CRITICAL BEHAVIOURS OF THE FIFTH PAINLEVÉ TRANSCENDENTS AND THE MONODROMY DATA

Shun SHIMOMURA

(Received 4 March 2016 and revised 22 April 2016)

Abstract. For the fifth Painlevé equation, we present families of convergent series solutions near the origin and the corresponding monodromy data for the associated isomonodromy linear system. These solutions are of complex power type, of inverse logarithmic type and of Taylor series type. For generic parameters the total set of these critical behaviours is almost complete. For the complex power type of solutions in the generic case, we clarify the structure of the analytic continuation on the universal covering around the origin, and examine the distribution of zeros, poles and 1-points. It is shown that two kinds of spiral domains including a sector as a special case are alternately arrayed; the domains of one kind contain sequences both of zeros and of poles, and those of the other kind sequences of 1-points.

1. Introduction

For the solutions of the sixth Painlevé equation, Guzzetti [11] provided the tables of their critical behaviours and parametric connection formulas. The solutions near each critical point are classified as follows: complex power type, logarithmic type, inverse oscillatory type, inverse logarithmic type, and Taylor series type. The total set of these solutions is almost complete, that is, it contains almost all possible critical behaviours. His tables consist of important formulas as nonlinear special functions, which are expected to be of great use in applications to a variety of problems in mathematics and mathematical physics.

The fifth Painlevé equation normalized in the form

$$\frac{d^2y}{dx^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{(y-1)^2}{8x^2} \left( (\theta_0 - \theta_x + \theta_\infty) y - \frac{(\theta_0 - \theta_x - \theta_\infty)^2}{y} \right)$$

\[ + (1 - \theta_0 - \theta_x) \frac{y}{x} - \frac{y(y+1)}{2(y-1)} \tag{V} \]

with $\theta_0$, $\theta_x$, $\theta_\infty \in \mathbb{C}$ follows from the isomonodromy deformation of a two-dimensional linear system of the form

$$\frac{dY}{d\lambda} = \left( \frac{A_0(x)}{\lambda} + \frac{A_x(x)}{\lambda-x} + \frac{J}{2} \right) Y, \tag{1.1}$$

$J = \text{diag}[1, -1]$, where $A_0(x)$ and $A_x(x)$ satisfy the following:

2010 Mathematics Subject Classification: Primary 34M55; Secondary 34M56, 34M35.

Keywords: fifth Painlevé equation; critical behaviour; isomonodromy deformation; monodromy data; Schlesinger equation.

© 2017 Faculty of Mathematics, Kyushu University
(a) the eigenvalues of $A_0(x)$ and $A_x(x)$ are $\pm \theta_0/2$ and $\pm \theta_x/2$, respectively;
(b) $(A_0(x) + A_x(x))_{11} = -(A_0(x) + A_x(x))_{22} = -\theta_\infty/2$

(cf. [14, Section 3] and [15, Appendix C]). Jimbo [14] studied the asymptotic behaviour of the $\tau$-function for (V) near $x = 0$ parametrized by its corresponding monodromy data for (1.1). Applying WKB analysis to (1.1), Andreev and Kitaev [2] obtained asymptotic solutions of (V) along the real axis near $x = 0$ and $x = \infty$ together with their monodromy data that yield the connection formulas between them. For more general integration constants, a family of solutions near $x = 0$ expanded into convergent series in spiral domains or sectors was given by the present author [22]. Kaneko and Ohyama [16] presented certain Taylor series solutions around $x = 0$ such that each corresponding linear system (1.1) is solvable in terms of hypergeometric functions and that the monodromy may be calculated explicitly. Various formal series solutions of (V) were listed by Bryuno and Parusnikova [4, 19], who computed them by the method of power geometry.

For the fifth Painlevé transcendents as well, it is preferable to give tables of critical behaviours like those of Guzzetti [11]. Toward this goal, in this paper, near the critical point $x = 0$ of (V), we present families of solutions expanded into convergent series of three types and the respective monodromy data parametrized by integration constants yielding analogues of the parametric connection formulas in [11]. The monodromy data are given under more general conditions on $\theta_0, \theta_x, \theta_\infty$ and the integration constants than those of [2] or [14]. These solutions, including degenerate cases, are of complex power type, of inverse logarithmic type and of Taylor series type (cf. Theorems 2.1 and 2.3–2.5). For these solutions, under certain conditions, it is possible to derive connection formulas relating to asymptotic solutions as $x \to +\infty$ in [2], and the total set of them is almost complete (cf. Sections 2.5 and 2.6). All of them are derived by a unified method. The key is finding suitable matrix solutions of the Schlesinger equation equivalent to (V) controlling the isomonodromy deformation of (1.1). Solutions of logarithmic type [24] are also obtained from those of inverse logarithmic type through a Bäcklund transformation found by Gromak [6] (cf. Remark 2.7 and Section 6).

In such a sense these inverse logarithmic solutions play the same role as that of the basic logarithmic solutions of the sixth Painlevé equation. If $\theta_0 - \theta_x = \theta_\infty = 0$, the complex power type of solutions have relatively simple oscillatory expressions although, in general, such expressions are complicated (cf. Theorem 2.2 and Remark 2.4). For the complex power type of solutions in the generic case, we clarify the structure of the analytic continuation on the universal covering around $x = 0$, and examine the distribution of zeros, poles and 1-points (note that $y = 0$, 1 and $\infty$ are singular values of equation (V)), where a 1-point means a point $x = x_1$ such that $y(x_1) = 1$. It is shown that the domains where each solution behaves like a complex power are separated by two kinds of spiral domains including a sector as a special case, which are alternately arrayed; the separating domains of one kind contain sequences both of zeros and of poles, and those of the other kind sequences of 1-points (cf. Remark 2.10). This situation is different from that of the sixth Painlevé transcendents, in which the separating domain with zeros and that with poles alternately appear (cf. [8, 9, 12, 13, 23]).

All the results above are described in Section 2. In Section 5 the families of solutions of three types are derived from the matrix solutions of the Schlesinger equation given in Section 3 by using the lemmas in Section 4. In Sections 6 and 7 we prove the result on the analytic continuation, those on the distribution of zeros, poles and 1-points, and Theorem 2.2.
on the special oscillatory expressions. In proving them, the Bäcklund transformation referred
to above is crucial. Section 8 is devoted to the summary of the argument in [14, Section 2]
concerning the limiting procedure applied to (1.1) and its monodromy data, and Section 9 to
the computation of the connection formulas for the related Whittaker and hypergeometric
systems. In non-generic cases of these linear systems it is also possible to compute the
monodromy data (cf. Remark 2.13 and Section 9.3). In the final section, using the material
above, we derive the results on the monodromy data for our solutions.

Throughout this paper we use the following symbols:
(1) for a ring $\mathbb{A}$, $M_2(\mathbb{A})$ is the ring of $2 \times 2$ matrices whose entries are in $\mathbb{A}$, and
$\text{GL}_2(\mathbb{A}) := \{ C \in M_2(\mathbb{A}); \; C^{-1} \in M_2(\mathbb{A}) \};$
(2) $I, J, \Delta, \Delta_-$ denote the matrices
\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Delta_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \]
(3) $\mathcal{R}(\mathbb{C} \setminus \{0\})$ denotes the universal covering of $\mathbb{C} \setminus \{0\}$;
(4) $\mathbb{Q}_\theta := \mathbb{Q}[\theta_0, \theta_x, \theta_\infty]$;
(5) for $A, B \subset \mathbb{C}$, $\text{cl}(A)$ denotes the closure of $A$, $\text{dist}(A, B)$ the distance between $A$ and $B$;
(6) $\psi(x) := \Gamma'(x)/\Gamma(x)$ is the di-Gamma function.

2. Main results

2.1. Solutions near $x = 0$

Set $\mathbb{Q}_\theta := \mathbb{Q}[\theta_0, \theta_x, \theta_\infty]$. For each $(\theta_0, \theta_x, \theta_\infty) \in \mathbb{C}^3$, we have solutions of complex power
type near $x = 0$.

**THEOREM 2.1.** Let $\Sigma_0$ be a bounded domain satisfying
\[ \Sigma_0 \subset \mathbb{C} \setminus \Sigma_* \quad \text{with} \quad \Sigma_* = \{ \sigma \leq -1 \} \cup \{ 0 \} \cup \{ \sigma \geq 1 \} \subset \mathbb{R} \]
and $\text{dist}(\Sigma_0, \Sigma_*) > 0$. Suppose that $(\sigma^2 - (\theta_0 \pm \theta_x)^2)(\sigma^2 - \theta_\infty^2) \neq 0$ for every $\sigma \in \text{cl}(\Sigma_0)$. Then (V) admits a two-parameter family of solutions $\{ y(\sigma, \rho, x); \; (\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\}) \}$
with the following properties.

(i) $y(\sigma, \rho, x)$ is holomorphic in $(\sigma, \rho, x) \in \Omega^+(\Sigma_0, \varepsilon_0) \cup \Omega^-(\Sigma_0, \varepsilon_0) \subset \Sigma_0 \times (\mathbb{C} \setminus \{0\}) \times \mathcal{R}(\mathbb{C} \setminus \{0\})$, where
\[ \Omega^+(\Sigma_0, \varepsilon_0) := \bigcup_{(\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\})} \{ (\sigma, \rho) \} \times \Omega_{\sigma, \rho}^+(\varepsilon_0), \]
\[ \Omega_{\sigma, \rho}^+(\varepsilon_0) := \{ x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); \; |\rho x^\sigma| < \varepsilon_0, \; |x(\rho x^\sigma)^{-1}| < \varepsilon_0 \}, \]
\[ \Omega_{\sigma, \rho}^-(\varepsilon_0) := \{ x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); \; |x(\rho x^\sigma)| < \varepsilon_0, \; |(\rho x^\sigma)^{-1}| < \varepsilon_0 \}, \]
$\varepsilon_0 = \varepsilon_0(\Sigma_0, \theta_0, \theta_x, \theta_\infty)$ being a sufficiently small number depending only on $\Sigma_0$ and
$(\theta_0, \theta_x, \theta_\infty)$. 

(ii) \( y(\sigma, \rho, x) \) is represented by the convergent series as follows:
\[
y_+(\sigma, \rho, x) := 1 + \frac{4\sigma^2(\theta_0 + \theta_x - \sigma)}{\sigma + \theta_\infty}(\theta_0^2 - (\sigma + \theta_x)^2)\rho x^\sigma + \frac{4\sigma^2(\theta_0 + \theta_x + \sigma)}{\sigma - \theta_\infty}(\theta_0^2 - (\sigma - \theta_x)^2)(\rho x^\sigma)^{-1}
\]
\[
+ \sum_{j \geq 2} c_j^+(\sigma)(\rho x^\sigma)^j + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c_{jn}^+(\sigma)(\rho x^\sigma)^{-n+j}
\]
in \( \Omega^+(\Sigma_0, \varepsilon_0) \), and
\[
y_-(\sigma, \rho, x) := 1 - \frac{4\sigma^2(\theta_0 + \theta_x + \sigma)}{\sigma - \theta_\infty}(\theta_0^2 - (\sigma - \theta_x)^2)(\rho x^\sigma)^{-1}
\]
\[
+ \sum_{j \geq 2} c_j^-(\sigma)(\rho x^\sigma)^{-j} + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c_{jn}^-(\sigma)(\rho x^\sigma)^{n-j}
\]
in \( \Omega^-(\Sigma_0, \varepsilon_0) \), where \( c_j^\pm(\sigma), c_{jn}^\pm(\sigma) \in \mathbb{Q}[\theta(\sigma)] \).

**Remark 2.1.** Note that \(|x| < \varepsilon_0^2 \) in \( \Omega_{\sigma, \rho}^\pm(\varepsilon_0) \). For each \((\sigma, \rho)\), \( \Omega_{\sigma, \rho}^+ \) is given by
\[
\text{Re } \sigma \cdot \log |x| + \log |\rho| + \log(\varepsilon_0^{-1}) < \text{Im } \sigma \cdot \arg x < (\text{Re } \sigma - 1) \log |x| + \log |\rho| - \log(\varepsilon_0^{-1}),
\]
and \( \Omega_{\sigma, \rho}^- \) by
\[
(\text{Re } \sigma + 1) \log |x| + \log |\rho| + \log(\varepsilon_0^{-1}) < \text{Im } \sigma \cdot \arg x < \text{Re } \sigma \cdot \log |x| + \log |\rho| - \log(\varepsilon_0^{-1}).
\]

**Remark 2.2.** The asymptotic solution with \( 0 < \text{Re } \sigma < 1 \) in [2, Theorem 6.1] coincides with \( y_+(\sigma, \rho, x) \).

**Remark 2.3.** For each \((\sigma, \rho)\), in the domain
\[
\Omega_{\sigma, \rho}(\varepsilon_0) := \{ x \in \mathbb{R}(\mathbb{C} \setminus \{0\}) : |x(\rho x^\sigma)| < \varepsilon_0, |x(\rho x^\sigma)^{-1}| < \varepsilon_0 \}
\]
\[
\supset \Omega_{\sigma, \rho}^+(\varepsilon_0) \cup \Omega_{\sigma, \rho}^-(\varepsilon_0),
\]
y(\( \sigma, \rho, x \)) is meromorphic and represented by the ratio of convergent series (see Sections 5.1 and 7). In a special case, such expressions of \( y(\sigma, \rho, x) \) are relatively simple formulas as in the following theorem, which may be regarded as counterparts of an elliptic representation for the sixth Painlevé transcendents [8, 9].

**Theorem 2.2.** Suppose that \( \theta_0 - \theta_x = \theta_\infty = 0 \) and \( \sigma \in \Sigma_0 \).
(1) If \( \sigma^2 - 4\theta_0^2 \neq 0 \) for every \( \sigma \in \text{cl}(\Sigma_0) \), then
\[
y(\sigma, \rho, x) = \tanh^2 \left( \frac{1}{2} \log(\tilde{\rho} x^\sigma) + \sum_{n=1}^\infty x^n \sum_{j=-n}^n c_{jn}(\sigma)(\tilde{\rho} x^\sigma)^j \right)
\]
with \( \tilde{\rho} = (2\theta_0 - \sigma)(2\theta_0 + \sigma)^{-1} \rho \), in which the series with \( c_{jn}(\sigma) \in \mathbb{Q}[\theta_0](\sigma) \) converges in \( \Omega_{\sigma, \tilde{\rho}}(\varepsilon_0) \).
where the series
\[
\Phi(x) = \sum_{n=1}^{\infty} x^n \sum_{j=-n}^{n} c_{jn}(\sigma)(\rho x^{\sigma + 1})^j, \quad c_{jn}(\sigma) \in \mathbb{Q}[\theta](\sigma)
\]
and \(\Sigma'(x) = (d/dx)\Sigma(x)\) converge in \(\Omega_{\sigma+1,\rho}(\varepsilon_0)\).

**Remark 2.4.** The expressions above describe oscillatory behaviours. Indeed, if \(|\rho x^{\sigma}|^{\pm 1}\) are bounded, then (2.1) is
\[
-\tan^2(-(i/2)\log|\rho x^{\sigma}| + O(x)) = -\tan^2((1/2)\arg|\rho x^{\sigma}| - (i/2)\log|\rho x^{\sigma}| + O(x)),
\]
and if \(|\rho x^{\sigma + 1}|^{\pm 1}\) are bounded, then (2.2) is
\[
1 - ((\sigma + 1)^{-1} + O(x))x \sin(\arg|\rho x^{\sigma + 1}| - i\log|\rho x^{\sigma + 1}| + O(x)).
\]
In the case where \(\theta_0 - \theta_x \neq 0\) or \(\theta_\infty \neq 0\) as well, \(y(\sigma, \rho, x)\) admits oscillatory expressions as follows (cf. Section 7):

(i) under the suppositions of Theorem 2.1, if \(|\rho x^{\sigma}|^{\pm 1}\) are bounded,
\[
y(\sigma, \rho, x) = \Phi_1(x)\Phi_2(x)F(x)G(x) + O(x)
\]

(ii) under certain generic conditions added to the suppositions of Theorem 2.1, if \(|\rho x^{\sigma + 1}|^{\pm 1}\) are bounded,
\[
y(\sigma, \rho, x) = \frac{1}{2x \rho^2(\sigma + 1)^2}\left[\Phi_1^2(x)\Phi_2^2(x)\Phi_3^2(x)\Psi_1^2(x)F^2(x)G^2(x) + O(x)\right].
\]
Here \(\Phi_1(x), \Phi_2(x), \Psi_1(x), \Psi_2(x), F(x), G(x)\) are as in Section 7.1, and, say \(\Phi_1^2(x)\), the result of the substitution
\[
(\sigma, \rho, \theta_0 - \theta_x, \theta_0 + \theta_x, \theta_\infty) \mapsto (\sigma + 1, \rho, 1 - \theta_\infty, 1 - \theta_0 + \theta_x, \theta_0 + \theta_x - 1)
\]
in \(\Phi_1(x)\) (cf. (2.4)), \(\rho\) being such that
\[
\rho = \frac{(\sigma - \theta_\infty)(2 - \theta_0 + \theta_x + \sigma)(\theta_0 + \theta_x - \sigma)}{8\sigma^2(\sigma + 1)^2}.
\]
(cf. the proof of Theorem 2.10). In the expressions above it seems that the cancellations of the factors \(F(x)G(x), \Phi_2^2(x)\Psi_1^2(x)F^2(x)G^2(x)\) occur, but we have not succeeded in proving them.

For each \((\theta_0, \theta_x, \theta_\infty)\) we have solutions of special complex power type.
Theorem 2.3. Suppose that $\theta_0, \theta_x \neq 0$. Let $\sigma_0 = \theta_0 \pm \theta_x$ or $\theta_x - \theta_0$ be such that $\sigma_0 \in \Sigma_+ := C \setminus ([\sigma \leq -1] \cup \mathbb{Z})$ and $\sigma_0^2 - \theta_0^2 \neq 0$. Then (V) admits a one-parameter family of solutions \( \{y_{\sigma_0}(\rho, x); \rho \in C\} \) with the following properties.

(i) We have that $y_{\sigma_0}(\rho, x)$ is holomorphic in $(\rho, x) \in \Omega^0(\varepsilon_0) \cup \Omega^-(\sigma_0, \varepsilon_0) \subset C \times \mathcal{R}(\mathbb{C} \setminus \{0\})$, where

\[
\Omega^0(\varepsilon_0) := \bigcup_{\rho \in \mathbb{C}} \{\rho\} \times \Omega^0_\rho(\varepsilon_0), \quad \Omega^-(\sigma_0, \varepsilon_0) := \bigcup_{\rho \in \mathbb{C} \setminus \{0\}} \{\rho\} \times \Omega^-_{\sigma_0, \rho}(\varepsilon_0),
\]

\[
\Omega^-_{\sigma_0, \rho}(\varepsilon_0) := \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}) \mid |x_0| < \varepsilon_0, |\rho x^\sigma_0| < \varepsilon_0\},
\]

\[
\varepsilon_0 = \varepsilon_0(\theta_0, \theta_x, \theta_\infty)
\]

being a sufficiently small number depending only on $(\theta_0, \theta_x, \theta_\infty)$.

(ii) We have that $y_{\sigma_0}(\rho, x)$ is represented by the convergent series as follows:

(a) if $\sigma_0 = \theta_0 + \theta_x$, then

\[
1 - \frac{1}{\theta_0 \theta_x^3} \rho x^\sigma_0 + \sum_{j \geq 2} c_j(\sigma_0)(\rho x^\sigma_0)^j + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c_{jn}(\sigma_0)(\rho x^\sigma_0)^j
\]

in $\Omega^0(\varepsilon_0)$, and

\[
1 - \frac{4\theta_0^2}{\sigma_0^2 - \theta_\infty^2} (\rho x^\sigma_0)^{-1} + \sum_{j \geq 2} c_j(\sigma_0)(\rho x^\sigma_0)^{-j} + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c_{jn}(\sigma_0)(\rho x^\sigma_0)^{n-j}
\]

in $\Omega^-(\sigma_0, \varepsilon_0)$;

(b) if $\sigma_0 = \theta_0 - \theta_x$, then

\[
\frac{\sigma_0 - \theta_\infty}{\sigma_0 + \theta_\infty} \left(1 - \frac{1}{\theta_0 \theta_x^3} \rho x^\sigma_0 + \sum_{j \geq 2} c_j(\sigma_0)(\rho x^\sigma_0)^j + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c_{jn}(\sigma_0)(\rho x^\sigma_0)^j\right)
\]

in $\Omega^0(\varepsilon_0)$, and

\[
1 - \frac{4\theta_0 \sigma_0}{\sigma_0^2 - \theta_\infty^2} (\rho x^\sigma_0)^{-1} + \sum_{j \geq 2} c_j(\sigma_0)(\rho x^\sigma_0)^{-j} + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c_{jn}(\sigma_0)(\rho x^\sigma_0)^{n-j}
\]

in $\Omega^-(\sigma_0, \varepsilon_0)$;

(c) if $\sigma_0 = \theta_x - \theta_0$, then

\[
\frac{\sigma_0 + \theta_\infty}{\sigma_0 - \theta_\infty} \left(1 - \frac{1}{\theta_0 \theta_x^3} \rho x^\sigma_0 + \sum_{j \geq 2} c_j(\sigma_0)(\rho x^\sigma_0)^j + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c_{jn}(\sigma_0)(\rho x^\sigma_0)^j\right)
\]

in $\Omega^0(\varepsilon_0)$, and

\[
1 - \frac{4\theta_x \sigma_0}{\sigma_0^2 - \theta_\infty^2} (\rho x^\sigma_0)^{-1} + \sum_{j \geq 2} c_j(\sigma_0)(\rho x^\sigma_0)^{-j} + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c_{jn}(\sigma_0)(\rho x^\sigma_0)^{n-j}
\]

in $\Omega^-(\sigma_0, \varepsilon_0)$.

Here $c_j(\sigma_0), c_{jn}(\sigma_0) \in \mathbb{Q}_0[\theta_0^{-1}, \theta_x^{-1}](\sigma_0)$ and $\tilde{c}_j(\sigma_0), \tilde{c}_{jn}(\sigma_0) \in \mathbb{Q}_0(\sigma_0)$. 

Remark 2.5. For each $\rho \neq 0$, $\Omega^0_0(\varepsilon_0)$ is given by

$$|x| < \varepsilon_0, \quad \text{Re } \sigma_0 \cdot \log|x| + \log|\rho| + \log(\varepsilon_0^{-1}) < \text{Im } \sigma_0 \cdot \arg x,$$

and $\Omega_{\sigma_0, \rho}(\varepsilon_0)$ by

$$(1 + \text{Re } \sigma_0) \log|x| + \log|\rho| + \log(\varepsilon_0^{-1}) < \text{Im } \sigma_0 \cdot \arg x$$

$$< \text{Re } \sigma_0 \cdot \log|x| + \log|\rho| - \log(\varepsilon_0^{-1}).$$

Remark 2.6. In $\Omega^0_0(\varepsilon_0)$, $\gamma_0(0, x)$ in each case is a Taylor series solution. If $\sigma_0 = \pm(\theta_0 - \theta_\infty)$, then $\gamma_0(0, x) = -((\theta_0 - \theta_\infty)/(\theta_0 - \theta_\infty + \Omega(x))$, and if $\sigma_0 = \theta_0 + \theta_\infty$, direct substitution into (V) yields $\gamma_0(0, x) = 1 + (1 - \theta_0 - \theta_\infty)\cdot x + O(x^2)$. Since $\sigma_0 \notin \mathbb{Z}$, the coefficients $c^0_{ij}(\sigma_0)$ of both solutions are uniquely determined, and they coincide with the solutions (II) and (III) in [16, Theorem 2], respectively.

The following are solutions of inverse logarithmic type, which correspond to the Chazy solutions of the sixth Painlevé equation [17].

**Theorem 2.4.** (1) Suppose that $\theta_\infty \neq 0$ and $\theta_0^2 - \theta_\infty^2 \neq 0$. Then (V) admits a one-parameter family of solutions $\{y^\log_\rho(\rho, x); \rho \in \mathbb{R}(\mathbb{C} \setminus \{0\})\}$ such that $y^\log_\rho(\rho, x)$ is holomorphic in $(\rho, x) \in \Omega^*\varepsilon_0(0, \Theta_0) \subset \mathbb{R}(\mathbb{C} \setminus \{0\})^2$ and is represented by the convergent series

$$y^\log_\rho(\rho, x) = 1 + \frac{4}{\theta_\infty(\theta_0 - \theta_\infty)} \log^{-2}(\rho x) + \sum_{j \geq 3} c_j \log^{-j}(\rho x)$$

$$+ \sum_{n=1}^\infty \sum_{j \geq 0} c_{jn} \log^{2n-j}(\rho x)$$

with $c_j, c_{jn} \in \mathbb{Q}[\theta_\infty^{-1}, \theta_0^2 - \theta_\infty^2]^{-1}$, where

$$\Omega^*\varepsilon_0(0, \Theta_0) := \bigcup_{\rho \in \mathbb{R}(\mathbb{C} \setminus \{0\})} \{\rho\} \times \Omega_\rho^*(\varepsilon_0, \Theta_0),$$

$\Theta_0$ being a given positive number and $\varepsilon_0 = \varepsilon_0(\Theta_0, \theta_0, \theta_\infty, \theta_\infty) \geq \theta_0$, a sufficiently small number depending only on $\Theta_0$ and $(\theta_0, \theta_\infty, \theta_\infty)$.

(2) Suppose that $\theta_\infty \neq 0$ and $\theta_0^2 - \theta_\infty^2 = 0$. If $\theta_0 = \theta_\infty \neq 0$ or if $\theta_0 = -\theta_\infty \neq 0$, then (V) admits one-parameter families of solutions $\{y^\log_\rho(\rho, x); \rho \in \mathbb{R}(\mathbb{C} \setminus \{0\})\}$ or $\{y^\log_\rho(\rho, x); \rho \in \mathbb{R}(\mathbb{C} \setminus \{0\})\}$, respectively, such that each solution is holomorphic in $(\rho, x) \in \Omega^*\varepsilon_0(0, \Theta_0)$ and is represented by the convergent series as follows:

(i) if $\theta_0 = \theta_\infty \neq 0$,

$$y^\pm_\log(\rho, x) = 1 \mp \frac{2}{\theta_\infty} \log^{-1}(\rho x) + \sum_{j \geq 2} c^\pm_j \log^{-j}(\rho x) + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c^\pm_{jn} \log^{n-j}(\rho x);$$

(ii) if $\theta_0 = -\theta_\infty \neq 0$,

$$y^\pm_\log(\rho, x) = 1 \mp \frac{2}{\theta_0 \theta_\infty} \log^{-2}(\rho x) + \sum_{j \geq 3} c^\pm_j \log^{-j}(\rho x) + \sum_{n=1}^\infty x^n \sum_{j \geq 0} c^\pm_{jn} \log^{n-j}(\rho x).$$

Here $c^\pm_j, c^\pm_{jn} \in \mathbb{Q}[\theta_0, \theta_0^{-1}, \theta_\infty, \theta_\infty^{-1}]$, in particular, in case $\theta_0 = \theta_\infty, c^\pm_j = 0$ for $j \geq 2$. 
(3) Suppose that $\theta_{\infty} = 0$. If $\theta_0^2 - \theta_x^2 \neq 0$ or if $\theta_0 = -\theta_x \neq 0$, then (V) admits one-parameter families of solutions $\{y^{(l)}_{\log}(\rho, x); \rho \in \mathcal{R}(\mathbb{C}\setminus\{0\}) \} (l = 1, 2)$ or $\{y^{(l)}_{\log}(\rho, x); \rho \in \mathcal{R}(\mathbb{C}\setminus\{0\}) \}$, respectively, such that each solution is holomorphic in $(\rho, x) \in \Omega^\ast(\varepsilon_0, \Theta_0)$ and is represented by the convergent series as follows:

(i) if $\theta_0^2 - \theta_x^2 \neq 0$,

$$y^{(l)}_{\log}(\rho, x) = 1 + \frac{(-1)^{l+1}2}{\theta_0 - \theta_x} \log^{-1}(\rho x) + \sum_{j=2}^{\infty} c_j^{(l)} \log^{-j}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j=0}^{\infty} c_{jn}^{(l)} \log^{2n-j}(\rho x);$$

(ii) if $\theta_0 = -\theta_x \neq 0$,

$$y^{(l)}_{\log}(\rho, x) = 1 + \frac{1}{\theta_0} \log^{-1}(\rho x) + \sum_{j=2}^{\infty} c_j^{(l)} \log^{-j}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j=0}^{\infty} c_{jn}^{(l)} \log^{2n-j}(\rho x).$$

Here $c_j^{(l)}, c_{jn}^{(l)} \in \mathbb{Q}[\theta_0, \theta_x, (\theta_0^2 - \theta_x^2)^{-1}] (l = 1, 2)$ and $c_j^{(1)}, c_{jn}^{(1)} \in \mathbb{Q}[\theta_0, \theta_0^{-1}]$, respectively.

Remark 2.7. Applying the Bäcklund transformation with $\pi$ in Lemma 6.1 to the inverse logarithmic solutions above except for $y^+_\log(\rho, x)$ with $\theta_0 = \theta_x$, we derive solutions of logarithmic type as follows: under the condition $\theta_0 + \theta_x \neq 1$, solutions satisfying

$$y_{\log}(\rho, x) \sim 1 - \frac{1}{2}(1 - \theta_0 - \theta_x)x \log^2(\rho x)$$

with $(1 - \theta_{\infty})(1 - \theta_0 + \theta_x) \neq 0$, with $\theta_{\infty} = 1$, $\theta_0 - \theta_x \neq 1$ and with $\theta_{\infty} \neq 1$, $\theta_0 - \theta_x = 1$ follow from $y_{\log}(\rho, x)$, from $y_{\log}^{-}(\rho, x)$ with $\theta_0 = \theta_x \neq 0$ and from $y^+_{\log}(\rho, x)$ with $\theta_0 = -\theta_x \neq 0$, respectively; and under the condition $\theta_0 + \theta_x = 1$, those satisfying

$$y^{(l)}_{\log}(\rho, x) \sim 1 + (-1)^{l+1}x \log(\rho x) \quad (l = 1, 2)$$

with $(1 - \theta_{\infty})(1 - \theta_0 + \theta_x) \neq 0$ and with $\theta_{\infty} \neq 1$, $\theta_0 - \theta_x = 1$ follow from $y^{(l)}_{\log}(\rho, x)$ $(l = 1, 2)$ with $\theta_0^2 - \theta_x^2 \neq 0$ and from $y^{(1)}_{\log}(\rho, x)$ with $\theta_0 = -\theta_x \neq 0$, respectively. These logarithmic solutions are studied in [24]. For the exceptional case of $y^+_{\log}(\rho, x)$ the denominator of the Bäcklund transformation is of the form $x(\cdots)$, and to compute the resultant solution we need to know some of the coefficients $c_{jn}^+.$

Theorem 2.5. Suppose that $\theta_{\infty} = 0$. If $\theta_0 = \theta_x$ or if $\theta_0 = -\theta_x$, then (V) has a one-parameter family of solutions $\{y^+_{\text{Taylor}}(a, x); a \in \mathbb{C}\setminus\{0\}\}$ or $\{y^-_{\text{Taylor}}(a, x); a \in \mathbb{C}\}$, respectively, represented by the convergent series

$$y_{\text{Taylor}}^{\pm}(a, x) = \sum_{n=0}^{\infty} c_n^{\pm}(a)x^n$$

with $c_0^{\pm}(a) = (a + \theta_0)/a$, $c_1^{\pm}(a) = (a + \theta_0)(1 - 2\theta_0)/a$, $c_n^{\pm}(a) \in \mathbb{Q}[a, a^{-1}, \theta_0]$, or $c_0^-(a) = c_1^-(a) = 1$, $c_2^-(a) = (1 - \theta_0 - 2a)/2$, $c_n^-(a) \in \mathbb{Q}[a, \theta_0]$ If $\theta_0 = \theta_x = 0$, then $c_n^+(a) = c_n^-(a)$ for every $n \geq 0.$
Remark 2.8. The formal series solutions around $x = 0$ by [4, 19], except for some cases of logarithmic type and of special complex power type, may be identified with our critical behaviours or special cases of them. The author believes that the exceptional cases are also derived by suitable Bäcklund transformations (cf. [24, Remark 2.3]).

2.2. Analytic continuation

Suppose that $(\sigma^2 - (\theta_0 \pm \theta_x)^2)(\sigma^2 - \theta_\infty^2) \neq 0$ for every $\sigma \in \text{cl}(\Sigma_0)$. Let us discuss the analytic continuation of $y(\sigma, \rho, x)$ on $R(C \setminus \{0\})$ around $x = 0$. If $\sigma \in \Sigma_0$ satisfies $0 < \sigma < 1$ (respectively, $-1 < \sigma < 0$), then the solution $y_+(\sigma, \rho, x)$ (respectively, $y_-(\sigma, \rho, x)$) in Theorem 2.1 converges in $\{x \in R(C \setminus \{0\}): |x| < \epsilon_0^0\}$ for some $\epsilon_0' > 0$ (in fact $y_+(\sigma, \rho, x) \equiv y_-(-\sigma, \rho^{-1}, x)$).

In what follows suppose that $(\sigma, \rho) \in \Sigma_0 \times (C \setminus \{0\})$ satisfies $\text{Im } \sigma \neq 0$. For $\nu \in \mathbb{Z}$ let $D_{\pm}(\sigma, \rho, \nu) \subset R(C \setminus \{0\})$ be domains given by

\[
D_+(\sigma, \rho, \nu): \begin{cases}
(\text{Re } \sigma - 2\nu) \log|x| + \log|\rho| + \log(\epsilon_0^{-1}) < \text{Im } \sigma \cdot \arg x \\
< (\text{Re } \sigma - 2\nu - 1) \log|x| + \log|\rho| - \log(\epsilon_0^{-1})
\end{cases}
\]

\[
D_-(\sigma, \rho, \nu): \begin{cases}
(\text{Re } \sigma - 2\nu + 1) \log|x| + \log|\rho| + \log(\epsilon_0^{-1}) < \text{Im } \sigma \cdot \arg x \\
< (\text{Re } \sigma - 2\nu) \log|x| + \log|\rho| - \log(\epsilon_0^{-1})
\end{cases}
\]

Furthermore, set

\[
D_{\text{even}}(\sigma, \rho, \nu): |x| < \epsilon_0,
\]

\[
-\log(\epsilon_0^{-1}) < (\text{Re } \sigma - 2\nu) \log|x| - \text{Im } \sigma \cdot \arg x + \log|\rho| < \log(\epsilon_0^{-1}),
\]

\[
D_{\text{odd}}(\sigma, \rho, \nu): |x| < \epsilon_0,
\]

\[
-\log(\epsilon_0^{-1}) < (\text{Re } \sigma - 2\nu + 1) \log|x| - \text{Im } \sigma \cdot \arg x + \log|\rho| < \log(\epsilon_0^{-1})
\]

(cf. Figure 1). In general these are spiral domains, and

\[
\bigcup_{\nu \in \mathbb{Z}} (\text{cl}(D_{\text{odd}}(\sigma, \rho, \nu)) \cup D_-(\sigma, \rho, \nu) \cup \text{cl}(D_{\text{even}}(\sigma, \rho, \nu)) \cup D_+(\sigma, \rho, \nu))
\]

contains $\{x \in R(C \setminus \{0\}): |x| < \epsilon_0^2\}$. For every $\nu \in \mathbb{Z}$, by Theorem 2.1, $y(\sigma - 2\nu, \rho, x)$ with $\sigma \in \Sigma_0$ is represented by $y_+(\sigma - 2\nu, \rho, x)$ in $D_+(\sigma, \rho, \nu)$ and by $y_-(\sigma - 2\nu, \rho, x)$ in $D_-(\sigma, \rho, \nu)$, as long as $((\sigma - 2\nu)^2 - (\theta_0 \pm \theta_x)^2)((\sigma - 2\nu)^2 - \theta_\infty^2) \neq 0$ for $\sigma \in \text{cl}(\Sigma_0)$.

Set

\[
c(\sigma) := \frac{4\sigma^2(\theta_0 + \theta_x - \sigma)}{(\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)}, \quad (2.3)
\]

To $c(\sigma)$ apply the substitution

\[
\pi: (\theta_0 - \theta_x, \theta_0 + \theta_x, \theta_\infty) \mapsto (1 - \theta_\infty, 1 - \theta_0 + \theta_x, \theta_0 + \theta_x - 1) \quad (2.4)
\]

and denote the result by $c^{\pi}(\sigma) = \tilde{c}(\sigma)$. Then we have the following relation, which gives the analytic continuation of $y(\sigma, \rho, x)$ on $\{x \in R(C \setminus \{0\}): |x| < \epsilon_0^2\}$. 

The fifth Painlevé equation 147
148 S. Shimomura

\[ D_{\text{odd}}(\sigma, \rho, \nu + 2) \]
\[ D_+(\sigma, \rho, \nu + 1) \]
\[ D_{\text{even}}(\sigma, \rho, \nu + 1) \]
\[ D_-(\sigma, \rho, \nu + 1) \]
\[ D_{\text{odd}}(\sigma, \rho, \nu + 1) \]
\[ D_+(\sigma, \rho, \nu) \]
\[ D_{\text{even}}(\sigma, \rho, \nu) \]
\[ D_-(\sigma, \rho, \nu) \]
\[ D_{\text{odd}}(\sigma, \rho, \nu) \]
\[ D_+(\sigma, \rho, \nu - 1) \]
\[ D_{\text{even}}(\sigma, \rho, \nu - 1) \]
\[ D_-(\sigma, \rho, \nu - 1) \]

**Figure 1.** $\log |x|$ and $\text{Im} \sigma \cdot \text{arg} \ x$ for each domain.

**Theorem 2.6.** For every $\nu \in \mathbb{Z}$,
\[ y(\sigma - 2\nu + 2, \rho, x) \equiv y(\sigma - 2\nu, \gamma(\sigma, \nu)\rho, x) \]

with
\[ \gamma(\sigma, \nu) = \frac{1}{4}(2\nu - \theta_{\infty} - \sigma)(\sigma - \theta_{\infty} - 2\nu + 2)c(2\nu - \sigma)c(\sigma - 2\nu + 2) \]
\[ \times \tilde{c}(\sigma - 2\nu + 1)\tilde{c}(2\nu - 1 - \sigma) \]
\[ = \frac{64(\sigma - 2\nu)^2(\sigma - 2\nu + 1)^4(\sigma - 2\nu + 2)^2}{(\theta_{\infty} - \sigma + 2\nu)(\theta_{\infty} + \sigma - 2\nu + 2)((\sigma - 2\nu - \theta_{\infty})^2 - \theta_{0}^2)((\sigma - 2\nu + 2 + \theta_{\infty})^2 - \theta_{0}^2)} \]
as long as $\gamma(\sigma, \nu) \neq 0, \infty$ for $\sigma \in \text{cl}(\Sigma_0)$.

2.3. **Distribution of poles, zeros and 1-points**

If $\theta_0 - \theta_1 = \theta_{\infty} = 0$ and $\text{Im} \sigma \neq 0$, then, by Remark 2.4, (2.1) in $D_{\text{even}}(\sigma, \tilde{\rho}, 0)$ has sequences of zeros and of poles lying asymptotically along the curve $\log|\tilde{\rho}x^\sigma| = \text{Re} \sigma \cdot \log|x| - \text{Im} \sigma \cdot \text{arg} \ x + \log|\tilde{\rho}| = 0$. Similarly, (2.2) in $D_{\text{odd}}(\sigma, \tilde{\rho}, 0)$ has sequences of 1-points asymptotically along the curve $\log|\tilde{\rho}x^{\sigma + 1}| = (\text{Re} \sigma + 1) \cdot \log|x| - \text{Im} \sigma \cdot \text{arg} \ x + \log|\tilde{\rho}| = 0$. These facts are generalized by the following theorems, in which
\[ L(r_0, \omega) = (1 + \text{Re} \sigma - \omega) \log|x| - \text{Im} \sigma \cdot \text{arg} x = r_0, \quad r_0, \omega \in \mathbb{R} \] (2.5)
is a spiral curve if $\text{Re} \sigma \neq \omega - 1$ or a ray if $\text{Re} \sigma = \omega - 1$. 
The fifth Painlevé equation

**THEOREM 2.7.** In addition to \((\sigma^2 - \theta_\infty^2)(\sigma^2 - (\theta_0 \pm \theta_x)^2) \neq 0\), suppose that
\[
\theta_x(\theta_0 + \theta_x - \theta_\infty)(\theta_0^2 - \theta_x^2 + \sigma^2 - 2\theta_0\theta_\infty) \neq 0
\]
for \(\sigma \in \text{cl}(\Sigma_0)\) and
\[
\theta_0 - \theta_x - \theta_\infty \neq 0.
\] (2.6)

Set
\[
r_0 := \log|\xi_0|, \quad \mu_0 := \arg \xi_0 \quad \text{with} \quad \xi_0 := \frac{-\sigma + \theta_0 + \theta_x}{\sigma - \theta_0 - \theta_x} \rho^{-1}.
\]

Then \(y(\sigma, \rho, x)\) admits a sequence of simple zeros \(\{\hat{x}_n^0\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma, \rho, 0)\) such that
\[
|\sigma|^2 \log|x_n^0| - r_0 \Re \sigma - \mu_0 \Im \sigma \sim -2\pi n |\Im \sigma|
\]
and \(\text{dist}(x_n^0, L(r_0, 1)_\sigma) = O(|x_n^0|^2)\). Furthermore, for
\[
\hat{\xi}_0 := -\frac{(\sigma + \theta_\infty)((\sigma + \theta_x)^2 - \theta_0^2)}{(\sigma - \theta_\infty)((\sigma - \theta_x)^2 - \theta_0^2)} \rho^{-1}
\]
there exists another sequence of simple zeros \(\{\hat{\xi}_n^0\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma, \rho, 0)\) with similar properties, that is,
\[
|\sigma|^2 \log|\hat{\xi}_n^0| - \hat{r}_0 \Re \sigma - \hat{\mu}_0 \Im \sigma \sim -2\pi n |\Im \sigma|
\]
and \(\text{dist}(\hat{\xi}_n^0, L(\hat{r}_0, 1)_\sigma) = O(|\hat{\xi}_n^0|^2)\), where \(\hat{r}_0 := \log|\hat{\xi}_0|\), \(\hat{\mu}_0 := \arg \hat{\xi}_0\).

**THEOREM 2.8.** In addition to \((\sigma^2 - \theta_\infty^2)(\sigma^2 - (\theta_0 \pm \theta_x)^2) \neq 0\), suppose that
\[
\theta_0(\theta_0 + \theta_x - \theta_\infty)(\theta_0^2 - \theta_x^2 - \sigma^2 + 2\theta_x\theta_\infty) \neq 0
\]
for \(\sigma \in \text{cl}(\Sigma_0)\) and
\[
\theta_0 - \theta_x + \theta_\infty \neq 0.
\] (2.7)

Set
\[
r_\infty := \log|\xi_\infty|, \quad \mu_\infty := \arg \xi_\infty \quad \text{with} \quad \xi_\infty := \frac{(\sigma + \theta_x)^2 - \theta_0^2}{(\sigma - \theta_x)^2 - \theta_0^2} \rho^{-1}.
\]

Then \(y(\sigma, \rho, x)\) admits a sequence of simple poles \(\{x_\infty^0\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma, \rho, 0)\) such that
\[
|\sigma|^2 \log|x_\infty^0| - r_\infty \Re \sigma - \mu_\infty \Im \sigma \sim -2\pi n |\Im \sigma|
\]
and \(\text{dist}(x_\infty^0, L(r_\infty, 1)_\sigma) = O(|x_\infty^0|^2)\). Another similar sequence of simple poles \(\{\hat{x}_\infty^0\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma, \rho, 0)\) exists for
\[
\hat{\xi}_\infty := -\frac{(\sigma + \theta_\infty)(\sigma + \theta_0 + \theta_x)}{(\sigma - \theta_\infty)(\sigma - \theta_0 - \theta_x)} \rho^{-1}.
\]

**THEOREM 2.9.** If \(\theta_0 - \theta_x - \theta_\infty = 0\) in place of (2.6), then \(\{x_n^0\}_{n \in \mathbb{N}} = \{\hat{x}_n^0\}_{n \in \mathbb{N}}\) is a sequence of double zeros. If \(\theta_0 - \theta_x + \theta_\infty = 0\) in place of (2.7), then \(\{x_n^\infty\}_{n \in \mathbb{N}} = \{\hat{x}_n^\infty\}_{n \in \mathbb{N}}\) is a sequence of double poles.

Note that the singular values of (V) are \(y = 0, 1, \infty\). The results above describe sequences of zeros and of poles in \(D_{\text{even}}(\sigma, \rho, 0)\). In \(D_{\text{odd}}(\sigma, \rho, 0)\) there exist sequences of 1-points.
Theorem 2.10. In addition to \((\sigma^2 - \theta_\infty^2)(\sigma^2 - (\theta_0 \pm \theta_x)^2) \neq 0\), suppose that
\[
(\theta_0 - 1)(\theta_0 + \theta_x - \theta_\infty)((2 - \theta_0 - \theta_\infty)^2 - \theta_x^2)((\sigma + 1)^2 - (\theta_0 \pm \theta_x - 1)^2)
\]
\[
\times ((\sigma + 1)^2 - (1 - \theta_\infty)^2)((\sigma + 1)^2 + (1 - \theta_0)^2 + 2(1 - \theta_\infty)(1 - \theta_0 - \theta_\infty^2) \neq 0 \ (2.8)
\]
for \(\sigma \in \text{cl}(\Sigma_0)\). Set
\[
\xi_1 := \log|\xi_1|, \quad \mu_1 := \arg\xi_1
\]
with
\[
\xi_1 := \frac{(\sigma + 2 - \theta_0 + \theta_x)(\sigma + \theta_\infty)c(-\sigma)c(\sigma + 1)}{2(\sigma + \theta_0 - \theta_x)} \rho^{-1} = \frac{8\sigma^2(\sigma + 1)^2}{(\theta_\infty - \sigma)(\theta_0^2 - (\theta_x + \sigma)^2)} \rho^{-1}.
\]
Then \(y(\sigma, \rho, x)\) admits a sequence of simple 1-points \(\{x_n^1\}_{n \in \mathbb{N}} \subset D_{\text{odd}}(\sigma, \rho, 0)\) such that
\[
|\sigma + 1|^2 \log|x_n^1| - r_1(\Re \sigma + 1) - \mu_1 \Im \sigma \sim -2\pi n|\Im \sigma|
\]
and \(\text{dist}(x_n^1, L(r_1, 0)\sigma) = O(|x_n^1|^2)\). Another similar sequence of simple 1-points \(\{\hat{x}_n^1\}_{n \in \mathbb{N}} \subset D_{\text{odd}}(\sigma, \rho, 0)\) exists for
\[
\hat{\xi}_1 := -\frac{\sigma + \theta_0 + \theta_x}{\sigma + 2 - \theta_0 - \theta_x}\xi_1.
\]
Remark 2.9. In the proof of Theorem 2.10 in Section 7, if we use (6.2) instead of (6.1), we obtain, for
\[
\hat{\xi}_1 := -\frac{2(\sigma - \theta_0 + \theta_x)}{(\sigma - 2 + \theta_0 - \theta_x)(\sigma - \theta_\infty)c(\sigma)c(1 - \sigma)} \rho^{-1} = \frac{(\theta_\infty + \sigma)(\theta_0^2 - (\theta_x + \sigma)^2)}{8\sigma^2(\sigma - 1)^2} \rho^{-1},
\]
a sequence of simple 1-points such that
\[
|\sigma - 1|^2 \log|x_n^1| - r_1(\Re \sigma - 1) - \mu_1 \Im \sigma \sim -2\pi n|\Im \sigma|
\]
and \(\text{dist}(x_n^1, L(r_1, 2)\sigma) = O(|x_n^1|^2)\), and a similar sequence for
\[
\hat{\xi}_1 := \frac{\sigma - 2 + \theta_0 + \theta_x}{\sigma - \theta_0 - \theta_x}\xi_1.
\]
Remark 2.10. Using Theorems 2.7–2.10 combined with the relation of Theorem 2.6, we may find sequences of zeros, of poles and of 1-points beyond \(\text{cl}(D_{\text{odd}}(\sigma, \rho, 0)) \cup D_-(\sigma, \rho, 0) \cup \text{cl}(D_{\text{even}}(\sigma, \rho, 0)) \cup D_+(\sigma, \rho, 0)\). As a result \(D_{\text{even}}(\sigma, \rho, v)\) contains sequences of zeros and of poles, and \(D_{\text{odd}}(\sigma, \rho, v)\) those of 1-points. They are lying asymptotically along the respective spiral curves or rays.

For the solutions of special complex power type we have the following.

Theorem 2.11. Under the same supposition as in Theorem 2.3 with \(\rho \neq 0\), set
\[
r_0^* := \log|\xi_0^*|, \quad \mu_0^* := \arg\xi_0^* \quad \text{with} \quad \xi_0^* := \frac{2\theta_0}{\sigma_0 + \theta_\infty} \rho^{-1},
\]
and
\[
\hat{\xi}_0^* := \frac{2\theta_x}{\sigma_0 - \theta_\infty} \rho^{-1}, \quad \hat{\xi}_\infty^* := -\frac{2\theta_x}{\sigma_0 + \theta_\infty} \rho^{-1}, \quad \hat{\xi}_\infty^* := -\frac{2\theta_0}{\sigma_0 - \theta_\infty} \rho^{-1}.
\]
The author believes that, in Remark 2.11. those in (2.2) with the 1-points given by Theorem 2.10) has no zeros, poles or 1-points other than indeed, solution (2.1) with the double zeros and poles given by Theorem 2.9 (respectively, excluding the possibility of its existence with the exception of the cases of \( \sigma \) odd
\[
|\sigma_0|^2 \log |x_n^{*0}| - r_0^* \Re \sigma_0 - \mu_0^* \Im \sigma_0 \sim -2\pi n |\Im \sigma_0|
\]
and \( \text{dist}(x_n^{*0}, L(r_0^*, 1)_{\zeta_0}) = O(|x_n^{*0}|^2) \), and a similar sequence of simple zeros \( \{ \hat{x}_n^{*0} \}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma_0, \rho, 0) \) for \( \hat{\xi}_0^* \). Under (2.7), there exist sequences of simple poles \( \{ x_n^{*\infty} \}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma_0, \rho, 0) \) for \( \xi_0^{*\infty} \) and \( \hat{\xi}_0^{*\infty} \), respectively. If \( \theta_0 - \theta_\infty = 0 \), then \( x_n^{*0} = \hat{x}_n^{*0} \) \( (n \in \mathbb{N}) \) are double zeros; and if \( \theta_0 - \theta_\infty = 0 \), then \( x_n^{*\infty} = \hat{x}_n^{*\infty} \) \( (n \in \mathbb{N}) \) are double poles.

(2) If \( \sigma_0 = \theta_0 - \theta_\infty \) and \( \theta_0 + \theta_\infty \neq \theta_\infty \), then \( y_{\sigma_0}(\rho, x) \) admits sequences of simple zeros \( \{ x_n^{*0} \}_{n \in \mathbb{N}} \) and of simple poles \( \{ \hat{x}_n^{*\infty} \}_{n \in \mathbb{N}} \) as in (1).

(3) If \( \sigma_0 = \theta_\infty \) and \( \theta_0 + \theta_\infty \neq \theta_\infty \), then \( y_{\sigma_0}(\rho, x) \) admits sequences of simple zeros \( \{ x_n^{*0} \}_{n \in \mathbb{N}} \) and of simple poles \( \{ \hat{x}_n^{*\infty} \}_{n \in \mathbb{N}} \) as in (1).

**Theorem 2.12.** In addition to the supposition of Theorem 2.3, suppose (2.8) with \( \sigma = \sigma_0 \).
Set
\[
r_1^* := \log |\xi_1^*|, \quad \mu_1^* := \arg \xi_1^* \quad \text{with} \quad \xi_1^* := \frac{2(\sigma_0 + 1)^2 c^*(\sigma_0)}{\sigma_0 + \theta_0 + \theta_\infty} \rho^{-1},
\]
where
\[
c^*(\sigma_0) := \begin{cases} 
\frac{4\sigma_0^2}{\sigma_0 - \theta^2_\infty} & \text{if } \sigma_0 = \theta_0 + \theta_\infty, \\
\frac{4\theta_0 \sigma_0}{\sigma_0^2 - \theta^2_\infty} & \text{if } \sigma_0 = \theta_0 - \theta_\infty, \\
\frac{4\theta_\infty \sigma_0}{\sigma_0^2 - \theta^2_\infty} & \text{if } \sigma_0 = \theta_\infty - \theta_0.
\end{cases}
\]
Then \( y_{\sigma_0}(\rho, x) \) admits a sequence of simple 1-points \( \{ x_n^{*1} \}_{n \in \mathbb{N}} \subset D_{\text{odd}}(\sigma_0, \rho, 0) \) such that
\[
|\sigma_0 + 1|^2 \log |x_n^{*1}| - r_1^*(\Re \sigma_0 + 1) - \mu_1^* \Im \sigma_0 \sim -2\pi n |\Im \sigma_0|
\]
and \( \text{dist}(x_n^{*1}, L(r_1^*, 0)_{\zeta_0}) = O(|x_n^{*1}|^2) \). Another similar sequence of simple 1-points \( \{ \hat{x}_n^{*1} \}_{n \in \mathbb{N}} \subset D_{\text{odd}}(\sigma_0, \rho, 0) \) exists for
\[
\hat{\xi}_1^* := \frac{-\sigma_0 + \theta_0 + \theta_\infty}{\sigma_0 + 2 - \theta_0 - \theta_\infty} \xi_1^*.
\]
**Remark 2.11.** The author believes that, in \( D_{\text{even}}(\cdots) \) and \( D_{\text{odd}}(\cdots) \), there exists no sequence of zeros, of poles or of 1-points other than those given in Theorems 2.7–2.12. Indeed, solution (2.1) with the double zeros and poles given by Theorem 2.9 (respectively, (2.2) with the 1-points given by Theorem 2.10) has no zeros, poles or 1-points other than those in \( D_{\text{even}}(\sigma, \rho, 0) \) (respectively, in \( D_{\text{odd}}(\sigma, \rho, 0) \)). However, we have not succeeded in excluding the possibility of its existence with the exception of the cases of \( y_{\sigma_0}(\rho, x) \) with \( \sigma_0 = \theta_0 + \theta_\infty \) in \( D_{\text{even}}(\sigma_0, \rho, 0) \) and of \( y(\sigma, \rho, x) \) with \( \theta_0 - \theta_\infty = \theta_\infty = 0 \).
2.4. Monodromy data

Linear system (1.1) given by

$$\frac{dY}{d\lambda} = \left( \frac{A_0(x)}{\lambda} + \frac{A_x(x)}{\lambda - x} + \frac{J}{2} \right) Y$$

with the properties (a) and (b) admits a fundamental matrix solution of the form

$$Y(\lambda, x) = (I + O(\lambda^{-1})) e^{(\lambda/2)J} \lambda^{-(\theta_\infty/2)J}$$

as \(\lambda \to \infty\) through the sector \(-\pi/2 < \arg \lambda < 3\pi/2\). Other matrix solutions \(Y_1(\lambda, x), Y_2(\lambda, x)\) with the same asymptotic representation through the sectors \(-3\pi/2 < \arg \lambda < \pi/2, \pi/2 < \arg \lambda < 5\pi/2\), respectively, are uniquely determined. According to [2] define the Stokes multipliers \(S_1 = I + s_1 \Delta_1\) and \(S_2 = I + s_2 \Delta_1\) by

$$Y(\lambda, x) = Y_1(\lambda, x) S_1, \quad Y_2(\lambda, x) = Y(\lambda, x) S_2,$$

and let \(M_0, M_x, M_\infty\) be monodromy matrices with respect to \(Y(\lambda, x)\) such that \(M_0, M_x, M_\infty\) are given by loops surrounding \(\lambda = 0, x, \infty\), respectively, in the positive sense, and that \(M_\infty M_x M_0 = I\) (cf. Figure 2). Observing that

$$Y_2(e^{2\pi i \lambda}, x) = (I + O(\lambda^{-1})) e^{(\lambda/2)J} \lambda^{-(\theta_\infty/2)J} e^{-\pi i \theta_\infty J}$$

if \(\pi/2 < \arg(e^{2\pi i \lambda}) < 5\pi/2\), i.e. \(-3\pi/2 < \arg \lambda < \pi/2\), and that

$$Y_1(\lambda, x) = (I + O(\lambda^{-1})) e^{(\lambda/2)J} \lambda^{-(\theta_\infty/2)J}$$

in the same sector, we have \(Y_1(\lambda, x) = Y_2(e^{2\pi i \lambda}, x) e^{\pi i \theta_\infty J}\), and hence

$$Y(\lambda, x) S_1^{-1} = Y(e^{2\pi i \lambda}, x) S_2 e^{\pi i \theta_\infty J} = Y(\lambda, x) M_x M_0 S_2 e^{\pi i \theta_\infty J}.$$

This yields \(M_\infty = M_0^{-1} M_x^{-1} = S_2 e^{\pi i \theta_\infty J} S_1\).

![Figure 2. Loops for \(M_0, M_x\) and \(M_\infty\).](image)

Then the isomonodromy deformation of (1.1) preserving the monodromy data \(M_0, M_x, M_\infty, S_1, S_2\) under a small change of \(x\), is controlled by the Schlesinger equation

$$x \frac{dA_0}{dx} = [A_x, A_0], \quad x \frac{dA_x}{dx} = [A_0, A_x] + \frac{x}{2} [J, A_x]$$

(2.10)
that is equivalent to (V) through
\[ y(x) = \frac{A_x(x)_{12}(A_0(x)_{11} + \theta_0/2)}{A_0(x)_{12}(A_x(x)_{11} + \theta_x/2)} \] (2.11)

(for details see [15]).

The following results give the monodromy data \( M_0, M_x \) related to each solution of (V) by the correspondence described above. Note that \( S_1 \) and \( S_2 \) are given by \( (M_x M_0)_{21} = -e^{-\pi i \theta_0} S_1, (M_x M_0)_{12} = -e^{-\pi i \theta_0} S_2 \) and that
\[ \text{tr}(M_x M_0) = 2 \cos \pi \theta_\infty + e^{-\pi i \theta_\infty} S_1 S_2. \]

**THEOREM 2.13.** Suppose that \( \theta_0, \theta_x \not\in \mathbb{Z} \). Then the monodromy data for \( y(\sigma, \rho, x) \) of Theorem 2.1 are given by
\[ M_0 = (C_0 C_\infty)^{-1} e^{\pi i \theta_0 J} C_0 C_\infty, \quad M_x = (C_x C_\infty)^{-1} e^{\pi i \theta_x J} C_x C_\infty, \]
where
\[ C_\infty = \begin{pmatrix} -e^{-\pi i (\sigma + \theta_\infty)/2} \Gamma(-\sigma) & \Gamma(-\sigma) \\ \Gamma(1 - (\sigma - \theta_\infty)/2) & \Gamma(1 - (\sigma + \theta_\infty)/2) \end{pmatrix}, \]
\[ C_0 = \tilde{C}_0 \begin{pmatrix} 1 & 0 \\ 0 & 2 \rho/(\sigma + \theta_\infty) \end{pmatrix}, \quad C_x = \tilde{C}_1 \begin{pmatrix} 1 & 0 \\ 0 & 2 \rho/(\sigma + \theta_\infty) \end{pmatrix}, \]
with
\[ \tilde{C}_0 = \begin{pmatrix} e^{\pi i (\sigma - \theta_0 + \theta_x)/2} \Gamma(1 - \sigma) \Gamma(-\theta_0) & e^{-\pi i (\sigma + \theta_0 - \theta_x)/2} \Gamma(1 + \sigma) \Gamma(-\theta_0) \\ e^{\pi i (\sigma + \theta_0 + \theta_x)/2} \Gamma(1 - \sigma) \Gamma(\theta_0) & e^{-\pi i (\sigma - \theta_0 + \theta_x)/2} \Gamma(1 + \sigma) \Gamma(\theta_0) \end{pmatrix}, \]
\[ \tilde{C}_1 = \begin{pmatrix} \Gamma(1 - \sigma) \Gamma(-1 - \theta_x) & \Gamma(1 + \sigma) \Gamma(-1 - \theta_x) \\ \Gamma(-\sigma + \theta_0 + \theta_x) \Gamma(1 + \theta_x) & \Gamma(1 + \sigma) \Gamma(1 + \theta_x) \end{pmatrix}, \]
\[ (C_0^{\sigma_0})_{ij} := (\tilde{C}_0)_{ij} |_{\sigma = \sigma_0}, \quad (C_1^{\sigma_0})_{ij} := (\tilde{C}_1)_{ij} |_{\sigma = \sigma_0}, \quad C_\infty^{\sigma_0} := C_\infty |_{\sigma = \sigma_0}. \]

Then the monodromy data for \( y(\sigma, \rho, x) \) of Theorem 2.3 are given by
\[ M_0 = (C_0^* C_\infty^{\sigma_0})^{-1} e^{\pi i \theta_0 J} C_0^* C_\infty^{\sigma_0}, \quad M_x = (C_x^* C_\infty^{\sigma_0})^{-1} e^{\pi i \theta_x J} C_x^* C_\infty^{\sigma_0} \]
with
\[ C_0^* = \tilde{C}_0^* \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, \quad C_x^* = \tilde{C}_1^* \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \]
Here \( \tilde{C}_0^* \) and \( \tilde{C}_1^* \) are as follows:
(i) if $\sigma_0 = \theta_0 + \theta_x$,

$$
\tilde{C}_0^* = \begin{pmatrix}
  (\tilde{C}_{0^0})_{11} & (\tilde{C}_{0^0})_{12} \\
  0 & (\tilde{C}_{0^0})_{22}
\end{pmatrix},
\tilde{C}_1^* = \begin{pmatrix}
  (\tilde{C}_{1^0})_{11} & (\tilde{C}_{1^0})_{12} \\
  0 & (\tilde{C}_{1^0})_{22}
\end{pmatrix}
$$

with

$$
(\tilde{C}_{0^0})_{12} = -\frac{\sigma_0 e^{-\pi i \theta_0}}{\theta_0 \Gamma(1 + \theta_x)}, \quad (\tilde{C}_{1^0})_{12} = -\frac{\sigma_0 \Gamma(1 + \sigma_0) \Gamma(-1 - \theta_x)}{\Gamma(1 + \theta_0)};
$$

(ii) if $\sigma_0 = \theta_0 - \theta_x$,

$$
\tilde{C}_0^* = \begin{pmatrix}
  (\tilde{C}_{0^0})_{11} & (\tilde{C}_{0^0})_{12}/\theta_x \\
  0 & (\tilde{C}_{0^0})_{22}
\end{pmatrix},
\tilde{C}_1^* = \begin{pmatrix}
  0 & (\tilde{C}_{1^0})_{12}/\theta_x \\
  0 & (\tilde{C}_{1^0})_{22}
\end{pmatrix}
$$

with

$$
(\tilde{C}_{0^0})_{22} = \frac{e^{\pi i \theta_0} \Gamma(1 + \sigma_0) \Gamma(\theta_0)}{\Gamma(1 + \theta_x)}, \quad (\tilde{C}_{1^0})_{12} = -\frac{\sigma_0 \Gamma(1 + \sigma_0) \Gamma(-1 - \theta_x)}{\Gamma(1 - \theta_0)}.
$$

Let $\psi(x) = \Gamma'(x)/\Gamma(x)$.

**Theorem 2.15.** Suppose that $\theta_0, \theta_x \notin \mathbb{Z}$. Then the monodromy data for $y_{\log}(\rho, x)$ or $y_{\log}^{(l)}(\rho, x)$ ($l = 1, 2$) of Theorem 2.4 with $\theta_0^2 - \theta_x^2 \neq 0$ are given by

$$
M_0 = (C_0^- C_\infty)^{-1} e^{\pi i \theta_0} J C_0^- C_\infty, \quad M_\infty = (C_\infty^- C_\infty)^{-1} e^{\pi i \theta_x} J C_\infty^- C_\infty,
$$

those for $y_{\log}^{\pm}(\rho, x)$ with $\theta_0^2 - \theta_x^2 = 0$ by

$$
M_0 = (C_0^\pm C_\infty)^{-1} e^{\pi i \theta_0} J C_0^\pm C_\infty, \quad M_\infty = (C_\infty^\pm C_\infty)^{-1} e^{\pi i \theta_x} J C_\infty^\pm C_\infty,
$$

and those for $y_{\log}^{(1)}(\rho, x)$ with $\theta_0 = -\theta_x \neq 0$ by

$$
M_0 = (C_0^+ C_\infty)^{-1} e^{\pi i \theta_0} J C_0^+ C_\infty, \quad M_\infty = (C_\infty^+ C_\infty)^{-1} e^{\pi i \theta_x} J C_\infty^+ C_\infty,
$$

where

$$
C_0^\pm = \tilde{C}_0^\pm \rho_0^{-\Delta}, \quad C_\infty^\pm = \tilde{C}_\infty^\pm \rho_0^{-\Delta}
$$

with $\rho_0 = \rho \exp(-2\theta_x(\theta_0^2 - \theta_x^2)^{-1})$ if $\theta_0^2 - \theta_x^2 \neq 0$, $\rho_0 = \rho$ otherwise. The matrices $C_\infty^\pm, \tilde{C}_0^\pm, \tilde{C}_\infty^\pm$ are as follows:

(i) if $\theta_\infty \neq 0$, then

$$
C_\infty = \begin{pmatrix}
  e^{-\pi i \theta_\infty/2}(\psi(1 + \theta_\infty/2) - 2\psi(1) - \pi i) & \psi(-\theta_\infty/2) - 2\psi(1) \\
  \Gamma(1 + \theta_\infty/2) e^{-\pi i \theta_\infty/2} & \Gamma(1 - \theta_\infty/2)
\end{pmatrix}
$$

and if $\theta_\infty = 0$, then $C_\infty = I - \psi(1) \Delta$ for $l = 1$, and $C_\infty = (1 - \pi i - \psi(1))(I + J)/2 + \Delta + \Delta_-$ for $l = 2$.

\[ (ii) \]
\[ \tilde{C}_0^\pm = \begin{pmatrix} e^{-\pi i (\theta_0 \pm \theta_x)/2} \Gamma(-\theta_0) & e^{-\pi i (\theta_0 \pm \theta_x)/2} \psi_1^0(0, \theta_x) \Gamma(-\theta_0) \\ \Gamma(-\theta_0 \mp \theta_x)/2 \Gamma(1 - \theta_0 \pm \theta_x)/2 & e^{\pi i (\theta_0 \mp \theta_x)/2} \psi_2^{02}(0, \theta_x) \Gamma(\theta_0) \end{pmatrix} \]

with
\[ \psi_{12}(\theta_0, \theta_x) = \psi(-\theta_0 \mp \theta_x)/2 + \psi(1 - \theta_0 \pm \theta_x)/2 - 2\psi(1) + \pi i, \]
\[ \psi_{22}(\theta_0, \theta_x) = \psi(1 + \theta_0 \mp \theta_x)/2 + \psi((\theta_0 \pm \theta_x)/2) - 2\psi(1) + \pi i, \]

and
\[ \tilde{C}_1^\pm = K^\pm \begin{pmatrix} \Gamma(-1 \pm \theta_x)/\Gamma((\theta_0 \pm \theta_x)/2) \Gamma(1 \mp \theta_x)/\Gamma(1 - (\theta_0 \pm \theta_x)/2) \\ \Gamma(1 \mp \theta_x)/\Gamma((\theta_0 \pm \theta_x)/2) \Gamma(1 - \theta_0 \pm \theta_x)/\Gamma(1 - (\theta_0 \pm \theta_x)/2) \end{pmatrix} \]

with \( K^- = I, K^+ = \Delta + \Delta_-, \)
\[ \psi_{12}^1(\theta_0, \theta_x) = \psi(-\theta_0 \mp \theta_x)/2 + \psi((\theta_0 \pm \theta_x)/2) - 2\psi(1), \]
\[ \psi_{22}^1(\theta_0, \theta_x) = \psi(1 + \theta_0 \mp \theta_x)/2 + \psi(1 - \theta_0 \pm \theta_x)/2 - 2\psi(1). \]

**Remark 2.12.** In \( \tilde{C}_0^\pm \) and \( \tilde{C}_1^\pm \) above, if \( (\theta_0 \mp \theta_x)/2 \in \mathbb{Z}, \) read \( \psi(-n)/\Gamma(-n) = (-1)^{n+1}n! \) for \( n = 0, 1, 2, \ldots \). For example, if \( (\theta_0 \mp \theta_x)/2 = 0, \) then
\[ \tilde{C}_0^\pm \begin{pmatrix} 0 \\ \pi i + \psi(\theta_0) - \psi(1) \end{pmatrix}, \quad \tilde{C}_1^\pm \begin{pmatrix} 0 \\ 1/(1 - \theta_0) \end{pmatrix}. \]

**Remark 2.13.** For \( \theta_0 \in \mathbb{Z} \) or \( \theta_x \in \mathbb{Z} \) as well, it is possible to compute the monodromy data \( M_0, M_x \) by using Propositions 9.6–9.8 instead of Propositions 9.3 and 9.5 in an argument of Section 10.

**Theorem 2.16.** If \( \theta_0 \neq 0, \) the monodromy data for \( y_{\text{Taylor}}(a, x) \) of Theorem 2.5 are given by \( M_0 = T e^{\pi i \theta_0} T^{-1}, \) \( M_x = T e^{-\pi i \theta_0} T^{-1}, \) and if \( \theta_0 = 0, \) then \( M_0 = T_0 e^{2\pi i A} T_0^{-1}, \) \( M_x = T_0 e^{-2\pi i A} T_0^{-1}, \) where
\[ T = \begin{pmatrix} 1 & 1 \\ a & \theta_0 + a \end{pmatrix}, \quad T_0 = \begin{pmatrix} 1 & 1 \\ a & \theta_0 + a \end{pmatrix}. \]

**Remark 2.14.** The transformation \( Y = DZ \) with an invertible matrix \( D \) preserves the form of (1.1) and its properties (a), (b) if and only if \( D \) is diagonal. Then (1.1) becomes
\[ \frac{dZ}{d\lambda} = \left( \frac{\tilde{A}_0(x)}{\lambda} + \frac{\tilde{A}_x(x)}{\lambda - x} + \frac{J}{2} \right) Z \quad (2.12) \]

with \( \tilde{A}_0(x) = D_0^{-1} A_0(x) D_0, \) \( \tilde{A}_x(x) = D_0^{-1} A_x(x) D_0, \) \( D_0 = \text{diag}(r, 1/r), \) \( r \neq 0, \) and we have
\[ y(x) = \begin{pmatrix} A(x)_{12}(\tilde{A}_0(x)_{11} + \theta_0/2) \\ A_0(x)_{12}(\tilde{A}_0(x)_{11} + \theta_0/2) \end{pmatrix} = \begin{pmatrix} \tilde{A}_x(x)_{12}(\tilde{A}_0(x)_{11} + \theta_0/2) \\ \tilde{A}_0(x)_{12}(\tilde{A}_x(x)_{11} + \theta_0/2) \end{pmatrix}. \]
that is, \( (A_0(x), A_x(x)) = (D_0^{-1} A_0(x) D_0, D_0^{-1} A_x(x) D_0) \) yields the same solution as that obtained from \( (A_0(x), A_x(x)) \) given in each of Theorems 2.1–2.5 (cf. Section 5). Hence, \( (M_0, M_x) \) for each solution above may be replaced by \( (D_0 M_0 D_0^{-1}, D_0 M_x D_0^{-1}) \), which is the monodromy for the matrix solution \( Z(\lambda, x) = (I + O(\lambda^{-1})) e^{(\lambda/2)J} \lambda^{-(\theta_\infty/2)J} \) of (2.12) (for example, cf. [2, Theorem 6.2]).

2.5. Remarks on connection formulas

Under the supposition of Theorem 2.13,

\[
2 \cos \pi \sigma = \text{tr}(M_x M_0) = 2 \cos \pi \theta_\infty + e^{-\pi i \theta_\infty} s_1 s_2
\]

(cf. the argument in Section 8 on (8.1) and (8.2)), which agrees with [2, (6.5)]. Since

\[
C_\infty M_x C_{\infty}^{-1} = \text{diag}[1, (\sigma + \theta_\infty)/(2 \rho)] \tilde{M}_x \text{ diag}[1, 2 \rho/(\sigma + \theta_\infty)]
\]

with \( \tilde{M}_x = \tilde{C}_1^{-1} e^{\pi i \theta_x} J \tilde{C}_1 \), we have

\[
\rho = \frac{(\sigma + \theta_\infty) (C_\infty M_x C_{\infty}^{-1})_{12}}{2(M_x)_{12}} = \frac{(\sigma + \theta_\infty) (C_\infty M_x C_{\infty}^{-1})_{12}}{4 \pi i \sigma \Gamma(\sigma)} \Gamma((\sigma + \theta_0 - \theta_x)/2)
\]

\[
	imes \Gamma((\sigma - \theta_0 - \theta_x)/2) \Gamma(1 + (\sigma - \theta_0 + \theta_x)/2) \Gamma(1 + (\sigma + \theta_0 + \theta_x)/2).
\]

Thus, for \( y(\sigma, \rho, x) \), the integration constants \( \sigma, \rho \) may be represented in terms of \( M_0, M_x \). Since \( M_0 \neq \pm I, M_x \neq \pm I \), by [2, Proposition 2.1] we get connection formulas that relate \( (\sigma, \rho) \) to, say, \( (\rho_0, \hat{v}) \) of the asymptotic solution as \( x \to +\infty \) [2, Theorem 3.1] under suitable suppositions. For other types of one-parameter solutions, such relations are obtained in a similar way; say, for \( y_{\text{log}}(\rho, x) \) with \( \theta_0^2 - \theta_x^2 \neq 0 \), by Theorem 2.15,

\[
(C_\infty M_x C_{\infty}^{-1})_{11} = (\tilde{M}_x^-)^1_{11} + (\tilde{M}_x^-)^{21}_{21} (\log \rho - 2 \theta_0, (\theta_0^2 - \theta_x^2)^{-1})
\]

with \( \tilde{M}_x^- = (\tilde{C}_1)^{-1} e^{\pi i \theta_x} J \tilde{C}_1^- \).

2.6. Almost-completeness of our critical behaviours

Let us discuss the completeness for each \( (\theta_0, \theta_x, \theta_\infty) \) (note that [11, Appendix B] discusses the completeness in the \( (\theta_0, \theta_x, \theta_\infty, M_0, M_x, M_1) \)-space for the sixth Painlevé transcendent). For fixed \( (\theta_0, \theta_x, \theta_\infty) \in \mathbb{C}^3 \), let \( \mathcal{M}_0 \) be the monodromy manifold defined by \( \mathcal{M}_0 := \mathcal{M}/\sim \), where \( \mathcal{M} \) is the family of monodromy data \((M_0, M_x) \in \text{SL}_2(\mathbb{C})^2 \) such that

\[
\text{tr } M_0 = 2 \cos \pi \theta_0, \quad \text{tr } M_x = 2 \cos \pi \theta_x, \quad (M_x M_0)_{11} = e^{-\pi i \theta_\infty}
\]

(cf. [2, (2.14)]), and \( \sim \) is the conjugate relation \((M_0, M_x) \sim (C^{-1} M_0 C, C^{-1} M_x C)\) by an invertible matrix \( C \). Then \( \dim_{\mathbb{C}} \mathcal{M}_0 = 2 \). For every \((M_0, M_x) \in \mathcal{M}_0\), the Riemann–Hilbert correspondence may be constructed [3, 20], that is,

\[
(\text{rh})^{-1}: \{y(x); \text{ solution of (V)}\} \to \mathcal{M}_0
\]

is surjective, and \( \text{rh} \) restricted to \( \mathcal{M}_0 \setminus \{(M_0, M_x); \ M_0 = \pm I \text{ or } M_x = \pm I \} \) is injective [2, Proposition 2.1]. As shown by [20, Section 3], for each \((\theta_0, \theta_x, \theta_\infty) \in \mathbb{C}^3 \), \( \mathcal{M}_0 \) is identified
with an affine cubic surface given by

\[ X_1 X_2 X_3 + X_1^2 + X_2^2 - (p_0 + p_x p_\infty)X_1 - (p_x + p_0 p_\infty)X_2 - p_\infty X_3 + p_0^2 + p_0 p_x p_\infty + 1 = 0 \]

with \( p_0 = 2 \cos \pi \theta_0, \ p_x = 2 \cos \pi \theta_x, \ p_\infty = e^{-\pi i \theta_\infty}. \) Although our definition of the Stokes multipliers is slightly different from that of [20], the equation of \( M_0 \) is essentially the same. If \((\theta_0, \theta_x, \theta_\infty) \in \mathbb{C}^3 \setminus (\Theta_\Sigma \cup \Theta_{ir}), \) then this surface has no singular point, and admits the local coordinates \((X_1, X_2, X_3) = ((M_0)_{11}, (M_x)_{11}, \text{tr}(M_x M_0)), \) where \( \Theta_\Sigma := \{(\theta_0, \theta_x, \theta_\infty); \ \theta_0 \in \mathbb{Z} \text{ or } \theta_x \in \mathbb{Z} \} \) and \( \Theta_{ir} \) is defined by

\[ \cos^2 \pi \theta_0 + \cos^2 \pi \theta_x + \cos^2 \pi \theta_\infty - 2 \cos \pi \theta_0 \cos \pi \theta_x \cos \pi \theta_\infty = 1. \]

If \((\theta_0, \theta_x, \theta_\infty) \in \Theta_\Sigma \cup \Theta_{ir}, \) then this surface contains one or two singular points [20, Table 3.1] and, around each regular point, it is described by the same local coordinates as above.

For \((\theta_0, \theta_x, \theta_\infty) \notin \Theta_\Sigma \cup \Theta_{ir}, \) the subset in \( M_0 \) corresponding to the complex power-type solutions \( y(\sigma, \rho, x) \) is characterized by

\[ \text{tr}(M_x M_0) \in \{2 \cos \pi \sigma; \ \sigma \in \mathbb{C} \setminus \Sigma_\ast, \ (\sigma^2 - (\theta_0 \pm \theta_x)^2)(\sigma^2 - \theta_\infty^2) \neq 0\}, \]

and that corresponding to the special complex power-type solutions \( y_{\sigma_0}(\rho, x) \) with \( \sigma_0 = \theta_0 \pm \theta_x \) or \( \theta_x - \theta_0 \) by \( \text{tr}(M_x M_0) = 2 \cos \pi \sigma_0, \) provided that \( \sigma_0 \in \Sigma_+, \ \sigma_0^2 - \theta_\infty^2 \neq 0 \) (for \( \Sigma_\ast \) and \( \Sigma_+, \) cf. Theorems 2.1 and 2.3, respectively). Furthermore, \((M_0, M_x)\) for each inverse logarithmic solution with \((\theta_0, \theta_x, \theta_\infty) \notin \Theta_\Sigma \cup \Theta_{ir} \) of Theorem 2.4 is characterized by \( \text{tr}(M_x M_0) = 2. \) Hence, if \((\theta_0, \theta_x, \theta_\infty) \notin \Theta_\Sigma \cup \Theta_{ir}, \) the closure of the union of these subsets coincides with \( M_0 \), which implies the almost-completeness of the set of our critical behaviours. It may be possible to extend it to be complete in the strict sense by adding some more solutions including logarithmic ones (cf. Remark 2.7). The Taylor series solutions of Theorem 2.5 are given for \((\theta_0, \pm \theta_0, 0) \in \Theta_\Sigma \cup \Theta_{ir}. \) If \((\theta_0, \theta_x, \theta_\infty) \in \Theta_\Sigma \cup \Theta_{ir}, \) then \((\text{rh})^{-1} \) is not necessarily injective and \( M_0 \) has singular points. However, it seems that the almost-completeness is valid for \((\theta_0, \theta_x, \theta_\infty) \in \Theta_\Sigma \cup \Theta_{ir} \) as well.

3. Solutions of the Schlesinger equation

We give matrix solutions of (2.10), from which Theorems 2.1, 2.3 and 2.4 are obtained by using (2.11).

Suppose that \( \Lambda_0, \Lambda_x \in M_2(\mathbb{Q}_0[\sigma, \sigma^{-1}]), \ T \in GL_2(\mathbb{Q}_0[\sigma, \sigma^{-1}]) \) and \( \Lambda := \Lambda_0 + \Lambda_x \) have the properties:

(P1) the eigenvalues of \( \Lambda_i \) (\( i = 0, x \)) are \( \pm \theta_i/2; \)
(P2) \( (\Lambda)_{11} = (\Lambda_0 + \Lambda_x)_{11} = - (\Lambda)_{22} = - (\Lambda_0 + \Lambda_x)_{22} = - \theta_\infty/2; \)
(P3) \( T^{-1} \Lambda T = (\sigma/2)J. \)

For \( \Sigma_0 \) and \( \Sigma_+ \) as in Theorems 2.1 and 2.3 we have the following result.
PROPOSITION 3.1. (1) System (2.10) possesses a two-parameter family of solutions 
\{(A_0(\sigma, \rho, x), A_\chi(\sigma, \rho, x)); (\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\})\} given by the convergent series

\[
A_0(\sigma, \rho, x) = (\rho x^\sigma)^{\Lambda_0/\sigma} \left( \Lambda_0 + \sum_{n=1}^{\infty} x^n \Pi_0^n(\sigma, \rho x^\sigma) \right) (\rho x^\sigma)^{-\Lambda_0/\sigma},
\]

\[
A_\chi(\sigma, \rho, x) = (\rho x^\sigma)^{\Lambda_\chi/\sigma} \left( \Lambda_\chi + \sum_{n=1}^{\infty} x^n \Pi_\chi^n(\sigma, \rho x^\sigma) \right) (\rho x^\sigma)^{-\Lambda_\chi/\sigma}
\]

with

\[
\Pi_\chi^n(\sigma, \xi) = \sum_{m=-n}^{n} C_{im}^n(\sigma) \xi^m, \quad C_{im}^n(\sigma) \in M_2(\mathbb{Q}_0(\sigma))
\]

\((i = 0, \chi),\) which are holomorphic in \((\sigma, \rho, x) \in \Omega(\Sigma_0, \epsilon_0) \subset \Sigma_0 \times (\mathbb{C} \setminus \{0\}) \times \mathcal{R}(\mathbb{C} \setminus \{0\})\) with

\[
\Omega(\Sigma_0, \epsilon_0) := \bigcup_{(\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\})} \{ (\sigma, \rho) \times \Omega_{\sigma, \rho}(\epsilon_0) \},
\]

\[
\Omega_{\sigma, \rho}(\epsilon_0) = \{ x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x| < \epsilon_0, |x(\rho x^\sigma)| < \epsilon_0, |x(\rho x^\sigma)|^{-1} < \epsilon_0 \},
\]

\(\epsilon_0\) being a sufficiently small number depending only on \(\Sigma_0\) and \((\theta_0, \theta_\chi, \theta_\infty)\).

(2) If \((T^{-1} \Lambda_0 T)_{21}\) vanishes at \(\sigma = \sigma_0 \in \Sigma_+\), then (2.10) admits a one-parameter family of solutions \(\{(A_0(\sigma_0, \rho, x), A_\chi(\sigma_0, \rho, x)); \rho \in \mathbb{C}\}\) given by the representations above restricted to \(\sigma = \sigma_0\) whose inner sums satisfy

\[
\xi^{\Lambda_0/\sigma_0} \Pi_0^n(\sigma_0, \xi) \xi^{-\Lambda_0/\sigma_0} = \sum_{m=0}^{n+1} \tilde{C}_{im}^n(\sigma_0) \xi^m, \quad \xi^{\Lambda_0/\sigma_0} \Lambda_\chi \xi^{-\Lambda_0/\sigma_0} = \tilde{C}_{00}(\sigma_0) + \tilde{C}_{11}(\sigma_0) \xi
\]

with \(\tilde{C}_{im}^n(\sigma_0) \in M_2(\mathbb{Q}_0(\sigma_0))\) \((i = 0, \chi)\) for \(n \geq 0\). Each entry of the solution is holomorphic in \((\rho, x) \in \Omega(\epsilon_0) \subset \mathbb{C} \times \mathcal{R}(\mathbb{C} \setminus \{0\})\), where

\[
\Omega(\epsilon_0) := \bigcup_{\rho \in \mathbb{C}} \{ \rho \} \times \Omega_{\rho}(\epsilon_0),
\]

\[
\Omega_{\rho}(\epsilon_0) := \{ x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x| < \epsilon_0, |x(\rho x^\sigma)| < \epsilon_0, |x(\rho x^\sigma)|^{-1} < \epsilon_0 \},
\]

\(\epsilon_0\) being a sufficiently small number depending only on \(\sigma_0\) and \((\theta_0, \theta_\chi, \theta_\infty)\).

Set \(\mathbb{Q}_0 := \mathbb{Q}_0[\theta_0^{-1}, \theta_\chi^{-1}, \theta_\infty^{-1}] = \mathbb{Q}[\theta_0, \theta_\chi, \theta_\infty, \theta_\infty^{-1}]\) if \(\theta_\infty \neq 0\), and \(\mathbb{Q}[\theta_0, \theta_\chi]\) if \(\theta_\infty = 0\). Let \(\Lambda_0, \Lambda_\chi \in M_2(\mathbb{Q}_0), T \in \text{GL}_2(\mathbb{Q}_0)\) and \(\Lambda := \Lambda_0 + \Lambda_\chi\) have the properties (P.1), (P.2) and:

(P.3') \(T^{-1} \Lambda T = \Delta\).

Then we have the following result.

PROPOSITION 3.2. System (2.10) possesses a one-parameter family of solutions of logarithmic type \(\{(A_0(\rho, x), A_\chi(\rho, x)); \rho \in \mathcal{R}(\mathbb{C} \setminus \{0\})\}\) given by the convergent series

\[
A_0(\rho, x) = (\rho x)^{\Lambda_0} \left( \Lambda_0 + \sum_{n=1}^{\infty} x^n \Pi_0^n(\log(\rho x)) \right) (\rho x)^{-\Lambda_0},
\]

\[
A_\chi(\rho, x) = (\rho x)^{\Lambda_\chi} \left( \Lambda_\chi + \sum_{n=1}^{\infty} x^n \Pi_\chi^n(\log(\rho x)) \right) (\rho x)^{-\Lambda_\chi}
\]
with

\[ \Pi^n_\imath(\xi) = \sum_{m=0}^{2n} C^{sn}_{im} \xi^m, \quad C^{sn}_{im} \in M_2(\mathbb{Q}_{\delta}) \]

\((\imath = 0, x)\), which are holomorphic in \((\rho, x) \in \Omega^\star(\varepsilon_0, \Theta_0) \subset \mathcal{R}(\mathbb{C} \setminus \{0\})^2\), where \(\Omega^\star(\varepsilon_0, \Theta_0)\) is the domain as in Theorem 2.4. Furthermore, if \((T^{-1} \Lambda_0 T)_{21} = 0\), then

\[ e^{A\xi} \Pi^n_\imath(\xi) e^{-A\xi} = \sum_{m=0}^{n+1} \tilde{C}^{sn}_{im} \xi^m, \quad e^{A\xi} \Lambda_0 e^{-A\xi} = \tilde{C}^{sn}_0 + \tilde{C}^{sn}_{11} \xi \]

with \(\tilde{C}^{sn}_{im} \in M_2(\mathbb{Q}_{\delta})(\imath = 0, x)\) for \(n \geq 0\).

System (2.10) corresponds to the Schlesinger equation associated with the sixth Painlevé equation studied in [23]. Propositions 3.1 and 3.2 are counterparts of [23, Theorems 2.1 and 2.2], respectively, and are proved by the same arguments as in their proofs in [23, Section 5]. We describe the outline of the proofs of these propositions. By the change of variables

\[ x = \kappa t, \quad A_0 = t^\Lambda (\Lambda_0 + U_0)t^{-\Lambda}, \quad A_0 + A_x = \Lambda + U_\infty, \]

where \(\Lambda = \Lambda_0 + \Lambda_x\) and \(\kappa \neq 0\), equation (2.10) is taken to

\[
\begin{align*}
\frac{dU_0}{dt} &= [t^{-\Lambda} U_\infty t^\Lambda, \Lambda_0 + U_0], \\
\frac{dU_\infty}{dt} &= \kappa t [J/2, t^\Lambda A_x t^{-\Lambda} - t^\Lambda U_0 t^{-\Lambda} + U_\infty],
\end{align*}
\]

since \(A_x = \Lambda + U_\infty - t^\Lambda (\Lambda_0 + U_0)t^{-\Lambda} = t^\Lambda (A_x - U_0 + t^{-\Lambda} U_\infty t^\Lambda)t^{-\Lambda}\). The form of system (3.2) is similar to that of [23, (5.2)].

To show Proposition 3.2 for each fixed \((\theta_0, \theta_x, \theta_\infty)\), we use the ring \(\widehat{\mathcal{L}}\) of formal series of the form

\[ \Phi = \Phi(\kappa, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{2n} C^n_m (\kappa t)^m \log^m t, \quad C^n_m \in M_2(\mathbb{Q}_{\delta}), \]

and the subring

\[ \mathcal{L}(D) := \{ \Phi \in \widehat{\mathcal{L}}; \| \Phi \| < \infty \text{ for } (\kappa, t) \in D\}, \]

which are defined in [23, Section 4.1]. Here, for \(\Phi \in \widehat{\mathcal{L}}\) as above, \(\| \Phi \|\) is the norm defined by

\[ \| \Phi \| := \sum_{n=1}^{\infty} \sum_{m=0}^{2n} \| C^n_m \| \| \kappa t \|^m \| t \|^{-m/4} \]

with the standard norm of the matrix \(\| C^n_m \| = \max_{i=1,2} \left\{|(C^n_m)_{i1}| + |(C^n_m)_{i2}|\right\}\), and \(D\) is a subdomain of \((\mathbb{C} \setminus \{0\}) \times \mathcal{R}(\mathbb{C} \setminus \{0\})\) such that \(\| \log t \| \leq |t|^{-1/4}\) for every \((\kappa, t) \in D\). Then the holomorphic nature of \(\Phi(\kappa, t) \in \mathcal{L}(D)\) in \(D\) is guaranteed by [23, Proposition 4.1]. For \((m, n) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}\) and \(C \in M_2(\mathbb{Q}_{\delta})\) let

\[ \mathcal{I}[C(\kappa t)^n \log^m t] := C \frac{(\kappa t)^n}{n} \left( \log^m t + \cdots + (-1)^j m! \frac{n^j (m-j)!}{n^j (m-j)!} \log^{m-j} t + \cdots + (-1)^m m! \right). \]

Then \(\mathcal{I}\) induces a linear operator \(\mathcal{I}: \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}\) assigning the formal primitive function of \(t^{-1} \Phi\) to each \(\Phi \in \widehat{\mathcal{L}}\), which satisfies \(\mathcal{I}[\Phi] \in \mathcal{L}(D), \ t(d/dt)\mathcal{I}[\Phi] = \Phi\) and \(\| \mathcal{I}[\Phi] \| \leq 2 \| \Phi \|\) for
\( \Phi \in \mathfrak{L}(D) \), provided that, for \((\kappa, t) \in D, |t| \) is sufficiently small. To construct a solution of (3.2) we define the sequence \([U_{0}^{(v)}, U_{\infty}^{(v)}] \in (\mathfrak{L})^{2}; \ v \geq 0\) by
\[
U_{0}^{(0)} = U_{0}^{(0)} = 0,
\]
\[
U_{\infty}^{(v+1)} = \mathcal{I}[\kappa t[J/2, t^{\Lambda} \Lambda x t^{-\Lambda} - t^{\Lambda} U_{0}^{(v)} t^{-\Lambda} + U_{\infty}^{(v)}]], \quad (3.3)
\]
\[
U_{0}^{(v+1)} = \mathcal{I}[\kappa t U_{\infty}^{(v+1)} t^{\Lambda}, \Lambda_{0} + U_{0}^{(v)}]
\]
with \(\Lambda_{0}, \Lambda_{x}\) and \(\Lambda\) satisfying (P.1), (P.2) and (P.3). This converges to a formal series solution of (3.2). Choosing \(D\) suitably, we may show that \([U_{0}^{\infty}, U_{\infty}^{\infty}] = \lim_{v \to \infty}(U_{0}^{(v)}, U_{\infty}^{(v)}) \in \mathfrak{L}(D)^{2}\) solves (3.2) and that \(t^{-\Lambda} U_{\infty}^{\infty} t^{\Lambda} \in \mathfrak{L}(D)\). Setting \(\kappa t = x\) and \(t = \rho x\) in
\[
A_{0} = t^{\Lambda}(\Lambda_{0} + U_{0}^{\infty}) t^{-\Lambda}, \quad A_{x} = t^{\Lambda}(\Lambda_{x} - U_{0}^{\infty} + t^{-\Lambda}U_{\infty}^{\infty}) t^{-\Lambda},
\]
we obtain the family of solutions \(\{(A_{0}(\sigma, \rho, x), A_{x}(\sigma, \rho, x))\}\) as in Proposition 3.2. If \((T^{-1} \Lambda_{0} T)_{21} = 0\), then \(\kappa t \cdot t^{\Lambda} U_{\infty}^{\infty} t^{-\Lambda}, U_{\infty}^{\infty} \in \mathfrak{L}^{*}\), where \(\mathfrak{L}^{*}\) is the subring of \(\mathfrak{L}\) consisting of formal series of the form
\[
\Phi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} C_{m}^{n}(\kappa t)^{n} \log^{m} t, \quad C_{m}^{n} \in M_{2}(\mathbb{Q}_{\bar{s}}).
\]
From this fact the remaining part of Proposition 3.2 follows (cf. [23, Section 5.2]).

To show Proposition 3.1 for each fixed \((\theta_{0}, \theta_{x}, \theta_{\infty})\), consider the ring \(\mathfrak{G}\) of formal series of the form
\[
\Phi = \Phi(\sigma, \kappa, t) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} C_{m}^{n}(\kappa t)^{n} t^{\sigma m}, \quad C_{m}^{n} = C_{m}^{n}(\sigma) \in M_{2}(\mathbb{Q}_{\bar{s}}(\sigma)),
\]
and the subring
\[
\mathfrak{G}(D(\Sigma_{0})) := \{\Phi \in \mathfrak{G}; \|\Phi\| < \infty \text{ for } (\sigma, \kappa, t) \in D(\Sigma_{0})\}
\]
as in [23, Section 4.2]. Here, for \(\Phi \in \mathfrak{G}\) as above,
\[
\|\Phi\| := \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \|C_{m}^{n}\| |\kappa t|^{n} |t|^{\sigma m},
\]
and \(D(\Sigma_{0})\) is a subdomain of \(\Sigma_{0} \times (\mathbb{C} \setminus \{0\}) \times R(\mathbb{C} \setminus \{0\})\). For \((m, n) \in \mathbb{Z} \times \mathbb{N}\) and \(C \in M_{2}(\mathbb{Q}_{\bar{s}}(\sigma))\), let \(\mathcal{I}\) be such that
\[
\mathcal{I}[C(\kappa t)^{n} t^{\sigma m}] := \frac{1}{n + \sigma m} C(\kappa t)^{n} t^{\sigma m}.
\]
This induces a linear operator \(\mathcal{I}: \mathfrak{G} \to \mathfrak{G}\) satisfying \(\mathcal{I}[\Phi] \in \mathfrak{G}(D(\Sigma_{0})), t(d/dt) \mathcal{I}[\Phi] = \Phi\) and \(\|\mathcal{I}[\Phi]\| \leq L_{0}\|\Phi\|\) for some \(L_{0} > 0\) if \(\Phi \in \mathfrak{G}(D(\Sigma_{0}))\). Then we define \([U_{0}^{(v)}, U_{\infty}^{(v)}] \in (\mathfrak{G})^{2}; \ v \geq 0\) by (3.3) with \(\Lambda_{0}, \Lambda_{x}\) and \(\Lambda\) satisfying (P.1), (P.2) and (P.3). Choosing \(D(\Sigma_{0})\) suitably, we may show that \([U_{0}^{\infty}, U_{\infty}^{\infty}] = \lim_{v \to \infty}(U_{0}^{(v)}, U_{\infty}^{(v)}) \in \mathfrak{G}(D(\Sigma_{0}))^{2}\) solves (3.2) and that \(t^{-\Lambda} U_{\infty}^{\infty} t^{\Lambda} \in \mathfrak{G}(D(\Sigma_{0}))\). Setting \(\kappa t = x\) and \(t^{\sigma} = \rho x^{\sigma}\) we obtain Proposition 3.1(1). The second assertion is proved by considering, for \(\sigma_{0} \in \Sigma_{+}\), the ring \(\mathfrak{G}^{+}(\sigma_{0})\) of formal series
\[
\Phi(\sigma_{0}, \kappa, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{m}^{n}(\kappa t)^{n} t^{\sigma_{0} m}, \quad C_{m}^{n} \in M_{2}(\mathbb{Q}_{\bar{s}}(\sigma_{0})),
\]
and by using the sequence \( \{ (Z^{(v)}_0, U^{(v)}_\infty)_{|\sigma = \sigma_0; \: v \geq 0} \} \) with \( Z^{(v)}_0 := t^\Lambda U^{(v)}_0 t^{-\Lambda} \) such that \( \kappa t Z^{(v)}_0, \: U^{(v)}_\infty \in \hat{\mathcal{S}}^+(\sigma_0) \) if \( (T^{-1} \Lambda_0 T)_{21|\sigma = \sigma_0} = 0 \). In a suitable domain, we get \( (U^{\infty}_\infty, \kappa t Z^{(v)}_0) = \lim_{v \to \infty} (U^{(v)}_\infty, \kappa t Z^{(v)}_0) \), from which the desired solution of (2.10) follows (cf. [23, Section 5.3]).

**Proposition 3.3.** The solution \( (A_0(x), A_x(x)) \) of (2.10) given by Proposition 3.1 or 3.2 satisfies the conditions (a) and (b) on (1.1), and the corresponding system (1.1) has the isomonodromy property.

**Proof.** Observing that \( (d/dx)(A_0(x) + A_x(x)) = [J, A_x(x)]/2 \), in which the diagonal part on the right-hand side vanishes identically, we deduce (b) from the fact that \( A_0(x) + A_x(x) \to \Lambda \) as \( x \to 0 \) along a suitable curve. The property (a) may be verified by the same argument as in the proof of [23, Proposition 3.1].

**4. Lemmas on matrices**

For \( \sigma \neq 0 \), we have the following (cf. [9, Lemma A.2]):

**Lemma 4.1.** The matrices

\[
\Lambda_0 = T \begin{pmatrix} \Lambda'_0_{11} & 1 \\ \Lambda'_0_{21} & -\Lambda'_0_{11} \end{pmatrix} T^{-1}, \quad \Lambda_x = T \begin{pmatrix} \Lambda'_x_{11} & -1 \\ \Lambda'_x_{21} & -\Lambda'_x_{11} \end{pmatrix} T^{-1},
\]

\[
\Lambda_0 = \frac{\sigma}{2} - \Lambda'_0_{11} = \frac{1}{4\sigma}(\sigma^2 + \theta_0^2 - \theta_x^2),
\]

\[
-\Lambda'_0_{21} = \Lambda'_x_{21} = \frac{1}{16\sigma^2}((\theta_0 + \theta_x)^2 - \sigma^2)((\theta_0 - \theta_x)^2 - \sigma^2),
\]

\[
T = \begin{pmatrix} (\sigma - \theta_\infty)/2 & 1 \\ (\sigma + \theta_\infty)/2 & -1 \end{pmatrix}, \quad \Lambda = \Lambda_0 + \Lambda_x = \begin{pmatrix} -\theta_\infty/2 & (\sigma - \theta_\infty)/2 \\ (\sigma + \theta_\infty)/2 & \theta_\infty/2 \end{pmatrix}
\]

have the properties (P.1), (P.2) and (P.3).

Using [10, Proposition 2.1, Jordan case], we have the following lemma.

**Lemma 4.2.** (1) If \( \theta_\infty \neq 0 \), then the matrices

\[
\Lambda_0 = T \begin{pmatrix} \pm \theta_x/2 \\ (\theta_0^2 - \theta_x^2)/4 \end{pmatrix} T^{-1}, \quad \Lambda_x = T \begin{pmatrix} \pm \theta_x/2 \\ (\theta_0^2 - \theta_x^2)/4 \end{pmatrix} T^{-1},
\]

\[
T = \begin{pmatrix} -\theta_\infty/2 & 1 \\ \theta_\infty/2 & 0 \end{pmatrix}, \quad \Lambda = \Lambda_0 + \Lambda_x = \theta_\infty/2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}
\]

have the properties (P.1), (P.2) and (P.3).

(2) If \( \theta_\infty = 0 \), then the matrices \( \Lambda_0, \Lambda_x \) given as above with

\[
T = I \quad \text{(respectively, } T = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix})
\]

and \( \Lambda = \Delta \) (respectively, \( \Lambda = \Delta_- \)) have the properties (P.1), (P.2) and (P.3).
LEMMA 4.3. Suppose that $\theta_\infty = 0$.

(1) If $\theta_0 = \pm \theta_x \neq 0$, then

$$\Lambda_0 = -\Lambda_x = T(\theta_0/2)JT^{-1} = \begin{pmatrix} \theta_0/2 + a & -1 \\ a(\theta_0 + a) & -\theta_0/2 - a \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ a & \theta_0 + a \end{pmatrix}$$

have the properties (P.1) and (P.2).

(2) If $\theta_0 = \theta_x = 0$, then

$$\Lambda_0 = -\Lambda_x = T\Delta T^{-1} = \begin{pmatrix} a & -1 \\ a^2 & -a \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ a & a - 1 \end{pmatrix}$$

have the properties (P.1) and (P.2).

5. Proofs of Theorems 2.1 and 2.3–2.5

5.1. Proof of Theorem 2.1

By Proposition 3.1 with $(\Lambda_0, \Lambda_x, \Lambda, \Lambda)$ in Lemma 4.1 we have, for $\sigma \in \Sigma_0$, $\rho \in \mathbb{C} \setminus \{0\}$,

$$A_0(\sigma, \rho, x) = T(\rho \sigma)^{J/2} \left( T^{-1} \Lambda_0 T + \sum_{n=1}^{\infty} x^n T^{-1} \Pi_0^n(\sigma, \rho \sigma) T \right) (\rho \sigma)^{-J/2} T^{-1}$$

$$= T \left( (\rho \sigma)^{J/2} T^{-1} \Lambda_0 T(\rho \sigma)^{-J/2} + \sum_{n=1}^{\infty} x^n \sum_{j=-n-1}^{n+1} C_{nj}(\rho \sigma)^j \right) T^{-1}$$

with $C_{nj}(\sigma) \in M_2(\mathbb{Q}_\theta(\sigma))$, and hence

$$(A_0)_{11} = -\sigma^{-1}(\theta_0(\Lambda_0')_{11} + \frac{1}{4}(\sigma^2 - \theta_0^2) \rho x^\sigma + (\Lambda_0')_{21}(\rho x^\sigma)) + (\cdots),$$

$$(A_0)_{12} = -\sigma^{-1}(\rho x^\sigma) + (\cdots),$$

where $\cdots$ denotes a series of the form $\sum_{n=1}^{\infty} x^n \sum_{j=-n-1}^{n+1} c_{nj}(\rho x^\sigma)^j$, $c_{nj}(\sigma) \in \mathbb{Q}_\theta(\sigma)$.

Similarly,

$$(A_x)_{11} = -\sigma^{-1}(\theta_\infty(\Lambda_x')_{11} + \frac{1}{4}(\sigma^2 - \theta_\infty^2) \rho x^\sigma + (\Lambda_x')_{21}(\rho x^\sigma)) + (\cdots),$$

$$(A_x)_{12} = -\sigma^{-1}(\rho x^\sigma) + (\cdots).$$

By Proposition 3.3 and (2.11), $y = Y_{11}Y_{12}$ with

$$Y_{11} = \frac{(A_0)_{11} + \theta_0/2}{(A_x)_{11} + \theta_x/2}, \quad Y_{12} = \frac{(A_x)_{12}}{(A_0)_{12}}$$

solves (V). Then we may write $Y_{ij} = (Y_{ij})_{\text{num}}/(Y_{ij})_{\text{den}} ((i, j) = (1, 1), (1, 2))$ with

$$(Y_{11})_{\text{num}} = 4\theta_0 \sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2) - 2\sigma(\sigma^2 - \theta_\infty^2) \rho x^\sigma$$

$$+ \frac{1}{2\sigma}(\theta_0 + \theta_x)^2 - 2\sigma((\theta_0 - \theta_x)^2 - \sigma^2)(\rho x^\sigma)^{-1} + \sigma^2(\cdots),$$

$$(Y_{11})_{\text{den}} = 4\theta_0 \sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2) + 2\sigma(\sigma^2 - \theta_\infty^2) \rho x^\sigma$$

$$- \frac{1}{2\sigma}(\theta_0 + \theta_x)^2 - 2\sigma((\theta_0 - \theta_x)^2 - \sigma^2)(\rho x^\sigma)^{-1} + \sigma^2(\cdots),$$

(5.2)
and
\[(Y_{12})_{\text{num}} = 2(\sigma - \theta_\infty)(\sigma^2 + \theta_0^2 - \theta_x^2) - 2\sigma(\sigma - \theta_\infty)^2 \rho x^\sigma \]
\[- \frac{1}{2\sigma}((\theta_0 + \theta_x)^2 - \sigma^2)((\theta_0 - \theta_x)^2 - \sigma^2)(\rho x^\sigma)^{-1} + \sigma^2(\cdots), \]
\[(Y_{12})_{\text{den}} = 2(\sigma - \theta_\infty)(\sigma^2 + \theta_0^2 - \theta_x^2) + 2\sigma(\sigma - \theta_\infty)^2 \rho x^\sigma \]
\[+ \frac{1}{2\sigma}((\theta_0 + \theta_x)^2 - \sigma^2)((\theta_0 - \theta_x)^2 - \sigma^2)(\rho x^\sigma)^{-1} + \sigma^2(\cdots). \]
Aiming at a solution of the form as in Theorem 2.1 we replace \(\rho\) by
\[
\frac{(\theta_0 - \theta_x + \sigma)(\theta_0 + \theta_x - \sigma)}{2\sigma(\sigma - \theta_\infty)} \rho. \]

Then \((Y_{ij})_{\text{num}}\) and \((Y_{ij})_{\text{den}}\) become
\[(Y_{11})_{\text{num}} = 4\theta_0\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2) - (\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \]
\[+ (\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} + (\cdots), \]
\[(Y_{11})_{\text{den}} = 4\theta_x\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 + \theta_x^2) + (\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \]
\[- (\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} + (\cdots), \]  
\[(Y_{12})_{\text{num}} = (\sigma + \theta_\infty)^{-1}(Y_{12})_{\text{num}}, (Y_{12})_{\text{den}} = (\sigma + \theta_\infty)^{-1}(Y_{12})_{\text{den}} \] with
\[(Y_{12})_{\text{num}} = 2(\sigma - \theta_\infty)(\sigma^2 + \theta_0^2 - \theta_x^2) - (\sigma - \theta_\infty)^2(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \]
\[- (\sigma + \theta_\infty)^2(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} + (\cdots), \]
\[(Y_{12})_{\text{den}} = 2(\sigma - \theta_\infty)(\sigma^2 + \theta_0^2 - \theta_x^2) + (\sigma - \theta_\infty)^2(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \]
\[+ (\sigma + \theta_\infty)^2(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} + (\cdots), \]
which are holomorphic in a domain on \(\mathcal{R}(\mathbb{C} \setminus \{0\})\) where \(|x(\rho x^\sigma)|\) and \(|x(\rho x^\sigma)^{-1}|\) are sufficiently small. A general solution \(y(\sigma, \rho, x)\) meromorphic in such a domain is represented in terms of (5.5) and (5.6). If \(\rho x^\sigma\) is also sufficiently small, then, under the supposition of Theorem 2.1, we may write, say \((Y_{11})_{\text{num}}\), in the form
\[(Y_{11})_{\text{num}} = (\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1}\left(1 + \frac{4\theta_0\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2)}{(\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)} \rho x^\sigma \right. \]
\[\left. + c_2^{(1)}(\sigma)(\rho x^\sigma)^2 + \sum_{n=1}^{\infty} x^n \sum_{j=-n-1}^{n+1} c_{nj}^{(1)}(\sigma)(\rho x^\sigma)^{j+1} \right) \] with \(c_2^{(1)}(\sigma), c_{nj}^{(1)}(\sigma) \in \mathcal{O}_\theta(\sigma).\) From this and analogous expressions of \((Y_{11})_{\text{den}}, (Y_{12})_{\text{num}}\) and \((Y_{12})_{\text{den}}\) we derive \(y_+ (\sigma, \rho, x)\) convergent in \(\Omega^+(\Sigma_0, \varepsilon_0).\) If \((\rho x^\sigma)^{-1}\) is sufficiently small, writing
\[(Y_{11})_{\text{num}} = -(\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \left(1 - \frac{4\theta_0\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2)}{(\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2)} (\rho x^\sigma)^{-1} \right. \]
\[\left. + c_2^{(1)}(\sigma)(\rho x^\sigma)^{-2} + \sum_{n=1}^{\infty} x^n \sum_{j=-n-1}^{n+1} c_{nj}^{(1)}(\sigma)(\rho x^\sigma)^{-j-1} \right) \]
with $c_2^{(11)}(\sigma)$, $c_n^{(11)}(\sigma) \in \mathbb{Q}_0(\sigma)$ and so on, we have $y_-(\sigma, \rho, x)$. Thus, we obtain Theorem 2.1.

5.2. Proof of Theorem 2.3

If $\sigma_0^2 = (\theta_0 \pm \theta_x)^2$, then $2\theta_0\sigma_0^2 - \theta_\infty(\sigma_0^2 + \theta_0^2 - \theta_x^2) = 2\theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty)$. Putting $\sigma = \sigma_0$ in $Y_{11}$ and $Y_{12}$ with (5.2) and (5.3), we have

$$Y_{11}|_{\sigma = \sigma_0} = \frac{2\theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty)}{2\theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty) + \sigma_0(\sigma_0^2 - \sigma_0^2 - \sigma_0^2)} \cdot \left( \begin{array}{c} 2 \theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty) - \sigma_0(\sigma_0^2 - \sigma_0^2 - \sigma_0^2) \rho x^\sigma_0 + \sigma_0^2(\sigma_0^2 - \sigma_0^2) + \sigma_0^2(\sigma_0^2 - \sigma_0^2) \\
\sigma_0(\sigma_0 - \theta_\infty) \rho x^\sigma_0 + \sigma_0^2(\sigma_0 - \theta_\infty)^{-1}(\sigma_0^2 - \sigma_0^2) 
\end{array} \right),$$

\begin{equation} \tag{5.7}
Y_{12}|_{\sigma = \sigma_0} = \frac{2\theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty)}{2\theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty) + \sigma_0(\sigma_0 - \theta_\infty) \rho x^\sigma_0 + \sigma_0^2(\sigma_0 - \theta_\infty)^{-1}(\sigma_0^2 - \sigma_0^2)} \cdot \left( \begin{array}{c} 2 \theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty) - \sigma_0(\sigma_0 - \theta_\infty) \rho x^\sigma_0 + \sigma_0^2(\sigma_0 - \theta_\infty)^{-1}(\sigma_0^2 - \sigma_0^2) \\
\sigma_0(\sigma_0 - \theta_\infty) \rho x^\sigma_0 + \sigma_0^2(\sigma_0 - \theta_\infty)^{-1}(\sigma_0^2 - \sigma_0^2) 
\end{array} \right),
\end{equation}

with $(\cdots)$ denoting a series of the form $\sum_{n=1}^{\infty} x^n \sum_{j=0}^{n+1} c_{nj}(\sigma_0)(\rho x^\sigma_0)^j$. Suppose that $\sigma_0 = \theta_0 \neq \theta_\infty$ and $\theta_0 \theta_x = 0$. If $\rho x^\sigma_0$ is sufficiently small,

$$Y_{11}Y_{12}|_{\sigma = \sigma_0} = \frac{1 - (\sigma_0 - \theta_\infty) \rho x^\sigma_0/(2\theta_\theta) + \cdots + (\sigma_0 - \theta_\infty) \rho x^\sigma_0/(2\theta_\theta) + \cdots}{1 + (\sigma_0 - \theta_\infty) \rho x^\sigma_0/(2\theta_\theta) + \cdots + (\sigma_0 - \theta_\infty) \rho x^\sigma_0/(2\theta_\theta) + \cdots} \cdot \left( \begin{array}{c} 1 - \frac{\sigma_0^2}{\theta_0 \theta_x} \rho x^\sigma_0 + \sum_{j=2}^{\infty} c_j^0(\sigma_0)(\rho x^\sigma_0)^j + \sum_{n=1}^{\infty} x^n \sum_{j=0}^{\infty} c_{jn}^0(\sigma_0)(\rho x^\sigma_0)^j \\
\sum_{n=1}^{\infty} x^n \sum_{j=0}^{\infty} \rho x^\sigma_0(\rho x^\sigma_0)^n 
\end{array} \right),$$

$$= 1 - \frac{4\sigma_0^2}{\sigma_0^2 - \sigma_0^2} \rho x^\sigma_0(\rho x^\sigma_0)^{-1} + \sum_{j=2}^{\infty} c_j^0(\sigma_0)(\rho x^\sigma_0)^{-j} + \sum_{n=1}^{\infty} x^n \sum_{j=0}^{\infty} c_{jn}^0(\sigma_0)(\rho x^\sigma_0)^{n-j}$$

with $c_j^0(\sigma_0), c_{jn}^0(\sigma_0) \in \mathbb{Q}_0[\theta_0^{-1}, \theta_x^{-1}]$. If $(\rho x^\sigma_0)^{-1}$ and $x(\rho x^\sigma_0)$ are sufficiently small,

$$Y_{11}Y_{12}|_{\sigma = \sigma_0} = \frac{1 - 2\theta_0(\rho x^\sigma_0)^{-1}/(\sigma_0 + \theta_\infty) + \cdots + 2\theta_0(\rho x^\sigma_0)^{-1}/(\sigma_0 - \theta_\infty) + \cdots}{1 + 2\theta_0(\rho x^\sigma_0)^{-1}/(\sigma_0 + \theta_\infty) + \cdots + 2\theta_0(\rho x^\sigma_0)^{-1}/(\sigma_0 - \theta_\infty) + \cdots} \cdot \left( \begin{array}{c} 1 - \frac{4\sigma_0^2}{\sigma_0^2 - \sigma_0^2} \rho x^\sigma_0(\rho x^\sigma_0)^{-1} + \sum_{j=2}^{\infty} c_j^0(\sigma_0)(\rho x^\sigma_0)^{-j} + \sum_{n=1}^{\infty} x^n \sum_{j=0}^{\infty} c_{jn}^0(\sigma_0)(\rho x^\sigma_0)^{n-j} \\
\sum_{n=1}^{\infty} x^n \sum_{j=0}^{\infty} \rho x^\sigma_0(\rho x^\sigma_0)^n 
\end{array} \right),$$

If $\sigma_0 = 0$, we have

$$\rho x^\sigma_0(\rho x^\sigma_0)^n = \sum_{n=0}^{\infty} \rho x(\rho x)^{n-j} + \sum_{n=1}^{\infty} x^n \sum_{j=0}^{\infty} \rho x^\sigma_0(\rho x^\sigma_0)^n$$

5.3. Proof of Theorem 2.4

Suppose that $\theta_\infty \neq 0$. By Proposition 3.2 with $(\Lambda_0, \Lambda_x, T, \Lambda)$ in Lemma 4.2(1), we have

$$A_0(\rho, x) = T \left( (\rho x)^{\Delta} T^{-1} \Lambda_0 T \rho x^{\Delta} A + \sum_{n=1}^{\infty} x^n (\rho x)^{\Delta} T^{-1} \Pi_0^n \log(\rho x) T \rho x^{\Delta} A \right) T^{-1}.$$
Here the sign ± is chosen according to \((T^{-1} \Lambda_0 T)_{11} = \mp \theta_\times /2\), and \((\cdots)\) denotes a series of the form \(\sum_{n=1}^{\infty} x^n \sum_{j=0}^{n(\theta)} c_{nj}^* \log^j(\rho x)\) with \(c_{nj}^* \in \mathbb{Q}_\theta\) \(n(\theta)\) being such that \(n(\theta) = 2n + 2\) if \(\theta_0^2 - \theta_\times^2 \neq 0\), and \(n + 1\) if \(\theta_0^2 - \theta_\times^2 = 0\). Note that

\[
-Y_{11} = 1 + \frac{\theta_0 + \theta_\times - \theta_\infty + (\cdots)}{2(A_0)_{11} - \theta_\times + \theta_\infty + (\cdots)}, \quad -Y_{12} = 1 + \frac{\theta_\infty + (\cdots)}{2(A_0)_{12} + (\cdots)}.
\]

If \(\theta_0^2 - \theta_\times^2 \neq 0\), then

\[
-Y_{11} = 1 + \frac{4(\theta_0 + \theta_\times - \theta_\infty)}{\theta_\times \theta_\infty (\theta_0^2 - \theta_\times^2)} \log^{-2}(\rho x) + \cdots, \\
-Y_{12} = 1 + \frac{4\theta_\infty}{\theta_\times \theta_\infty (\theta_0^2 - \theta_\times^2)} \log^{-2}(\rho x) + \cdots;
\]

and if \(\theta_0^2 = \theta_\times^2 \neq 0\), then

\[
-Y_{11} = 1 \mp \frac{\theta_0 + \theta_\times - \theta_\infty}{\theta_\times \theta_\infty} \log^{-1}(\rho x) \left(1 + \frac{\theta_\infty}{\theta_\times \theta_\infty} \log^{-1}(\rho x) + \cdots\right), \\
-Y_{12} = 1 \mp \frac{\theta_\infty}{\theta_\times \theta_\infty} \log^{-1}(\rho x) \left(1 + \frac{2\theta_\infty}{\theta_\times \theta_\infty} \log^{-1}(\rho x) + \cdots\right).
\]

From these formulas, we derive \(y_{\log}(\rho, x)\) and \(y_{\log}^\pm(\rho, x)\). In the case where \(\theta_0^2 - \theta_\times^2 \neq 0\), apparently, there exist two kinds of inverse logarithmic solutions depending on the sign of \(\mp \theta_\times\), but by the following proposition verified later we may replace \(\rho\) suitably to derive \(y_{\log}(\rho, x)\) as in (1) independent of the sign; indeed \(A_0^*(\rho, x)\) in the proposition has entries as follows:

\[
(A_0^*)_{11} = \frac{\theta_\infty}{8} (\theta_0^2 - \theta_\times^2) \log^2(\rho x) - \frac{1}{4} (\theta_0^2 - \theta_\times^2) \log(\rho x) - \frac{\theta_\infty \theta_\times^2}{2} (\theta_0^2 - \theta_\times^2)^{-1} - \frac{\theta_\infty}{2} + \cdots, \\
(A_0^*)_{12} = (A_0^*)_{11} - \frac{1}{4} (\theta_0^2 - \theta_\times^2) \log(\rho x) + \frac{1}{2 \theta_\infty} (\theta_0^2 - \theta_\times^2) + \cdots.
\]

**Proposition 5.1.** Suppose that \(\theta_0^2 - \theta_\times^2 \neq 0\). Let \(A_0^-(\rho, x)\) denote \(A_0(\rho, x)\) in the case where \((T^{-1} \Lambda_0 T)_{11} = \theta_\times /2\). Then \(A_0^*(\rho, x) := A_0^-(\rho \exp(-2\theta_\times (\theta_0^2 - \theta_\times^2)^{-1}), x)\) is represented by

\[
\frac{1}{4} T \left( \begin{array}{c}
(\theta_0^2 - \theta_\times^2) \log(\rho x) \\
\theta_0^2 - \theta_\times^2
\end{array} \right) + \sum_{n=1}^{\infty} x^n \sum_{j=0}^{n(\theta)} A_{jn}^* \log^j(\rho x)
\]

with \(A_{jn}^* \in M_2(\mathbb{Q}_\theta[(\theta_0^2 - \theta_\times^2)^{-1}]),\) which is independent of the sign \(\pm\).

If \(\theta_0 = \theta_\times \neq 0\), from

\[
Y_{11} Y_{12} = \frac{(2(A_0)_{11} + \theta_0 + (\cdots))(2(A_0)_{12} + \theta_\infty + (\cdots))}{(2(A_0)_{11} - \theta_\times + \theta_\infty + (\cdots))(2(A_0)_{12} + (\cdots))}
\]
we derive the expression
\[ y_{\log}^+(\rho, x) = 1 - \frac{2}{\theta_\infty} \log^{-1}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_n^+ \log^{n-j}(\rho x). \]

Let us suppose that \( \theta_\infty = 0 \). Then we use Proposition 5.1. If \( \Lambda = \Delta \),
\[ (A_0)_{11} = \frac{\theta_x}{2} + \frac{1}{4}(\theta_0^2 - \theta_x^2) \log(\rho x) + (\cdots)_{11}, \]
\[ (A_0)_{12} = 1 \pm \theta_x \log(\rho x) - \frac{1}{4}(\theta_0^2 - \theta_x^2) \log^2(\rho x) + (\cdots)_{12}, \]
\[ (A_x)_{11} = -(A_0)_{11} + (\cdots)_{11}, \quad (A_x)_{12} = -(A_0)_{12} + 1 + (\cdots)_{12}, \]
and, hence, under the condition \( \theta_0^2 - \theta_x^2 \neq 0 \) or \( \theta_x \neq 0 \),
\[ -Y_{11} = 1 + \frac{\theta_0 + \theta_x + (\cdots)_{11}}{2(A_0)_{11} - \theta_x + (\cdots)_{11}}, \quad -Y_{12} = 1 - \frac{1 + (\cdots)_{12}}{(A_0)_{12} + (\cdots)_{12}}, \]
where \((\cdots)_{11}\) (respectively, \((\cdots)_{12}\)) denotes the sum of \( c_{n,j}^+ x^n \log^j(\rho x) \) for \( n \geq 1 \) and for \( 0 \leq j \leq n(\theta) - 1 \) (respectively, \( 0 \leq j \leq n(\theta) - 2 \)). If \( \Lambda = \Delta_\pm \),
\[ (A_0)_{11} = \frac{1}{4}(\theta_0^2 - \theta_x^2) \pm \frac{\theta_x}{2} - \frac{1}{4}(\theta_0^2 - \theta_x^2) \log(\rho x) + (\cdots)_{11}, \]
\[ (A_0)_{12} = \frac{1}{4}(\theta_0^2 - \theta_x^2) + (\cdots)_{12}, \]
\[ (A_x)_{11} = -(A_0)_{11} + (\cdots)_{11}, \quad (A_x)_{12} = -(A_0)_{12} + (\cdots)_{12}, \]
and, hence, under the condition \( \theta_0^2 - \theta_x^2 \neq 0 \),
\[ -Y_{11} = 1 + \frac{\theta_0 + \theta_x + (\cdots)_{11}}{2(A_0)_{11} - \theta_x + (\cdots)_{11}}, \quad -Y_{12} = 1 + (\cdots)_{12}, \]
where \((\cdots)_{11}\) (respectively, \((\cdots)_{12}\)) denotes the sum of \( c_{n,j}^+ x^n \log^j(\rho x) \) for \( n \geq 1 \) and for \( 0 \leq j \leq n(\theta) - 1 \) (respectively, \( 0 \leq j \leq n(\theta) - 2 \)). The solutions \( y_{\log}^{(1)}(\rho, x) \) and \( y_{\log}^{(2)}(\rho, x) \) in (3) are derived from the formulas above for \( \Lambda = \Delta \) and for \( \Lambda = \Delta_\pm \), respectively. In case \( \theta_0^2 - \theta_x^2 \neq 0 \), we use Proposition 5.1. If \( \Lambda = \Delta_\pm, \theta_0 + \theta_x = 0 \) and \( (A_0)_{11} = -\theta_x/2 + (\cdots)_{11} \), then \( y_{\log}^{(1)}(\rho, x) \) follows. Thus, we obtain the expressions in Theorem 2.4.

**Proof of Proposition 5.1.** Note that
\[ t^\Lambda \Lambda_0 t^{-\Lambda} = T \hat{\Lambda}_0(\tau)T^{-1}, \quad t^\Lambda \Lambda_x t^{-\Lambda} = T(\Delta - \hat{\Lambda}_0(\tau))T^{-1}, \]
where \( \tau = t \exp(\mp 2\theta_x(\theta_0^2 - \theta_x^2)^{-1}) \) and
\[ \hat{\Lambda}_0(\tau) := t^\Lambda T^{-1} \Lambda_0 T_{\tau}^{-\Lambda} \]
\[ = \Delta + \frac{1}{4} \begin{pmatrix} (\theta_0^2 - \theta_x^2) \log \tau & -(\theta_0^2 - \theta_x^2) \log^2 \tau + 4\theta_x^2(\theta_0^2 - \theta_x^2)^{-1} \\ \theta_0^2 - \theta_x^2 & -(\theta_0^2 - \theta_x^2) \log \tau \end{pmatrix} \]
\[ = \tau^\Delta \begin{pmatrix} 0 & 1 + \theta_x^2(\theta_0^2 - \theta_x^2)^{-1} \\ (\theta_0^2 - \theta_x^2)/4 & 0 \end{pmatrix} \tau^{-\Delta}. \]
Putting \( t = c_\pm \tau, \kappa' = c_\pm \kappa \) with \( c_\pm = \exp(\pm 2\theta_x (\theta_0^2 - \theta_x^2)^{-1}) \) in (3.3), we have
\[
U^{(0)}(t) = U^{(0)}(0) = 0, \\
U^{(v+1)}(t) = T[\kappa' t [J/2, T(\Delta - \hat{\Lambda}_0(\tau))] T^{-1} \cdot (c_\pm \tau)^{\Lambda}U^{(v)}(c_\pm \tau)^{-\Lambda} + U^{\infty}(v)], \\
(c_\pm \tau)^{\Lambda}U^{(v+1)}_0(c_\pm \tau)^{-\Lambda} = \tau^{\Lambda}T[\tau^{-\Lambda}[U^{(v+1)}(t), T\hat{\Lambda}_0(\tau)] T^{-1} \cdot (c_\pm \tau)^{\Lambda}U^{(v)}(c_\pm \tau)^{-\Lambda} \tau^{-\Lambda}],
\]
since
\[
\left[ t^{-\Lambda}U^{(v+1)}(t), \Lambda_0 + U^{(v)}(t) \right] = t^{-\Lambda}[U^{(v+1)}(t), t^{\Lambda}(\Lambda_0 + U^{(v)}(t))^{-\Lambda}].
\]

Using this new recursive relation, we may inductively show that, for every integer \( v, \) \((c_\pm \tau)^{\Lambda}U^{(v)}_0(c_\pm \tau)^{-\Lambda}, U^{\infty}(v))\) does not depend on the choice of the sign \( \pm. \)

Write \( \hat{U}^{\infty}_0(k') = \lim_{v \to \infty}(c_\pm \tau)^{\Lambda}U^{(v)}_0(c_\pm \tau)^{-\Lambda}. \) Setting \( A^*_0(\rho, x) = T\hat{\Lambda}_0(\rho x) T^{-1} + \hat{U}^{\infty}_0(1/\rho, \rho x), \) which is equal to \( A_0(c_\pm \rho, x) = A^*_0(c_\pm \rho, x), \) we arrive at the conclusion of Proposition 5.1. \( \Box \)

### 5.4. Proof of Theorem 2.5

Suppose that \( \theta_\infty = 0. \) Let \( \Lambda_0 = -\Lambda_\xi \) be as in Lemma 4.3. Then \( U_0 \) and \( U_\infty \) such that \( A_0 = A_0 + U_0, A_0 + A_\xi = U_\infty \) satisfy
\[
\begin{align*}
\tau \frac{dU_0}{dt} &= [U_\infty, \Lambda_0 + U_0], \\
\tau \frac{dU_\infty}{dt} &= t[J, -\Lambda_0 - U_0 + U_\infty]
\end{align*}
\]
with \( t = x/2 \) (cf. (3.1) and (3.2)).

**PROPOSITION 5.2.** System (5.8) admits a solution \((U_0, U_\infty) = (U^*_0(t), U^*_\infty(t))\) with
\[
U^*_0(t) = \sum_{j=1}^{\infty} U^0_j t^j, \quad U^*_\infty(t) = \sum_{j=1}^{\infty} U^\infty_j t^j
\]
holomorphic around \( t = 0. \) Here \( U^0_j, U^\infty_j \in M_2(\mathbb{Q}[\theta_0, a])\), and \( (U^*_\infty(t))_{11} = (U^*_\infty(t))_{22} \equiv 0. \)

**Proof.** For \( V = \sum_{j=0}^{\infty} V_j t^j \in M_2(\mathbb{Q}[\theta_0, a])[[t]] \) and for \( n \in \mathbb{N} \) write \( V = O(t^n) \) if \( V = \sum_{j=n}^{\infty} V_j t^j \). If \( V = O(t) \), we may set \( \mathcal{I}[V] := \sum_{j=1}^{\infty} (V_j / j) t^j \in M_2(\mathbb{Q}[\theta_0, a])[[t]], \) and then \( t(d/dt)\mathcal{I}[V] = V. \) By induction on \( n \) we may define \( U^{(n)}_0, U^{(n)}_\infty \in M_2(\mathbb{Q}[\theta_0, a])[[t]] \) \((n \geq 1)\) by
\[
\begin{align*}
U^{(0)}_\infty &\equiv 0, \quad U^{(0)}_0 \equiv 0, \\
U^{(n+1)}_\infty &= \mathcal{I}[t[J, -\Lambda_0 - U^{(n)}_0 + U^{(n)}_\infty]], \quad U^{(n+1)}_0 = \mathcal{I}[[U^{(n+1)}_\infty, \Lambda_0 + U^{(n)}_0]].
\end{align*}
\]
Indeed, supposing \( U^{(n)}_\infty = O(t) \) and \( U^{(n)}_0 = O(t) \) we easily show \( U^{(n+1)}_\infty = O(t) \) and \( U^{(n+1)}_0 = O(t) \). Furthermore, since

\[
U^{(n+1)}_\infty - U^{(n)}_\infty = \mathcal{I}[I, (U^{(n)}_\infty - U^{(n-1)}_0) + (U^{(n)}_\infty - U^{(n-1)}_\infty)], \\
U^{(n+1)}_0 - U^{(n)}_0 = \mathcal{I}[[U^{(n+1)}_\infty - U^{(n)}_\infty, \Lambda_0 + U^{(n)}_\infty] + [U^{(n)}_\infty, U^{(n)}_0 - U^{(n)}_\infty]], \\
U^{(n+1)}_\infty - U^{(n)}_\infty = O(t), \quad U^{(n+1)}_0 - U^{(n)}_0 = O(t),
\]

we have \( U^{(n)}_\infty - U^{(n-1)}_\infty = O(t^n) \), \( U^{(n)}_0 - U^{(n-1)}_0 = O(t^n) \) for \( n \geq 1 \). Hence, \( U^{(n)}_\infty(t) := \lim_{n \to \infty} U^{(n)}_\infty \), \( U^{(n)}_0(t) := \lim_{n \to \infty} U^{(n)}_0 \) are in \( M_2(\mathbb{Q}[\theta_0, a])[[t]] \), and \((U^{(n)}_\infty(t), U^{(n)}_0(t))\) formally solves (5.8). By [7, Theorem A], \( U^{(n)}_\infty(t) \) and \( U^{(n)}_0(t) \) are convergent around \( t = 0 \).

It is easy to see that the diagonal part of \( U^{(n)}_\infty(t) \) vanishes identically. \( \square \)

From the relations for \( n = 0, 1 \) in the proof above we have

\[
(U^{(n)}_\infty)_1 = 0, \quad (U^{(n)}_\infty)_2 = 2, \quad (U^{(n)}_0)_1 = 4(a(\theta_0 + a)), \quad (U^{(n)}_0)_2 = -2(\theta_0 + 2a), \\
(U^{(n)}_\infty)_1 = 0, \quad (U^{(n)}_\infty)_2 = 2(\theta_0 + 2a + 1).
\]

Note that \((A_0, A_x) = (\Lambda_0 + U^*_0(x/2), -\Lambda_0 - U^*_0(x/2) + U^*_\infty(x/2))\) solves (2.10). Here

\[
(A_0)_1 = -(A_x)_1 = \theta_0/2 + a + 2a(\theta_0 + a)x + \cdots, \quad (A_0)_2 = -1 - (\theta_0 + 2a)x + \cdots, \\
(A_x)_1 = -(A_0)_2 + x + (1/2)(\theta_0 + 2a + 1)x^2 + \cdots,
\]

and, hence, \(-Y_1 = 1 + x + (1/2)(1 - \theta_0 - 2a)x^2 + \cdots\). If \( \theta_0 = -\theta_\xi \), then \(-Y_1 = 1\); and if \( \theta_0 = \theta_\xi \), then \(-Y_1 = (a + \theta_0)a^{-1}(1 - 2\theta_0x + \cdots)\). From these series we obtain \( y_Taylor^\pm(a, x) \).

### 6. Proof of Theorem 2.6

Recall the Bäcklund transformation for (V) by Gromak [6] (see also [7, Section 39] and [18]).

**Lemma 6.1.** Let \( y \) be a given solution of (V) and let \( \pi \) be the substitution defined by (2.4), that is,

\[
\pi: (\theta_0 - \theta_x, \theta_0 + \theta_x, \theta_\infty) \mapsto (1 - \theta_\infty, 1 - \theta_0 + \theta_x, \theta_0 + \theta_x - 1).
\]

Set

\[
\hat{y} = \hat{B}(y) := 1 - \frac{2xy}{xy^\prime - (\theta_0 - \theta_x + \theta_\infty)y^2/2 + (\theta_\infty + x)y + (\theta_0 - \theta_x - \theta_\infty)/2}
\]

\[
= 1 - \frac{2xy}{2(y - 1)^2(A_x)_1 + \theta_x y^2 + 2xy - \theta_x}.
\]

Then \( \hat{y}^\pi \), which is the result of application of \( \pi \) to \( \hat{y} \), solves (V), that is,

\[
\hat{y}^\pi = \hat{B}(y)^\pi = B(y^\pi) := 1 - \frac{2xy^\pi}{B_{\text{den}}},
\]

\[
B_{\text{den}} = x(y^\pi) - \frac{1}{2}((\theta_0 + \theta_x - \theta_\infty)(y^\pi)^2 + (\theta_0 + \theta_x - 1 + x)y^\pi + 1 - \frac{1}{2}((\theta_0 + \theta_x + \theta_\infty)
\]

is also a solution of (V).
The second expression of \( \hat{B}(y) \) follows from

\[
2(A_1)_{11}(y - 1)^2 = xy' - (\theta_0 + \theta_\infty)(y - 1)^2 - xy + (1/2)(y - 1)((\theta_0 - \theta_x + \theta_\infty)y - (3\theta_0 + \theta_x + \theta_\infty))
\]

(cf. \([2, 1.2), (1.3), (1.4)]\) and \([15]\)).

Concerning the uniqueness of a solution of \((V)\) near \(x = 0\) we have the following.

**Lemma 6.2.** Let \(\sigma, \rho_0 \in \mathbb{C} \setminus \{0\}\) with \(\text{Im } \sigma \neq 0\). Let \(L(r_0, \omega)\) be the curve defined by \((2.5)\). If \(0 < \omega < 1\) (respectively, \(1 < \omega < 2\)), then a solution of \((V)\) such that

\[
y(x) = 1 + \rho_0 x^{-\sigma}(1 + o(1))\quad \text{(respectively, } 1 + \rho_0 x^{\sigma}(1 + o(1)))
\]

as \(x \to 0\) along \(L(r_0, \omega)_\sigma\) is uniquely determined.

**Proof.** By \(y = \tanh^2(u/2)\), \((V)\) is changed into \(x(xu')' = f(x, e^{-u}, xe^u)\) with

\[
f(x, e^{-u}, xe^u) = \frac{1}{8} \left( (\theta_0 - \theta_x + \theta_\infty)^2 \frac{\sinh(u/2)}{\cosh^3(u/2)} - (\theta_0 - \theta_x - \theta_\infty)^2 \frac{\cosh(u/2)}{\sinh^3(u/2)} \right) + \frac{1}{2} (1 - \theta_0 - \theta_x)x \sinh u + \frac{x^2}{8} \sinh(2u),
\]

where \(f(x, \xi, \eta)\) is holomorphic around \(x = \xi = \eta = 0\) and \(f(0, 0, 0) = 0\). Note that \(|\rho_0 x^{1+\sigma}| = O(|x|^{\omega})\) along \(L(r_0, \omega)_\sigma\). Suppose that \(1 < \omega < 2\) and that \(y(x) = 1 + \rho_0 x^{\sigma}(1 + o(1)) = 1 + O(|x|^{\omega-1})\) along \(L(r_0, \omega)_\sigma\). Let \(u(x)\) and \(v(x)\) be such that \(u(x) = -\sigma \log x - \log(-\rho_0/4) - v(x)\) with \(y(x) = \tanh^2(u(x)/2) = 1 - 4e^{-u(x)}(1 + O(e^{-u(x)}))\). They satisfy \((y(x) - 1)(\rho_0 x^{\sigma})^{-1} = e^{v(x)}(1 + O(e^{-u(x)})) = e^{v(x)}(1 + O(|x|^{\omega-1}))\), which implies \(v(x) = o(1)\) along \(L(r_0, \omega)_\sigma\). Then \(v = v(x)\) solves

\[
x(xu')' = g(x, \rho_0 x^{\sigma}, \rho_0^{-1} x^{1-\sigma}, v),
\]

where \(g(x, \xi, \eta, v) = f(x, e^v, e^{-v})\). The function \(g(x, \xi, \eta, v)\) is holomorphic around \(x = \xi = \eta = v = 0\) and satisfies \(g(x, \xi, \eta, v) = O(|x| + |\xi| + |\eta|)\) and \(g(x, \xi, \eta, \tilde{v}) - g(x, \xi, \eta, v) = O(|x| + |\xi| + |\eta|)|\tilde{v} - v|\) if \(|v|\) and \(|\tilde{v}|\) are small. For \(x, x_0 \in L(r_0, \omega)\)

\[
x_0 u'(x_0) - x v'(x) = \int_{L(x_0) \setminus L(x)} g(t, \rho_0 r^\sigma, \rho_0^{-1} \tau^{1-\sigma}, v(t)) \frac{dt}{t},
\]

where \(L(x) \subset L(r_0, \omega)_\sigma\) is a curve joining 0 to \(x\) given by \(t = \tau e^{i\theta(\tau)}, \tau = |t|, 0 < \tau \leq |x|\) with \(\theta(\tau) = ((1 + \text{Re } \sigma - \omega) \log \tau - r_0)/\text{Im } \sigma\). Observing that \(dt/d\tau = O(1)\) and \(g(t, \rho_0 r^\sigma, \rho_0^{-1} \tau^{1-\sigma}, v(t)) = O(e^{\omega-1} + \tau^{2-\omega})\) along \(L(r_0, \omega)_\sigma\), we have \(xv'(x) \to c_0\) as \(x \to 0\) for some \(c_0 \in \mathbb{C}\). Since \(v(x) = o(1)\), we have \(c_0 = 0\) and hence,

\[
v(x) = \int_{L(x)} \int_{L(s)} g(t, \rho_0 r^\sigma, \rho_0^{-1} \tau^{1-\sigma}, v(t)) \frac{dt}{T} ds.
\]

If \(v_1(x), v_2(x) = o(1)\) are solutions of this equation, then \(\phi(x) = \sup_{t \in L(x)} |v_2(t) - v_1(t)|\) satisfies \(\phi(x) = O(|x|^{\omega-1} + |x|^{2-\omega})\phi(x)\), which implies the uniqueness of \(y(x)\). \(\square\)
By Remark 2.1, $\Omega_{\sigma, \rho}^{-}(\varepsilon_{0})$ and $\Omega_{\sigma, \rho}^{+}(\varepsilon_{0})$ are spanned by $L(r_{0}, \omega)_{\sigma}$ with $0 < \omega < 1$ and $1 < \omega < 2$, respectively. Hence,

$$y(\sigma, \rho, x) = \begin{cases} y^{*}(\sigma, \rho, x) \sim 1 + c(-\sigma)\rho^{-1}x^{-\sigma} & \text{on } L(r_{0}, \omega)_{\sigma} \text{ with } 0 < \omega < 1, \\ y^{*}(\sigma, \rho, x) \sim 1 + c(\sigma)\rho x^{\sigma} & \text{on } L(r_{0}, \omega)_{\sigma} \text{ with } 1 < \omega < 2 \end{cases}$$

with $c(\sigma)$ given by (2.3). To $y(\sigma, \rho, x)$, $y_{\pm}(\sigma, \rho, x)$ and $c(\sigma)$, apply $\pi$ of Lemma 6.1, and denote the results by $y^{\pi}(\sigma, \rho, x)$, $y_{\pm}^{\pi}(\sigma, \rho, x)$ and $\tilde{c}(\sigma) := c^{\pi}(\sigma)$. Then the results of the Bäcklund transformation $y^{*}(\sigma, \rho, x) := B(y^{\pi}(\sigma, \rho, x))$, $y_{\pm}^{*}(\sigma, \rho, x) := B(y_{\pm}^{\pi}(\sigma, \rho, x))$ also solve (V) and satisfy

$$y^{*}(\sigma, \rho, x) = \begin{cases} y^{*}(\sigma, \rho, x) \sim 1 + \frac{2\rho x^{1+\sigma}}{(1 - \theta_{\infty} + \sigma)\tilde{c}(\sigma)} & \text{on } L(r_{0}, \omega)_{\sigma} \text{ with } 0 < \omega < 1, \\ y^{*}(\sigma, \rho, x) \sim 1 + \frac{2\rho^{-1}x^{-1-\sigma}}{(1 - \theta_{\infty} - \sigma)\tilde{c}(\sigma)} & \text{on } L(r_{0}, \omega)_{\sigma} \text{ with } 1 < \omega < 2. \end{cases}$$

From the facts above, for every $\nu \in \mathbb{Z}$ we derive the following expressions along the curve $L(r_{0}, \omega)_{\sigma} = L(r_{0}, \omega - 2\nu)_{\sigma - 2\nu} = L(r_{0}, \omega - 2\nu + 1)_{\sigma - 2\nu + 1}$:

1. If $2\nu < \omega < 2\nu + 1$,
   $$y(\sigma - 2\nu, \rho, x) \sim 1 + c(2\nu - \sigma)\rho^{-1}x^{2\nu - \sigma};$$

2. If $2\nu + 1 < \omega < 2\nu + 2$,
   $$y(\sigma - 2\nu, \rho, x) \sim 1 + c(2\nu - \sigma)\rho x^{\sigma - 2\nu};$$

3. If $2\nu - 1 < \omega < 2\nu$,
   $$y^{*}(\sigma - 2\nu + 1, \hat{\rho}, x) \sim 1 + \frac{2\hat{\rho} x^{2\nu + 2}}{(2 - \theta_{\infty} - 2\nu + \sigma)\tilde{c}(2\nu - 1 - \sigma)};$$

4. If $2\nu < \omega < 2\nu + 1$,
   $$y^{*}(\sigma - 2\nu + 1, \hat{\rho}, x) \sim 1 + \frac{2\hat{\rho}^{-1}x^{2\nu - \sigma}}{(2\nu - \theta_{\infty} - \sigma)\tilde{c}(\sigma - 2\nu + 1)}.$$

By Lemma 6.2, matching (1) with (4), we derive

$$y(\sigma - 2\nu, \rho, x) \equiv y^{*}(\sigma - 2\nu + 1, \hat{\rho}, x) \quad (6.1)$$

if $c(2\nu - \sigma)\rho^{-1} = 2\hat{\rho}^{-1}(2\nu - \theta_{\infty} - \sigma)^{-1}\tilde{c}(\sigma - 2\nu + 1)^{-1}$. Similarly, from (2) and (3) we obtain

$$y(\sigma - 2\nu, \rho, x) \equiv y^{*}(\sigma - 2\nu - 1, \hat{\rho}, x) \quad (6.2)$$

if $c(\sigma - 2\nu)\rho = 2\hat{\rho}(\sigma - \theta_{\infty} - 2\nu)^{-1}\tilde{c}(2\nu + 1 - \sigma)^{-1}$. From these relations Theorem 2.6 follows.

Remark 6.1. We have $y^{*}(\sigma - 2\nu + 1, \rho, x) \equiv y^{*}(\sigma - 2\nu - 1, \gamma^{*}(\sigma, \nu)\rho, x)$, where $\gamma^{*}(\sigma, \nu)$ is given by

$$\frac{1}{4}(\sigma - \theta_{\infty} - 2\nu)(2\nu - \theta_{\infty} - \sigma)c(\sigma - 2\nu)c(2\nu - \sigma)\tilde{c}(2\nu + 1 - \sigma)\tilde{c}(\sigma - 2\nu + 1).$$
7. Proofs of Theorems 2.2 and 2.7–2.12

7.1. Proofs of Theorems 2.7–2.12

By (5.5) and (5.6) we write

\[ Y_1 = \frac{F(x)\Phi_1(x) + O(x)}{G(x)\Psi_1(x) + O(x)}, \quad Y_2 = \frac{F(x)\Phi_2(x) + O(x)}{F(x)\Psi_2(x) + O(x)} \]

in \( D_{\text{even}}(\sigma, \rho, 0) = \{ x \in \mathbb{R}(\mathbb{C} \setminus \{0\}) : |x| < \epsilon_0, \epsilon_0 < |\rho x^\sigma| < \epsilon_0^{-1} \} \), where

\[ F(x) = (\sigma - \theta_\infty)(\sigma + \theta_0 - \theta_x)(\rho x^\sigma)^{1/2} - (\sigma + \theta_\infty)(\sigma - \theta_0 + \theta_x)(\rho x^\sigma)^{-1/2}, \]
\[ G(x) = (\sigma - \theta_\infty)(\rho x^\sigma)^{1/2} + (\sigma + \theta_\infty)(\rho x^\sigma)^{-1/2}, \]
\[ \Phi_1(x) = (\sigma - \theta_0 - \theta_x)(\rho x^\sigma)^{1/2} + (\sigma + \theta_0 + \theta_x)(\rho x^\sigma)^{-1/2}, \]
\[ \Psi_1(x) = (\sigma - \theta_\infty)(\rho x^\sigma)^{1/2} - (\sigma + \theta_\infty)(\rho x^\sigma)-1)^{1/2}, \]
\[ \Phi_2(x) = (\sigma - \theta_\infty)(\sigma - \theta_x)(\rho x^\sigma)^{1/2} + (\sigma + \theta_\infty)(\sigma + \theta_x)^2 - \theta_0^2)(\rho x^\sigma)^{-1/2}, \]
\[ \Psi_2(x) = (\sigma - \theta_\infty)(\sigma - \theta_0 - \theta_x)(\rho x^\sigma)^{1/2} - (\sigma + \theta_\infty)(\sigma + \theta_0 + \theta_x)(\rho x^\sigma)^{-1/2}. \]

Let us seek the zeros of

\[ \Phi_1(x) = (\sigma - \theta_0 - \theta_x)(\rho x^\sigma)^{-1/2}(x^\sigma - \xi_0), \quad \xi_0 = -\frac{\sigma + \theta_0 + \theta_x}{\sigma - \theta_0 - \theta_x} \rho^{-1}. \]

Set \( r_0 = \log|\xi_0| \) and \( \mu_0 = \arg \xi_0 \). Then we have

\[ x^\sigma - \xi_0 = -2i \exp((\sigma \log x + r_0 + i\mu_0)/2) \sin(i\sigma \log x + \mu_0 - ir_0)/2), \]

in which, along \( L(r_0, 1) \sigma \),

\[ \frac{1}{2}(i\sigma \log x + \mu_0 - ir_0) = -\frac{1}{2 \text{Im} \sigma}(|\sigma|^2 \log|x| - r_0 \text{Re} \sigma - \mu_0 \text{Im} \sigma). \]

Hence, \( \Phi_1(x) \) has zeros \( x_n, n \in \mathbb{N} \) such that \( |\sigma|^2 \log|x_n| - r_0 \text{Re} \sigma - \mu_0 \text{Im} \sigma = -2\pi |\text{Im} \sigma|/n \) on \( L(r_0, 1) \sigma \). Under the supposition of Theorem 2.7 the ratio of \( \Phi_1(x) \) to any of the other five functions is not a constant, and then \( F(x_n), \ldots, \Psi_2(x_n) \) other than \( \Phi_1(x_n) \) are non-zero numbers independent of \( n \), since \( x_n^\sigma = \xi_0 \). Using Rouche’s theorem, by the same argument as that of [23, Section 2.2.2] we may prove the existence of the sequence of zeros \( \{x_n^0\}_{n \in \mathbb{N}} \) in Theorem 2.7. The other sequence \( \{\tilde{x}_n^0\}_{n \in \mathbb{N}} \) is obtained from zeros of \( \Phi_2(x) \).

Theorem 2.8 is proved by using \( \Psi_1(x) \) and \( \Psi_2(x) \). Theorem 2.11 is also proved by using (5.7) by the same argument as above. Theorem 2.9 follows from the facts that \( \Phi_1(x)/\Phi_2(x) \) is a constant if \( \theta_0 - \theta_x - \theta_\infty = 0 \) and that then every zero of a solution of (V) is double, and from analogous facts on poles.

To prove Theorem 2.10 we use the Bäcklund transformation. Let \( y \) and \( \hat{y} = \hat{B}(y) \) be as in Lemma 6.1. Then

\[ \frac{1}{\hat{y}} - 1 = \frac{2xy}{2(y - 1)^2(A_x)_{11} + \theta_x(y^2 - 1)}. \]

For \( x \in D_{\text{even}}(\sigma, \rho, 0) \) substitute

\[ y = y(\sigma, \rho, x) = \frac{\Phi_1(x)\Phi_2(x)F(x)G(x) + O(x)}{\Psi_1(x)\Psi_2(x)F(x)G(x) + O(x)} \tag{7.1} \]
into the right-hand side. Observing that

\[(A_x)_{11} + \frac{\theta_x}{2} = \frac{1}{8\sigma^2}(Y_{11})_{\text{den}} = -\frac{1}{8\sigma^2}(G(x)\Psi_1(x) + O(x))\]

with \((Y_{11})_{\text{den}}\) as in (5.5), that \(\Phi_1(x)\Phi_2(x) - \Psi_1(x)\Psi_2(x) = 4\sigma^2(\sigma^2 - (\theta_0 + \theta_x)^2)\), and that \((\sigma^2 - (\theta_0 + \theta_x)^2)G(x) - 2\theta_x\Psi_2(x) = \Phi_2(x)\), we obtain

\[1 - \frac{1}{y} = \frac{x(\Phi_1(x)\Phi_2(x)\Psi_1(x)\Psi_2(x)F(x)^2G(x)^2 + O(x))}{2\sigma^2(\sigma^2 - (\theta_0 + \theta_x)^2)\Phi_2(x)\Psi_1(x)F(x)^2G(x)^2 + O(x)}. \tag{7.2}\]

Under the supposition

\[
\theta_0(\theta_0 + \theta_x - \theta_\infty)((\theta_0 - \theta_x)^2 - \theta_\infty^2)(\sigma^2 - \theta_\infty^2)(\sigma^2 - (\theta_0 - \theta_x)^2) \\
\times (\theta_0^2 - \theta_x^2 + \sigma^2 - 2\theta_0\theta_0) \neq 0, \tag{7.3}
\]

which is a part of those of Theorems 2.7 and 2.8, the ratio of \(\Phi_1(x)\) or of \(\Psi_2(x)\) to any of the other five functions is not a constant and its zeros have the same property as that of \(x\xi\) above.

Setting \(\sigma = \hat{\sigma} - 1\). From the result of the application of \(\pi\) to (7.2) with \((\hat{\sigma}, \hat{\rho})\) we conclude that \(y^*(\hat{\sigma}, \hat{\rho}, x) = B(y^*(\hat{\sigma}, \hat{\rho}, x)) = \hat{y}^*(\hat{\sigma}, \hat{\rho}, x)\) solving (V) has 1-points near the zeros of \(\Phi_1^T(x)\) and \(\Psi_2^T(x)\). By (6.1) with \(v = 0\) we have \(y(\sigma, \rho, x) \equiv y^*(\hat{\sigma}, \hat{\rho}, x)\) if

\[
\hat{\sigma} = \sigma + 1, \quad \hat{\rho} = -\frac{1}{\rho}(\theta_\infty + \sigma)\hat{c}(\sigma + 1)\hat{c}(-\sigma)\rho^{-1}/2.
\]

Observing that \(\Phi_1^T(x)\) (respectively, \(\Psi_2^T(x)\)) vanishes if

\[
\chi = \xi_1 = -\frac{\hat{\sigma} + 1 - \theta_0 + \theta_x}{\hat{\sigma} - 1 + \theta_0 - \theta_x} \hat{\rho}^{-1} \quad \text{respectively, } \chi = \hat{\xi}_1 = -\frac{\hat{\sigma} - 1 + \theta_0 + \theta_x}{\hat{\sigma} - 1 - \theta_0 - \theta_x} \hat{\xi}_1,
\]

and that \(L(r_1, 1)_{\sigma} = L(r_1, 1)_{\sigma + 1} = L(r_1, 0)_{\sigma}\) with \(r_1 = \log|\xi_1|\), we obtain Theorem 2.10. By the expressions of \(y_{\sigma_0}(\rho, x)\) in \(\Omega^{-}\)(\(\sigma_0, \varepsilon_0\)) of Theorem 2.3, \(y_{\sigma_0}(\rho, x) \sim 1 + c^*(\sigma_0)\rho^{-1}x^{-\sigma_0}\) on \(L(\rho_0, \omega)\), with \(0 < \omega < 1\) in each case, \(r_0\) being a fixed number. On the other hand

\[
y^*(\sigma_0 + 1, \hat{\rho}, x) = y^*_1(\sigma_0 + 1, \hat{\rho}, x) \sim 1 - \frac{2\hat{\rho}^{-1}x^{-\sigma_0}}{(\theta_\infty + \sigma_0)\hat{c}(\sigma_0 + 1)}
\]

on \(L(\rho_0, \omega + 1)_{\sigma_0 + 1} = L(\rho_0, \omega)_{\sigma_0}\) with \(0 < \omega < 1\) (cf. Section 6). Hence, \(y_{\sigma_0}(\rho, x) \equiv y^*(\sigma_0 + 1, \hat{\rho}, x)\) with \(-\theta_\infty + \sigma_0)c^*(\sigma_0)\hat{c}(\sigma_0 + 1)\hat{\rho} = 2\rho\). Using this fact we may show Theorem 2.12.

### 7.2. Proof of Theorem 2.2

If \(\theta_0 - \theta_x = \theta_\infty = 0\), then by [22, Theorem 5.6], equation (V) admits a family of solutions given by

\[
tanh^2\left(\left(\frac{1}{2} - \hat{\sigma}\right)\log x + \frac{1}{2} \log \hat{\rho} + s_\nu(\hat{\sigma}, x, \hat{\rho}^{-1/2}x^{\hat{\sigma}}, \hat{\rho}^{1/2}x^{1-\hat{\sigma}})\right)
\]

with \(\hat{\sigma} \in \hat{\Sigma} \subset C \setminus ([\hat{\sigma} \leq 0] \cup [\hat{\sigma} \geq 1])\), \(\hat{\rho} \in C \setminus \{0\}\), \(\hat{\Sigma}\) being a bounded domain such that \(\text{dist}(\hat{\Sigma}, [\hat{\sigma} \leq 0] \cup [\hat{\sigma} \geq 1]) > 0\). The series

\[
s_\nu(\hat{\sigma}, x, \xi, \eta) = \sum_{i \geq 1} c_{i0}^\nu(\hat{\sigma})x^i + \sum_{i \geq 0, j \geq 1} c_{ij}^\nu(\hat{\sigma})x^i\xi^j + \sum_{i \geq 0, j \geq 1} c_{ij}^2(\hat{\sigma})x^i\eta^{2j}
\]
with $c_i^0(\tilde{\sigma})$, $c_{ij}^1(\tilde{\sigma})$, $c_{ij}^2(\tilde{\sigma}) \in \mathbb{Q}[\theta_0](\tilde{\sigma})$ converges if $|x|$, $|\xi|$, $|\eta|$ are sufficiently small. Note that we have replaced $(\xi, \eta)$ in [22, Theorem 5.6] by $(\xi^2, \eta^2)$ since, under the condition $\theta_0 - \theta_\infty = 0$, the local equation in [22, Section 5] equivalent to (V) is written in terms of $F(x, \xi, \eta) = (1/32)(\eta^2 - \xi^2)(4(1 - \theta_0 - \theta_\infty) + (\eta^2 + \xi^2))$. Putting $\sigma = 1 - 2\tilde{\sigma} \in \Sigma_0$, we have

$$\tanh^2\left(\frac{1}{2} \log(\tilde{\rho}x^\sigma) + s_\nu((1 - \sigma)/2, x), (\tilde{\rho} - 1)x^{1-\sigma})^{1/2}, (\tilde{\rho}x^{1+\sigma})^{1/2}\right),$$

which satisfies

$$= \begin{cases} 
1 - 4\tilde{\rho}x^\sigma (1 + o(1)) & \text{in } D_+(\sigma, \tilde{\rho}, 0), \\
1 - 4(\tilde{\rho}x^\sigma)^{-1}(1 + o(1)) & \text{in } D_- (\sigma, \tilde{\rho}, 0).
\end{cases}$$

Since $((\tilde{\rho}^{\pm 1}x^{1\pm\sigma})^{1/2})^{2j} = x^j(\tilde{\rho}x^\sigma)^{\pm j}$, $s_\nu(\cdots)$ has the form $\sum_{n=1}^\infty x^n \sum_{j=-n}^n c_{nj}(\tilde{\rho}x^\sigma)^j$.

Noting that $c(\sigma) = -4(2\theta_0 - \sigma)(2\theta_0 + \sigma)^{-1}$, we compare the behaviour above with that of $y(\sigma, \rho, x)$ to obtain (2.1). By Lemma 6.1, $\hat{y} = \hat{B}(y) = 1 - 2y/(y' + y)$. Substitution of (2.1) with $(\sigma, \rho) = (\tilde{\sigma}, \tilde{\rho})$ yields

$$\hat{y}(\tilde{\sigma}, \tilde{\rho}, x) = 1 - \frac{2x \sinh(\log(\tilde{\rho}_-x^\tilde{\sigma}) + \Sigma_\nu(x))}{2\tilde{\sigma} + 2x \Sigma'_\nu(x) + x \sinh(\log(\tilde{\rho}_-x^\tilde{\sigma}) + \Sigma_\nu(x))},$$

with $\tilde{\rho}_- = (2\theta_0 - \tilde{\sigma})(2\theta_0 + \tilde{\sigma})^{-1} \hat{\rho}$, where

$$\Sigma_\nu(x) = 2\nu((1 - \tilde{\sigma})/2, x), (\tilde{\rho}^{-1}_-x^{1-\tilde{\sigma}})^{1/2}, (\tilde{\rho}^{-1}_-x^{1+\tilde{\sigma}})^{1/2}$$

and $\Sigma'_\nu(x) = (d/dx)\Sigma_\nu(x)$. Apply $\pi$ to both sides. Observing that $\hat{\rho} := (\tilde{\rho}_-)^\pi = (2 - \tilde{\sigma})(2 + \tilde{\sigma})^{-1} \hat{\rho}$, and that $y^\pi(\tilde{\sigma}, \tilde{\rho}, x) = \hat{y}^\pi(\tilde{\sigma}, \tilde{\rho}, x)$ coincides with $y(\sigma, \rho, x)$ if $\tilde{\sigma} = \sigma + 1$ and

$$\hat{\rho}^{-1} = -\sigma \tilde{c}(\sigma + 1)c(-\sigma)\frac{\rho^{-1}}{2} = \frac{8\sigma(\sigma + 1)^2}{(\sigma + 2)(2\theta_0 - \sigma)} \rho^{-1},$$

we obtain (2.2).

8. Monodromy data for the isomonodromy deformation

Let $(A_0(x), A_\lambda(x))$ be the solution in Proposition 3.1 or 3.2 yielding each solution of (V) in Section 2 (cf. Proposition 3.3 and Section 5). The associated linear system (1.1) admits the matrix solution

$$Y(\lambda, x) = (I + O(\lambda^{-1}))e^{(\lambda/2)}(\lambda - (\theta_\infty/2))^{\lambda} \rightarrow \infty, \quad -\pi/2 < \arg \lambda < 3\pi/2$$

having the isomonodromy property. Set $\tilde{\rho} = \rho_\sigma^{1/\sigma}$ with

$$\rho_\sigma := \frac{(\theta_0 - \theta_\xi + \sigma)(\theta_0 + \theta_\xi - \sigma)}{2\sigma(\sigma + \theta_\infty)} \rho$$

if $(A_0(x), A_\lambda(x))$ yields $y(\sigma, \rho, x)$ (cf. (5.4)), $\tilde{\rho} = \rho^{1/\sigma_0}$ if it yields $y_{\sigma_0}(\rho, x)$, and $\tilde{\rho} = \rho$ if $\sigma = 0$. Then by Proposition 3.1 or 3.2, $A_0$ and $A_\lambda$ satisfy

$$x^{-\Lambda}A_0x^\Lambda \rightarrow \Lambda_0 := \tilde{\rho}^\Lambda \Lambda_0 \tilde{\rho}^{-\Lambda}, \quad x^{-\Lambda}A_\lambda x^\Lambda \rightarrow \Lambda_\lambda := \tilde{\rho}^\Lambda \Lambda_\lambda \tilde{\rho}^{-\Lambda}, \quad A_0 + A_\lambda \rightarrow \Lambda$$

as $x \rightarrow 0$, the eigenvalues of $\Lambda$ being $\pm \sigma/2$ with $\sigma \in (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\}$. 
Let us apply the argument of [14, Section 2] to our case. By [21]

\[ \hat{Y}(\lambda) := \lim_{x \to 0} Y(\lambda, x) \quad \text{and} \quad \tilde{Y}(\lambda) := \lim_{x \to 0} x^{-\lambda} Y(x\lambda, x) \]

solve

\[ \frac{d\hat{Y}}{d\lambda} = \left( \frac{\lambda}{\lambda} + \frac{J}{2} \right) \hat{Y} \quad (8.1) \]

and

\[ \frac{d\tilde{Y}}{d\lambda} = \left( \frac{\tilde{\lambda}}{\lambda} + \frac{\tilde{\lambda}}{\lambda - 1} \right) \tilde{Y}, \quad (8.2) \]

respectively. Since \( \sigma \in (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\} \), we may choose a matrix solution of (8.1) such that

\[ \hat{Y}_0(\lambda) = (I + O(\lambda^{-1}))e^{(\lambda/2)J} \lambda^{-(\theta_{\infty}/2)J} \lambda \to \infty, \]

\[ = (I + O(\lambda))\lambda^\Delta C \lambda \to 0, \]

and of (8.2) such that

\[ \tilde{Y}_0(\lambda) = (I + O(\lambda^{-1}))\lambda^\Delta \lambda \to \infty, \]

\[ = G_0(I + O(\lambda)\lambda^{(\theta_0/2)J} \lambda^{\Delta_0} C^{(0)}) \lambda \to 0, \]

\[ = G_x(I + O(\lambda - 1)(\lambda - 1)^{(\theta_x/2)J} (\lambda - 1)^\Delta_x C^{(x)}) \lambda \to 1 \]

for \( 0 < \arg \lambda < \pi, \quad 0 < \arg(\lambda - 1) < \pi \), where \( C, C^{(0)}, C^{(x)}, G_0, G_x \) are some invertible matrices, and, for each \( \iota = 0, x \), the matrix \( \Delta_\iota \) equals zero if \( \theta_\iota \notin \mathbb{Z}, \epsilon_\iota \Delta_\iota \) if \( \theta_\iota \in \mathbb{N} \cup \{0\} \), and \( \epsilon_\iota \Delta_\iota \) if \( \theta_\iota \in \mathbb{N} \). The multiplier \( \epsilon_\iota = \epsilon_\iota(\lambda_0, \Lambda_x) \) is some complex number depending only on \( \lambda_0, \Lambda_x \). Then by the same argument as in the proof of [14, Proposition 2.1]

\[ \hat{Y}(\lambda) = \lim_{x \to 0} Y(\lambda, x) = \hat{Y}_0(\lambda), \quad \tilde{Y}(\lambda) = \lim_{x \to 0} x^{-\lambda} Y(x\lambda, x) = \tilde{Y}_0(\lambda)C \]

as long as \( |x^{1+\sigma}|, |x^{1-\sigma}| \to 0 \), and

\[ Y(\lambda, x) = G_0(x)(I + O(\lambda))\lambda^{(\theta_0/2)J} \lambda^{\Delta_0} C^{(0)} \lambda \to 0, \]

\[ = G_x(x)(I + O(\lambda - x))(\lambda - x)^{(\theta_x/2)J} (\lambda - x)^\Delta_x C^{(x)} \lambda \to x, \]

where \( G_0(x) \) and \( G_x(x) \) are invertible matrices. Therefore the monodromy matrices \( M_0, M_x \) defined in Section 2.4 are written as follows:

\[ M_\iota = \begin{cases} (C^{(i)}C)^{-1}e^{\pi i \theta_\iota J} C^{(i)}C & \theta_\iota \notin \mathbb{Z}, \\ (-1)^{\theta_\iota} (C^{(i)}C)^{-1}e^{2\pi i \Delta_\iota} C^{(i)}C & \theta_\iota \in \mathbb{Z} \end{cases} \quad (8.4) \]

(\( \iota = 0, x \)). Monodromy data for each solution will be computed by using this fact.

Remark 8.1. The matrices \( C^{(i)}(\iota = 0, x) \) are not determined uniquely. Indeed, say, if \( \theta_\iota \notin \mathbb{Z} \), we may take \( G_\iota(x)D_0^{-1}, D_0C^{(i)} \) with any invertible diagonal matrix \( D_0 \) instead of \( G_\iota(x), C^{(i)} \).

Remark 8.2. In our setting of isomonodromy deformation for system (1.1) with \( x \neq 0 \), the monodromy data and \( y \) are the same as those in [2], and the unknown variables corresponding to \( z = z_{AK} \) and \( u = u_{AK} \) in [2] are \( z_{AK} \) and \( x^{-\theta_{\infty}}u_{AK} \), respectively.
9. Connection formulas for the Whittaker and the hypergeometric systems

9.1. The Whittaker system

Since \((A_0(x), A_x(x))\) mentioned in Section 8 is obtained by using \(\Lambda_0, \Lambda_x, T, \Lambda\) in Lemma 4.1 or 4.2, our concern is (8.1) corresponding to such matrices. The connection matrix \(C\) for (8.1) is given by the following.

**Proposition 9.1.** (1) Suppose that \(\sigma \notin \mathbb{Z}\). For \(\Lambda, T\) as in Lemma 4.1, \(C = TC_\infty\) with

\[
C_\infty = \begin{pmatrix} \frac{-e^{-\pi i(\sigma + \theta_\infty)/2} \Gamma(-\sigma)}{\Gamma(1 - (\sigma - \theta_\infty)/2)} & \frac{\Gamma(-\sigma)}{\Gamma(1 - (\sigma + \theta_\infty)/2)} \\ \frac{e^{\pi i(\theta_\infty)/2} \Gamma(1 + \theta_\infty/2)}{\Gamma(1 - \theta_\infty/2)} & 1 \\ \frac{\Gamma(\sigma)}{\Gamma((\sigma - \theta_\infty)/2)} & \frac{\Gamma(\sigma)}{\Gamma((\sigma + \theta_\infty)/2)} \end{pmatrix}.
\]

(2) Suppose that \(\sigma = 0\). For \(\Lambda, T\) as in Lemma 4.2, \(C = TC_\infty\) with \(C_\infty\) given as follows:

(i) if \(\theta_\infty \neq 0\),

\[
C_\infty = \begin{pmatrix} \frac{e^{-\pi i \theta_\infty/2} (\psi(1 + \theta_\infty/2) - 2\psi(1) - \pi i)}{\Gamma(1 + \theta_\infty/2)} & \frac{\psi(-\theta_\infty/2) - 2\psi(1)}{\Gamma(1 - \theta_\infty/2)} \\ \frac{e^{-\pi i \theta_\infty/2}}{\Gamma(1 + \theta_\infty/2)} & 1 \\ \frac{\Gamma(\sigma)}{\Gamma((\sigma - \theta_\infty)/2)} & \frac{\Gamma(\sigma)}{\Gamma((\sigma + \theta_\infty)/2)} \end{pmatrix};
\]

(ii) if \(\theta_\infty = 0\) and \(\Lambda = \Delta\), then \(C_\infty = I - \psi(1)\Delta\);

(iii) if \(\theta_\infty = 0\) and \(\Lambda = \Delta_+\), then \(C_\infty = (1 - \pi i - \psi(1))(I + J)/2 + \Delta + \Delta_+\).

**Proof.** If \((\sigma, \theta_\infty) \neq (0, 0)\), then system (8.1) with \(\Lambda\) as in Lemma 4.1 or 4.2 has the matrix solution \(\hat{Y}(\lambda)\) given by

\[
\left(\begin{array}{cc}
e^{-\pi i (1-\theta_\infty)/2} W_{(1-\theta_\infty)/2, \sigma/2} \left(e^{-\pi i \lambda}\right) \\

\frac{1}{2} (\sigma + \theta_\infty) e^{-\pi i (1-\theta_\infty)/2} W_{-(1+\theta_\infty)/2, \sigma/2} \left(e^{-\pi i \lambda}\right)
\end{array}\right) \lambda^{-1/2}
\]

(cf.\[14, (3.10)\]), which behaves as

\[
\hat{Y}(\lambda) = (I + O(\lambda^{-1})) e^{(\lambda/2)J} \lambda^{-\theta_\infty/2} J \lambda \to \infty, \ 0 < \arg \lambda < \pi.
\]

Here \(W_{k,\mu}(z)\) is the Whittaker function such that \(W_{k,\mu}(z) \sim e^{-z/2} z^\kappa, \ |\arg z| < \pi\). If \(\sigma \notin \mathbb{Z}\), we have, around \(\lambda = 0\),

\[
\hat{Y}(\lambda) = (I + O(\lambda)) \lambda^\Delta C = T (I + O(\lambda)) \lambda^{(\sigma/2)J} T^{-1} C,
\]

where \(T\) is as in Lemma 4.1. Using the connection formula

\[
e^{z/2} z^{-1/2} W_{k,\sigma/2} (z) = \frac{\Gamma(-\sigma) z^{\sigma/2}}{\Gamma((1 - \sigma)/2 - \kappa)} (1 + O(z)) + \frac{\Gamma(\sigma) z^{-\sigma/2}}{\Gamma((1 + \sigma)/2 - \kappa)} (1 + O(z))
\]

(cf. \[1, 13.1.3\] and \[13.1.33\]), we compare the behaviours around \(\lambda = 0\) to derive \(C\) as in (1). If \(\sigma = 0\) and \(\theta_\infty \neq 0\), then, around \(\lambda = 0\),

\[
\hat{Y}(\lambda) = (I + O(\lambda)) \lambda^\Delta C = T (I + O(\lambda)) \lambda^\Delta T^{-1} C,
\]
where $T$ and $\Lambda$ are as in Lemma 4.2. Note that (1,1)- and (1,2)-entries of $T(I + O(\lambda))^{\lambda,\Delta}$ are $-\theta_{\infty}/2 + O(\lambda)$ and $-(\theta_{\infty}/2) \log \lambda + 1 + O(\lambda)$, respectively. On the other hand, using

$$e^{z/2}z^{-1/2}W_{\kappa,0}(z) = -\frac{1}{\Gamma(1/2 - \kappa)}((1 + O(z)) \log z + \psi(1/2 - \kappa) - 2\psi(1) + O(z))$$

(cf. [1, 13.1.6]), we can also see how $\tilde{Y}(\lambda)$ behaves around $\lambda = 0$. Comparison of these leads us to $C$ as in (2)(i). Under the supposition $\sigma = \theta_{\infty} = 0$, if $\Lambda = \Delta$ (respectively, $\Lambda = \Delta_-)$, then (8.1) admits the matrix solution

$$\begin{pmatrix} e^{\lambda/2} & -\lambda^{-1/2}W_{-1/2,0}(\lambda) \\ 0 & e^{-\lambda/2} \end{pmatrix} \text{respectively, } \begin{pmatrix} e^{\lambda/2} & 0 \\ -\pi i/2\lambda^{-1/2}W_{-1/2,0}(e^{-\pi i \lambda}) & e^{-\lambda/2} \end{pmatrix}.$$  

Using the connection formula above together with $T$ in each case, we find the matrices in (ii) and (iii).

9.2. The hypergeometric system

Let us begin with

$$\frac{du}{dz} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 - \gamma \end{pmatrix} + \frac{1}{z-1} \begin{pmatrix} 0 & 0 \\ -\alpha \beta & \gamma - \alpha - \beta - 1 \end{pmatrix}u, \quad u = \begin{pmatrix} u \\ zu' \end{pmatrix},$$

in which $u$ solves the hypergeometric equation $z(1-z)u'' + (\gamma - (\alpha + \beta + 1)z)u' - \alpha\beta u = 0$. The eigenvalues of the residue matrices are 0, $1 - \gamma$ at $z = 0$, $\gamma - \alpha - \beta - 1$ at $z = 1$ and $\alpha, \beta$ at $z = \infty$. Under the supposition $\alpha - \beta \notin \mathbb{Z}$, diagonalizing the residue matrix at $z = \infty$ and shifting the eigenvalues to $\pm(1-\gamma)/2, \pm(\gamma - \alpha - \beta - 1)/2, \pm(\beta - \alpha)/2$, we obtain the system

$$\frac{d\Psi}{dz} = \left( \frac{B_0}{z} + \frac{B_1}{z-1} \right) \Psi,$$

with $R \neq 0$, which has the following property.

**Proposition 9.2.** Suppose that $\alpha - \beta \notin \mathbb{Z}$. System (9.2) admits the matrix solution

$$\Psi(z) = z^{-(1-\gamma)/2}(z-1)^{-(\gamma - \alpha - \beta - 1)/2}\text{diag}[1, R][v_\alpha v_\beta]\text{diag}[1, 1/R]$$

with

$$[v_\alpha v_\beta] = P^{-1} \begin{pmatrix} u_\alpha & u_\beta \\ zu'_\alpha & zu'_\beta \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -\alpha & -\beta \end{pmatrix},$$

$$u_\alpha := z^{-\alpha}F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, z^{-1}),$$

$$u_\beta := z^{-\beta}F(\beta, \beta - \gamma + 1, \beta - \alpha + 1, z^{-1}),$$

which satisfies

$$\Psi(z) = (I + O(z^{-1}))z^{(\beta - \alpha)/2} \quad z \to \infty.$$
We substitute the connection formulas representing \( u_\alpha \) and \( u_\beta \) as linear combinations of hypergeometric series around \( z = 0 \) or \( z = 1 \) (cf. [5, Section 2.9]) and choose gauge matrices suitably to obtain the following proposition.

**PROPOSITION 9.3.** Suppose that \( \alpha - \beta, \gamma, \gamma - \alpha - \beta \notin \mathbb{Z} \). Then

\[
\Psi(z)|_{R=1} = G_0(I + O(z))z^{(1-\gamma)/2} \tilde{C}_0 \quad z \to 0, \\
= G_1(I + O(z - 1))(z - 1)^{(\gamma - \alpha - \beta - 1)/2} \tilde{C}_1 \quad z \to 1
\]

for \(|\arg z - \pi| < \pi, |\arg(z - 1) - \pi| < \pi\), where

\[
\tilde{C}_0 = \left( \begin{array}{cc}
\frac{\Gamma(1 - \beta + \alpha) \Gamma(\gamma + 1)}{\Gamma(1 - \gamma + \alpha) \Gamma(1 - \beta)} & \frac{\Gamma(\gamma - \alpha - 1) \Gamma(1 - \alpha + \beta) \Gamma(\gamma + 1)}{\Gamma(1 - \gamma + \alpha) \Gamma(1 - \beta)} \\
\frac{\Gamma(\gamma - \alpha - 1) \Gamma(1 - \alpha + \beta) \Gamma(\gamma + 1)}{\Gamma(1 - \gamma + \alpha) \Gamma(1 - \beta)} & \frac{\Gamma(\gamma - \alpha) \Gamma(1 - \alpha + \beta) \Gamma(\gamma + 1)}{\Gamma(1 - \gamma + \alpha) \Gamma(1 - \beta)}
\end{array} \right), \\
\tilde{C}_1 = \left( \begin{array}{cc}
\frac{\Gamma(1 - \beta + \alpha) \Gamma(\gamma - \alpha - 1)}{\Gamma(1 - \gamma + \alpha) \Gamma(\gamma - \alpha)} & \frac{\Gamma(1 - \alpha + \beta) \Gamma(\gamma - \alpha - 1)}{\Gamma(1 - \gamma + \alpha) \Gamma(\gamma - \alpha)} \\
\frac{\Gamma(1 - \beta + \alpha) \Gamma(\gamma - \alpha - 1)}{\Gamma(1 - \gamma + \alpha) \Gamma(\gamma - \alpha)} & \frac{\Gamma(1 - \alpha + \beta) \Gamma(\gamma - \alpha - 1)}{\Gamma(1 - \gamma + \alpha) \Gamma(\gamma - \alpha)}
\end{array} \right).
\]

In the case where \( \alpha = \beta \), we transform the system in such a way that the residue matrix at \( z = \infty \) becomes \( \Delta_1 \). The result of this procedure is

\[
\frac{d\Psi}{dz} = \left( \frac{B_0}{z} + \frac{B_1}{z - 1} \right) \Psi, \quad (9.2)
\]

\[
B_0 = \begin{pmatrix} (\gamma - 2\alpha - 1)/2 & 1 \\ \alpha(\gamma - \alpha - 1) & - (\gamma - 2\alpha - 1)/2 \end{pmatrix},
\]

\[
B_1 = \begin{pmatrix} (2\alpha - \gamma + 1)/2 & 0 \\ \alpha(\alpha - \gamma + 1) & - (2\alpha - \gamma + 1)/2 \end{pmatrix}.
\]

Write, for \( m \in \mathbb{N} \),

\[
F_{\log}(\alpha, \beta, m, z) := F(\alpha, \beta, m, z) \log z + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(m)_n n!} \psi_n(\alpha, \beta, m) z^n
\]

\[
- \sum_{n=1}^{m-1} \frac{(n - 1)!(1 - m)_n}{(1 - \alpha)_n (1 - \beta)_n} z^{-n}
\]

with

\[
\psi_n(\alpha, \beta, m) = \psi(\alpha + n) - \psi(\alpha) + \psi(\beta + n) - \psi(\beta) \\
- \psi(m + n) + \psi(m) - \psi(1 + n) + \psi(1).
\]
Then $z^{-\alpha} F_{\log}(\alpha, \alpha - \gamma + 1, 1, z^{-1})$ is a logarithmic hypergeometric function near $z = \infty$, which satisfies, for $|\arg z - \pi| < \pi$,

$$
- z^{-\alpha} F_{\log}(\alpha, \alpha - \gamma + 1, 1, z^{-1}) = \frac{\Gamma(\alpha) \Gamma(\gamma - \alpha)}{\Gamma(\gamma)} e^{-\pi i \alpha} F(\alpha, \alpha, \gamma, z)
$$

$$
- (2\psi(1) - \psi(\alpha) - \psi(\gamma - \alpha - \pi i) z^{-\alpha} F(\alpha, \alpha - \gamma + 1, 1, z^{-1})
$$

(9.3)

(proposition 9.4. System (9.2) admits the matrix solution

$$
\Psi(z) = z^{-(1-\gamma)/2} (z - 1)^{-(\gamma - 2\alpha - 1)/2} [v_\alpha \ v_\alpha \log]
$$

with

$$
[v_\alpha \ v_\alpha \log] = P^{-1} \begin{pmatrix} u_\alpha & u_\alpha \log \\ z u'_\alpha & z u'_\alpha \log \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix},
$$

$$
u_\alpha := z^{-\alpha} F(\alpha, \alpha - \gamma + 1, 1, z^{-1}),
$$

$$
u_\alpha \log := - z^{-\alpha} F_{\log}(\alpha, \alpha - \gamma + 1, 1, z^{-1}),
$$

and $\Psi(z)$ satisfies

$$
\Psi(z) = (I + O(z^{-1})) z^\Delta \quad z \to \infty.
$$

In [5, Section 2.10, (3)] replacing $z$ by $1 - z$ and applying limiting procedure, we get, for $|\arg z| < \pi$,

$$
F(\alpha, \alpha, 2\alpha - \gamma + 1, 1 - z)
$$

$$
= \frac{\Gamma(2\alpha - \gamma + 1)}{\Gamma(\alpha) \Gamma(\alpha - \gamma + 1)} z^{-\alpha} (- F_{\log}(\alpha, \alpha - \gamma + 1, 1, z^{-1})
$$

$$
- (\psi(\alpha) + \psi(\alpha - \gamma + 1) - 2\psi(1)) F(\alpha, \alpha - \gamma + 1, 1, z^{-1})).
$$

(9.4)

Using (9.3) and this relation, we have the following:

(i) if $\gamma \notin \mathbb{Z}$, then, for $|\arg z - \pi| < \pi$,

$$
z^{-\alpha} F(\alpha, \alpha - \gamma + 1, 1, z^{-1})
$$

$$
= e^{-\pi i \alpha} \frac{\Gamma(1 - \gamma)}{\Gamma(1 - \alpha) \Gamma(\alpha - \gamma + 1)} f_0(z) - e^{\pi i (\gamma - \alpha)} \frac{\Gamma(\gamma - 1)}{\Gamma(\gamma - \alpha) \Gamma(1 - \alpha)} g_0(z),
$$

$$
- z^{-\alpha} F_{\log}(\alpha, \alpha - \gamma + 1, 1, z^{-1})
$$

$$
= - e^{-\pi i \alpha} \frac{2\psi(1) - \psi(1 - \alpha) - \psi(\alpha - \gamma + 1) - \pi i \Gamma(1 - \gamma)}{\Gamma(1 - \alpha) \Gamma(\alpha - \gamma + 1)} f_0(z)
$$

$$
+ e^{\pi i (\gamma - \alpha)} \frac{2\psi(1) - \psi(\alpha) - \psi(\gamma - \alpha) - \pi i \Gamma(\gamma - 1)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} g_0(z)
$$

with

$$
f_0(z) = F(\alpha, \alpha, \gamma, z), \quad g_0(z) = z^{1-\gamma} F(\alpha - \gamma + 1, \alpha - \gamma + 1, 2 - \gamma, z);\]
Then we have the following proposition.

**PROPOSITION 9.5.** Suppose that \( \alpha = \beta \) and that \( \gamma, \gamma - 2\alpha \notin \mathbb{Z} \). Then
\[
\Psi(z) = G_0(I + O(z))z^{(1-\gamma)J/2}\tilde{C}_0, \quad \text{as } z \to 0,
\]
\[
= G_1(I + O(z-1))(z-1)^{(\gamma-2\alpha)J/2}\tilde{C}_1, \quad \text{as } z \to 1
\]
for \( 0 < \arg z < \pi, |\arg(z-1) - \pi| < \pi \), where
\[
\tilde{C}_0 = \begin{pmatrix}
    \frac{\Gamma(\gamma - \alpha)}{\Gamma(\alpha)} & \frac{\Gamma(\gamma - 1)}{\Gamma(\gamma + 1)} \\
    -\frac{\Gamma(\gamma - \alpha)}{\Gamma(\alpha)} & \frac{\Gamma(\gamma - 1)}{\Gamma(\gamma + 1)}
\end{pmatrix},
\]
\[
\tilde{C}_1 = \begin{pmatrix}
    \frac{\Gamma(2\alpha - \gamma)}{\Gamma(\alpha)} & \frac{\Gamma(2\alpha - \gamma)}{\Gamma(\alpha)} \\
    \frac{\Gamma(\gamma - \alpha)}{\Gamma(\alpha)} & \frac{\Gamma(\gamma - \alpha)}{\Gamma(\alpha)}
\end{pmatrix}
\]
with
\[
\hat{\psi}_{12}^0(\alpha, \gamma) = 2\psi(1) - \psi(\alpha) - \psi(\gamma - \alpha) - \pi i,
\]
\[
\hat{\psi}_{22}^0(\alpha, \gamma) = 2\psi(1) - \psi(1 - \alpha) - \psi(\alpha - \gamma + 1) - \pi i,
\]
\[
\hat{\psi}_{12}^1(\alpha, \gamma) = \psi(\alpha) + \psi(\alpha - \gamma + 1) - 2\psi(1),
\]
\[
\hat{\psi}_{22}^1(\alpha, \gamma) = \psi(1 - \alpha) + \psi(\gamma - \alpha) - 2\psi(1).
\]

9.3. **Non-generic cases**

Suppose that \( \alpha - \beta \notin \mathbb{Z} \). In the case where \( \gamma \in \mathbb{Z} \), the hypergeometric function behaves logarithmically around \( z = 0 \). Under the condition \( \alpha, \beta \notin \mathbb{Z} \), for \( |\arg z - \pi| < \pi \), application
of limiting procedure to connection formulas in the generic case yields the following (cf. [5, Section 2.10] and [1, 15.5.17, 15.5.19 and 15.5.21]):

(i) if \( 1 - \gamma = l = 0, 1, 2, \ldots \),

\[
z^{-\alpha} F(\alpha, \alpha + l, \alpha - \beta + 1, z^{-1}) = a_-(\alpha, \beta, l)(z^l F_{\log}(\alpha + l, \beta + l, 1 + l, z) + b_-(\alpha, \beta, l)z^l F(\alpha + l, \beta + l, 1 + l, z)),
\]

where

\[
a_-(\alpha, \beta, l) = \frac{(-1)^l e^{-\pi i\alpha} \Gamma(\alpha - \beta + 1) \Gamma(\beta + l)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(1 - \beta) l!},
\]

\[
b_-(\alpha, \beta, l) = \psi(\alpha + l) + \psi(\beta + l) - \psi(1 - l) - \psi(1) - \psi(1 - \beta) - \pi i;
\]

(ii) if \( 1 - \gamma = l = -1, -2, -3, \ldots \),

\[
\begin{align*}
&z^{-\alpha} F(\alpha, \alpha + l, \alpha - \beta + 1, z^{-1}) = a_+(\alpha, \beta, l)(F_{\log}(\alpha, \beta, 1 - l, z) + b_+(\alpha, \beta, l)F(\alpha, \beta, 1 - l, z)), \\
&\text{where}
\end{align*}
\]

\[
a_+(\alpha, \beta, l) = -\frac{e^{-\pi i\alpha} \Gamma(\alpha - \beta + 1) \Gamma(\beta)}{\Gamma(\alpha + l) \Gamma(\beta + l) \Gamma(1 - \beta - l)(-l)!},
\]

\[
b_+(\alpha, \beta, l) = \psi(\alpha) + \psi(\beta - \psi(1 - l) - \psi(1) - \psi(1 - \beta) + \pi i.
\]

In these cases, \( \Psi(z) |_{R=1} \) for (9.2) around \( z = 0 \) has the form \( G_0(I + O(z))E(z)\tilde{C}_0 \), where \( E(z) \) is \( z^{l/2}e^z \) if \( 1 - \gamma = l = 0, 1, 2, \ldots \), and \( z^{l/2}e^{-z} \) if \( 1 - \gamma = l = -1, -2, -3, \ldots \), and hence we have the following proposition.

**PROPOSITION 9.6.** Suppose that \( \alpha, \beta, \alpha - \beta \notin \mathbb{Z} \) and \( 1 - \gamma = l \in \mathbb{Z} \). For \( |\arg z - \pi| < \pi \),

\[
\tilde{C}_0 = \begin{cases}
\begin{pmatrix} b_-(\alpha, \beta, l) & b_-(\alpha, \beta, l) \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} a_-(\alpha, \beta, l) & 0 \\
0 & a_-(\beta, \alpha, l) \end{pmatrix} \text{ if } l = 0, 1, 2, \ldots, \\
\begin{pmatrix} 1 & 1 \\ b_+(\alpha, \beta, l) & b_+(\alpha, \beta, l) \end{pmatrix} & \begin{pmatrix} a_+(\alpha, \beta, l) & 0 \\
0 & a_+(\beta, \alpha, l) \end{pmatrix} \text{ if } l = -1, -2, -3, \ldots.
\end{cases}
\]

From \( F(\alpha, \beta, \gamma, z) = (1 - z)^{-\alpha} F(\alpha, \gamma - \beta, \gamma, z/(z - 1)) \) (see [1, 15.3.4]), it follows that

\[
z^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, z^{-1}) = e^{\pi i\alpha} \zeta^{-\alpha} F(\alpha, \gamma - \beta, \alpha - \beta + 1, \zeta^{-1})
\]

for \( |\arg(1 - z) - \pi| < \pi \), \( |\arg z - \pi| < \pi \), where \( \zeta = 1 - z = e^{\pi i}(z - 1) \). In the case where \( \gamma - \alpha - \beta \in \mathbb{Z} \), using this relation, from the connection formulas above we immediately obtain the following:

(i) if \( \gamma - \alpha - \beta = l = 0, 1, 2, \ldots \),

\[
z^{-\alpha} F(\alpha, 1 - l - \beta, \alpha - \beta + 1, z^{-1}) = e^{\pi i\alpha} a_-(\alpha, \beta, l)((1 - z)^l F_{\log}(\alpha + l, \beta + l, 1 + l, 1 - z) + b_-(\alpha, \beta, l)(1 - z)^l F(\alpha + l, \beta + l, 1 + l, 1 - z));
\]
(ii) if \( \gamma - \alpha - \beta = l = -1, -2, -3, \ldots \),
\[
z^{-\alpha} F(\alpha, 1 - l - \beta, \alpha - \beta + 1, z^{-1}) = e^{\pi i a_+(\alpha, \beta, l)}(F_{\log}(\alpha, \beta, 1 - l, 1 - z) + b_+(\alpha, \beta, l)F(\alpha, \beta, 1 - l, 1 - z)).
\]
Observing that \((1 - z)^{1/2}(1 - z)^{\Delta} = (-1)^l(I + O(z - 1))(z^{-1})^{1/2}(z - 1)^{\Delta}\), we have the following proposition.

**PROPOSITION 9.7.** Suppose that \( \alpha, \beta, \alpha - \beta \notin \mathbb{Z} \) and \( \gamma - \alpha - \beta = l \in \mathbb{Z} \). For \( |\arg z - \pi| < \pi \), \( |\arg(z - 1)| < \pi \),
\[
\begin{aligned}
\tilde{C}_l &= e^{\pi i a_+(\alpha, \beta, l)}
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & a_+(\beta, \alpha, l) \\
0 & a_+(\beta, \alpha, l) & 0
\end{pmatrix}
\end{aligned}
\]
\[
\begin{aligned}
&\text{if } l = 0, 1, 2, \ldots,
\end{aligned}
\]
\[
\begin{aligned}
&\text{if } l = -1, -2, -3, \ldots,
\end{aligned}
\]
For \( \alpha \notin \mathbb{Z} \), putting \( \alpha = \beta \), we have the following:
(i) if \( 1 - \gamma = l = 0, 1, 2, \ldots \),
\[
z^{-\alpha} F(\alpha, \alpha + l, 1, z^{-1}) = a_-(\alpha, \alpha, l)(z^lF_{\log}(\alpha + l, \alpha + l, 1 + l, z) + b_-(\alpha, \alpha, l)z^lF(\alpha + l, \alpha + l, 1 + l, z));
\]
(ii) if \( 1 - \gamma = l = -1, -2, -3, \ldots, \),
\[
z^{-\alpha} F(\alpha, \alpha + l, 1, z^{-1}) = a_+(\alpha, \alpha, l)(F_{\log}(\alpha, \alpha, 1 - l, z) + b_+(\alpha, \alpha, l)F(\alpha, \alpha, 1 - l, z)).
\]
Combining these formulas with (9.3), we have another relation in the case where \( \alpha \notin \mathbb{Z} \), \( \gamma \in \mathbb{Z} \). For example, if \( 1 - \gamma = l = 0, 1, 2, \ldots \), observing that
\[
\frac{1}{\Gamma(\gamma)} F(\alpha, \alpha, \gamma, z) \bigg|_{\gamma = -l + 1} = \frac{\Gamma(\alpha + l)^2}{\Gamma(\alpha)^{2l}} z^l F(\alpha + l, \alpha + l, 1 + l, z),
\]
we get
\[
- z^{-\alpha} F_{\log}(\alpha, \alpha + l, 1, z^{-1}) = \frac{\Gamma(\alpha + l)^2\Gamma(1 - \alpha - l)}{\Gamma(\alpha)l!} e^{-\pi i a_+} z^l F(\alpha + l, \alpha + l, 1 + l, z)
\]
\[
\times (2\psi(1) - \psi(\alpha) - \psi(1 - \alpha - l) - \pi i) a_- (\alpha, \alpha, l)
\]
\[
\times (z^l F_{\log}(\alpha + l, \alpha + l, 1 + l, z) + b_-(\alpha, \alpha, l)z^l F(\alpha + l, \alpha + l, 1 + l, z))
\]
for \( |\arg z - \pi| < \pi \). On the other hand, in the case where \( \gamma - 2\alpha \in \mathbb{Z} \), we have the relations:
(i) if \( \gamma - 2\alpha = l = 0, 1, 2, \ldots \),
\[
z^{-\alpha} F(\alpha, 1 - l - \alpha, 1, z^{-1}) = e^{\pi i a_- (\alpha, \alpha, l)}((1 - z)^lF_{\log}(\alpha + l, \alpha + l, 1 + l, 1 - z)
\]
\[
+ b_-(\alpha, \alpha, l)(1 - z)^l F(\alpha + l, \alpha + l, 1 + l, 1 - z));
\]
if \( \gamma - 2\alpha = l = -1, -2, -3, \ldots \),
\[
z^{-\alpha} F(\alpha, 1 - l - \alpha, 1, z^{-1}) = e^{\pi i \alpha} a_+(\alpha, \alpha, l) (F_{\text{log}}(\alpha, \alpha, 1 - l, 1 - z) + b_+(\alpha, \alpha, l) F(\alpha, \alpha, 1 - l, 1 - z)).
\]

The connection formula for \(-z^{-\alpha} F_{\text{log}}(\alpha, 1 - l - \alpha, 1, z^{-1})\) is obtained by an analogous argument, in which we use (9.4). Thus we have the following.

**Proposition 9.8.** (1) Suppose that \( \alpha = \beta \not\in \mathbb{Z} \) and \( 1 - \gamma = l \in \mathbb{Z} \). For \( |\arg z - \pi| < \pi \),
\[
\tilde{c}_0 = \begin{pmatrix} b_-(\alpha, \alpha, l) & b_-(\alpha, \alpha, l) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_-(\alpha, \alpha, l) & 0 \\ 0 & \psi_0(\alpha, l) a_-(\alpha, \alpha, l) \end{pmatrix}
+ \frac{\Gamma(\alpha + l)^2 \Gamma(1 - \alpha - l)}{\Gamma(\alpha)!} e^{-\pi i \alpha} \Delta \quad \text{if } l = 0, 1, 2, \ldots,
\]
\[
= \begin{pmatrix} 1 & 1 \\ b_+(\alpha, \alpha, l) & b_+(\alpha, \alpha, l) \end{pmatrix} \begin{pmatrix} a_+(\alpha, \alpha, l) & 0 \\ 0 & \psi_0(\alpha, l) a_+(\alpha, \alpha, l) \end{pmatrix}
+ \frac{\Gamma(\alpha) \Gamma(1 - \alpha - l)}{2\Gamma(1 - l)} e^{-\pi i \alpha} (I - J) \quad \text{if } l = -1, -2, -3, \ldots,
\]
where \( \psi_0(\alpha, l) = \psi(\alpha) + \psi(1 - \alpha - l) - 2\psi(1) + \pi i \).

(2) Suppose that \( \alpha = \beta \not\in \mathbb{Z} \) and \( \gamma - 2\alpha = l \in \mathbb{Z} \). For \( 0 < |\arg z - \pi| < \pi \), \( |\arg(z - 1)| < \pi \),
\[
\tilde{c}_1 = e^{\pi i \alpha} \begin{pmatrix} b_-(\alpha, \alpha, l) & b_-(\alpha, \alpha, l) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_-(\alpha, \alpha, l) & 0 \\ 0 & \psi_1(\alpha, l) a_-(\alpha, \alpha, l) \end{pmatrix}
+ \frac{\Gamma(\alpha + l)^2 \Gamma(1 - \alpha - l)}{\Gamma(\alpha)!} \Delta \quad \text{if } l = 0, 1, 2, \ldots,
\]
\[
= e^{\pi i \alpha} \begin{pmatrix} 1 & 1 \\ b_+(\alpha, \alpha, l) & b_+(\alpha, \alpha, l) \end{pmatrix} \begin{pmatrix} a_+(\alpha, \alpha, l) & 0 \\ 0 & \psi_1(\alpha, l) a_+(\alpha, \alpha, l) \end{pmatrix}
+ \frac{\Gamma(\alpha) \Gamma(1 - \alpha - l)}{2\Gamma(1 - l)} (I - J) \quad \text{if } l = -1, -2, -3, \ldots,
\]
where \( \psi_1(\alpha, l) = \psi(\alpha) + \psi(1 - \alpha - l) - 2\psi(1) \).

### 10. Proofs of Theorems 2.13–2.16

**10.1. Proofs of Theorems 2.13 and 2.14**

In (9.2) and Proposition 9.2 set \((1 - \gamma)/2 = \theta_0/2, (\gamma - \alpha - \beta - 1)/2 = \theta_x/2, (\beta - \alpha)/2 = \sigma/2\), that is, \( \alpha = -\beta + 2\theta_x, \beta = (\sigma - \theta_0 - \theta_x)/2, \gamma = 1 - \theta_0, \) and \( R = \beta(\beta - \gamma + 1)/(\alpha - \beta) = (\theta_0^2 - (\sigma - \theta_x)^2)/(4\sigma) \). Then we have \( T B_0 T^{-1} = \Lambda_0, T B_1 T^{-1} = \Lambda_x, T(\sigma J/2) T^{-1} = \Lambda, \) where \( \Lambda_0, \Lambda_x, T, \Lambda \) are as in Lemma 4.1, and \( Z = T \Psi(\lambda) \) satisfies
\[
\frac{dZ}{d\lambda} = \left( \frac{\Lambda_0}{\lambda} + \frac{\Lambda_x}{\lambda - 1} \right) Z.
\]
Furthermore, by
\[
Z = \tilde{z}^{-\Lambda} Y = (\rho_x^{1/\sigma})^{T(\sigma J/2) T^{-1}} Y = T \rho_x^{J/2} T^{-1} Y
\]
this is changed into a system of the form (8.2)

\[
\frac{dY}{d\lambda} = \left( \frac{\tilde{\alpha}}{\lambda} + \frac{\tilde{\alpha}}{\lambda - 1} \right) Y,
\]

which has a matrix solution such that

\[
Y = T\rho_{\ast}^{J/2} T^{-1} Z = T\rho_{\ast}^{J/2} \Psi(\lambda) = T\rho_{\ast}^{J/2} (I + O(\lambda^{-1}))\lambda^J T\rho_{\ast}^{J/2}
\]

\[
= T(I + O(\lambda^{-1}))\lambda^J T\rho_{\ast}^{J/2} = (I + O(\lambda^{-1}))\lambda^J T\rho_{\ast}^{J/2}
\]

near \(\lambda = \infty\). Hence the connection matrices \(C^{(0)}, C^{(x)}\) are those for

\[
\tilde{Y} = Y(T\rho_{\ast}^{J/2})^{-1} = T\rho_{\ast}^{J/2} \Psi(\lambda)(T\rho_{\ast}^{J/2})^{-1} = (I + O(\lambda^{-1}))\lambda^A
\]

corresponding to (8.3). By Proposition 9.3, the connection matrices for \(\Psi(\lambda)\) are \(\tilde{C}_{i/x} = \text{diag}[1, 1/R] \) (\(i = 0, x\)) under the supposition \(\gamma, -\alpha - \beta \notin \mathbb{Z}\), that is, \(\theta_0, \theta_x \notin \mathbb{Z}\), and we may choose \(C^{(i)} = C(i) T\) in such a way that

\[
C_i = \rho_{\ast}^{J/2} \tilde{C}_{i/x} \text{diag}[1, 1/R] \rho_{\ast}^{-J/2} \quad (i = 0, x)
\]

with \(\alpha, \beta, \gamma, R\) set as above (cf. Remark 8.1). Combining \(C\) in Proposition 9.1 with \(C^{(0)}, C^{(x)}\), from (8.4) we obtain \(M_0, M_x\) in Theorem 2.13. Theorem 2.14 is shown by the limiting procedure \(\sigma = 0\). In (10.1), replacing \(\rho_{\ast}\) by \(\rho\) and letting \(\sigma = \theta_0 \pm \theta_x, \theta_x - \theta_0\), we obtain \(C^0_0, C^0_x\) as in Theorem 2.14 other than \(C^0_0\) for \(\sigma_0 = \theta_x - \theta_0\). By Remark 8.1, we may choose \(\text{diag}[(\sigma + \theta_0 - \theta_x)/2, ((\sigma + \theta_0 - \theta_x)/2)^{-1}]C_0^*\) instead of \(\tilde{C}_0\). In (10.1) with such a matrix, letting \(\sigma = \theta_x - \theta_0\) we derive \(C^*_0\) with \(\tilde{C}^*_x\) in (iii).

**10.2. Proof of Theorem 2.15**

Suppose that \(\theta_\infty \neq 0\). In (9.2) and Proposition 9.4 set \(1 - \gamma = \theta_0, \gamma - 2\alpha - 1 = \mp \theta_x\), that is, \(\alpha = -(\theta_0 \mp \theta_x)/2, \gamma = 1 - \theta_0\). Then, for \(\Lambda_0, \Lambda_x, T, \Lambda\) as in Lemma 4.2, we have \(TB_0 T^{-1} = \Lambda_0, T B_1 T^{-1} = \Lambda_x, T \Delta T^{-1} = \Lambda, \) and, near \(\lambda = \infty\),

\[
Y = T \rho^\Delta \Psi(\lambda) = T \rho^\Delta (I + O(\lambda^{-1}))\lambda^\Delta
\]

\[
= T(I + O(\lambda^{-1}))\lambda^\Delta \rho^\Delta = (I + O(\lambda^{-1}))\lambda^\Delta T \rho^\Delta
\]

solves system (8.2). Hence the connection matrices \(C^{(0)\pm}, C^{(x)\pm}\) for

\[
\tilde{Y} = Y(T\rho^\Delta)^{-1} = T \rho^\Delta \Psi(\lambda)(T\rho^\Delta)^{-1} = (I + O(\lambda^{-1}))\lambda^\Delta
\]

are the desired ones, which are written as \(C^{\pm}_i T^{-1}\) with \(C^{\pm}_i = \tilde{C}^{\pm}_{i/x} \rho^{-\Delta}(i = 0, x)\), where \(\tilde{C}^{\pm}_{i/x}\) are given by Proposition 9.5. For \(y_{\text{ilog}}(\rho, x)\) and \(y^{(l)}(\rho, x)\) with \(\theta_0^2 - \theta_x^2 \neq 0\), we restrict to \(\alpha = -(\theta_0 + \theta_x)/2\) and replace \(\rho\) by \(\rho \exp(-2\theta_x(\theta_0^2 - \theta_x^2)^{-1})\) according to Proposition 5.1. In the case where \(\alpha = -(\theta_0 - \theta_x)/2\), \(\tilde{C}^+_1\) should be replaced by \(K^+ \tilde{C}^+_1\), because the local solution of (9.2) around \(\lambda = 1\) has the form \(G_1(I + O(\lambda - 1))(\lambda - 1)^{-\theta_x/2} J\). If \(\theta_\infty = 0\), in the argument above the matrix \(T\) is to be replaced by those in (2) of Lemma 4.2. Thus, we obtain the monodromy matrices in Theorem 2.15.
10.3. Proof of Theorem 2.16

The systems corresponding to (8.1) and (8.2) are

\[ \frac{d\hat{Y}}{d\lambda} = \frac{J}{2} \hat{Y}, \quad \frac{d\tilde{Y}}{d\lambda} = \left( \frac{\Lambda_0}{\lambda} - \frac{\Lambda_0}{\lambda - 1} \right) \tilde{Y} \]

having the matrix solutions \( \hat{Y} = \exp((J/2)\lambda) \) and

\[
\tilde{Y} = \begin{cases} 
T \begin{pmatrix} \lambda^{b_0/2}(\lambda - 1)^{-b_0/2} & 0 \\ 0 & \lambda^{-b_0/2}(\lambda - 1)^{b_0/2} \end{pmatrix} T^{-1} & \text{if } \theta_0 \neq 0, \\
T \begin{pmatrix} 1 & \log(\lambda/(\lambda - 1)) \\ 0 & 1 \end{pmatrix} T^{-1} & \text{if } \theta_0 = 0,
\end{cases}
\]

respectively, where \( \Lambda_0 \) and \( T \) are as in Lemma 4.3. From these facts Theorem 2.16 immediately follows.

Acknowledgement. The author is deeply grateful to the referee for valuable comments and for bringing to his attention the reference [20]. Motivated by the comments, the author could make the important remarks on connection formulas and the almost-completeness of our critical behaviours.

REFERENCES

The fifth Painlevé equation


Shun Shimomura
Department of Mathematics
Keio University
3-14-1 Hiyoshi Kohoku-ku
Yokohama 223-8522
Japan
(E-mail: shimomur@math.keio.ac.jp)