ON THE RICCI TENSOR OF SUBMANIFOLDS IN CONFORMAL KENMOTSU MANIFOLDS

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Abstract. In this paper, we obtain several interesting results on submanifolds of conformal Kenmotsu manifolds. In addition to this we consider submanifolds of a conformal Kenmotsu manifold of which the Ricci tensor is parallel, Lie ξ -parallel or recurrent. We also present an illustration example of a three-dimensional conformal Kenmotsu manifold that is not Kenmotsu.

1. Introduction

Let (M, J, g) be an almost Hermitian manifold of dimension $2n$, where J denotes the almost complex structure and g the Hermitian metric. Then (M, J, g) is called a locally conformal Kaehler manifold if for each point p of M there exists an open neighborhood U of p and a positive function f_U on U so that the local metric $g_U = \exp(-f)g|_U$ is Kaehlerian. If $U = M$, then the manifold (M, J, g) is said to be a globally conformal Kaehler manifold. The 1-form $\omega = df$ is called the Lee form and its metrically equivalent vector field $\omega^\sharp = \text{grad } f$, where $\sharp$ means the rising of the indices with respect to g , namely $g(X, \omega^\sharp) = \omega(X)$ for all X tangent to M , is called the Lee vector field [9]. Submanifolds of locally conformal Kaehler manifolds with parallel Lee form have been studied by several authors (see, for instance, [6, 10, 14]).

In [11], Kenmotsu defined and studied a new class of almost contact manifolds called Kenmotsu manifolds.

We introduced conformal Kenmotsu manifolds by using an idea of globally conformal Kaehler manifolds (see [1] and [2]). Also we give an example of a conformal Kenmotsu manifold that is not Kenmotsu. Hence, the category of conformal Kenmotsu manifolds and Kenmotsu manifolds is completely different.

In [13], Kobayashi proved: let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M , then

$$\nabla_X \xi = X - \eta(X)\xi, \quad h(X, \xi) = 0, \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X$$

for all X and Y tangent to M , where ∇ , h and R are the Riemannian connection, the second fundamental form and the curvature tensor of M , respectively.

In this paper, as a generalization of these results we state Lemmas 3.1, 3.2, 3.3, 3.4, 3.6 and 3.8 for a submanifold of a conformal Kenmotsu manifold.

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A Riemannian manifold $(M^n, g)(n > 2)$ is said to be an Einstein manifold if $Ric = (r/n)g$ holds on M where Ric and r denote the Ricci tensor and the scalar curvature of M , respectively [3]. Einstein manifolds play an important role in Riemannian geometry as well as in physics. Every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y), \quad (1.1)$$

where a and b are smooth functions of which $b \neq 0$ and A is a non-zero 1-form such that $g(X, U) = A(X)$ for each vector field X and U being a unit vector field which is called the generator of the manifold.

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be a quasi-Einstein manifold [7] if its Ricci tensor Ric is not identically zero and satisfies the condition (1.1). Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations. For example, the Robertson–Walker space-time is a quasi-Einstein manifold [8]. So, quasi-Einstein manifolds have some importance in general theory of relatively.

Motivated by the importance of Einstein and quasi-Einstein manifolds, in this paper we present the following problem:

Can we characterize submanifolds in a conformal Kenmotsu manifold with Lie ξ -parallel, recurrent or parallel Ricci tensor such that $\omega^\sharp$ is tangent (normal) to the submanifold?

Corresponding to this problem we give the following theorems.

THEOREM 1. *Let $\hat{M}^m$ be a submanifold of a conformal Kenmotsu manifold M with parallel or recurrent Ricci tensor such that $\omega^\sharp$ is normal to $\hat{M}$. Then $\hat{M}$ is an Einstein submanifold.*

THEOREM 2. *Let $\hat{M}^m$ be a submanifold of a conformal Kenmotsu manifold M with parallel Ricci tensor such that $\omega^\sharp$ is tangent to $\hat{M}$. Then $\hat{M}$ is a quasi-Einstein submanifold.*

THEOREM 3. *There is not any submanifolds $\hat{M}^m$ of a conformal Kenmotsu manifold M with Lie ξ -parallel Ricci tensor when $\omega^\sharp$ is tangent to $\hat{M}$.*

THEOREM 4. *Let $\hat{M}^m$ be a submanifold of a conformal Kenmotsu manifold M with recurrent Ricci tensor such that $\omega^\sharp$ is tangent to $\hat{M}$. Then the Ricci tensor of $\hat{M}$ cannot be parallel.*

The present paper is organized as follows. In Section 2, we recall some preliminary definitions on Kenmotsu manifolds, then we define conformal Kenmotsu manifolds. Furthermore, we give an example of a three-dimensional conformal Kenmotsu manifold that is not Kenmotsu. Section 3 gives some preliminary lemmas on submanifolds of a conformal Kenmotsu manifold tangent to ξ . In Section 4, we consider submanifolds of a conformal Kenmotsu manifold with parallel Ricci tensor such that $\omega^\sharp$ is normal (tangent) to the submanifold. We also give a non-existence theorem for submanifolds in conformal Kenmotsu manifolds with Lie ξ -parallel Ricci tensor when $\omega^\sharp$ is tangent to the submanifold. In Section 5, we consider submanifolds of a conformal Kenmotsu manifold with recurrent Ricci tensor, namely $\hat{\nabla} Ric = \alpha \otimes Ric$ where $\hat{\nabla}$ and Ric are the Riemannian connection and the Ricci tensor respectively and α is a 1-form on the submanifold. We obtain values of α when $\omega^\sharp$ is normal (tangent) to submanifolds with recurrent Ricci tensor in conformal Kenmotsu manifolds.

2. Conformal Kenmotsu manifolds

A $(2n + 1)$ -dimensional differentiable manifold M is an almost contact metric manifold, if it admits an almost contact metric structure (φ, ξ, η, g) consisting of a tensor field φ of type $(1, 1)$, a vector field ξ , a 1-form η and a Riemannian metric g and satisfying following properties:

$$\begin{aligned}\varphi^2 &= -\text{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \\ \varphi\xi &= 0, \quad \eta\varphi = 0, \quad \eta(X) = g(X, \xi)\end{aligned}\quad (2.1)$$

for all vector fields X, Y on M , (2.1) is a consequence of the other conditions [4, 5].

An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be a Kenmotsu manifold if the relation

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X \quad (2.2)$$

holds on M , where ∇ denotes the Riemannian connection of g . From (2.2) for a Kenmotsu manifold, we have

$$\nabla_X \xi = X - \eta(X)\xi. \quad (2.3)$$

For a Kenmotsu manifold, we also have

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \quad (2.4)$$

for all vector fields X, Y tangent to M , where R is the curvature tensor of M (see [11]).

A $(2n + 1)$ -dimensional smooth manifold M with almost contact metric structure (φ, η, ξ, g) is called a conformal Kenmotsu manifold if there exists a positive smooth function $f : M \rightarrow \mathbb{R}$ so that

$$\tilde{g} = \exp(f)g, \quad \tilde{\xi} = (\exp(-f))^{1/2}\xi, \quad \tilde{\eta} = (\exp(f))^{1/2}\eta, \quad \tilde{\varphi} = \varphi$$

is a Kenmotsu structure on M (see [2]).

Let M be a conformal Kenmotsu manifold, with $\tilde{\nabla}$ and ∇ denoting the Riemannian connections of M with respect to metrics $\tilde{g}$ and g , respectively. Using the Koszul formula, one simply can obtain the following relation between $\tilde{\nabla}$ and ∇ :

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X, Y)\omega^\sharp\} \quad (2.5)$$

for all vector fields X, Y on M , where $\omega(X) = g(\text{grad } f, X) = X(f)$. Note that the vector field $\omega^\sharp = \text{grad } f$ is called the Lee vector field of the conformal Kenmotsu manifold M . Then from $\eta(X) = g(X, \xi)$, we have the equality $\eta(\omega^\sharp) = \omega(\xi)$. Although $\omega(\omega^\sharp) = \|\omega^\sharp\|^2$, it is not necessarily $\|\omega^\sharp\|^2 = 1$ (unit vector field).

Assuming that $\tilde{R}$ and R are, respectively, the curvature tensors of $(M, \varphi, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ and $(M, \varphi, \eta, \xi, g)$. We have the following relation between $\tilde{R}$ and R :

$$\begin{aligned}\exp(-f)\tilde{g}(\tilde{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) \\ &+ \frac{1}{2}\{B(X, Z)g(Y, W) - B(Y, Z)g(X, W) \\ &+ B(Y, W)g(X, Z) - B(X, W)g(Y, Z)\} \\ &+ \frac{1}{4}\|\omega^\sharp\|^2\{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\}\end{aligned}\quad (2.6)$$

for all vector fields X, Y, Z, W on M and B satisfies

$$B := \nabla\omega - \frac{1}{2}\omega \otimes \omega. \quad (2.7)$$

Obviously, B is a symmetric tensor field of type $(0, 2)$. On the other hand, from equations (2.2), (2.3) and (2.5), we get

$$\begin{aligned} (\nabla_X \varphi)Y &= (\exp(f))^{1/2} \{-g(X, \varphi Y)\xi - \eta(Y)\varphi X\} \\ &\quad - \frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)\varphi X + g(X, Y)\varphi\omega^\sharp - g(X, \varphi Y)\omega^\sharp\}, \end{aligned} \quad (2.8)$$

$$\nabla_X \xi = (\exp(f))^{1/2} \{X - \eta(X)\xi\} - \frac{1}{2}\{\omega(\xi)X - \eta(X)\omega^\sharp\} \quad (2.9)$$

for all vector fields X, Y on M .

2.1. Example

We give an example of a conformal Kenmotsu manifold but not necessarily a Kenmotsu manifold. We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ with the linearly independent vector fields

$$e_1 = x \frac{\partial}{\partial z}, \quad e_2 = x \frac{\partial}{\partial y}, \quad e_3 = -(\exp(x))^{1/2} x \frac{\partial}{\partial x}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_2) &= g(e_1, e_3) = g(e_2, e_3) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = \exp(-x), \quad g(e_3, e_3) = 1. \end{aligned}$$

Let η be the 1-form defined by

$$\eta(e_3) = 1, \quad \eta(e_2) = 0, \quad \eta(e_1) = 0.$$

We define the $(1, 1)$ tensor field φ as $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi e_3 = 0$. Then using the linearity of φ and g we have

$$\varphi^2 X = -X + \eta(X)e_3, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all X, Y on M . Thus, for $e_3 = \xi$, (φ, ξ, η, g) defines an almost contact metric structure on M .

On the other hand, we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = (\exp(x))^{1/2} e_1, \quad [e_2, e_3] = (\exp(x))^{1/2} e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

which is known as the Koszul formula. By using the Koszul formula, we obtain

$$\begin{aligned} \nabla_{e_1} e_1 &= -\frac{1}{2}(\exp(-x))^{1/2}(x+2)e_3, \quad \nabla_{e_1} e_2 = 0, \\ \nabla_{e_1} e_3 &= \frac{1}{2}(\exp(x))^{1/2}(x+2)e_1, \quad \nabla_{e_2} e_1 = 0, \\ \nabla_{e_2} e_2 &= -\frac{1}{2}(\exp(-x))^{1/2}(x+2)e_3, \quad \nabla_{e_2} e_3 = \frac{1}{2}(\exp(x))^{1/2}(x+2)e_2, \\ \nabla_{e_3} e_1 &= \frac{1}{2}(\exp(x))^{1/2} x e_1, \quad \nabla_{e_3} e_2 = \frac{1}{2}(\exp(x))^{1/2} x e_2, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

By the following contact transformation

$$\tilde{g} = \exp(x)g, \quad \tilde{\xi} = (\exp(-x))^{1/2}\xi, \quad \tilde{\eta} = (\exp(x))^{1/2}\eta, \quad \tilde{\varphi} = \varphi,$$

$(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Kenmotsu manifold (see [15]). Hence, $(M, \varphi, \xi, \eta, g)$ is a conformal Kenmotsu manifold but is not Kenmotsu, since we have

$$(\nabla_X \varphi)Y \neq -g(X, \varphi Y)\xi - \eta(Y)\varphi X$$

for some vector fields X, Y on M (for instance, $(\nabla_{e_2} \varphi)e_1 \neq -g(e_2, \varphi e_1)\xi - \eta(e_1)\varphi e_2$). By using above results, we can easily obtain the following:

$$\begin{aligned} R(e_1, e_2)e_2 &= -\frac{1}{4}(x+2)^2 e_1, & R(e_2, e_3)e_3 &= -\exp(x)e_2, \\ R(e_1, e_2)e_3 &= 0, & R(e_1, e_3)e_3 &= -\exp(x)e_1, \\ R(e_3, e_1)e_1 &= -e_3, & R(e_3, e_2)e_1 &= 0, & R(e_3, e_1)e_2 &= 0, \\ R(e_2, e_1)e_1 &= -\frac{1}{4}(x+2)^2 e_2, & R(e_3, e_2)e_2 &= -e_3. \end{aligned}$$

In view of above relations, we get the following results:

$$K(X, e_3) = -\exp(x), \quad K(X, Y) = -\frac{1}{4}\exp(x)(x+2)^2$$

for all vector fields X, Y orthogonal to e_3 . Note that manifold M with almost contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Kenmotsu manifold of constant curvature -1 (see [15]).

3. Submanifolds of conformal Kenmotsu manifolds

Let $(\acute{M}, \acute{g})$ be an m -dimensional submanifold into a $(2n+1)$ -dimensional conformal Kenmotsu manifold (M, g) . The Gauss and Weingarten formulas are given by

$$\nabla_X Y = \acute{\nabla}_X Y + h(X, Y), \quad \nabla_X N = -A_N X + \nabla_X^\perp N$$

for all vector fields X, Y tangent to $\acute{M}$ and normal vector field N on $\acute{M}$, where $\acute{\nabla}$ is the Riemannian connection of $\acute{M}$ determined by the induced metric $\acute{g}$ and $\nabla^\perp$ is the normal connection on $T^\perp \acute{M}$ of $\acute{M}$. It is known that $g(h(X, Y), N) = \acute{g}(A_N X, Y)$, where A_N is the shape operator of $\acute{M}$ with respect to unit normal vector field N .

The Gauss equation is given by

$$\begin{aligned} \exp(-f)\tilde{g}(\tilde{R}(X, Y)Z, W) &= \acute{g}(\acute{R}(X, Y)Z, W) \\ &\quad - \acute{g}(h(X, W), h(Y, Z)) + \acute{g}(h(Y, W), h(X, Z)) \\ &\quad + \frac{1}{2}\{B(X, Z)\acute{g}(Y, W) - B(Y, Z)\acute{g}(X, W) \\ &\quad + B(Y, W)\acute{g}(X, Z) - B(X, W)\acute{g}(Y, Z)\} \\ &\quad + \frac{1}{4}\|\omega^\sharp\|^2\{\acute{g}(X, Z)\acute{g}(Y, W) - \acute{g}(Y, Z)\acute{g}(X, W)\} \end{aligned} \quad (3.1)$$

for all X, Y, Z, W tangent to $\acute{M}$ where $\acute{R}$ is the curvature tensor of $\acute{M}$.

In this paper, we assume that ξ is tangent to $\acute{M}$.

LEMMA 3.1. *Let $\dot{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is normal to $\dot{M}$. Then*

$$B(X, Y) = -\omega(h(X, Y)), \quad (3.2)$$

$$h(X, \xi) = \frac{1}{2}\eta(X)\omega^\sharp, \quad (3.3)$$

$$\dot{\nabla}_X \xi = (\exp(f))^{1/2}\{X - \eta(X)\xi\} \quad (3.4)$$

for all vector fields X, Y tangent to $\dot{M}$.

Proof. From (2.7) we have

$$B(X, Y) = (\nabla_X \omega)Y - \frac{1}{2}\omega(X)\omega(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y) - \frac{1}{2}\omega(X)\omega(Y)$$

for all X, Y tangent to $\dot{M}$. Since $\omega^\sharp$ is normal to $\dot{M}$, the above equation can be written as

$$B(X, Y) = -\omega(\nabla_X Y)$$

for all X, Y on $\dot{M}$. Then by the use of the Gauss formula we obtain (3.2).

Taking $Y = \xi$ in the Gauss formula and using (2.9) we have

$$\dot{\nabla}_X \xi + h(X, \xi) = \nabla_X \xi = (\exp(f))^{1/2}\{X - \eta(X)\xi\} - \frac{1}{2}\{\omega(\xi)X - \eta(X)\omega^\sharp\}$$

for each X tangent to $\dot{M}$. Since $\omega^\sharp$ is normal to $\dot{M}$, we compare the tangential part and the normal part in the above equation. Then we obtain (3.3) and (3.4). $\square$

LEMMA 3.2. *Let $\dot{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is tangent to $\dot{M}$. Then*

$$B(X, Y) = \dot{g}(\dot{\nabla}_X \omega^\sharp, Y) - \frac{1}{2}\omega(X)\omega(Y), \quad (3.5)$$

$$h(X, \xi) = 0, \quad (3.6)$$

$$\dot{\nabla}_X \xi = (\exp(f))^{1/2}\{X - \eta(X)\xi\} - \frac{1}{2}\{\omega(\xi)X - \eta(X)\omega^\sharp\} \quad (3.7)$$

for all vector fields X, Y tangent to $\dot{M}$.

Proof. Similarly to Lemma 3.1, equations (3.5), (3.6) and (3.7) are immediate results from (2.7), (2.9) and the Gauss formula. $\square$

LEMMA 3.3. *Let $\dot{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is normal to $\dot{M}$. Then*

$$\dot{R}(X, Y)\xi = \exp(f)\{\eta(X)Y - \eta(Y)X\}, \quad (3.8)$$

$$\dot{R}(X, \xi)Y = \exp(f)\{\dot{g}(X, Y)\xi - \eta(Y)X\} \quad (3.9)$$

for all vector fields X, Y tangent to $\dot{M}$.

Proof. Equation (3.8) follows from (2.4), (3.1), (3.2) and (3.3). From (3.8) and the symmetric property of $\dot{R}$ we get (3.9). $\square$

LEMMA 3.4. *Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is tangent to $\acute{M}$. Then*

$$\begin{aligned} \acute{R}(X, Y)\xi &= (\exp(f) - \frac{1}{4}\|\omega^\sharp\|^2)\{\eta(X)Y - \eta(Y)X\} \\ &\quad - \frac{1}{2}\{\eta(\acute{\nabla}_X\omega^\sharp)Y - \frac{1}{2}\omega(X)\omega(\xi)Y - \eta(\acute{\nabla}_Y\omega^\sharp)X + \frac{1}{2}\omega(Y)\omega(\xi)X \\ &\quad + \eta(X)\acute{\nabla}_Y\omega^\sharp - \frac{1}{2}\eta(X)\omega(Y)\omega^\sharp - \eta(Y)\acute{\nabla}_X\omega^\sharp + \frac{1}{2}\eta(Y)\omega(X)\omega^\sharp\} \end{aligned} \quad (3.10)$$

for all vector fields X, Y tangent to $\acute{M}$.

Proof. Equations (2.4), (3.1), (3.5) and (3.6) yield (3.10). $\square$

COROLLARY 3.5. *Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is tangent to $\acute{M}$. If $\omega^\sharp$ is parallel on $\acute{M}$, then*

$$\begin{aligned} \acute{R}(X, Y)\xi &= (\exp(f) - \frac{1}{4}\|\omega^\sharp\|^2)\{\eta(X)Y - \eta(Y)X\} \\ &\quad + \frac{1}{4}\{\omega(X)\omega(\xi)Y - \omega(Y)\omega(\xi)X + \eta(X)\omega(Y)\omega^\sharp - \eta(Y)\omega(X)\omega^\sharp\}, \end{aligned} \quad (3.11)$$

$$\begin{aligned} \acute{R}(X, \xi)Y &= (\exp(f) - \frac{1}{4}\|\omega^\sharp\|^2)\{\acute{g}(X, Y)\xi - \eta(Y)X\} \\ &\quad + \frac{1}{4}\{\omega(\xi)\acute{g}(X, Y)\omega^\sharp - \omega(\xi)\omega(Y)X + \omega(X)\omega(Y)\xi - \omega(X)\eta(Y)\omega^\sharp\} \end{aligned} \quad (3.12)$$

for all vector fields X, Y tangent to $\acute{M}$.

Proof. Equation (3.11) is an immediate result of (3.10). We obtain (3.12) by (3.11) and the symmetric property of $\acute{R}$. $\square$

LEMMA 3.6. *Let $\acute{M}^m$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is normal to $\acute{M}$. Then*

$$\text{Ric}(X, \xi) = -(m-1)\exp(f)\eta(X) \quad (3.13)$$

for each vector field X tangent to $\acute{M}$ where Ric denotes the Ricci tensor of $\acute{M}$.

Proof. Suppose that $\{e_i, \xi \mid i = 1, \dots, m-1\}$ is an orthonormal basis on $\acute{M}$. The definition of the Ricci tensor in $\acute{M}^m$ yields

$$\text{Ric}(X, \xi) = \sum_{i=1}^{m-1} \acute{g}(\acute{R}(e_i, X)\xi, e_i).$$

By using (3.8), we get (3.13). $\square$

COROLLARY 3.7. *Let $\acute{M}^m (m > 1)$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is normal to $\acute{M}$. Then $\acute{M}$ cannot be Ricci flat.*

Proof. Let $\acute{M}$ be Ricci flat. Putting $X = \xi$ in (3.13), one simply get $\exp(f) = 0$ that is a contradiction. $\square$

LEMMA 3.8. *Let $\acute{M}^m$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is tangent to $\acute{M}$. If $\omega^\sharp$ is parallel on $\acute{M}$, then*

$$\begin{aligned} \text{Ric}(X, \xi) &= (-(m-1)\exp(f) + \frac{1}{4}(m-2)\|\omega^\sharp\|^2)\eta(X) \\ &\quad - \frac{1}{4}(m-2)\omega(\xi)\omega(X) \end{aligned} \quad (3.14)$$

for each vector field X tangent to $\acute{M}$.

Proof. Suppose that $\{e_i, \xi \mid i = 1, \dots, m-1\}$ is an orthonormal basis on $\dot{M}$. The definition of the Ricci tensor in $\dot{M}^m$ implies that

$$Ric(X, \xi) = \sum_{i=1}^{m-1} \acute{g}(\acute{R}(e_i, X)\xi, e_i).$$

By using (3.11), we get

$$\begin{aligned} Ric(X, \xi) &= -(m-1)\exp(f) + \frac{1}{4}(m-1)\|\omega^\sharp\|^2\eta(X) - \frac{1}{4}(m-1)\omega(\xi)\omega(X) \\ &\quad + \frac{1}{4}\sum_{i=1}^{m-1}\{\omega(e_i)\omega(\xi)\acute{g}(X, e_i) - \eta(X)\omega(e_i)^2\}. \end{aligned} \quad (3.15)$$

On the other hand, we have

$$\acute{g}(X, Y) = \sum_{i=1}^{m-1} \acute{g}(X, e_i)\acute{g}(Y, e_i) + \acute{g}(X, \xi)\acute{g}(Y, \xi) \quad (3.16)$$

for all vector fields X, Y on $\dot{M}$. Then equations (3.15) and (3.16) yield (3.14). $\square$

From now on, we suppose that the Lee vector field $\omega^\sharp$ is non-zero.

LEMMA 3.9. *Let $\dot{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is tangent to $\dot{M}$. If $\omega^\sharp$ is parallel on $\dot{M}$, then*

$$\omega(\xi) \neq 0, \quad (3.17)$$

$$(\exp(f))^{1/2} - \frac{1}{2}\omega(\xi) \neq 0, \quad (3.18)$$

$$\alpha \exp(f) + \beta(\|\omega^\sharp\|^2 - \omega(\xi)^2) \neq 0. \quad (3.19)$$

In (3.19), α and β are some non-zero constants.

Proof. The proof of the relations (3.17), (3.18) and (3.19) are given by contradiction.

Suppose $\omega(\xi) = 0$. Taking the covariant differentiation of $\omega(\xi) = 0$ with respect to ξ and using $\acute{\nabla}\omega^\sharp = 0$, we obtain

$$\acute{g}(\acute{\nabla}_\xi \xi, \omega^\sharp) = 0.$$

Using (3.7) in the above equation, we get

$$\|\omega^\sharp\|^2 = \omega(\xi)^2.$$

Since we suppose that $\omega(\xi) = 0$, from the above equation it follows that $\|\omega^\sharp\|^2 = 0$ which contradicts the hypothesis $\omega^\sharp \neq 0$. Hence (3.17) holds on $\dot{M}$.

Now, we assume

$$(\exp(f))^{1/2} - \frac{1}{2}\omega(\xi) = 0. \quad (3.20)$$

Taking the covariant differentiation of (3.20) along vector field ξ , we have

$$\frac{1}{2}\omega(\xi)(\exp(f))^{1/2} - \frac{1}{2}\xi(\omega(\xi)) = 0. \quad (3.21)$$

Using (3.20) and $\acute{\nabla}\omega^\sharp = 0$ in (3.21), we get $2\acute{g}(\acute{\nabla}_\xi \xi, \omega^\sharp) = \omega(\xi)^2$. Then by the use of (3.7) in $2\acute{g}(\acute{\nabla}_\xi \xi, \omega^\sharp) = \omega(\xi)^2$, we find $2\omega(\xi)^2 = \|\omega^\sharp\|^2$. As $\omega^\sharp$ is parallel on $\dot{M}$, it follows that

$\|\omega^\sharp\|^2$ is constant on $\dot{M}$. Thus, $\omega(\xi)^2$ is constant on $\dot{M}$. Then from (3.21) we have $\omega(\xi) = 0$ which is a contradiction in view of (3.17).

Finally, we suppose

$$\alpha \exp(f) + \beta(\|\omega^\sharp\|^2 - \omega(\xi)^2) = 0. \quad (3.22)$$

Taking the covariant differentiation of (3.22) along vector field ξ and using $\dot{\nabla}\omega^\sharp = 0$, we find

$$\alpha\omega(\xi)\exp(f) - 2\beta\omega(\xi)\dot{g}(\dot{\nabla}_\xi\xi, \omega^\sharp) = 0.$$

Making use of (3.7) and (3.17) in the above equation, we get

$$\alpha \exp(f) + \beta(\omega(\xi)^2 - \|\omega^\sharp\|^2) = 0. \quad (3.23)$$

Summing (3.22) to (3.23), we have $\alpha \exp(f) = 0$ which is a contradiction.

Hence (3.17), (3.18) and (3.19) hold on $\dot{M}$. $\square$

COROLLARY 3.10. *Let $\dot{M}^m$ ($m > 1$) be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp$ is tangent to $\dot{M}$. Suppose that $\omega^\sharp$ is parallel on $\dot{M}$. Then $\dot{M}$ cannot be Ricci flat.*

Proof. Suppose $\dot{M}$ is Ricci flat. Putting $X = \omega^\sharp$ in (3.14), we obtain

$$(m - 1) \exp(f) \omega(\xi) = 0.$$

Using (3.17) in the above equation, we get $\exp(f) = 0$ that is a contradiction. $\square$

4. Submanifolds with parallel Ricci tensor

The notion of Einstein manifolds and quasi-Einstein manifolds was introduced in [3] and [7], respectively.

Therefore, a submanifold $(\dot{M}^m, \dot{g})(m > 2)$ of a conformal Kenmotsu manifold (M, g) is an Einstein manifold if its Ricci tensor Ric satisfies the condition $Ric = \lambda \dot{g}$, where λ is a non-zero constant.

Moreover, a submanifold $(\dot{M}^m, \dot{g})(m > 2)$ of a conformal Kenmotsu manifold (M, g) is a quasi-Einstein manifold if the condition

$$Ric(X, Y) = a\dot{g}(X, Y) + bA(X)A(Y) \quad (4.1)$$

is fulfilled on $\dot{M}$, where a, b are smooth functions of which $b \neq 0$ and A is non-zero 1-form such that

$$\dot{g}(X, U) = A(X)$$

for all vector fields X and U being a unit vector field which is called the generator of $\dot{M}$.

In [13], Kobayashi showed that a submanifold of a Kenmotsu manifold has parallel second fundamental form if and only if the submanifold is totally geodesic. As a generalization of this result, we consider submanifolds of a conformal Kenmotsu manifold with parallel Ricci tensor.

Now we state the following theorems.

THEOREM 4.1. *Let $\dot{M}^m$ ($m > 2$) be a submanifold of a conformal Kenmotsu manifold M with parallel Ricci tensor such that $\omega^\sharp$ is normal to $\dot{M}$. Then $\dot{M}$ is an Einstein submanifold.*

Proof. Taking the covariant differentiation of (3.13) with respect to vector field Y , we obtain

$$\text{Ric}(X, \nabla_Y \xi) = -(m-1) \exp(f) \acute{g}(X, \nabla_Y \xi).$$

Making use of (3.4) and (3.13) in the above equation, it follows that

$$\text{Ric}(X, Y) = -(m-1) \exp(f) \acute{g}(X, Y) \quad (4.2)$$

for all X, Y on $\acute{M}$. Hence, $\acute{M}$ is an Einstein submanifold. $\square$

THEOREM 4.2. *Let $\acute{M}^m (m > 2)$ be a submanifold of a conformal Kenmotsu manifold M with parallel Ricci tensor such that $\omega^\sharp$ is tangent to $\acute{M}$. Then $\acute{M}$ is a quasi-Einstein submanifold.*

Proof. Suppose that $\omega^\sharp$ is parallel on $\acute{M}$. Taking the covariant differentiation of (3.14) and using (3.7), we have

$$\begin{aligned} & ((\exp(f))^{1/2} - \frac{1}{2} \omega(\xi)) \{ \text{Ric}(X, Y) - (-(m-1) \exp(f) \\ & + \frac{1}{4} (m-2) \|\omega^\sharp\|^2) \acute{g}(X, Y) + \frac{1}{4} (m-2) \omega(X) \omega(Y) \} = 0 \end{aligned} \quad (4.3)$$

for all vector fields X, Y orthogonal to ξ . By the use of (3.18) in (4.3) we get

$$\begin{aligned} \text{Ric}(X, Y) = & \{ -(m-1) \exp(f) + \frac{1}{4} (m-2) \|\omega^\sharp\|^2 \} \acute{g}(X, Y) \\ & - \frac{1}{4} (m-2) \omega(X) \omega(Y), \end{aligned} \quad (4.4)$$

where X, Y are orthogonal to ξ . We get (4.4) for all vector fields X, Y in view of (3.14). Since $\omega^\sharp \neq 0$, (4.4) can be written as

$$\begin{aligned} \text{Ric}(X, Y) = & \left\{ -(m-1) \exp(f) + \frac{1}{4} (m-2) \|\omega^\sharp\|^2 \right\} \acute{g}(X, Y) \\ & - \frac{1}{4} (m-2) \|\omega^\sharp\|^2 \acute{g} \left(X, \frac{\omega^\sharp}{\|\omega^\sharp\|} \right) \acute{g} \left(Y, \frac{\omega^\sharp}{\|\omega^\sharp\|} \right) \end{aligned}$$

for all X, Y tangent to $\acute{M}$. Hence, by (4.1) $\acute{M}$ is a quasi-Einstein submanifold. $\square$

THEOREM 4.3. *There is not any submanifolds $\acute{M}^m (m > 1)$ of a conformal Kenmotsu manifold M with Lie ξ -parallel Ricci tensor when $\omega^\sharp$ is tangent to $\acute{M}$.*

Proof. Since the Ricci tensor of $\acute{M}$ is Lie ξ -parallel, we can write

$$0 = L_\xi \text{Ric}(X, \xi) - \text{Ric}(L_\xi X, \xi)$$

for each X on $\acute{M}$ where L_ξ is the Lie derivative. Let $\omega^\sharp$ be parallel on $\acute{M}$. Putting (3.14) into the above equation, we have

$$\begin{aligned} 0 = & \{ -(m-1) \exp(f) + \frac{1}{4} \|\omega^\sharp\|^2 (m-2) \} (L_\xi \acute{g})(X, \xi) \\ & - (m-1) \exp(f) \omega(\xi) \eta(X) + \frac{1}{4} (m-2) \omega(\xi) \acute{g}(X, \nabla_{\omega^\sharp} \xi). \end{aligned}$$

Then by the use of (3.7), we get

$$\begin{aligned} 0 = & \{ -(m-1) \exp(f) + \frac{1}{4} \|\omega^\sharp\|^2 (m-2) \} (L_\xi \acute{g})(X, \xi) \\ & - (m-1) \exp(f) \omega(\xi) \eta(X) + \frac{1}{4} (m-2) (\exp(f))^{1/2} \omega(\xi) (\omega(X) - \omega(\xi) \eta(X)) \end{aligned}$$

for each X on $\dot{M}$. Taking $X = \xi$ in the above equation, we obtain

$$\exp(f)\omega(\xi) = 0. \quad (4.5)$$

Making use of relation (3.17) in (4.5), we have $\exp(f) = 0$ that is a contradiction. Hence, the proof of the theorem is completed. $\square$

5. Submanifolds with recurrent Ricci tensor

The notion of a recurrent tensor field of type (r, s) on a differentiable manifold M with a linear connection ∇ was introduced in [12] and [16]. A non-zero tensor field K of type (r, s) on M is recurrent if there is a 1-form α such that

$$\nabla K = \alpha \otimes K.$$

Therefore, a submanifold $\dot{M}$ of a conformal Kenmotsu manifold is said to be with recurrent Ricci tensor if there exists a 1-form α on $\dot{M}$ such that

$$(\dot{\nabla}_X Ric)(Y, Z) = \alpha(X) Ric(Y, Z) \quad (5.1)$$

for all vector fields X, Y, Z on $\dot{M}$ where Ric denotes the Ricci tensor of $\dot{M}$.

Now, we state the following results.

LEMMA 5.1. *Let $\dot{M}^m (m > 1)$ be a submanifold of a conformal Kenmotsu manifold M with recurrent Ricci tensor such that $\omega^\sharp$ is normal to $\dot{M}$. Then the Ricci tensor of $\dot{M}$ is parallel.*

Proof. Taking $X = \xi$ in (3.13), we have

$$Ric(\xi, \xi) = -(m - 1) \exp(f). \quad (5.2)$$

Taking covariant differentiation of (5.2) along vector field X , we obtain

$$(\dot{\nabla}_X Ric)(\xi, \xi) + 2 Ric(\dot{\nabla}_X \xi, \xi) = -(m - 1)X(f) \exp(f).$$

Since Ric is recurrent and $X(f) = \omega(X)$, from (5.1) the above equation can be written as

$$\alpha(X) Ric(\xi, \xi) + 2 Ric(\dot{\nabla}_X \xi, \xi) = -(m - 1) \exp(f) \omega(X).$$

On the other hand $\omega^\sharp$ is orthogonal to $\dot{M}$. Hence, by the use of (3.13) and (5.2) in the above equation we have $\alpha(X) = 0$ for each vector field X on $\dot{M}$. Thus, by (5.1), the Ricci tensor of $\dot{M}$ is parallel. $\square$

THEOREM 5.2. *Let $\dot{M}^m (m > 2)$ be a submanifold of a conformal Kenmotsu manifold M with recurrent Ricci tensor such that $\omega^\sharp$ is normal to $\dot{M}$. Then $\dot{M}$ is an Einstein submanifold.*

Proof. It is an immediate result of Theorem 4.1 and Lemma 5.1. $\square$

LEMMA 5.3. *Let $\dot{M}^m (m > 1)$, be a submanifold of a conformal Kenmotsu manifold M with recurrent Ricci tensor such that $\omega^\sharp$ is tangent to $\dot{M}$. Then*

$$\alpha(X) = \frac{-4(m - 1) \exp(f)}{-4(m - 1) \exp(f) + (m - 2)(\|\omega^\sharp\|^2 - \omega(\xi)^2)} \omega(X) \quad (5.3)$$

for each vector field X tangent to $\dot{M}$.

Proof. Suppose that $\omega^\sharp$ is parallel on $\acute{M}$. Putting $X = \xi$ in (3.14), we have

$$\text{Ric}(\xi, \xi) = -(m-1)\exp(f) + \frac{1}{4}(m-2)\|\omega^\sharp\|^2 - \frac{1}{4}(m-2)\omega(\xi)^2. \quad (5.4)$$

Taking covariant differentiation of (5.4) along vector field X and using (5.1), we obtain

$$\begin{aligned} \alpha(X) \text{Ric}(\xi, \xi) + 2 \text{Ric}(\acute{\nabla}_X \xi, \xi) &= -(m-1)\exp(f)\omega(X) \\ &\quad + \frac{1}{4}(m-2)X(\|\omega^\sharp\|^2) - \frac{1}{2}(m-2)\omega(\xi)\omega(\acute{\nabla}_X \xi). \end{aligned}$$

As $\omega^\sharp$ is parallel on $\acute{M}$, $\|\omega^\sharp\|^2$ is constant on $\acute{M}$. Thus using (3.14), (3.19) and (5.4), we get (5.3). $\square$

THEOREM 5.4. *Let $\acute{M}^m (m > 1)$ be a submanifold of a conformal Kenmotsu manifold M with recurrent Ricci tensor such that $\omega^\sharp$ is tangent to $\acute{M}$. Then the Ricci tensor of $\acute{M}$ cannot be parallel.*

Proof. Assuming that the Ricci tensor of $\acute{M}$ is parallel and using (5.1) we will have

$$\alpha(X) \text{Ric}(Y, Z) = 0.$$

Suppose that $\omega^\sharp$ is parallel on $\acute{M}$. Taking $Y = Z = \xi$ in the above equation and using (3.14) and (3.19), we get $\alpha = 0$. Then from (5.3), we have $\omega^\sharp = 0$ that is a contradiction. $\square$

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