TOROIDAL GROUPS WITHOUT NON-CONSTANT MEROMORPHIC FUNCTIONS

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(Received 3 March 2016)

Abstract. We study the fundamental properties of toroidal groups, complex line bundles and the meromorphic functions on them. As an application, we show that every toroidal group with zero Néron–Severi group has no non-constant meromorphic functions on it. Further, we shall give some examples of such toroidal groups.

1. Introduction

We talked about toroidal groups without non-constant meromorphic functions [5] at the *15th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications*, which was held at Osaka City University, Osaka, Japan, 2007, but the detailed proof had been left unpublished. In this paper we give proofs of the main results.

In Section 2, we study the fundamental properties of toroidal groups. Some facts are written in [2], but we try to explain these facts more geometrically, using real coordinates in toroidal groups. See, for example, Proposition 2.4 and Corollary 2.7.

In Section 3, the main result is Proposition 3.1. This was obtained by Abe [1]. In this paper we prove this by more direct methods.

In Section 4, we show that toroidal groups with zero Néron–Severi group have no non-constant meromorphic functions on them (Theorem 4.3). Further, we characterize toroidal groups with zero Néron–Severi group and construct some examples of toroidal groups without non-constant meromorphic functions. Siegel [10] constructed an example of a complex torus which has no non-constant meromorphic functions on it (Example 1). Our results are generalizations of Siegel’s result to toroidal groups.

2. Toroidal groups

Definition 2.1. A connected complex Lie group X is called a toroidal group if

$$H^0(X, \mathcal{O}) = \mathbb{C}.$$

By definition, X is a complex abelian Lie group and there exists a discrete subgroup Γ of $\mathbb{C}^n$ such that $X \cong \mathbb{C}^n / \Gamma$, where n is a complex dimension of X . We denote by π the natural projection $\mathbb{C}^n \rightarrow \mathbb{C}^n / \Gamma$. Put $r = \text{rank } \Gamma$, then there exist $\mathbb{R}$ -linearly independent vectors $\lambda_1, \dots, \lambda_r \in \mathbb{C}^n$ satisfying $\Gamma = \mathbb{Z}\{\lambda_1, \dots, \lambda_r\}$. A matrix $P = [\lambda_1, \dots, \lambda_r]$ is called a period matrix for $\mathbb{C}^n / \Gamma$. We have the following (cf. [2]).

2010 Mathematics Subject Classification: Primary 32M05; Secondary 32A20.

Keywords: toroidal groups; Néron–Severi group; meromorphic functions.

PROPOSITION 2.2. *A complex abelian Lie group $\mathbb{C}^n/\Gamma$ is a toroidal group if and only if the following condition is satisfied:*

$$\text{for any } \sigma \neq 0 \in \mathbb{C}^n, {}^t\sigma P \notin \mathbb{Z}^{1 \times r}. \quad (2.1)$$

Hence, if a complex abelian Lie group $\mathbb{C}^n/\Gamma$ is a toroidal group, we have $n+1 \leq r \leq 2n$. Put $\mathbb{R}_\Gamma = \mathbb{R}\{\lambda_1, \dots, \lambda_r\}$, then $K := \mathbb{R}_\Gamma/\Gamma$ is a maximal compact subgroup of $\mathbb{C}^n/\Gamma$. Let $\mathbb{C}_\Gamma := \mathbb{R}_\Gamma \cap \sqrt{-1}\mathbb{R}_\Gamma$ be the maximal complex subspace of $\mathbb{R}_\Gamma$. By the result of Matsushima and Morimoto [7], a complex abelian Lie group $\mathbb{C}^n/\Gamma$ is a Stein group if and only if $\mathbb{C}_\Gamma = \{0\}$. Hence, for a toroidal group $\mathbb{C}^n/\Gamma$, we have $\dim_{\mathbb{C}} \mathbb{C}_\Gamma > 0$.

Definition 2.3. A toroidal group $\mathbb{C}^n/\Gamma$ is said to be type q if $\dim_{\mathbb{C}} \mathbb{C}_\Gamma = q$ ($q > 0$).

Then, a toroidal group $\mathbb{C}^n/\Gamma$ is type q if and only if $\text{rank } \Gamma = n + q$.

Let $\mathbb{C}^n/\Gamma$ be a complex abelian Lie group with $\text{rank } \Gamma = n + q$. By a suitable linear change of $\mathbb{C}^n$, a period matrix P can be written as $P = [I_n, V]$, where $I_n = [e_1, \dots, e_n]$ is an identity matrix and $V = [v_{ij}; 1 \leq i \leq n, 1 \leq j \leq q] = [v_1, \dots, v_q]$ is an $n \times q$ matrix. Then $\Gamma = \mathbb{Z}\{e_1, \dots, e_n, v_1, \dots, v_q\}$ and $\mathbb{R}_\Gamma = \mathbb{R}\{e_1, \dots, e_n, v_1, \dots, v_q\}$. Put $V_1 = [v_{ij}; 1 \leq i, j \leq q]$ and $V_2 = [v_{ij}; q+1 \leq i \leq n, 1 \leq j \leq q]$. We may assume $\det(\text{Im } V_1) \neq 0$. Further, put $v_i = \sqrt{-1}e_i$ ($q+1 \leq i \leq n$), $\hat{V} = [v_{ij}; 1 \leq i, j \leq n] = [v_1, \dots, v_n]$, $A = \text{Re } \hat{V} = [\alpha_{ij}; 1 \leq i, j \leq n]$, $B = \text{Im } \hat{V} = [\beta_{ij}; 1 \leq i, j \leq n] = [\beta_1, \dots, \beta_n]$, $A_1 = [\alpha_{ij}; 1 \leq i, j \leq q]$, $A_2 = [\alpha_{ij}; q+1 \leq i \leq n, 1 \leq j \leq q]$, $B_1 = [\beta_{ij}; 1 \leq i, j \leq q]$ and $B_2 = [\beta_{ij}; q+1 \leq i \leq n, 1 \leq j \leq q]$.

By the definition, $\beta_1, \dots, \beta_n$ are $\mathbb{C}$ -linearly independent. We have two coordinates of $z \in \mathbb{C}^n$:

$$\begin{aligned} z &= z_1\beta_1 + \dots + z_n\beta_n \\ &= t_1e_1 + \dots + t_ne_n + t_{n+1}v_1 + \dots + t_{2n}v_n. \end{aligned} \quad (2.2)$$

We write $z = {}^t(z_1, \dots, z_n)$ and $t = {}^t(t_1, \dots, t_{2n})$ in place of (2.2).

Put $t^1 = {}^t(t_1, \dots, t_n)$, $t^2 = {}^t(t_{n+1}, \dots, t_{2n})$, $t^{1'} = {}^t(t_1, \dots, t_q)$, $t^{1''} = {}^t(t_{q+1}, \dots, t_n)$, $t^{2'} = {}^t(t_{n+1}, \dots, t_{n+q})$ and $t^{2''} = {}^t(t_{n+q+1}, \dots, t_{2n})$.

Put

$$P'_\beta = B^{-1}P = \begin{bmatrix} B_1^{-1} & O & B_1^{-1}V_1 \\ -B_2B_1^{-1} & I_{n-q} & -B_2B_1^{-1}A_1 + A_2 \end{bmatrix} \quad \text{and} \quad P_\beta = \begin{bmatrix} O & T_\beta \\ I_{n-q} & R \end{bmatrix},$$

where $T_\beta = [B_1^{-1}, B_1^{-1}V_1]$ and $R = [R_1, R_2] = [-B_2B_1^{-1}, -B_2B_1^{-1}A_1 + A_2]$.

By the commutative diagram

$$\begin{array}{ccc} z & \xrightarrow{\quad} & t \\ \downarrow & & \downarrow \\ \sqrt{-1}z & \xrightarrow{\quad} & Jt \end{array}$$

we have

$$Jt = \begin{bmatrix} B & -A \\ O & I_n \end{bmatrix} \begin{bmatrix} O & -I_n \\ I_n & O \end{bmatrix} \begin{bmatrix} B & -A \\ O & I_n \end{bmatrix}^{-1} \begin{bmatrix} t^1 \\ t^2 \end{bmatrix},$$

for $t = \begin{bmatrix} t^1 \\ t^2 \end{bmatrix} \in \mathbb{R}^{2n}$.

Since $t \in \mathbb{R}_\Gamma$ if and only if $t = t^1 + Vt^{2'}$, we get the following.

PROPOSITION 2.4. *Let $\mathbb{C}^n/\Gamma$ be a complex abelian Lie group with a period matrix of the form $P = [I_n, V]$ and $t \in \mathbb{R}_\Gamma$. Then $t \in \mathbb{C}_\Gamma$ if and only if*

$$t^{1''} + R_1 t^{1'} + R_2 t^{2'} = 0. \quad (2.3)$$

Proof. For $t = t^1 + Vt^{2'} \in \mathbb{R}_\Gamma$, $t \in \mathbb{C}_\Gamma$ if and only if $Jt \in \mathbb{R}_\Gamma$. We see that $Jt \in \mathbb{R}_\Gamma$ is equivalent to (2.3) from a direct computation. $\square$

For $z \in \mathbb{C}^n$, we write $z = \begin{bmatrix} z' \\ z'' \end{bmatrix}$, where ${}^t z' = (z_1, \dots, z_q)$ and ${}^t z'' = (z_{q+1}, \dots, z_n)$. Put $R = [r_1, \dots, r_{2q}]$, then $r_1, \dots, r_{2q} \in \mathbb{R}\{e''_{q+1}, \dots, e''_n\}$. Further, we put

$$\Gamma_\beta = \mathbb{Z}\{e_{q+1}, \dots, e_n, B^{-1}e_1, \dots, B^{-1}e_q, B^{-1}v_1, \dots, B^{-1}v_q\}$$

and

$$\Gamma''_\beta = \mathbb{Z}\{e''_{q+1}, \dots, e''_n, r_1, \dots, r_{2q}\}.$$

PROPOSITION 2.5. *Let $\mathbb{C}^n/\Gamma$ be a complex abelian Lie group with a period matrix of the form $P = [I_n, V]$, where V is an $n \times q$ matrix. Then $\mathbb{C}^n/\Gamma$ is a toroidal group of type q if and only if*

$$\text{for any } m'' \in \mathbb{Z}^{n-q} \setminus \{0\}, {}^t m'' R \notin \mathbb{Z}^{1 \times 2q}. \quad (2.4)$$

Proof. Suppose $\mathbb{C}^n/\Gamma$ is a toroidal group of type q . For any $\sigma \in \mathbb{C}^n \setminus \{0\}$, ${}^t \sigma P \notin \mathbb{Z}^{1 \times (n+q)}$. Hence, for any $\sigma \in \mathbb{C}^n \setminus \{0\}$, ${}^t \sigma P_\beta \notin \mathbb{Z}^{1 \times (n+q)}$. Put $\sigma = \begin{bmatrix} 0 \\ m'' \end{bmatrix} \in \mathbb{C}^n \setminus \{0\}$, where $m'' \in \mathbb{Z}^{n-q} \setminus \{0\}$, then (2.4) follows from ${}^t \sigma P_\beta = [{}^t m'', {}^t m'' R]$.

Conversely, suppose (2.4) holds. If $\mathbb{C}^n/\Gamma$ is not a toroidal group, then there exists $\sigma \in \mathbb{C}^n \setminus \{0\}$, such that ${}^t \sigma P_\beta = [{}^t \sigma'', {}^t \sigma' T_\beta + {}^t \sigma'' R] \in \mathbb{Z}^{1 \times (n+q)}$. Then $\text{Im } {}^t \sigma' T_\beta = 0$. Hence, $\sigma' = 0$ and $\sigma'' \in \mathbb{Z}^{n-q}$, then ${}^t \sigma'' R \in \mathbb{Z}^{1 \times 2q}$. This contradicts the assumption. Hence, $\mathbb{C}^n/\Gamma$ is a toroidal group. $\square$

Hence, we get the following.

COROLLARY 2.6. *Let $\mathbb{C}^n/\Gamma$ be a complex abelian Lie group with a period matrix of the form $P = [I_n, V]$, where V is an $n \times q$ matrix. Then $\mathbb{C}^n/\Gamma$ is a toroidal group of type q if and only if*

$$\Gamma''_\beta \text{ is dense in } \mathbb{R}^{n-q} = \mathbb{R}\{e''_{q+1}, \dots, e''_n\}.$$

Proof. Suppose $\mathbb{C}^n/\Gamma$ is a toroidal group of type q . Let $(\Gamma''_\beta)^* := \{u = ({}^t u_1, \dots, {}^t u_{n-q}) \in \mathbb{R}^{n-q} \mid {}^t u x \in \mathbb{Z} \text{ for any } x \in \Gamma''_\beta\}$. Then $u_i \in \mathbb{Z}$ ($1 \leq i \leq n-q$). If $u \neq 0$, by (2.4) there exists j such that ${}^t u r_j \notin \mathbb{Z}$. This contradicts the fact that $u \in (\Gamma''_\beta)^*$. Hence, $u = 0$. Then $(\Gamma''_\beta)^* = 0$ and $((\Gamma''_\beta)^*)^* = \overline{\Gamma''_\beta} = \mathbb{R}^{n-q}$ [3, Ch. 7]. Namely, Γ''_β is dense in $\mathbb{R}^{n-q}$.

Conversely, suppose $\overline{\Gamma''_\beta} = ((\Gamma''_\beta)^*)^* = \mathbb{R}^{n-q}$, and then we get $(\Gamma''_\beta)^* = 0$. It is equivalent to (2.4), and hence $\mathbb{C}^n/\Gamma$ is a toroidal group. $\square$

Define the projection $\pi_\Gamma: \mathbb{R}_\Gamma \longrightarrow \mathbb{R}_\Gamma/\Gamma = T^{n+q}$ by

$$\pi_\Gamma(t) = ({}^t(\exp(2\pi\sqrt{-1}t_1), \dots, \exp(2\pi\sqrt{-1}t_{n+q}))).$$

COROLLARY 2.7. (Cf. [1, p. 6]) *Let $\mathbb{C}^n/\Gamma$ be a complex abelian Lie group with rank $\Gamma = n+q$. Then $\mathbb{C}^n/\Gamma$ is a toroidal group if and only if*

$$\pi_\Gamma(\mathbb{C}_\Gamma) \text{ is dense in } \mathbb{R}_\Gamma/\Gamma.$$

Proof. Let $t \in \mathbb{C}_\Gamma$. Then by Proposition 2.4

$$\begin{aligned} \pi_\Gamma(t) &= {}^t(\exp(2\pi\sqrt{-1}t^{1'}), \exp(2\pi\sqrt{-1}(-R_1t^{1'} - R_2t^{2'})), \exp(2\pi\sqrt{-1}t^{2'})) \\ &= {}^t(\exp(2\pi\sqrt{-1}t^{1'}), \exp(2\pi\sqrt{-1}(-R_1t^{1'} - R_2t^{2'})) \\ &\quad \times \exp(2\pi\sqrt{-1}(-R_1k - R_2l)), \exp(2\pi\sqrt{-1}t^{2'})) \\ &\in T^q \times T^{n-q} \times T^q, \quad \text{where } k, l \in \mathbb{Z}^q. \end{aligned}$$

From Corollary 2.6, we see that $\mathbb{C}^n/\Gamma$ is a toroidal group of type q if and only if $\{\exp(2\pi\sqrt{-1}(-R_1k - R_2l)) \mid k, l \in \mathbb{Z}^q\}$ is dense in T^{n-q} . Hence the corollary holds. $\square$

3. Complex line bundles on toroidal groups

Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group of type q . Put $H = H^0(\mathbb{C}^n, \mathcal{O})$ and $H^* = H^0(\mathbb{C}^n, \mathcal{O}^*)$. Then we have isomorphisms (cf. [8, 9])

$$H^1(X, \mathcal{O}) \cong H^1(\Gamma, H) \quad \text{and} \quad H^1(X, \mathcal{O}^*) \cong H^1(\Gamma, H^*).$$

We denote by $Z^1(\Gamma, H)$ and $Z^1(\Gamma, H^*)$ the cocycles which define $H^1(\Gamma, H)$ and $H^1(\Gamma, H^*)$, respectively. By the definition any cocycle $e_\lambda(z) \in Z^1(\Gamma, H^*)$ satisfies

$$e_\lambda(z + \mu)e_\mu(z) = e_{\lambda+\mu}(z). \quad (3.1)$$

Let L be a complex line bundle on X . Then L is defined by a cocycle $e_\lambda(z) \in Z^1(\Gamma, H^*)$. Namely, L is the quotient space of $\mathbb{C}^n \times \mathbb{C}$ by the action of Γ given by $(z, \xi) \rightarrow (z + \lambda, e_\lambda(z)\xi)$. On the other hand, there exist an open covering $\{U_\alpha\}$ of X and $g_\alpha(z) \in H^0(\pi^{-1}(U_\alpha), \mathcal{O}^*)$ such that for each α , the mapping $\pi^*(L)|_{\pi^{-1}(U_\alpha)} \ni (z, \xi_\alpha) \rightarrow (z, g_\alpha(z)\xi_\alpha) \in \mathbb{C}^n \times \mathbb{C}$ gives a global trivialization $\pi^*(L) \rightarrow \mathbb{C}^n \times \mathbb{C}$ and

$$e_\lambda(z) = g_\alpha(z + \lambda)g_\alpha(z)^{-1}, \quad z \in \pi^{-1}(U_\alpha). \quad (3.2)$$

For $z \in \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ and $\pi(z) = x \in U_\alpha \cap U_\beta$,

$$\xi_{\alpha\beta}(x) := g_\alpha(z)^{-1}g_\beta(z) \quad (3.3)$$

is a transition function for L relative to $\{U_\alpha\}$. Let $s_\alpha(x) \in H^0(X, \mathcal{O}(L))$ be a holomorphic section of L . Then for each $x \in U_\alpha \cap U_\beta$, we have $s_\alpha(x) = \xi_{\alpha\beta}(x)s_\beta(x)$. Put

$$s(z) := g_\alpha(z)s_\alpha(\pi(z)), \quad z \in \pi^{-1}(U_\alpha). \quad (3.4)$$

Then $s(z) \in H^0(\mathbb{C}^n, \mathcal{O})$ and we have

$$s(z + \lambda) = e_\lambda(z)s(z). \quad (3.5)$$

We call a holomorphic function $s(z)$ on $\mathbb{C}^n$ satisfying (3.5) a theta function for the complex line bundle L . From (3.4), the theta functions for L are in a one-to-one correspondence with the holomorphic sections of L . The following fact is proved by Abe [1] in a more general condition. Here we prove from a different point of view.

PROPOSITION 3.1. *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group of type q and let L be a topologically trivial line bundle over X . Suppose there exists a non-zero holomorphic section of L , then L is analytically trivial.*

Proof. Let $s_\alpha(x) \in H^0(X, \mathcal{O}(L))$ be a non-zero holomorphic section and let $s(z) \in H^0(\mathbb{C}^n, \mathcal{O})$ be a theta function for L defined by (3.4). We denote by $\mathcal{A}$ the sheaf of germs of C^∞ functions, $\mathcal{A}^*$ the sheaf of germs of non-zero C^∞ functions and $A^* := H^0(\mathbb{C}^n, \mathcal{A}^*)$. Since L is topologically trivial, there exists $c(z) \in H^0(\Gamma, A^*)$ satisfying

$$s(z + \lambda) = \frac{c(z + \lambda)}{c(z)} s(z) \quad \text{for any } z \in \mathbb{C}^n \text{ and } \lambda \in \Gamma.$$

Then $t(x) = s(z)/c(z)$ is a global C^∞ function on X , where $x = \pi(z)$. If $s(z)$ is non-vanishing on $\mathbb{C}^n$, $s_\alpha(x)$ is a non-vanishing holomorphic section of L and L is analytically trivial and the proposition holds. Suppose there exists $z_0 \in \mathbb{C}^n$ satisfying $s(z_0) = 0$. Put $x_0 = \pi(z_0)$. For any $a \in \mathbb{C}_\Gamma$, we put $a^* = \pi(a)$. Then $t(x_0 + a^*) = s(z_0 + a)/c(z_0 + a)$ is a C^∞ function on $\pi(\mathbb{C}_\Gamma)$ for fixed $x_0 \in X$. Since $t(x) \in H^0(X, \mathcal{A})$ and $\overline{\pi(\mathbb{C}_\Gamma)} = \mathbb{R}_\Gamma / \Gamma$ is a compact subset of X , $t(x_0 + a^*)$ is bounded on $\overline{\pi(\mathbb{C}_\Gamma)}$. Then $s(z_0 + a)$ is a bounded holomorphic function on $\mathbb{C}_\Gamma$ and constant on $\mathbb{C}_\Gamma$. Since $s(z_0) = 0$, $s(z_0 + a) = 0$ for any $a \in \mathbb{C}_\Gamma$. Hence, for any $x \in \overline{\pi(\mathbb{C}_\Gamma)}$, $t(x_0 + x) = 0$ and $s(z_0 + z) = 0$ for any $z \in \mathbb{R}_\Gamma$. Then $s(z) = 0$ on $\mathbb{C}^n$. Thus $s_\alpha(x) \equiv 0$ on X . It is a contradiction to our assumption. Hence, $s_\alpha(x)$ is a non-vanishing section of L and L is analytically trivial. $\square$

4. Toroidal groups without non-constant meromorphic functions

Let $X = \mathbb{C}^n / \Gamma$ be a toroidal group of type q . A meromorphic function on X is a meromorphic function on $\mathbb{C}^n$ which has a period group containing Γ as a subgroup.

PROPOSITION 4.1. *Let $X = \mathbb{C}^n / \Gamma$ be a toroidal group and let $f(z)$ be a meromorphic function on X . Then there exist a line bundle L on X and theta functions $h(z)$ and $k(z)$ for L satisfying*

$$f(z) = \frac{h(z)}{k(z)}. \quad (4.1)$$

Proof. Since $f(z)$ is a meromorphic function on $\mathbb{C}^n$, there exist $h, k \in H^0(\mathbb{C}^n, \mathcal{O})$ such that $(h, k) = 1$ and $f(z) = h(z)/k(z)$, for all $z \in \delta(f)$, where $\delta(f)$ is the set of regular points of $f(z)$. For each $\lambda \in \Gamma$, set, for all $z \in \mathbb{C}^n$, $h_\lambda(z) := h(z + \lambda)$ and $k_\lambda(z) := k(z + \lambda)$. For all $z \in \delta(f)$, $h(z)/k(z) = f(z) = f(z + \lambda) = h_\lambda(z)/k_\lambda(z)$. Since $(h_\lambda, k_\lambda) = 1$, there exists $v_\lambda \in H^0(\mathbb{C}^n, \mathcal{O})$ such that $h(z + \lambda) = h(z) \exp(v_\lambda(z))$ and $k(z + \lambda) = k(z) \exp(v_\lambda(z))$. Hence, we have

$$\exp(v_{\lambda_1 + \lambda_2}(z)) = \exp(v_{\lambda_2}(z + \lambda_1)) \exp(v_{\lambda_1}(z)).$$

Namely, $\{\exp(v_\lambda(z))\}$ defines a line bundle L on X and $h(z), k(z)$ are theta functions for L . Hence the proposition is proved. $\square$

Consider the exact sequence

$$\rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{l} H^2(X, \mathcal{O}) \rightarrow .$$

Definition 4.2. $\text{NS}(X) := c_1 H^1(X, \mathcal{O}^*)$ is called a Néron–Severi group.

THEOREM 4.3. *Let $X = \mathbb{C}^n / \Gamma$ be a toroidal group. If the Néron–Severi group $\text{NS}(X) = 0$, there are no non-constant meromorphic functions on X .*

Proof. Suppose $f(z)$ is a meromorphic function on X . By Proposition 4.1, there exist a line bundle on X and theta functions $h(z)$ and $k(z)$ for L satisfying $f(z) = h(z)/k(z)$. Since $\text{NS}(X) = 0$, L is topologically trivial. If $h(z)$ is not identically zero, then from Proposition 3.1, L is analytically trivial. Hence, $h(z)$ and $k(z)$ are holomorphic functions on X . Then these are constant and $f(z)$ is constant. Hence, the theorem is proved. $\square$

Suppose $E \in H^2(X, \mathbb{Z})$. Since E is a $\mathbb{Z}$ -valued skew-symmetric $(n+q, n+q)$ -matrix, we write

$$E = [E_{ij}; 1 \leq i, j \leq n+q] = \begin{bmatrix} E_1 & E_2 \\ -{}^t E_2 & E_3 \end{bmatrix}, \quad (4.2)$$

where $E_1 \in \mathbb{Z}^{n \times n}$ and $E_3 \in \mathbb{Z}^{q \times q}$.

The following theorem characterizes the Néron–Severi group on a toroidal group.

THEOREM 4.4. *Let $X = \mathbb{C}^n / \Gamma$ be a toroidal group of type q with a period matrix $P = [I_n, V]$. Then $E \in H^2(X, \mathbb{Z})$ belongs to $\text{NS}(X)$ if and only if*

$${}^t V E_1 V + {}^t E_2 V - {}^t V E_2 + E_3 = 0. \quad (4.3)$$

Proof. Using the coordinate (2.2), we can identify E with the 2-form

$$\frac{1}{2} \sum_{1 \leq i, j \leq n+q} E_{ij} dt_i \wedge dt_j. \quad (4.4)$$

and we use the same symbol E for (4.4). From (2.2), we get

$$\begin{aligned} dt_i &= \frac{1}{2\sqrt{-1}} \left(- \sum_{j=1}^q \bar{v}_{ij} dz_j + \sum_{j=1}^q v_{ij} d\bar{z}_j \right) \quad \text{for } i = 1, \dots, q, \\ dt_i &\sim \frac{1}{2\sqrt{-1}} \left(- \sum_{j=1}^q \bar{v}_{ij} dz_j + 2\sqrt{-1} dz_i + \sum_{j=1}^q v_{ij} d\bar{z}_j \right) \quad \text{for } i = q+1, \dots, n, \\ dt_{n+i} &= \frac{1}{2\sqrt{-1}} (dz_i - d\bar{z}_i) \quad \text{for } i = 1, \dots, n, \end{aligned} \quad (4.5)$$

and in particular

$$dz_i \sim d\bar{z}_i \quad \text{for } i = q+1, \dots, n.$$

Here, for any d -closed forms φ_1 and φ_2 we write $\varphi_1 \sim \varphi_2$ when φ_1 and φ_2 are d -cohomologous. Substituting (4.5) in to (4.4), we get a unique constant 2-form

$$E_C \in \bigwedge^2 \mathbb{C}\{dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_q\}$$

which is d -cohomologous to E . According to [11, Section 3], we write

$$\begin{aligned} E_C &= \frac{1}{2} \sum_{1 \leq i, j \leq n} A_{ij} dz_i \wedge dz_j + \sum_{1 \leq i \leq n, 1 \leq j \leq q} B_{ij} dz_i \wedge d\bar{z}_j \\ &\quad + \frac{1}{2} \sum_{1 \leq i, j \leq q} C_{ij} d\bar{z}_i \wedge d\bar{z}_j. \end{aligned}$$

Since $\iota(E) \in H^2(X, \mathcal{O})$, $\iota(E) = \frac{1}{2} \sum_{1 \leq i, j \leq q} C_{ij} d\bar{z}_i \wedge d\bar{z}_j$. We put $C = [C_{ij}; 1 \leq i, j \leq q]$. From [11, Section 3] we get

$$C = -\frac{1}{4}({}^t V E_1 V + {}^t E_2 V - {}^t V E_2 + E_3).$$

By the definition, E belongs to $\text{NS}(X)$ if and only if $\iota(E) = 0$ in $H^2(X, \mathcal{O})$; namely $\iota(E)$ is a constant $\bar{\partial}$ -exact $(0, 2)$ -form. Hence, $\iota(E) = 0$ (cf. [4, Lemma 2.1]). Then we get (4.3), and the theorem is proved. $\square$

Using Theorem 4.3 and Theorem 4.4, we shall construct examples of toroidal groups which have no non-constant meromorphic functions on them.

Definition 4.5. Let α and β be algebraic numbers.

$\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are said to be linearly disjoint over $\mathbb{Q}$ if the canonical $\mathbb{Q}$ -homomorphism of $\mathbb{Q}(\alpha) \otimes_{\mathbb{Q}} \mathbb{Q}(\beta)$ into $\mathbb{Q}(\alpha, \beta)$, which maps $x \otimes_{\mathbb{Q}} y$ onto xy , is injective.

Definition 4.6. Let α_i , $i = 1, \dots, k$ be algebraic numbers.

$\mathbb{Q}(\alpha_1), \dots, \mathbb{Q}(\alpha_k)$ are said to be linearly disjoint over $\mathbb{Q}$ if $\mathbb{Q}(\alpha_1, \dots, \alpha_{\ell-1})$ and $\mathbb{Q}(\alpha_{\ell})$ are linearly disjoint over $\mathbb{Q}$, for $\ell = 2, \dots, k$.

LEMMA 4.7. Let α_i , $i = 1, \dots, k$ be algebraic numbers.

$\mathbb{Q}(\alpha_1), \dots, \mathbb{Q}(\alpha_k)$ are linearly disjoint over $\mathbb{Q}$ if and only if the canonical $\mathbb{Q}$ -homomorphism of $\mathbb{Q}(\alpha_1) \otimes_{\mathbb{Q}} \dots \otimes_{\mathbb{Q}} \mathbb{Q}(\alpha_k)$ into $\mathbb{Q}(\alpha_1, \dots, \alpha_k)$, which maps $x_1 \otimes_{\mathbb{Q}} \dots \otimes_{\mathbb{Q}} x_k$ onto $x_1 \dots x_k$, is injective.

THEOREM 4.8. Let

$$V = [v_{ij}; 1 \leq i \leq n, 1 \leq j \leq q] = [\sqrt{-1}\beta_{ij}; 1 \leq i \leq n, 1 \leq j \leq q, \beta_{ij} \in \mathbb{R}]$$

be a $n \times q$ matrix satisfying:

- (1) $q \geq 2$ and β_{ij} are algebraic numbers;
- (2) $\mathbb{Q}(\beta_{ij})$, $1 \leq i \leq n, 1 \leq j \leq q$ are linearly disjoint.

Put $V = [v_1, \dots, v_q]$ and $\Gamma = \mathbb{Z}\{e_1, \dots, e_n, v_1, \dots, v_q\}$.

Then $\text{rank}_{\mathbb{C}} V = q$ and the complex abelian Lie group $X = \mathbb{C}^n / \Gamma$ is a toroidal group of type q and $\text{NS}(X) = 0$. Hence, X has no non-constant meromorphic functions on it.

Proof. Since $\mathbb{Q}(\beta_{11}, \dots, \beta_{1q})$ and $\mathbb{Q}(\beta_{ij}; 2 \leq i \leq q)$ are linearly disjoint, $\det[\beta_{ij}; 1 \leq i, j \leq q] \neq 0$. Hence $\text{rank}_{\mathbb{C}} V = q$. Next take a non-zero $\sigma \in \mathbb{C}^n$ and suppose ${}^t\sigma[I_n, V] \in \mathbb{Z}^{1 \times (n+q)}$. Then $\sigma \in \mathbb{Z}^n$ and ${}^t\sigma V = 0$. Hence $\sigma = 0$. Thus X is a toroidal group of type q . Finally, we shall show $\text{NS}(X) = 0$. Let E be a $\mathbb{Z}$ -valued skew-symmetric $(n+q, n+q)$ -matrix. We use the same symbols as (4.2). Then the (i, j) components of ${}^t V E_1 V$, ${}^t E_2 V$, ${}^t V E_2$ and E_3 are

$$\sum_{k, \ell=1}^n E_{k\ell} \beta_{kj} \beta_{\ell i}, \quad \sqrt{-1} \sum_{k=1}^n E_{kn+i} \beta_{kj}, \quad \sqrt{-1} \sum_{k=1}^n E_{kn+j} \beta_{ki}, \quad E_{n+i, n+j},$$

respectively.

Suppose ${}^t V E_1 V + {}^t E_2 V - {}^t V E_2 + E_3 = 0$, then for any $1 \leq i, j \leq q$,

$$\sum_{k, \ell=1}^n E_{k\ell} \beta_{kj} \beta_{\ell i} + E_{n+i, n+j} + \sqrt{-1} \sum_{k=1}^n (E_{kn+i} \beta_{kj} - E_{kn+j} \beta_{ki}) = 0.$$

Since $\{1, \beta_{ai}, \beta_{bj}, \beta_{ck} \mid 1 \leq a, b, c \leq n, 1 \leq i, j, k \leq q\}$ are linearly independent over $\mathbb{Q}$ and $q \geq 2$, we have $E_{ab} = 0$, for $1 \leq a, b \leq n + q$. Hence $\text{NS}(X) = 0$, and the theorem is proved. $\square$

The following are examples of toroidal groups without non-constant meromorphic functions, which satisfy conditions (1) and (2) of Theorem 4.8.

Example 1. $X = \mathbb{C}^2 / \Gamma$ is a toroidal group (complex torus) with a period matrix

$$P = \begin{bmatrix} 1 & 0 & \sqrt{-2} & \sqrt{-5} \\ 0 & 1 & \sqrt{-3} & \sqrt{-7} \end{bmatrix}.$$

Example 1 was first given by Siegel [10, p. 104]. We can show that X has no non-constant meromorphic functions on it by another method, namely using Theorem 4.8.

Example 2. $X = \mathbb{C}^3 / \Gamma$ is a toroidal group with a period matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & \sqrt{-2} & \sqrt{-7} \\ 0 & 1 & 0 & \sqrt{-3} & \sqrt{-11} \\ 0 & 0 & 1 & \sqrt{-5} & \sqrt{-13} \end{bmatrix}.$$

Similarly to Example 1 and Example 2, we can construct examples of toroidal groups which have no non-constant meromorphic functions on them for any complex dimension greater than 2. We note that Abe and Kopfermann [2, p. 103] gave an example of a toroidal group as one which does not have a non-constant meromorphic function on it. But Kim [6] showed that the toroidal group given by Abe and Kopferman is a quasi-Abelian variety and hence it has many non-constant meromorphic functions on it.

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