_i \) to \( u_i \) by

\[
t_i = zu_j + (1 - zu_j) u_i, \quad i = 1, \ldots, j - 1,
\]

\[
dt_i = (1 - zu_j) u_j \quad \text{d}u_i, \quad j = 1, \ldots, n + 1,
\]

implies that

\[
I_j(z) = \int_{D_j} u_{D_j} (t) \, dt
\]

\[
= \int_{0 < t_j < \cdots < t_n < z, t_j < t_{j-1} < \cdots < t_1 < 1} \prod_{i=1}^{n} t_i^{\lambda_i} \prod_{i=1}^{j} (t_i - t_j)^{\lambda_{i-1,j}} \prod_{i=j+1}^{n+1} (t_i - t_{j-1})^{\lambda_{i-1,j}} \, dt
\]

\[
= z^{n-j+1} \sum_{1 \leq i \leq n} (\lambda_i + \lambda_{i,i+1})
\]

\[
\times \int_{0 < u_j < \cdots < u_n < 1, 0 < u_{j-1} < \cdots < u_1 < 1} \prod_{i=1}^{j-1} (zu_j + (1 - zu_j) u_i)^{\lambda_i} \prod_{i=j}^{n} u_i^{\lambda_i}
\]

\[
\times \left\{ \prod_{i=1}^{j-1} ((1 - zu_j)(u_{i-1} - u_i))^{\lambda_{i-1,j}} \right\}
\]

\[
\times ((1 - zu_j)u_{j-1})^{\lambda_{j-1,j}} \left\{ \prod_{i=j+1}^{n+1} (u_i - u_{i-1})^{\lambda_{i-1,j}} \right\} du
\]

for \( j = 1, \ldots, n \), since

\[
t_{i-1} - t_i = (1 - zu_j)(u_{i-1} - u_i), \quad i = 1, \ldots, j - 1,
\]

\[
t_{j-1} - t_j = (1 - zu_j)u_{j-1},
\]

\[
t_i - t_{i-1} = z(u_i - u_{i-1}), \quad i = j + 1, \ldots, n + 1,
\]

where \( t_0 = 1, t_{n+1} = z, u_0 = 1, u_{n+1} = 1, dt = dt_1 \cdots dt_n \) and \( du = du_1 \cdots du_n \).
Similarly, the change of variables from $t_i$ to $u_i$ by

$$t_i = z + (1 - z) u_i, \quad i = 1, \ldots, n,$$

$$dt_i = (1 - z) \, du_i, \quad i = 1, \ldots, n$$

implies that

$$I_{n+1}(z) = \int_{D_{n+1} = \{z < t_n < \cdots < t_1 < 1\}} \prod_{i=1}^{n} t_i^{\lambda_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\lambda_{i-1,i}} \, dt$$

$$= (1 - z)^{n+\sum_{1 \leq i \leq n+1} \lambda_{i-1,i}}$$

$$\times \int_{0 < u_n < \cdots < u_1 < 1} \prod_{i=1}^{n} (z + (1 - z) u_i)^{\lambda_i} \prod_{i=1}^{n+1} (u_i - u_{i-1})^{\lambda_{i-1,i}} \, du,$$

since

$$t_i - t_{i-1} = (1 - z) (u_i - u_{i-1}), \quad i = 1, \ldots, n + 1,$$

where $t_0 = 1$, $t_{n+1} = z$, $u_0 = 1$, $u_{n+1} = 0$, $dt = dt_1 \cdots dt_n$ and $du = du_1 \cdots du_n$.

In what follows, we set

$$I_j(z) = z^{n+1-j+\rho_j} F_j(z), \quad j = 1, \ldots, n,$$

$$I_{n+1}(z) = (1 - z)^{n+\rho_{n+1}} F_{n+1}(z),$$

where

$$\rho_j = \sum_{j \leq i \leq n} (\lambda_i + \lambda_{i,i+1}) = \lambda_{0 \leq j = \cdots = t_n} + \lambda_{n,n+1}$$

$$= j - n - \beta_j, \quad j = 1, \ldots, n,$$

$$\rho_{n+1} = \sum_{1 \leq i \leq n+1} \lambda_{i-1,i}$$

$$= -n + \sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n+1} \alpha_i.$$

Then, we have

$$W(z)$$

$$= \det(\partial_{\xi_i-1}^j I_j(z))_{1 \leq i, j \leq n+1}$$

$$= \det\left(\begin{array}{cccccc}
z^{n+\rho_1} F_1 & z^{n-1+\rho_2} F_2 & \cdots & z^{1+\rho_n} F_n & (1 - z)^{n+\rho_{n+1}} F_{n+1} \\
\frac{\partial}{\partial z} (z^{n+\rho_1} F_1) & \frac{\partial}{\partial z} (z^{n-1+\rho_2} F_2) & \cdots & \frac{\partial}{\partial z} (z^{1+\rho_n} F_n) & \frac{\partial}{\partial z} ((1 - z)^{n+\rho_{n+1}} F_{n+1}) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial^m}{\partial z^m} (z^{n+\rho_1} F_1) & \frac{\partial^m}{\partial z^m} (z^{n-1+\rho_2} F_2) & \cdots & \frac{\partial^m}{\partial z^m} (z^{1+\rho_n} F_n) & \frac{\partial^m}{\partial z^m} ((1 - z)^{n+\rho_{n+1}} F_{n+1})
\end{array}\right)$$

$$= z^{1 + 2 + \cdots + n + \rho_1 + \cdots + \rho_n}$$

$$\times \det\left(\begin{array}{cccccc}
\frac{\partial}{\partial z} (z^{n+\rho_1} F_1) & \frac{\partial}{\partial z} (z^{n-1+\rho_2} F_2) & \cdots & \frac{\partial}{\partial z} (z^{1+\rho_n} F_n) & \frac{\partial}{\partial z} ((1 - z)^{n+\rho_{n+1}} F_{n+1}) \\
\frac{\partial^m}{\partial z^m} (z^{n+\rho_1} F_1) & \frac{\partial^m}{\partial z^m} (z^{n-1+\rho_2} F_2) & \cdots & \frac{\partial^m}{\partial z^m} (z^{1+\rho_n} F_n) & \frac{\partial^m}{\partial z^m} ((1 - z)^{n+\rho_{n+1}} F_{n+1})
\end{array}\right)$$
which implies that

\[
\lim_{z \to 0} z^{-\rho_1 - \cdots - \rho_n} \det(\partial_{z}^{-1} I_{j}(z))_{1 \leq i, j \leq n+1} = (-)^{n+2} \prod_{i=1}^{n+1} \left\{ \lim_{z \to 0} F_i \right\} \times \det \begin{pmatrix}
\frac{\partial_{z} (z^{n+1})}{z^{n-1+\rho_1}} & \frac{\partial_{z} (z^{n-1+\rho_2})}{z^{n-2+\rho_2}} & \cdots & \frac{\partial_{z} (z^{1+\rho_n})}{z^{\rho_n}} \\
\frac{\partial_{z} (z^{n+1})}{z^{n-1+\rho_1}} & \frac{\partial_{z} (z^{n-1+\rho_2})}{z^{n-2+\rho_2}} & \cdots & \frac{\partial_{z} (z^{1+\rho_n})}{z^{\rho_n}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial_{z} (z^{n+1})}{z^{n-1+\rho_1}} & \frac{\partial_{z} (z^{n-1+\rho_2})}{z^{n-2+\rho_2}} & \cdots & \frac{\partial_{z} (z^{1+\rho_n})}{z^{\rho_n}}
\end{pmatrix}
\]

\[
\times \det \left( \frac{\partial_{z} (z^{n+1-j+\rho_j})}{z^{n+1-i-j+\rho_j}} \right)_{1 \leq i, j \leq n} = (-)^{n+2} \prod_{i=1}^{n+1} \left\{ \lim_{z \to 0} F_i \right\} \times \det \left( \prod_{s=0}^{i-1} (n+1-j-s+\rho_j) \right)_{1 \leq i, j \leq n}.
\]

(1.7)

On the other hand, we have the following:

\[
\det \left( \prod_{s=0}^{i-1} (n+1-j-s+\rho_j) \right)_{1 \leq i, j \leq n} = \prod_{j=1}^{n} (n+1-j+\rho_j) \det(\prod_{s=1}^{i-1} (n+1-j-s+\rho_j))_{1 \leq i, j \leq n} = \prod_{j=1}^{n} (n+1-j+\rho_j) \det((n-j+\rho_j)^{i-1})_{1 \leq i, j \leq n} = \prod_{j=1}^{n} (n+1-j+\rho_j) \prod_{1 \leq i < j \leq n} ((n-j+\rho_j) - (n-i+\rho_i))
\]
\[
\begin{align*}
(-)^{(2)} \prod_{j=1}^{n} (n + 1 - j + \rho_j) \prod_{1 \leq i < j \leq n} (\rho_i - \rho_j + j - i) \\
(-)^{(2)} \prod_{j=1}^{n} (n + 1 - j + \lambda_{0=t_j=\ldots=t_n} + \lambda_{n+1}) \\
\times \prod_{1 \leq i < j \leq n} (j - i + \lambda_{0=t_j=\ldots=t_{j-1}+\lambda_{j-1}, j}) \\
(-)^{(2)} \prod_{1 \leq i < j \leq n+1} (j - i + \lambda_{0=t_j=\ldots=t_{j-1}+\lambda_{j-1}, j}) \\
(-)^{(2)} \prod_{1 \leq i < j \leq n+1} (\beta_j - \beta_i).
\end{align*}
\]

For
\[
F_j(z) = \int_{0 < u_j < \ldots < u_n < 1} \int_{0 < u_{j-1} < \ldots < u_1 < 1} \prod_{i=1}^{j-1} (zu_j + (1 - zu_j)u_i)^{\lambda_i} \prod_{i=j}^{n} u_i^{\lambda_i} \\
\times \left\{ \prod_{i=1}^{j-1} ((1 - zu_{i-1} - u_i)^{\lambda_{i-1}, i}) \right\} \left\{ ((1 - zu_j)u_{j-1} - u_j)^{\lambda_{j-1}, j} \right\} \\
\times \left\{ \prod_{i=j+1}^{n+1} (u_i - u_{i-1})^{\lambda_{j-1}, i} \right\} du, \quad j = 1, \ldots, n,
\]

where \(u_0 = 1\) and \(u_{n+1} = 1\), there exists a constant \(M\) satisfying
\[
|F_j(z)| \leq M \int_{0 < u_j < \ldots < u_n < 1} \int_{0 < u_{j-1} < \ldots < u_1 < 1} \prod_{i=1}^{j-1} u_i^{Re(\lambda_i)} \prod_{i=j}^{n} u_i^{Re(\lambda_i)} \\
\times \left\{ \prod_{i=1}^{j-1} (u_i - u_{i-1})^{Re(\lambda_{i-1}, i)} \right\} u_{j-1}^{Re(\lambda_{j-1}, j)} \left\{ \prod_{i=j+1}^{n+1} (u_i - u_{i-1})^{Re(\lambda_{i-1}, i)} \right\} du \\
= M \int_{0 < u_{j-1} < \ldots < u_1 < 1} \prod_{i=1}^{j-1} u_i^{Re(\lambda_{j-1}, j)} \prod_{i=j}^{n} u_i^{Re(\lambda_i)} \left\{ \prod_{i=1}^{j-1} (u_i - u_{i-1})^{Re(\lambda_{i-1}, i)} \right\} du_{j-1} \cdots du_{j+1} \\
\times \int_{0 < u_j < \ldots < u_n < 1} \prod_{i=j}^{n} u_i^{\lambda_i} \left\{ \prod_{i=j+1}^{n+1} (u_i - u_{i-1})^{Re(\lambda_{i-1}, i)} \right\} du_{j+1} \cdots du_n \\
= M \prod_{i=1}^{j-1} B \left( j - i + \sum_{k=1}^{j-1} Re(\lambda_k + \lambda_{k+1}, 1 + Re(\lambda_{j-1}, i)) \right) \\
\times \prod_{i=j}^{n} B \left( i - j + 1 + \sum_{k=j}^{i} Re(\lambda_k) + \sum_{k=j}^{i-1} Re(\lambda_{k+1}, 1 + Re(\lambda_{i, i+1})) \right),
\]
Here the last equality follows from Lemma 1.4 below and

\[ \tilde{\lambda}_i = \begin{cases} 0 & \text{if } \Re(\lambda_i) \geq 0, \\ \lambda_i & \text{if } \Re(\lambda_i) < 0. \end{cases} \]

Hence, when

\[ \Re(1 + \lambda_{i-1,i}) > 0, \quad i = 1, \ldots, \hat{j}, \ldots, n + 1, \]

\[ \Re\left(j - k + \sum_{s=k}^{j-1} \tilde{\mu}_s\right) > 0, \quad k = 1, \ldots, j - 1, \tag{1.9} \]

\[ \Re\left(k + 1 - j + \lambda_k + \sum_{s=j}^{k-1} \mu_s\right) > 0, \quad k = j, \ldots, n, \]

where

\[ \tilde{\mu}_i = \tilde{\lambda}_i + \lambda_{i,i+1}, \quad i = 1, \ldots, j - 1, \]

\[ \mu_i = \lambda_i + \lambda_{i,i+1}, \quad i = j, \ldots, n - 1, \]

the function \(F_j(z)\) is uniformly convergent on \(0 \leq z \leq \delta\) for enough small \(\delta > 0\). Therefore, under the above condition (1.9), we obtain

\[
\lim_{z \to 0} F_j(z) = \int_{0 < u_{j-1} < \cdots < u_1 < 1} u_j^{\lambda_j-1} \prod_{i=1}^{j-1} (u_i^\lambda (u_{i-1} - u_i)^{\lambda_{i-1,i}}) du_1 \cdots du_{j-1} \\
\times \int_{0 < u_j < \cdots < u_n < 1} \prod_{i=j}^{n} (u_i^\lambda (u_{i+1} - u_i)^{\lambda_{i,i+1}}) du_j \cdots du_n \\
= \prod_{i=1}^{j-1} B\left(j - i + \sum_{k=i}^{j-1} (\lambda_k + \lambda_{k,k+1}), 1 + \lambda_{i-1,i}\right) \\
\times \prod_{i=j}^{n} B\left(i - j + 1 + \sum_{k=j}^{i} \lambda_k + \sum_{k=j}^{i-1} \lambda_{k,k+1}, 1 + \lambda_{i,i+1}\right) \\
= \prod_{i=1}^{j-1} B(j - i + \lambda_{0=t_i=\cdots=t_{j-1} + \lambda_{j-1,j}, 1 + \lambda_{i-1,i}}) \\
\times \prod_{i=j}^{n} B(i - j + 1 + \lambda_{0=t_j=\cdots=t_i, 1 + \lambda_{i,i+1}}) \\
= \prod_{i=1}^{j-1} B(\beta_j - \beta_i, \beta_i - \alpha_i) \prod_{i=j}^{n} B(1 + \alpha_{i+1} - \beta_j, \beta_{i+1} - \alpha_{i+1}), \quad j = 1, \ldots, n. \tag{1.10} \]

Here the second equality follows from Lemma 1.4 and \(u_0 = u_{n+1} = 1\).

Similarly, for

\[
F_{n+1}(z) = \int_{0 < u_n < \cdots < u_1 < 1} \prod_{i=1}^{n} (z + (1-z)u_i)^\lambda_i \prod_{i=1}^{n+1} (u_{i-1} - u_i)^{\lambda_{i-1,i}} du, \]
where \( u_0 = 1, u_{n+1} = 0 \), there exists a constant \( M \) satisfying

\[
|F_{n+1}(z)| \leq M \int_{0 < u_0 < \cdots < u_1 < 1} u_1^{\Re(\tilde{\lambda}_i)} \prod_{i=1}^{n+1} (u_i - u_{i-1})^{\Re(\lambda_{i-1,i})} \, du
\]

\[
= \prod_{i=1}^{n} B \left( n + 1 - i + \sum_{k=i}^{n} \Re(\tilde{\lambda}_k + \lambda_{k,k+1}), 1 + \Re(\lambda_{i-1,i}) \right).
\]

where

\[
\tilde{\lambda}_i = \begin{cases} 
0 & \text{if } \Re(\lambda_i) \geq 0, \\
\lambda_i & \text{if } \Re(\lambda_i) < 0.
\end{cases}
\]

Hence, when

\[
\Re(1 + \lambda_{i-1,i}) > 0, \quad i = 1, \ldots, n,
\]

\[
\Re \left( n + 1 - i + \sum_{s=i}^{n} \tilde{\mu}_s \right) > 0, \quad i = 1, \ldots, n,
\]

(1.11)

where \( \tilde{\mu}_i = \tilde{\lambda}_i + \lambda_{i,i} \), the function \( F_{n+1}(z) \) is uniformly convergent on \( 0 \leq z \leq \delta \) for enough small \( \delta > 0 \). Therefore, under the above condition (1.11), we obtain

\[
\lim_{z \to 0} F_{n+1}(z) = \int_{0 < u_0 < \cdots < u_1 < 1} u_1^{\lambda_{n,n+1}} \prod_{i=1}^{n} \{ u_i^{\lambda_i} (u_i - u_i) \} \, du
\]

\[
= \prod_{i=1}^{n} B \left( n + 1 - i + \sum_{k=i}^{n} (\lambda_k + \lambda_{k,k+1}), 1 + \lambda_{i-1,i} \right)
\]

\[
= \prod_{i=1}^{n} B(n + 1 - i + \lambda_{0=1} = \cdots = \lambda_n, n+1, 1 + \lambda_{i-1,i})
\]

\[
= \prod_{i=1}^{n} B(1 - \beta_i, \beta_i - \alpha_i).
\]

(1.12)

Combination of (1.7), (1.8), (1.10) and (1.12) implies that

\[
\det(3^{j-1} L_j(z))_{1 \leq i, j \leq n+1}
\]

\[
= (-1)^{n+2} \prod_{i=1}^{n+1} \left\{ \lim_{z \to 0} F_i \right\} \times \det \left( \prod_{s=0}^{i-1} (n + 1 - j - s + \rho_j) \right)_{1 \leq i, j \leq n+1}
\]

\[
= (-1)^{n+1} \prod_{1 \leq i < j \leq n+1} (\beta_j - \beta_i)
\]

\[
\times \prod_{j=1}^{n+1} \left\{ \prod_{i=1}^{j-1} B(\beta_i - \alpha_i, \beta_j - \beta_i) \prod_{i=j}^{n} B(\beta_{i+1} - \alpha_{i+1}, 1 + \alpha_{i+1} - \beta_j) \right\}
\]

\[
= (-1)^{n+1} \prod_{1 \leq i < j \leq n+1} \Gamma(1 + \alpha_j - \beta_i) \prod_{1 \leq i \leq n+1} \Gamma(\beta_j - \alpha_i)^n \prod_{1 \leq i < j \leq n+1} \Gamma(\beta_j - \alpha_i)
\]

(1.13)
As an aside, (1.9) and (1.11) are satisfied by the conditions

\[
\Re(1 + \lambda_{i-1,i}) > 0, \quad i = 1, \ldots, n + 1,
\]
\[
\Re(1 + \mu_i) > 0, \quad i = 1, \ldots, n,
\]
\[
\Re(1 + \lambda_i) > 0, \quad i = 1, \ldots, n;
\]
in other words,
\[
\Re(\beta_i - \alpha_i) > 0, \quad i = 1, \ldots, n + 1,
\]
\[
\Re(1 + \alpha_{i+1} - \beta_i) > 0, \quad i = 1, \ldots, n.
\]
Moreover, equality (1.13) with analytic continuation shows that the conditions \(\Re(\beta_{i+1} - \beta_i) > 0, \quad i = 1, \ldots, n\), can be dropped, and the conditions
\[
\Re(\beta_i - \alpha_i) > 0, \quad i = 1, \ldots, n + 1,
\]
\[
\Re(1 + \alpha_{i+1} - \beta_i) > 0, \quad i = 1, \ldots, n.
\]
can be eased to
\[
\beta_i - \alpha_i \notin \mathbb{Z} \leq 0, \quad i = 1, \ldots, n + 1,
\]
\[
1 + \alpha_{i+1} - \beta_i \notin \mathbb{Z} \leq 0, \quad i = 1, \ldots, n.
\]
This completes the proof. \(\square\)

**Remark 1.3.** Equality (1.6) in Proposition 1.2 is equivalent to
\[
\det(\partial^{i-1} I_j(z))_{1 \leq i, j \leq n+1}
\]
\[
= (-1)^{z_{n+1}} \prod_{i=1}^{n+1} \Gamma(1 + \lambda_{i-1,i}) \prod_{1 \leq i \leq j \leq n} \frac{\Gamma(j - i + 1 + \lambda_{0=t_0=\ldots=t_j})}{\Gamma(j - i + 2 - \lambda_{t_0=\ldots=t_j=\infty})}
\]
\[
\times z^{\sum_{i=1}^{n+1} \lambda_{i-1,i}} (1 - z)^{\sum_{i=1}^{n+1} \lambda_{i-1,i}},
\]
where \(\mu_i = \lambda_i + \lambda_{i+1}\).

**Lemma 1.4.** If \(\Re(j + \sum_{r=1}^{j} \lambda_r + \sum_{r=1}^{j-1} \lambda_{r, r+1}) > 0\) and \(\Re(1 + \lambda_{j,j+1}) > 0\) for \(1 \leq j \leq n\), we have
\[
\int_{0 < u_1 < u_2 < \cdots < u_n < 1} \prod_{i=1}^{n} (u_i^{\lambda_i} (u_{i+1} - u_i)^{\lambda_{i+1}}) \ du_1 \cdots du_n
\]
\[
= \prod_{j=1}^{n} B\left(j + \sum_{r=1}^{j} \lambda_r + \sum_{r=1}^{j-1} \lambda_{r, r+1}, 1 + \lambda_{j,j+1}\right),
\]
(1.14)
where \(u_{n+1} = 1\).

**Proof.** The change of variables such as
\[ u_s = v_n v_{n-1} \cdots v_s, \quad du_s = v_n v_{n-1} \cdots v_{s+1} d v_s, \quad s = 1, \ldots, n, \]
implies that the left-hand side of (1.14) is equal to
\[
\int_{0<v_1<1,...,0<v_n<1} \prod_{j=1}^{n} \{ v_j^{j-1+\sum_{r=1}^{j} \lambda_r + \sum_{r=1}^{j-1} \lambda_{r,r+1}} (1-v_j)^{\lambda_{j,j+1}} \} \, dv_1 \cdots dv_n
\]
\[
= \prod_{j=1}^{n} \int_{0<v_j<1} v_j^{j-1+\sum_{r=1}^{j} \lambda_r + \sum_{r=1}^{j-1} \lambda_{r,r+1}} (1-v_j)^{\lambda_{j,j+1}} \, dv_j
\]
\[
= \prod_{j=1}^{n} B \left( j + \sum_{r=1}^{j} \lambda_r + \sum_{r=1}^{j-1} \lambda_{r,r+1}, 1 + \lambda_{j,j+1} \right).
\]
This completes the proof.

Combination of Propositions 1.1 and 1.2 implies the following theorem.

**Theorem 1.5.** Suppose that
\[
1 + \alpha_j - \beta_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i < j \leq n + 1, \quad \beta_i - \alpha_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i \leq n + 1,
\]
and
\[
\beta_j - \alpha_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i < j \leq n + 1,
\]
where \( \beta_{n+1} = 1 \). Then \( I_1(z), \ldots, I_{n+1}(z) \) are linearly independent, and give a fundamental set of solutions of \( n+1 E_n \).

**Remark 1.6.** Theorem 1.5 can be restated as follows. Suppose that
\[
j - i + 1 + \lambda_{0=t_i=\ldots=t_i} \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i \leq j \leq n, \quad \text{and} \quad 1 + \lambda_{i,j+1} \notin \mathbb{Z}_{\leq 0}, \quad 0 \leq i \leq n,
\]
and
\[
j - i + 2 - \lambda_{t_i=\ldots=t_i} = \infty, \quad 1 \leq i \leq j \leq n.
\]
Then \( I_1(z), \ldots, I_{n+1}(z) \) are linearly independent, and thus give a fundamental set of solutions of \( n+1 E_n \).

**Remark 1.7.** Equation (1.15) is the same as (1.5), and (1.17) is the same as (1.4).

**Remark 1.8.** The fundamental set of solutions used in [M2] is linearly independent under the conditions
\[
1 + \alpha_i - \beta_j \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i \neq j \leq n + 1, \quad \beta_i - \alpha_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i \leq n + 1,
\]
and
\[
\beta_i - \beta_j \notin \mathbb{Z}, \quad 1 \leq i < j \leq n + 1.
\]

2. **Matrix elements of the representation**

Fix a base point \( \bar{z}^0 \in \mathbb{C} \setminus \{0, 1\} \) to be real and \( 0 < \bar{z}^0 < 1 \). Assume the conditions (1.15) and (1.16). Let \( \{I_1(z), I_2(z), \ldots, I_{n+1}(z)\} \) be a fundamental set of solutions of \( n+1 E_n \) studied in the previous section. Then, we consider the monodromy representation with respect to
Figure 1. Paths \( \gamma_0 \) and \( \gamma_1 \).

\((I_1(z), I_2(z), \ldots, I_{n+1}(z)):\)

\[ \rho : \pi_1(\mathbb{C}\setminus\{0,1\}, z^0) \longrightarrow \text{GL}(n+1, \mathbb{C}), \quad \gamma \mapsto M(\gamma) = (m_{ij}(\gamma)), \]

which is defined by

\[ \rho(\gamma)I_j(z) = \sum_{1 \leq i \leq n+1} I_i(z)m_{ij}(\gamma), \]

where \( \rho(\gamma)f(z) \) denotes the analytic continuation of \( f(z) \) along the path \( \gamma \). In particular, we express explicitly the matrix elements of \( \rho(\gamma_0) \) and \( \rho(\gamma_1) \); in other words, the matrix elements of the circuit matrices \( M(\gamma_0) \) and \( M(\gamma_1) \). Here, the symbols \( \gamma_0 \) and \( \gamma_1 \) designate the paths as in Figure 1. It is known that these paths give a set of generators of the fundamental group \( \pi_1(\mathbb{C}\setminus\{0,1\}, z^0) \) with the base point \( z^0 \).

The path \( \gamma_k \) induces the action \( \gamma_k^* \) of \( \pi_1(\mathbb{C}\setminus\{0,1\}, z^0) \) on the homology groups \( H_1^H(T_{z^0}, \mathbb{L}_{z^0}) \) and \( H_n(T_{z^0}, \mathbb{L}_{z^0}) \). To obtain the matrix elements of \( \rho(\gamma_k) \), we first consider the action of \( \gamma_k^* \) on \( H_1^H(T_{z^0}, \mathbb{L}_{z^0}) \) and determine the matrix elements of \( \gamma_k^* \) with respect to \((D_1, \ldots, D_{n+1})\). The relations in terms of \( I_j(z), 1 \leq j \leq n+1 \), will be obtained from them.

For example, the relations

\[ \gamma_0^*(D_1) = e(-2\beta_1)D_1, \]

\[ \gamma_1^*(D_{n+1}) = e\left(2\left(\sum_{s=1}^n \beta_s - \sum_{s=1}^{n+1} \alpha_s\right)\right)D_{n+1} \]

among the elements of \( H_1^H(T_{z^0}, \mathbb{L}_{z^0}) \) imply

\[ \rho(\gamma_0)(I_1(z)) = e(-2\beta_1)I_1(z), \]

\[ \rho(\gamma_1)(I_{n+1}(z)) = e\left(2\left(\sum_{s=1}^n \beta_s - \sum_{s=1}^{n+1} \alpha_s\right)\right)I_{n+1}(z). \]

Before proceeding to deriving the expression of the matrix elements of \( \gamma_0^* \) and \( \gamma_1^* \), we explain the notation which will be used in our calculation.

When we consider the twisted chain \((0, 1) \otimes t^a(1-t)^b \) or \([0 < t < 1] \otimes t^a(1-t)^b \), we tacitly consider the integral \( \int_0^1 t^a(1-t)^b \ dt \). To express the domain \([0 < t < 1] \) graphically, we write it as

\[
\begin{align*}
\begin{cases}
\text{t-space} \\
0 & 1
\end{cases} & \text{or} \quad \begin{cases}
\text{t-space} \\
0 & 1
\end{cases}
\end{align*}
\]

or

\[
\begin{cases}
\text{t-space} \\
0 & 1
\end{cases}.
\]
Here, the orientation of each domain of $\mathbb{R}\backslash\{0, 1\}$ is induced from the natural orientation of $\mathbb{C}\backslash\{0, 1\}$.

Similarly, when we consider the twisted chain $\{0 < t_2 < t_1 < 1\} \otimes t_1^a t_2^b (t_1 - t_2)^d (1 - t_1)^b$, we tacitly consider the integral

$$\int_{0 < t_2 < t_1 < 1} t_1^a t_2^b (t_1 - t_2)^d (1 - t_1)^b \, dt_1 \, dt_2 = \int_0^1 dt_1 \int_0^{t_1} dt_2 t_1^a t_2^b (t_1 - t_2)^d (1 - t_1)^b$$

$$= \int_0^1 t_1^a (1 - t_1)^b \left( \int_0^{t_1} t_2^d (t_1 - t_2)^d \, dt_2 \right) \, dt_1.$$ 

As a reflection of such relations, we denote $\{0 < t_2 < t_1 < 1\} \otimes t_1^a t_2^b (t_1 - t_2)^d (1 - t_1)^b$ by

$$\left\{(0 < t_1 < 1) \times (0 < t_2 < t_1)\right\} \otimes t_1^a t_2^b (t_1 - t_2)^d (1 - t_1)^b$$

or

$$\left\{(0 < t_1 < 1) \otimes t_1^a (1 - t_1)^b\right\} \times \left\{(0 < t_2 < t_1) \otimes t_2^b (t_1 - t_2)^d\right\}.$$ 

To express the domain $\{0 < t_2 < t_1 < 1\}$ graphically, we write it as

$$\left\{ \begin{array}{c}
0 \\
\downarrow t_2
\end{array} \right\} \text{ or } \left\{ \begin{array}{c}
0 \\
\downarrow t_1
\end{array} \right\} \left\{ \begin{array}{c}
t_1 \\
\downarrow t_2
\end{array} \right\} \text{ or } \left\{ \begin{array}{c}
t_1 \text{-space} \\
\downarrow t_1 \\
\downarrow t_2
\end{array} \right\} \left\{ \begin{array}{c}
t_2 \text{-space} \\
\downarrow 0 \\
\downarrow 1
\end{array} \right\}.$$ 

Here, the orientation of each domain of $\mathbb{R}^2\backslash\{t_1 = 0\} \cup \{t_2 = 0\} \cup \{t_1 = 1\} \cup \{t_1 = t_2\}$ is induced from the natural orientation of $\mathbb{C}^2\backslash\{t_1 = 0\} \cup \{t_2 = 0\} \cup \{t_1 = 1\} \cup \{t_1 = t_2\}$. Twisted chains in the $n$-dimensional space are treated in the same way.

Let us start to derive the action of $\gamma_0$ and $\gamma_1$ on

$$D_j = \{0 < t_j < \cdots < t_n < z, \ t_j < t_{j-1} < \cdots < t_1 < 1\}.$$ 

We recall that the symbol to express a bounded domain $D$ of $(T_z)_{\mathbb{R}}$ also means that both the bounded domain $D$ itself and the bounded domain loaded with the coefficient $u_D$ standardly (the twisted chain), i.e. $D \otimes u_D$, where the orientation of $D$ is induced from $(T_z)_{\mathbb{R}}$.

First, we consider the action of $\gamma_0$.

**Lemma 2.1.** We have

$$\gamma_0^n(D_{n+1}) = D_{n+1} + \sum_{i=1}^n (-1)^{i-1} \langle e(\lambda_i) \rangle e \left( \sum_{r=i}^n \mu_r + \sum_{r=i+1}^n \lambda_r \right) D_i \in H_n^\mathbb{R}(T_{z^0}, L_{z^0}).$$

**Proof.** Induction implies the assertion.
(i) The case \( n = 1 \) follows from
\[
\gamma_0^* (D_2) = \gamma_0^* \left( \begin{array}{ccc}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
t_1\text{-space} & \cdots & 0 \\
1 & \cdots & 1
\end{array} \right) \otimes u_{1-\epsilon}(t) = \left\{ \begin{array}{ccc}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
t_1\text{-space} & \cdots & 0 \\
1 & \cdots & 1
\end{array} \right) \otimes u_{1-\epsilon}(t)
\]
\[
= \left\{ \begin{array}{ccc}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
t_1\text{-space} & \cdots & 0 \\
1 & \cdots & 1
\end{array} \right) \otimes u_{1-\epsilon}(t)
\]
\[
= \{ z < t_1 < 1 \} \otimes u_{1-\epsilon}(t) + e(\lambda_{12})(1 - e(2\lambda_1))(0 < t_1 < z) \otimes u_{0+\epsilon}(t)
\]
\[
= D_2 - e(\lambda_1 + \lambda_{12})(e(\lambda_1))D_1.
\]

(ii) The case \( n(\geq 2) \) is derived under the assumption in the case \( n - 1 \):
\[
\gamma_0^* (D_{n+1})
\]
\[
= \gamma_0^* \left( \{ z < t_n < \cdots < t_1 < 1 \} \otimes u_{1-\epsilon < t_n < \cdots < t_1 < 1}(t) \right)
\]
\[
= \gamma_0^* \left( \{ t_n\text{-space} \} \otimes u_{1-\epsilon < t_n < \cdots < t_1 < 1}(t) \right)
\]
\[
\quad = \{ t_n\text{-space} \} \otimes u_{1-\epsilon < t_n < \cdots < t_1 < 1}(t)
\]
\[
\quad + e(\lambda_{n,n+1}) \left[ \{ t_n\text{-space} \} \otimes u_{1-\epsilon < t_n < \cdots < t_1 < 1, 0 < t_n < \epsilon}(t) \right]
\]
\[
\quad + e(2\lambda_n + \lambda_{n,n+1}) \left[ \{ t_n\text{-space} \} \otimes u_{1-\epsilon < t_n < \cdots < t_1 < 1, 0 < t_n < \epsilon}(t) \right]
\]
\[ D_{n+1} = e(\lambda_{n, n+1}) D_n, \]

\[ + e(\lambda_n + \lambda_{n+1}) \]

\[ \otimes u \{ t_n < t_{n-1} < \cdots < t_1 < 1 \} \]

\[ + e(2\lambda_n + \lambda_{n, n+1}) \]

\[ \otimes u \{ t_n < t_{n-1} < \cdots < t_1 < 1 \}. \]

The assumption implies that

\[ \gamma^*_0(D_{n+1} | n \rightarrow n-1) \mid z \rightarrow t_n \]

\[ = \{ \begin{array}{c} t_{n-1} \\ \circ \end{array} \} \]

\[ \otimes u \{ t_n < t_{n-1} < \cdots < t_1 < 1 \} \]

\[ + \sum_{i=1}^{n-1} (-1)^{i-n} (e(\lambda_i)) e \left( \sum_{r=i}^{n-1} \mu_r + \sum_{r=i+1}^{n-1} \lambda_r \right) \{ 0 < t_i < \cdots < t_{n-1} < t_n, t_i < \cdots < t_1 < 1 \} \]

\[ \otimes u \{ t_n < t_{n-1} < \cdots < t_1 < 1 \} \mid z \rightarrow t_n. \]

Hence, we have

\[ \gamma^*_0(D_{n+1}) \]

\[ = D_{n+1} + e(\lambda_{n, n+1}) D_n \]

\[ - e(2\lambda_n + \lambda_{n, n+1}) \{ 0 < t_n < z \} \]

\[ + \sum_{i=1}^{n-1} (-1)^{i-n} (e(\lambda_i)) \]

\[ \times e \left( \sum_{r=i}^{n-1} \mu_r + \sum_{r=i+1}^{n-1} \lambda_r \right) \{ 0 < t_i < \cdots < t_{n-1} < t_n, t_i < \cdots < t_1 < 1 \} \]

\[ \otimes u \{ t_n < t_{n-1} < \cdots < t_1 < 1 \} \mid z \rightarrow t_n. \]

\[ = D_{n+1} - e(\lambda_n + \lambda_{n+1}) (e(\lambda_n)) D_n \]

\[ + \sum_{i=1}^{n-1} (-1)^{i-n-1} (e(\lambda_i)) e \left( \sum_{r=i}^{n-1} \mu_r + \sum_{r=i+1}^{n-1} \lambda_r \right) \{ 0 < t_i < \cdots < t_n < z, t_i < \cdots < t_1 < 1 \} \]

\[ \otimes u \{ t_n < t_{n-1} < \cdots < t_1 < 1 \} \mid z \rightarrow t_n. \]
Generalized hypergeometric equation

\[ D_{n+1} + \sum_{i=1}^{n} (-1)^{i-n-1} (e(\lambda_i)) e\left(\sum_{r=i}^{n} \mu_r + \sum_{r=i+1}^{n} \lambda_r\right) D_i. \]

This completes the proof. \( \Box \)

**Lemma 2.2.** We have

\[ \gamma_0^* (D_j) = e\left(2 \sum_{r=j}^{n} \mu_r\right) \left[ D_j + \sum_{i=1}^{j-1} (-1)^{j-i} (e(\lambda_i)) e\left(\sum_{r=i}^{j-1} \mu_r + \sum_{r=i+1}^{j-1} \lambda_r\right) D_i \right] \]

\( \in \mathcal{H}_n(T_{z^0}, L_{z^0}), \ j = 1, \ldots, n + 1. \)

**Proof.** The case \( j = 1 \) is trivially obtained as stated before. The case \( j = n + 1 \) is shown in Lemma 2.1. Hence, we suppose that \( j \) satisfies \( 1 < j < n + 1 \). Then, we have

\[ \gamma_0^* (D_j) = \gamma_0^* (\{0 < t_j < \cdots < t_n < z, \ t_j < t_{j-1} < \cdots < t_1 < 1\}) \]

\( \otimes u_{[0 < t_j < \cdots < t_n < \epsilon, \ 1-\epsilon < t_{j-1} < \cdots < t_1 < 1]}(t) \)

\[ = e\left(2 \sum_{r=j}^{n} \mu_r\right) \left\{0 < t_j < \cdots < t_n < z\right\} \left[ \begin{array}{c} 0 \\ t_j \\ \vdots \\ t_{j-1} \\ t_1 \\ 1 \end{array} \right] \]

\( \otimes u_{[0 < t_j < \cdots < t_n < \epsilon, \ 1-\epsilon < t_{j-1} < \cdots < t_1 < 1]}(t). \)

On the other hand, from Lemma 2.1,

\[ \gamma_0^* (D_{n+1} | n \to j-1) | z \to t_j \]

\[ = \left\{ \begin{array}{c} 0 \\ t_j \\ \vdots \\ t_{j-1} \\ t_1 \\ 1 \end{array} \right\} \otimes u_{[1-\epsilon < t_{j-1} < \cdots < t_1]}(t_1, \ldots, t_{j-1}) | z \to t_j \]

\[ = \{t_j < t_{j-1} < \cdots < t_1 < 1\} \otimes u_{[1-\epsilon < t_{j-1} < \cdots < t_1]}(t_1, \ldots, t_{j-1}) | z \to t_j \]

\[ + \sum_{i=1}^{j-1} (-1)^{j-i} (e(\lambda_i)) e\left(\sum_{r=i}^{j-1} \mu_r + \sum_{r=i+1}^{j-1} \lambda_r\right) \{0 < t_i < \cdots < t_{j-1} < t_j, \ t_i < \cdots < t_1 < 1\} \]

\( \otimes u_{[0 < t_i < \cdots < t_{j-1} < \epsilon, \ 1-\epsilon < t_{i-1} < \cdots < t_1 < 1]}(t_1, \ldots, t_{j-1}) | z \to t_j. \)

Therefore, we obtain the result. \( \Box \)

Next, we consider the action of \( \gamma_1. \)

**Lemma 2.3.** We have

\[ \gamma_1^* (D_1) = D_1 + (-)^n (e(\lambda_0)) e\left(\sum_{r=1}^{n+1} \lambda_{r-1, r}\right) D_{n+1} \in \mathcal{H}_n(T_{z^0}, L_{z^0}). \]

**Proof.** We divide the cases into the one \( n = 1 \) and the other.
(i) The case \( n = 1 \) follows from
\[
\gamma_1^*(D_1) = \gamma_1^* \left( \left\{ \begin{array}{c}
t_1-\text{space} \\
0 \\
z \\
1 \\
\end{array} \right\} \otimes u_{0+\epsilon}(t) \right) = \left\{ \begin{array}{c}
0 \\
z \\
1 \\
\end{array} \right\} \otimes u_{0+\epsilon}(t)
\]
\[
= \{0 < t_1 < z\} \otimes u_{0+\epsilon}(t) + e(\lambda_{12})(1 - e(2\lambda_{01}))\{z < t_1 < 1\} \otimes u_{z+\epsilon}(t)
\]
\[
= D_1 - e(\lambda_{01} + \lambda_{12})(e(\lambda_{01}))D_2.
\]

(ii) For the case \( n \) is greater than 1:
\[
\gamma_1^*(D_1)
\]
\[
= \gamma_1^* \left( \{0 < t_1 < \cdots < t_n < z\} \otimes u_{D_1(0)} \right)
\]
\[
= \gamma_1^* \left( \left\{ \begin{array}{c}
0 \\
z \\
1 \\
t_1 \\
t_2 \\
\vdots \\
t_n \\
\end{array} \right\} \otimes u_{(e^n,e^{n-1},\ldots,e)}(t) \right)
\]
\[
= \left\{ \begin{array}{c}
0 \\
z \\
1 \\
t_1 \\
t_2 \\
\vdots \\
t_n \\
\end{array} \right\} \otimes u_{(e^n,e^{n-1},\ldots,e)}(t)
\]
\[
= \left[ \begin{array}{c}
t_1-\text{space} \\
0 \\
z \\
1 \\
t_1 \\
t_2 \\
\vdots \\
t_n \\
\end{array} \right] \otimes u_{(e^n,e^{n-1},\ldots,e)}(t)
\]
\[
+ \begin{array}{c}
t_1-\text{space} \\
0 \\
z \\
1 \\
\end{array} \otimes u_{(e^n,e^{n-1},\ldots,e)}(t)
\]
\[
\times (-)^{n-1} e \left( \sum_{s=2}^{n+1} \lambda_{s-1,s} \right) \left\{ \begin{array}{c}
t_n \\
t_2 \\
\vdots \\
0 \\
\end{array} \right\} \otimes u_{[1-\epsilon < t_n < \cdots < t_1 < 1]}(t)
\]
\[
+ e(2\lambda_{01}) \left[ \begin{array}{c}
t_1-\text{space} \\
0 \\
z \\
1 \\
\end{array} \right] \otimes u_{[1-\epsilon < t_n < \cdots < t_1 < 1]}(t)
\]
\[
\times (-)^{n-1} e \left( \sum_{s=2}^{n+1} \lambda_{s-1,s} \right) \left\{ \begin{array}{c}
t_n \\
t_2 \\
\vdots \\
0 \\
\end{array} \right\} \otimes u_{[1-\epsilon < t_n < \cdots < t_1 < 1]}(t)
\]
\[
\begin{align*}
&= \left\{ \begin{array}{ccc}
0 & \cdots & 1 \\
\tilde{t}_1 & \cdots & \tilde{t}_n
\end{array} \right\} \\
&\quad + (-1)^{n-1} e(\lambda_{01}) e\left(\sum_{r=1}^{n+1} \lambda_{r-1,r}\right) (1 - e(2\lambda_{01})) \left\{ \begin{array}{ccc}
0 & \cdots & 1 \\
\tilde{t}_n & \cdots & \tilde{t}_1
\end{array} \right\}
\end{align*}
\]

\[
= D_1 + (-1)^n e(\lambda_{01}) e(\lambda_{12} + \cdots + \lambda_{n,n+1}) D_{n+1}.
\]

This completes the proof. \qed

**Lemma 2.4.** We have

\[
\gamma_1^u(D_n) = D_n - \left( e\left(\sum_{r=1}^{n} \lambda_{r-1,r}\right) e\left(\sum_{r=1}^{n+1} \lambda_{r-1,r}\right) \right) D_{n+1} \in H^n_{\lf}(T_{z_0}, L_{z_0}).
\]

**Proof.**

\[
\gamma_1^u(D_n)
= \gamma_1^u\left( \left\{ \begin{array}{ccc}
0 < \tilde{t}_n < z, \\
\tilde{t}_n < \cdots < \tilde{t}_1 < 1
\end{array} \right\} \otimes u(0 < \tilde{t}_n < 1 - \epsilon < \tilde{t}_{n-1} < \cdots < \tilde{t}_1 < 1) (t) \right)
\]

\[
= \left\{ \begin{array}{ccc}
\tilde{t}_n \text{-space} \\
0 & \cdots & 1 \\
\tilde{t}_n
\end{array} \right\} \left\{ \begin{array}{ccc}
0 & \cdots & 1 \\
\tilde{t}_n & \cdots & \tilde{t}_1
\end{array} \right\}
\]

\[
\quad \otimes u(0 < \tilde{t}_n < 1 - \epsilon < \tilde{t}_{n-1} < \cdots < \tilde{t}_1 < 1) (t)
\]

\[
= \left\{ \begin{array}{ccc}
\tilde{t}_n \text{-space} \\
0 & \cdots & 1 \\
0 & \cdots & 1
\end{array} \right\} \left\{ \begin{array}{ccc}
0 & \cdots & 1 \\
\tilde{t}_n & \cdots & \tilde{t}_1
\end{array} \right\}
\]

\[
\quad \otimes u(0 < \tilde{t}_n < 1 - \epsilon < \tilde{t}_{n-1} < \cdots < \tilde{t}_1 < 1) (t)
\]

\[
+ e(\lambda_{n,n+1}) \left\{ \begin{array}{ccc}
\tilde{t}_n \text{-space} \\
0 & \cdots & 1 \\
0 & \cdots & 1
\end{array} \right\} \left\{ \begin{array}{ccc}
0 & \cdots & 1 \\
\tilde{t}_n - 1 & \cdots & \tilde{t}_1
\end{array} \right\}
\]

\[
\quad \otimes u(1 - \epsilon < \tilde{t}_n < \cdots < \tilde{t}_1 < 1) (t)
\]

\[
= D_n - e\left(\sum_{r=1}^{n+1} \lambda_{r-1,r}\right) e\left(\sum_{r=1}^{n} \lambda_{r-1,r}\right) D_{n+1} \in H^n_{\lf}(T_{z_0}, L_{z_0}). \quad \qed
\]

**Lemma 2.5.** We have

\[
\gamma_1^u(D_{n+1}) = e\left(\sum_{r=1}^{n+1} \lambda_{r-1,r}\right) D_{n+1} \in H^n_{\lf}(T_{z_0}, L_{z_0}).
\]
LEMMA 2.6. We have
\[ \gamma_k^*(D_j) = D_j + (-)^{n-j+1} e^{\left( \sum_{r=1}^{n+1} \lambda_{r-1,r} \right) \left( e^{\left( \sum_{r=1}^{j} \lambda_{r-1,r} \right)} \right)} D_{n+1} \in H^H_n(T_{c0}, L_{c0}) \]
for \( j = 1, \ldots, n \).

Proof. We prove here only the cases such that \( 1 < j < n \), since the cases \( j = 1 \) and \( j = n \) have been obtained.

\[ \gamma_k^*(D_j) \]
\[ \otimes u_{\{0 < t_j < \cdots < t_n < 1 - \epsilon < t_{j-1} < \cdots < t_1 < 1\}}(t) \]
\[ = \left\{ \begin{array}{c}
0 \\
t_j \\
t_{j+1} \\
\vdots \\
t_n \\
\end{array} \right\} \otimes \left\{ \begin{array}{c}
0 \\
t_j \\
t_{j+1} \\
\vdots \\
t_n \\
\end{array} \right\} \]
\[ \times \left\{ \begin{array}{c}
0 \\
t_j \\
t_{j-1} \\
\vdots \\
t_1 \\
\end{array} \right\} \otimes u_{\{0 < t_j < \cdots < t_n < 1 - \epsilon < t_{j-1} < \cdots < t_1 < 1\}}(t) \]
\[ = \left\{ \begin{array}{c}
0 \\
t_{j+1} \cdots t_n \\
\vdots \\
t_j \\
0 \\
\end{array} \right\} \otimes \left\{ \begin{array}{c}
0 \\
t_j \\
t_{j-1} \\
\vdots \\
t_1 \\
\end{array} \right\} \]
\[ \otimes u_{\{0 < t_j < \cdots < t_n < 1 - \epsilon < t_{j-1} < \cdots < t_1 < 1\}}(t) \]
\[ \times \left\{ \begin{array}{c}
0 \\
t_{j+1} \cdots t_n \\
\vdots \\
t_j \\
0 \\
\end{array} \right\} \otimes \left\{ \begin{array}{c}
0 \\
t_j \\
t_{j-1} \\
\vdots \\
t_1 \\
\end{array} \right\} \]
\[ = D_j + (-)^{n-j+1} e^{\left( \sum_{r=1}^{n+1} \lambda_{r-1,r} \right) \left( e^{\left( \sum_{r=1}^{j} \lambda_{r-1,r} \right)} \right)} D_{n+1}. \]

Therefore, we arrive at the following:
\[ \gamma_k^*(D_j) = \sum_{i=1}^{n+1} D_i m_{ij}^{(k)}, \quad k = 0, 1 \text{ and } j = 1, \ldots, n + 1, \]
where

\[
m_{ij}^{(0)} = \begin{cases} 
  e\left(2 \sum_{r=1}^{n} \mu_r \right), & 1 \leq i = j \leq n, \\
  1, & i = j = n + 1, \\
  (-1)^{i-j} e\left(\sum_{r=1}^{n} \mu_r + \sum_{r=i+1}^{j-1} \lambda_r + \sum_{r=j}^{n} \mu_r \right) \langle e(\lambda_i) \rangle, & 1 \leq i < j \leq n + 1, \\
  0, & \text{otherwise}
\end{cases}
\]

with \(\mu_r = \lambda_r + \lambda_r, r+1,\) and

\[
m_{ij}^{(1)} = \begin{cases} 
  1, & 1 \leq j = i \leq n, \\
  e\left(2 \sum_{r=0}^{n} \lambda_{r, r+1} \right), & j = i = n + 1, \\
  (-1)^{n+1-j} e\left(\sum_{r=0}^{n} \lambda_{r, r+1} \right) \langle e\left(\sum_{r=0}^{j-1} \lambda_{r, r+1} \right) \rangle, & i = n + 1, 1 \leq j \leq n, \\
  0, & \text{otherwise}
\end{cases}
\]

The expressions of the circuit matrices are obtained from these combined with Theorem 1.5.

**Proposition 2.7.** Suppose that

\[
\begin{align*}
  j - i + 1 + \lambda_{0=\ldots=i} \notin \mathbb{Z}_{\leq 0}, & \quad 1 \leq i \leq j \leq n, \quad 1 + \lambda_{i,i+1} \notin \mathbb{Z}_{\leq 0}, \quad 0 \leq i \leq n, \quad (2.1) \\
  j - i + 2 - \lambda_{i=\ldots=j=\infty} \notin \mathbb{Z}_{\leq 0}, & \quad 1 \leq i \leq j \leq n. \quad (2.2)
\end{align*}
\]

Then we have

\[
\rho(\gamma_k) I_j(z) = \sum_{i=1}^{n+1} I_i(z) m_{ij}^{(k)}, \quad k = 0, 1 \text{ and } j = 1, \ldots, n + 1,
\]

where

\[
m_{ij}^{(0)} = \begin{cases} 
  e\left(2 \sum_{r=1}^{n} \mu_r \right), & 1 \leq i = j \leq n, \\
  1, & i = j = n + 1, \\
  (-1)^{i-j} e\left(\sum_{r=1}^{n} \mu_r + \sum_{r=i+1}^{j-1} \lambda_r + \sum_{r=j}^{n} \mu_r \right) \langle e(\lambda_i) \rangle, & 1 \leq i < j \leq n + 1, \\
  0, & \text{otherwise}
\end{cases}
\]

with \(\mu_r = \lambda_r + \lambda_r, r+1,\) and

\[
m_{ij}^{(1)} = \begin{cases} 
  1, & 1 \leq j = i \leq n, \\
  e\left(2 \sum_{r=0}^{n} \lambda_{r, r+1} \right), & j = i = n + 1, \\
  (-1)^{n+1-j} e\left(\sum_{r=0}^{n} \lambda_{r, r+1} \right) \langle e\left(\sum_{r=0}^{j-1} \lambda_{r, r+1} \right) \rangle, & i = n + 1, 1 \leq j \leq n, \\
  0, & \text{otherwise}
\end{cases}
\]
Proposition 2.7 with substitutions
\[ \lambda_i = \alpha_{i+1} - \beta_i, \quad \lambda_{i-1.i} = \beta_i - \alpha_i - 1, \quad t_0 = 1, \quad t_{n+1} = z \quad \text{and} \quad \beta_{n+1} = 1 \]
implies the following theorem.

**THEOREM 2.8.** Suppose that
\[ 1 + \alpha_j - \beta_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i < j \leq n + 1, \quad \beta_i - \alpha_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i \leq n + 1, \quad (2.3) \]
and
\[ \beta_j - \alpha_i \notin \mathbb{Z}_{\leq 0}, \quad 1 \leq i < j \leq n + 1, \quad (2.4) \]
where \( \beta_{n+1} = 1 \). Then we have
\[ \rho(\gamma_k) I_j(z) = \sum_{i=1}^{n+1} I_i(z) m_{ij}^{(k)}, \quad k = 0, 1 \text{ and } j = 1, \ldots, n + 1, \]
where
\[
m_{ij}^{(0)} = \begin{cases} e(-2\beta_i), & 1 \leq i = j \leq n, \\ 1, & i = j = n + 1, \\ e\left(\sum_{r=i+2}^{j} \alpha_r - \sum_{r=i}^{j} \beta_r\right)(e(\alpha_{i+1} - \beta_i)), & 1 \leq i < j \leq n + 1, \\ 0, & \text{otherwise} \end{cases}
\]
and
\[
m_{ij}^{(1)} = \begin{cases} 1, & 1 \leq j = i \leq n, \\ e\left(2\sum_{r=1}^{n} \beta_r - \sum_{r=1}^{n+1} \alpha_r\right), & j = i = n + 1, \\ -e\left(\sum_{r=1}^{n} \beta_r - \sum_{r=1}^{n+1} \alpha_r\right)e\left(\sum_{r=1}^{j} (\beta_r - \alpha_r)\right), & i = n + 1, 1 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}
\]

**Remark 2.9.** In [BH], Beukers and Heckman showed that monodromy representations associated with \( n+1 E_n \) are irreducible if and only if (1.3) holds. Even if the representation is not irreducible, the expressions of the matrix elements in Theorem 2.8 are valid, and thus include some examples of reducible representations.

**Remark 2.10.** If (1.3) holds, Theorem 2.8 shows that the linear map \( \rho(\gamma_1) - \text{Id} \) has rank one; in other words, the linear map \( \rho(\gamma_1) \) is a complex reflection. Hence, we obtain an independent proof of Proposition 2.10 of [BH].

**Remark 2.11.** To obtain the matrix elements in Theorem 2.8, it is enough to assume only the condition (2.3). The condition (2.4) guarantees the linear independence of the \( I_j(z) \).
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