TAKEUCHI’S EQUALITY FOR THE LEVI FORM OF THE FUBINI–STUDY DISTANCE TO COMPLEX SUBMANIFOLDS IN COMPLEX PROJECTIVE SPACES

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Abstract. A. Takeuchi showed that the negative logarithm of the Fubini–Study boundary distance function of pseudoconvex domains in the complex projective space $\mathbb{CP}^n$, $n \in \mathbb{N}$, is strictly plurisubharmonic and solved the Levi problem for $\mathbb{CP}^n$. His estimate from below of the Levi form is nowadays called the ‘Takeuchi’s inequality.’ In this paper, we give the ‘Takeuchi’s equality,’ i.e. an explicit representation of the Levi form of the negative logarithm of the Fubini–Study distance to complex submanifolds in $\mathbb{CP}^n$.

0. Introduction

Let $D \subseteq \mathbb{CP}^n$, $n \in \mathbb{N}$, be a pseudoconvex domain and denote by $\delta_{\partial D}(P)$ the Fubini–Study distance from $P \in D$ to the boundary $\partial D$ of $D$. Takeuchi [21] found that the strict subharmonicity of the function $-\log \tan^{-1}|z|$ on $\mathbb{C} \setminus \{0\}$ leads the strict plurisubharmonicity of the function $-\log \delta_{\partial D}$ on $D$, and solved the Levi problem for $\mathbb{CP}^n$. The inequality

$$i\partial \bar{\partial}(-\log \delta_{\partial D}) \geq \frac{1}{3} \omega_{FS} \quad \text{on } D$$

is nowadays called the ‘Takeuchi’s inequality’ (cf. [1, 7, 9, 20, 22]).

Recently, many mathematicians have been interested in the following problem: ‘Is there a smooth closed Levi-flat real hypersurface in $\mathbb{CP}^n$ if $n \geq 2$?’, where a real hypersurface $M \subset \mathbb{CP}^n$ is said to be Levi-flat if its complement $\mathbb{CP}^n \setminus M$ is locally pseudoconvex or equivalently locally Stein. When $n \geq 3$, Lins Neto [11] proved the non-existence in the real analytic case, and Siu [19] proved it in the smooth case. When $n = 2$, the non-existence problem is still open even in the real analytic case. Then Takeuchi’s inequality is one of the key points to approach the non-existence problem or related topics. His paper [21] is frequently cited even though he wrote it over 50 years ago (for example, see Adachi [1], Adachi and Brinkschulte [2], Brinkschulte [5], Brunella [6], Fu and Shaw [8], Harrington and Shaw [10], Ohsawa [16, 17], and Ohsawa and Sibony [18]).

It follows from Takeuchi’s theorem that if $S$ is a complex hypersurface in $\mathbb{CP}^n$ and if $\delta_S$ denotes the Fubini–Study distance to $S$, then the function $-\log \delta_S$ is strictly plurisubharmonic.

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Moreover, the real-valued functions $-\log \delta_S$ more precisely for nonsingular $S$.

In what follows, we denote the inhomogeneous coordinate system on $\mathbb{C}^2 \subset \mathbb{C}P^2$ by $(z, w)$ (see Section 1). The main results in this paper are the following.

**Theorem 1.** Let $S$ be a smooth complex hypersurface in $\mathbb{C}P^2$ locally defined by

$$S = \{(t, f(t)) \mid t \in V\}$$

for open $V \subset \mathbb{C}$ and holomorphic $f : V \to \mathbb{C}$. Suppose $0 \in V$ and $f(0) = f'(0) = 0$. Let $\delta_S$ be the Fubini–Study distance to $S$. Then there exists $\varepsilon > 0$ such that:

(i) $$\frac{\partial^2 (-\log \delta_S)}{\partial z \partial \bar{z}}(0, w)dz^2 = \frac{|w|}{2 \cdot \tan^{-1}|w|} \cdot \frac{1 + |f''(0)|^2}{1 - |f''(0)|^2 |w|^2} \cdot |dz|^2;$$

(ii) $$\frac{\partial^2 (-\log \delta_S)}{\partial w \partial \bar{w}}(0, w)|dw|^2 = \frac{|w| - \tan^{-1}|w| + |w|^2 \tan^{-1}|w|}{4 \cdot |w| (\tan^{-1}|w|)^2} \cdot \frac{|dw|^2}{(1 + |w|^2)^2};$$

(iii) $$\frac{\partial^2 (-\log \delta_S)}{\partial z \partial \bar{w}}(0, w)dz d\bar{w} = 0$$

for $0 < |w| < \varepsilon$.

**Corollary 1.** With the same notation as above, the eigenvalues of the Levi form of the function $-\log \delta_S$ with respect to the Fubini–Study metric $ds^2$ on $\mathbb{C}P^2$ are

$$\tan \delta_S \cdot \frac{1 + |f''(0)|^2}{2 \cdot \delta_S} \cdot \frac{\delta_S - \delta_S + \tan^2 \delta_S \cdot \delta_S}{4 \cdot \tan \delta_S \cdot \delta_S^2}$$

at $(0, w) \in \mathbb{C}^2 \subset \mathbb{C}P^2$ for $0 < |w| < \varepsilon$.

If we put

$$\Phi(x) := \frac{x}{2 \cdot \tan^{-1} x} \cdot \frac{1 + |f''(0)|^2}{1 - |f''(0)|^2 x^2}, \quad \Psi(x) := \frac{x - \tan^{-1} x + x^2 \tan^{-1} x}{4 \cdot x (\tan^{-1} x)^2}$$

for $x \in \mathbb{R} \setminus \{0\}$, the eigenvalues in the corollary above are written as $\Phi(\tan \delta_S)$ and $\Psi(\tan \delta_S)$. The functions $\Phi$ and $\Psi$ have some interesting properties. For example,

$$\lim_{x \to +0} \Phi(x) = \frac{1 + |f''(0)|^2}{2}, \quad \lim_{x \to +0} \Psi(x) = \frac{1}{3}.$$ 

Moreover, the real-valued functions $\Phi$ and $\Psi$ are convex, that is, $\Phi'(x) > 0, \Phi''(x) > 0$ for $0 < x < 1/|f''(0)|$, and $\Psi'(x) > 0, \Psi''(x) > 0$ for $x > 0$ (see Section 6, Lemmas 3, 4, 5, and 6). Therefore, we have the following proposition.

**Proposition.** With the same notation as above, the functions $\Phi(|z|)$ and $\Psi(|z|)$ are strictly subharmonic on $0 < |z| < 1/|f''(0)|$ and on $\mathbb{C} \setminus \{0\}$, respectively.

We can also extend Theorem 1 to the case where $M$ is a complex submanifold of general dimension in $\mathbb{C}P^n$ (see Section 4, Theorem 2). Although Theorem 1 is a special case of Theorem 2 logically, we think the essential part is Theorem 1 and so we will begin to prove Theorem 1 first.
Remark 1. For the Euclidian distance on $\mathbb{C}^n$, the corresponding results to Theorems 1 and 2 are Matsumoto–Ohsawa [15, Lemma], for $n = 2$ and Matsumoto [14, Theorem 1], in general, respectively.

1. Fubini–Study distance on $\mathbb{CP}^n$

Throughout this paper, let $\mathbb{CP}^n$, $n \in \mathbb{N}$, be the complex projective space equipped with the Fubini–Study metric $ds^2$, and denote by $\delta_n(P, Q)$ the Fubini–Study distance of $P, Q \in \mathbb{CP}^n$ with respect to the metric $ds^2$. In Section 1, we recall some fundamental properties of the distance.

Let $[\zeta_0 : \zeta_1 : \cdots : \zeta_n]$ be a homogeneous coordinate system on $\mathbb{CP}^n$. If we put $U_0 := \{[\zeta_0 : \zeta_1 : \cdots : \zeta_n] \in \mathbb{CP}^n \mid \zeta_0 \neq 0\}$ and put $z_i := \zeta_i/\zeta_0$ for $1 \leq i \leq n$, then $(z_1, \ldots, z_n)$ is a complex coordinate system on $U_0 \cong \mathbb{C}^n$. This is called an inhomogeneous coordinate system on $U_0$ of $\mathbb{CP}^n$. In this paper, we will denote by the same symbol $\mathbb{C}^n$ the subset $U_0$ of $\mathbb{CP}^n$ equipped with the inhomogeneous coordinate system $(z_1, \ldots, z_n)$.

For a fixed point $P_0 \in \mathbb{CP}^n$, by a suitable isometric transformation of $\mathbb{CP}^n$ we may suppose that $P_0$ is the origin $0 = (0, \ldots, 0) \in \mathbb{C}^n \subset \mathbb{CP}^n$. Then, real lines through the origin, i.e., $\{(a_1 t, \ldots, a_n t) \mid t \in \mathbb{R}\} \subset \mathbb{C}^n \subset \mathbb{CP}^n$ for $(a_1, \ldots, a_n) \in \mathbb{C}^n \setminus \{0\}$, are geodesics in $\mathbb{CP}^n$. Moreover, if $P_0 \in L_1 \subset L_2 \subset \cdots \subset L_{n-1} \subset \mathbb{C}^n \subset \mathbb{CP}^n$ is a sequence of complex linear subspaces $L_i$ of dimension $i$, $1 \leq i \leq n - 1$, by a suitable isometric transformation of $\mathbb{CP}^n$ we may suppose that each $L_i$ is defined by $z_{i+1} = \cdots = z_n = 0$ in $\mathbb{C}^n$ (see Takeuchi [21, Lemma 1]).

The Fubini–Study metric $ds^2$ is written as

$$ds^2 = \frac{\sum_{i=1}^n |dz_i|^2}{1 + \sum_{i=1}^n |z_i|^2} \frac{\sum_{i,j=1}^n \bar{z}_i z_j d\bar{z}_i dz_j}{(1 + \sum_{i=1}^n |z_i|^2)^2}$$

(1.1)

on $\mathbb{C}^n \subset \mathbb{CP}^n$. For two points $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ of $\mathbb{C}^n \subset \mathbb{CP}^n$, their Fubini–Study distance is written as

$$\delta_n(z, w) := \cos^{-1} \left( \frac{1 + \sum_{i=1}^n \bar{z}_i \bar{w}_i}{\sqrt{1 + \sum_{i=1}^n |z_i|^2} \sqrt{1 + \sum_{i=1}^n |w_i|^2}} \right).$$

(1.2)

When $n = 1$, it is rewritten as

$$\delta_1(z, w) := \cos^{-1} \left( \frac{|1 + z\bar{w}|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}} \right) = \sin^{-1} \frac{|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}$$

(1.3)

for $z, w \in \mathbb{C} \subset \mathbb{CP}^1$, and particularly

$$\delta_1^*(z) := \delta_1(z, 0) = \cos^{-1} \left( \frac{1}{\sqrt{1 + |z|^2}} \right) = \tan^{-1}|z|.$$  

(1.4)

Lemma 1. Let $\delta_1^*(z) := \tan^{-1}|z|$ for $z \in \mathbb{C}$. Then, for $z \in \mathbb{C} \setminus \{0\}$,

(i) \[ \frac{\partial \delta_1^*(z)}{\partial \bar{z}} = \frac{\bar{z}}{2 \cdot |z|(1 + |z|^2)}, \quad \frac{\partial^2 \delta_1^*(z)}{\partial \bar{z} \partial z} = \frac{1 - |z|^2}{4 \cdot |z|(1 + |z|^2)^2}, \]

(ii) \[ \frac{\partial^2 (-\log \delta_1^*)}{\partial \bar{z} \partial z}(z) = \frac{|z| - \tan^{-1}|z| + |z|^2 \tan^{-1}|z|}{4 \cdot |z|(\tan^{-1}|z|)^2} \cdot \frac{1}{(1 + |z|^2)^2}. \]
Proof. The proof is done by direct differentiations.

After Takeuchi [21, Lemma 4], it is well known that the function \(-\log \delta_1^t(z) := -\log \tan^{-1}|z|\) is strictly subharmonic on \(C \setminus \{0\}\). In the lemma above, we can also check that \(|z| - \tan^{-1}|z| + |z|^2 \tan^{-1}|z| > 0\) for \(z \neq 0\) directly. Moreover,

\[
\frac{|z| - \tan^{-1}|z| + |z|^2 \tan^{-1}|z|}{4 \cdot |z|(|\tan^{-1}|z|)^2} \leq \frac{1}{3}
\]

as \(|z| \to +0\) (Section 6, Lemma 6). In Section 6, we will study the function more precisely.

2. Proof of Theorem 1 and Corollary 1

In Sections 2 and 3, let \(S\) be a smooth complex hypersurface through a point \(P_0\) of \(C^2\). By a suitable isometric transformation of \(C^2\) we may suppose that \((z, w)\) is an inhomogeneous coordinate system on \(C^2 \subset C^3\), \(P_0\) is the origin \((0, 0)\), and \(S\) is locally defined by

\[
S := \{(t, f(t)) \mid t \in V\}
\]  \hspace{1cm} (2.1)

for open \(0 \in V \subset C\) and holomorphic \(f : V \to C\). Moreover, we may suppose that

\[
f(0) = f'(0) = 0,
\]  \hspace{1cm} (2.2)

that is, the complex hypersurface \(S\) is defined by \(w = f(z)\) near \((0, 0)\) and the tangent space of \(S\) at \((0, 0)\) is written as \(w = 0\).

We denote by

\[
\delta_S(z, w) := \inf_{t \in V} \delta_2((z, w), (t, f(t)))
\]  \hspace{1cm} (2.3)

the Fubini–Study distance to \(S\) from \((z, w) \in C^2 \subset C^3\), where \(\delta_2\) is the Fubini–Study distance on \(C^3\) defined in (1.2). Then the function \(\delta_S\) is real analytic in \(\Delta \setminus S\) for some open \(\Delta\) with \(P_0 \in \Delta \subset V \times C\), and there exists for each \((z, w) \in \Delta\) only one \((t, f(t)) \in S\) such that \(\delta_S(z, w) = \delta_2((z, w), (t, f(t)))\). In particular, by the condition (2.2), the nearest point of \(S\) is the origin \((0, 0)\) for \((0, w) \in \Delta \subset C^2\). We choose \(\varepsilon > 0\) so that \((0, w) \in \Delta\) if \(|w| < \varepsilon\). Then we have

\[
\delta_S(0, w) = \delta_2((0, w), (0, 0)) = \delta_1^w(0) = \tan^{-1}|w|
\]  \hspace{1cm} (2.4)

for \(|w| < \varepsilon\).

Now, since \(\delta_S(0, w) = \tan^{-1}|w|\), it follows directly from Lemma 1 that

\[
\frac{\partial \delta_S}{\partial w}(0, w) = \frac{\dot{w}}{2|w|(1 + |w|^2)}
\]  \hspace{1cm} (2.5)

and Theorem 1(ii). To prove Theorem 1(i) and (iii) it is enough to prove the following.

Lemma 2. For \(0 < |w| < \varepsilon\),

(i) \(\frac{\partial \delta_S}{\partial z}(0, w) = 0\),

(ii) \(\frac{\partial^2 \delta_S}{\partial z \partial \bar{z}}(0, w) = \frac{-|w|(1 + |f''(0)|^2)}{2(1 + |w|^2)(1 - |f''(0)|^2|w|^2)}\), \(\frac{\partial^2 \delta_S}{\partial \bar{z} \partial \dot{w}}(0, w) = 0\).
In fact, since
\[
\frac{\partial^2(-\log \delta_S)}{\partial z \partial \bar{z}} = -\frac{1}{\delta_S} \cdot \frac{\partial^2 \delta_S}{\partial z \partial \bar{z}} + \frac{1}{\delta_S^2} \cdot \left| \frac{\partial \delta_S}{\partial z} \right|^2
\]
and \( \delta_S(0, w) = \tan^{-1}|w| \), we can obtain Theorem 1(i) from Lemma 2(i), and similarly Theorem 1(iii). We will prove Lemma 2 in the next section.

If \((z, w)\) is the inhomogeneous coordinate system on \( \mathbb{C}^2 \subset \mathbb{CP}^2 \), the Fubini–Study metric \( ds^2 \) is written by
\[
ds^2 = \frac{|dz|^2}{1 + |w|^2} + \frac{|dw|^2}{(1 + |w|^2)^2}
\]
at \( P := (0, w) \in \mathbb{C}^2 \subset \mathbb{CP}^2 \). Since \( \delta_S(P) = \tan^{-1}|w| \), Corollary 1 follows from Theorem 1.

### 3. Proof of Lemma 2

Now we begin to prove Lemma 2. The Fubini–Study distance of \((z, w) \in \mathbb{C}^2\) and \((t, f(t)) \in S\), where \( t \in V \subset \mathbb{C} \), is written by (1.2) as
\[
\delta_2((z, w), (t, f(t))) = \cos^{-1}\left(\frac{|1 + t\bar{z} + f(t)\bar{w}|}{\sqrt{1 + |t|^2} \sqrt{1 + |f(t)|^2} \sqrt{1 + |z|^2 + |w|^2}}\right).
\]
If we put
\[
\alpha(z, w, t) := \frac{|1 + t\bar{z} + f(t)\bar{w}|^2}{(1 + |t|^2 + |f(t)|^2)(1 + |z|^2 + |w|^2)}
\]
for \((z, w) \in \mathbb{C}^2\) and \( t \in V \subset \mathbb{C} \), then
\[
\frac{\partial^2 \alpha}{\partial t \partial \bar{t}}(0, w, 0) = \frac{f''(0) \bar{w}}{1 + |w|^2}, \quad \frac{\partial^2 \alpha}{\partial t \partial t}(0, w, 0) = \frac{-1}{1 + |w|^2}
\]
and
\[
H(0, w, 0) := \det \begin{pmatrix} \frac{\partial^2 \alpha}{\partial t \partial t} & \frac{\partial^2 \alpha}{\partial t \partial \bar{t}} \\ \frac{\partial^2 \alpha}{\partial \bar{t} \partial \bar{t}} & \frac{\partial^2 \alpha}{\partial \bar{t} \partial t} \end{pmatrix}(0, w, 0) = \frac{|f''(0)|^2 |w|^2 - 1}{(1 + |w|^2)^2}.
\]
Since \( H(0, 0, 0) = -1 \neq 0 \), it follows by the implicit function theorem that one can find a \( C^0\)-function \( t = t(z, w) \) defined in some neighborhood \( \Delta (\subset V \times \mathbb{C}) \) of \( P_0 = (0, 0) \in S \) which satisfies
\[
\frac{\partial \alpha}{\partial t}(z, w, t(z, w)) = \frac{\partial \alpha}{\partial t}(z, w, t(z, w)) = 0
\]
for \((z, w) \in \Delta \). We choose \( \varepsilon > 0 \) so that \((0, w) \in \Delta \) if \( |w| < \varepsilon \). Then by condition (2.2) we have \( t(0, w) = 0 \) for \( |w| < \varepsilon \).

If we put
\[
\beta(z, w) := \alpha(z, w, t(z, w))
\]
then
\[
\delta_S(z, w) = \cos^{-1}\sqrt{\beta(z, w)}
\]
for \((z, w) \in \Delta \).
For a moment, we put \((z_1, z_2) := (z, w)\) for simplicity. By differentiating (3.5) and applying (3.4) we have

\[
\frac{\partial \beta}{\partial z_i} = \frac{\partial \alpha}{\partial z_i} + \frac{\partial \alpha}{\partial t} \frac{\partial t}{\partial z_i} + \frac{\partial \alpha}{\partial \bar{z}_i} \frac{\partial \bar{z}_i}{\partial z_i} = \frac{\partial \alpha}{\partial z_i}
\]

(3.7) and

\[
\frac{\partial^2 \beta}{\partial z_i \partial \bar{z}_j} = \frac{\partial^2 \alpha}{\partial z_i \partial \bar{z}_j} + \frac{\partial^2 \alpha}{\partial t \partial \bar{z}_j} \frac{\partial t}{\partial z_i} + \frac{\partial^2 \alpha}{\partial \bar{z}_i \partial \bar{z}_j} \frac{\partial \bar{z}_i}{\partial z_i} = \frac{\partial^2 \alpha}{\partial z_i \partial \bar{z}_j}
\]

(3.8) for \(i, j = 1, 2\). Moreover, by differentiating (3.4) we have

\[
\begin{cases}
\frac{\partial^2 \alpha}{\partial t^2} \frac{\partial t}{\partial z_i} + \frac{\partial^2 \alpha}{\partial t \partial \bar{z}_i} \frac{\partial \bar{z}_i}{\partial t} = 0, \\
\frac{\partial^2 \alpha}{\partial \bar{z}_i \partial t} + \frac{\partial^2 \alpha}{\partial \bar{z}_i \partial \bar{z}_i} \frac{\partial \bar{z}_i}{\partial \bar{z}_i} = 0,
\end{cases}
\]

for \(i = 1, 2\), and hence

\[
\begin{pmatrix}
\frac{\partial^2 \alpha}{\partial t \partial \bar{z}} \\
\frac{\partial^2 \alpha}{\partial \bar{z} \partial \bar{w}} \\
\frac{\partial^2 \alpha}{\partial \bar{z} \partial \bar{w}}
\end{pmatrix}
= -\begin{pmatrix}
\frac{\partial^2 \alpha}{\partial t \partial z} \\
\frac{\partial^2 \alpha}{\partial \bar{z} \partial w} \\
\frac{\partial^2 \alpha}{\partial \bar{z} \partial w}
\end{pmatrix}.
\]

(3.9)

Now, by definition (3.1) and condition (2.2), it is easy to see that

\[
\frac{\partial \alpha}{\partial t}(z, w, 0) = \frac{\bar{z}}{1 + |z|^2 + |w|^2},
\]

and hence

\[
\begin{cases}
\frac{\partial^2 \alpha}{\partial t \partial \bar{z}}(0, w, 0) = \frac{1}{1 + |w|^2}, \\
\frac{\partial^2 \alpha}{\partial \bar{z} \partial w}(0, w, 0) = \frac{\partial^2 \alpha}{\partial \bar{z} \partial \bar{w}}(0, w, 0) = 0.
\end{cases}
\]

(3.10)

Therefore, by substituting (3.2) and (3.10) into equation (3.9), we first obtain

\[
\begin{cases}
\frac{\partial t}{\partial z}(0, w) = \frac{1}{1 - |f''(0)|^2 |w|^2}, \\
\frac{\partial \bar{t}}{\partial \bar{z}}(0, w) = \frac{f''(0) \bar{w}}{1 - |f''(0)|^2 |w|^2},
\end{cases}
\]

(3.11)

for \(|w| < \varepsilon\).

Next, by definition (3.1) and condition (2.2), since

\[
\alpha(z, w, 0) = \frac{1}{1 + |z|^2 + |w|^2},
\]

it is also easy to see that

\[
\begin{cases}
\frac{\partial \alpha}{\partial \bar{z}}(0, w, 0) = 0, \\
\frac{\partial \alpha}{\partial w}(0, w, 0) = \frac{-\bar{w}}{(1 + |w|^2)^2}
\end{cases}
\]

(3.12)

and

\[
\begin{cases}
\frac{\partial^2 \alpha}{\partial z \partial \bar{z}}(0, w, 0) = \frac{-1}{(1 + |w|^2)^2}, \\
\frac{\partial^2 \alpha}{\partial \bar{z} \partial \bar{w}}(0, w, 0) = 0.
\end{cases}
\]

(3.13)
Therefore, by substituting (3.12) into (3.7), we have
\[
\frac{\partial \beta}{\partial z}(0, w) = 0, \quad \frac{\partial \beta}{\partial w}(0, w) = \frac{-\bar{w}}{(1 + |w|^2)^2},
\] (3.14)
and by substituting (3.13), (3.10), and (3.11) into (3.8), we have
\[
\frac{\partial^2 \beta}{\partial z \partial \bar{z}}(0, w, 0) = \frac{|w|^2(1 + |f''(0)|^2)}{(1 + |w|^2)^2(1 - |f''(0)|^2|w|^2)^2}, \quad \frac{\partial^2 \beta}{\partial z \partial w}(0, w, 0) = 0.
\] (3.15)

Now, if \( \delta_S \) is the Fubini–Study distance to the complex hypersurface \( S \), it follows from (3.6) that
\[
\beta(z, w) = \cos^2 \delta_S(z, w)
\] (3.16)
for \((z, w) \in \Delta\). We note here that if \( \delta^+ \) \( \wedge 1 \) := \( \tan^{-1} |w| \), then
\[
\cos \delta^+(w) = \frac{1}{\sqrt{1 + |w|^2}}, \quad \sin \delta^+(w) = \frac{|w|}{\sqrt{1 + |w|^2}}
\] (3.17)
and
\[
\beta(0, w) = \alpha(0, w, 0) = \frac{1}{1 + |w|^2} = \cos^2 \delta^+(w)
\]
for \( w \in \mathbb{C} \). Moreover, \( \delta_S(0, w) = \delta^+(w) = \tan^{-1} |w| \) if \( |w| < \varepsilon \).

If we differentiate (3.16), then
\[
\frac{\partial \beta}{\partial z}(z, w) = -2 \cos \delta_S(z, w) \sin \delta_S(z, w) \frac{\partial \delta_S}{\partial z}(z, w)
\] (3.18)
for \((z, w) \in \Delta\). Therefore, by (3.14), (3.17), and (3.18) we first obtain
\[
\frac{\partial \delta_S}{\partial z}(0, w) = 0
\] (3.19)
for \( 0 < |w| < \varepsilon \), which proves Lemma 1(i). Next, it follows from (3.17), (3.18), and (3.19) that
\[
\frac{\partial^2 \beta}{\partial z \partial \bar{z}}(0, w) = \frac{-2|w|}{1 + |w|^2}, \quad \frac{\partial^2 \delta_S}{\partial z \partial \bar{z}}(0, w, 0) = \frac{-2|w|}{1 + |w|^2} \cdot \frac{\partial^2 \delta_S}{\partial z \partial \bar{w}}(0, w, 0),
\]
and hence it follows from (3.15) that
\[
\frac{\partial^2 \delta_S}{\partial z \partial \bar{z}}(0, w) = \frac{-|w|(1 + |f''(0)|^2)}{2(1 + |w|^2)(1 - |f''(0)|^2|w|^2)}, \quad \frac{\partial^2 \delta_S}{\partial z \partial \bar{w}}(0, w) = 0
\]
for \( 0 < |w| < \varepsilon \). This completes the proof of Lemma 2.

4. The results for the general case

In Sections 4 and 5, let \( M \) be a complex submanifold of dimension \( r \) through a point \( P_0 \in \mathbb{C}P^n, \ n \geq 2 \). Then, by a suitable isometric transformation of \( \mathbb{C}P^n \), we can choose an inhomogeneous coordinate system \((z_1, \ldots, z_n)\) on \( \mathbb{C}n \subset \mathbb{C}P^n \) such that \( P_0 \) is the origin and \( M \) is locally defined by
\[
M = \{(t, f(t)) \mid t = (t_1, \ldots, t_r) \in V\}
\] (4.1)
for open $V \subset \mathbb{C}^r$ and holomorphic $f = (f_1, \ldots, f_q) : V \to \mathbb{C}^q$. Moreover, we may suppose that $0 \in V$ and
\[ f_{\mu}(0) = 0, \quad \frac{\partial f_{\mu}}{\partial z_i}(0) = 0 \quad (4.2) \]
for $1 \leq i \leq r$ and $1 \leq \mu \leq q$. Then the tangent space of $S$ at $P_0$ is written by $z_{r+1} = \cdots = z_n = 0$.

In what follows, we put $q := n - r$ and $w_{\mu} := z_{r+\mu}$ for $1 \leq \mu \leq q$. Let $z := (z_1, \ldots, z_r) \in \mathbb{C}^r$, $w := (w_1, \ldots, w_q) = (z_{r+1}, \ldots, z_n) \in \mathbb{C}^q$, and denote by $0 := (0, \ldots, 0)$ the origin of $\mathbb{C}^r$ or $\mathbb{C}^q$. Moreover, for $z \in V \subset \mathbb{C}^r$ and $w \in \mathbb{C}^q$, we use the following notation:
\[ F_{\mu}(z) := \left( \frac{\partial^2 f_{\mu}}{\partial z_i \partial z_j}(z) \right)_{1 \leq i, j \leq r}, \quad \mathcal{F}(w) := \sum_{\mu=1}^{q} F_{\mu}(0) w_{\mu}. \quad (4.3) \]
Then $F_{\mu}(z)$, $1 \leq \mu \leq q$, and $\mathcal{F}(w)$ are symmetric matrices of degree $r$. Moreover, we put
\[ \delta_q^*(w) := \delta_q(0, w) = \cos^{-1} \frac{1}{\sqrt{1 + \sum_{\mu=1}^{q} |w_{\mu}|^2}} = \tan^{-1} \left( \sum_{\mu=1}^{q} |w_{\mu}|^2 \right) \quad (4.4) \]
for $w \in \mathbb{C}^q$. Theorem 1 is now extended as follows.

**THEOREM 2.** Let $q, r, n \in \mathbb{N}$ and $q + r = n$. Let $M$ be a complex submanifold of codimension $q$ in $\mathbb{C}^n$ locally defined by
\[ M = \{(t, f(t)) \mid t = (t_1, \ldots, t_r) \in V\} \]
for open $V \subset \mathbb{C}^r$ and holomorphic $f = (f_1, \ldots, f_q) : V \to \mathbb{C}^q$. Suppose $0 \in V$ and $f_{\mu}(0) = \partial f_{\mu}/\partial z_i(0) = 0$ for $1 \leq i \leq r$ and $1 \leq \mu \leq q$. Let $\delta_M$ be the Fubini–Study distance to $M$. Then there exists $\varepsilon > 0$ such that the complex Hessian matrix of the function $\varphi(z, w) := -\log \delta_M(z, w)$ at $(0, w)$ is as follows:
\[ \left( \begin{array}{cc} (\partial^2 \varphi/\partial z_i \partial z_j) & (\partial^2 \varphi/\partial z_i \partial \bar{w}_j) \\ (\partial^2 \varphi/\partial w_{\mu} \partial z_j) & (\partial^2 \varphi/\partial w_{\mu} \partial \bar{w}_j) \end{array} \right)(0, w) = \left( \begin{array}{cc} A_r(w) & O \\ O & B_q(w) \end{array} \right) \]
for $0 < \|w\| < \varepsilon$, where $A_r(w)$ is the Hermitian matrix of degree $r$ defined by
\[ A_r(w) = \frac{\mathcal{F}(w) \mathcal{F}(w) + (\sum_{\mu=1}^{q} |w_{\mu}|^2) E_r}{2 \cdot \delta_q^*(w) \sqrt{\sum_{\mu=1}^{q} |w_{\mu}|^2 (1 + \sum_{\mu=1}^{q} |w_{\mu}|^2)}} \]
and $B_q(w)$ is the Hermitian matrix of degree $q$ defined by
\[ B_q(w) = \left( \frac{\partial^2 (-\log \delta_q^*)}{\partial w_{\mu} \partial \bar{w}_v}(w) \right)_{1 \leq \mu, v \leq q}. \]
The function $\delta_q^*(w)$ and the matrix $\mathcal{F}(w)$ are defined in (4.4) and (4.3) respectively. Here $E_r$ is the identity matrix of degree $r$.

We will prove Theorem 2 in the next section. As its corollary, we have the following.
COROLLARY 2. With the same notation as above, the Levi form of the function $-\log \delta_M$ with respect to the Fubini–Study metric $ds^2$ on $\mathbb{C}P^n$ has $n-q+1$ positive eigenvalues $a_1, \ldots, a_{n-q}, a_{n-q+1}$ and $q-1$ negative eigenvalues $b_{n-q+2}, \ldots, b_n$ for $0 < \delta < \tan^{-1} \varepsilon$, where

$$a_{n-q+1} = \frac{\tan \delta_M - \delta_M + \tan^2 \delta_M \cdot \delta_M}{4 \cdot \tan \delta_M \cdot \delta_M^2}, \quad b_{n-q+2} = \cdots = b_n = \frac{-1}{2 \cdot \tan \delta_M \cdot \delta_M^2}.$$  

Moreover, $a_1, \ldots, a_{n-q}$ are equal to the eigenvalues of the Hermitian matrix

$$\frac{\tan \delta_M}{2 \cdot \delta_M} \cdot \begin{bmatrix} E_r + F_q(0)F_q(0) \end{bmatrix} \left[ E_r - F_q(0)F_q(0) \cdot \tan^2 \delta_M \right]^{-1}$$

at $(0, \ldots, 0, z_n) \in \mathbb{C}^n \subset \mathbb{C}P^n$ for $0 < |z_n| < \varepsilon$, where $r := n-q$ and $F_q(0)$ is the symmetric matrix of degree $r$ defined in (4.3).

**Proof.** If we substitute $w_1 = \cdots = w_{q-1} = 0$ into $A_r(w)$ and $B_q(w)$ in Theorem 2, we have

$$A_r(0, \ldots, 0, w_q) = \frac{|w_q| \cdot \left[ E_r + F_q(0)F_q(0) \right] \left[ E_r - F_q(0)F_q(0) \cdot |w_q|^2 \right]^{-1}}{2 \cdot \tan^{-1}|w_q|(1 + |w_q|^2)}$$

and

$$B_q(0, \ldots, 0, w_q) = \begin{pmatrix} b(q)q_{q-1} & O_{q-1,1} \\ O_{1,q-1} & c(w_q) \end{pmatrix}$$

for $0 < |w_q| < \varepsilon$, where

$$b(w) := \frac{-1}{2 \cdot |w| \tan^{-1}|w|(1 + |w|^2)}, \quad c(w) := \frac{|w| - \tan^{-1}|w| + |w|^2 \tan^{-1}|w|}{4 \cdot |w| \tan^{-1}|w|^2(1 + |w|^2)^2}$$

for $0 \neq w \in \mathbb{C}$. If $(z_1, \ldots, z_r, z_{r+1}, \ldots, z_n) = (z_1, \ldots, z_r, w_1, \ldots, w_q)$ is the inhomogeneous coordinate system on $\mathbb{C}^n \subset \mathbb{C}P^n$, the Fubini–Study metric $ds^2$ is written as

$$ds^2 = \sum_{i=1}^n \frac{|dz_i|^2}{1 + |z_n|^2} \cdot \frac{|dz_n|^2}{(1 + |z_n|^2)^2} = \sum_{i=1}^r \frac{|dz_i|^2}{1 + |w_q|^2} + \sum_{\mu=1}^{q-1} \frac{|dw_{\mu}|^2}{1 + |w_q|^2} + \frac{|dw_q|^2}{(1 + |w_q|^2)^2}$$

at $P := (z_1, \ldots, z_{n-1}, z_n) = (0, \ldots, 0, w_q)$. Since $\delta_M(P) = \tan^{-1}|w_q|$, Corollary 2 follows from Theorem 2. \hfill $\square$

**Remark 2.** A real-valued $C^2$-function $\varphi$ is said to be $q$-convex in the sense of Andreotti and Grauert [3] if its complex Hessian matrix has at most $q-1$ negative eigenvalues at each point. It follows from Theorem 2 and its corollary that the function $-\log \delta_q^* = q$-convex on $\mathbb{C}^n \setminus \{0\}$ and the function $-\log \delta_M$ is $q$-convex on $B(\varepsilon) \setminus M \subset \mathbb{C}P^n$, where $B(\varepsilon) := \{z \in \mathbb{C}^n \mid \|z\| < \varepsilon\}$ (see Barth [4], and Matsumoto [12, 13] for related results).

5. **Proof of Theorem 2**

Now we begin to prove Theorem 2. The proof is quite similar to that of Theorem 1, but we have to calculate some matrices to do it.
Let $M$ be a complex submanifold of dimension $r$ in $\mathbb{C}P^n$ defined in (4.1) with condition (4.2). If we put
\[
\alpha(z, w, t) := \frac{|1 + \sum_{i=1}^{r} t_i z_i + \sum_{\mu=1}^{q} f_\mu(t) \bar{w}_\mu|^2}{1 + \sum_{i=1}^{r} |t_i|^2 + \sum_{\mu=1}^{q} |f_\mu(t)|^2(1 + \sum_{i=1}^{r} |z_i|^2 + \sum_{\mu=1}^{q} |w_\mu|^2)}
\]
for $(z, w) \in \mathbb{C}^r \times \mathbb{C}^q$ and $t \in V \subset \mathbb{C}^r$, then
\[
\frac{\partial^2 \alpha}{\partial t_i \partial t_j}(0, w, 0) = \frac{\sum_{\mu=1}^{q} (\partial^2 f_\mu / \partial t_i \partial t_j)(0) \bar{w}_\mu}{1 + \sum_{\mu=1}^{q} |w_\mu|^2}, \quad \frac{\partial^2 \alpha}{\partial t_i \partial t_j}(0, w, 0) = -\delta_{ij}
\]
for $1 \leq i, j \leq r$, where $\delta_{i,j}$ denotes the Kronecker symbol.

By the implicit function theorem, there exists a $C^\omega$-function $t = t(z, w) = (t_1(z, w), \ldots, t_r(z, w))$ defined in some neighborhood $\Delta (\subset V \times \mathbb{C}^q)$ of $P_0 = (0, 0) \in M$ which satisfies
\[
\frac{\partial \alpha}{\partial t_i}(z, w, t(z, w)) = \frac{\partial \alpha}{\partial t_i}(z, w, t(z, w)) = 0
\]
for $1 \leq i \leq r$ and $(z, w) \in \Delta$. If we put $\beta(z, w) := \alpha(z, w, t(z, w))$, then
\[
\delta_M(z, w) = \cos^{-1} \sqrt{\beta(z, w)}
\]
for $(z, w) \in \Delta$. We choose $\epsilon > 0$ so that $(0, w) \in \Delta$ if $\|w\| < \epsilon$. Then by condition (4.2) we have $t(0, w) = 0$ for $\|w\| < \epsilon$, where $\|w\|^2 := \sum_{\mu=1}^{q} |w_\mu|^2$.

At first, since
\[
\delta_M(0, w) = \delta_M((0, 0), (0, 0)) = \delta_M^\ast(w) = \cos^{-1} \frac{1}{\sqrt{1 + \sum_{\mu=1}^{q} |w_\mu|^2}}
\]
for $\|w\| < \epsilon$, we have
\[
\left( \frac{\partial^2 (-\log \delta_M)}{\partial w_\mu \partial \bar{w}_v}(0, w) \right)_{1 \leq \mu, v \leq q} = \left( \frac{\partial^2 (-\log \delta_M^\ast)}{\partial w_\mu \partial \bar{w}_v}(w) \right)_{1 \leq \mu, v \leq q}
\]
for $\|w\| < \epsilon$. To prove Theorem 2, we need to know
\[
\left( \frac{\partial^2 (-\log \delta_M)}{\partial z_i \partial \bar{z}_j}(0, w) \right)_{1 \leq i, j \leq r}, \quad \left( \frac{\partial^2 (-\log \delta_M)}{\partial z_i \partial \bar{w}_v}(0, w) \right)_{1 \leq i \leq r, 1 \leq v \leq q}
\]
for $\|w\| < \epsilon$.

By differentiating the function $\beta = \beta(z, w) = \beta(z_1, \ldots, z_r; \bar{z}_{r+1}, \ldots, z_n)$ and applying (5.2) we have
\[
\frac{\partial \beta}{\partial z_i} = \frac{\partial \alpha}{\partial z_i} + \sum_{k=1}^{r} \frac{\partial \alpha}{\partial k} \frac{\partial t_k}{\partial z_i} + \sum_{k=1}^{r} \frac{\partial \alpha}{\partial \bar{k}} \frac{\partial \bar{t}_k}{\partial z_i} = \frac{\partial \alpha}{\partial z_i}
\]
and
\[
\frac{\partial^2 \beta}{\partial z_i \partial \bar{z}_j} = \frac{\partial^2 \alpha}{\partial z_i \partial \bar{z}_j} + \sum_{k=1}^{r} \frac{\partial^2 \alpha}{\partial z_i \partial t_k} \frac{\partial t_k}{\partial \bar{z}_j} + \sum_{k=1}^{r} \frac{\partial^2 \alpha}{\partial z_i \partial \bar{k}} \frac{\partial \bar{t}_k}{\partial \bar{z}_j}
\]
for $1 \leq i, j \leq n$. Moreover, by differentiating (5.2) we have

$$\begin{align*}
\frac{\partial^2 \alpha}{\partial t_i \partial z_j} + \sum_{k=1}^{r} \left( \frac{\partial^2 \alpha}{\partial t_i \partial t_k} \frac{\partial t_k}{\partial z_j} + \frac{\partial^2 \alpha}{\partial t_i \partial t_k} \frac{\partial t_k}{\partial z_j} \right) &= 0, \\
\frac{\partial^2 \alpha}{\partial t_j \partial z_j} + \sum_{k=1}^{r} \left( \frac{\partial^2 \alpha}{\partial t_j \partial t_k} \frac{\partial t_k}{\partial z_j} + \frac{\partial^2 \alpha}{\partial t_j \partial t_k} \frac{\partial t_k}{\partial z_j} \right) &= 0
\end{align*}$$

for $1 \leq i \leq r$ and $1 \leq j \leq n := r + q$. If we use the notation of matrices so that

$$\frac{\partial^2 \alpha}{\partial t_i \partial t_j} := \left( \frac{\partial^2 \alpha}{\partial t_i \partial t_j} \right)_{1 \leq i, j \leq r}, \quad \frac{\partial^2 \alpha}{\partial t_i \partial t} := \left( \frac{\partial^2 \alpha}{\partial t_i \partial t} \right)_{1 \leq i, j \leq r},$$

$$\frac{\partial t}{\partial z} := \left( \frac{\partial t_i}{\partial z_j} \right)_{1 \leq i, j \leq r}, \quad \frac{\partial t}{\partial w} := \left( \frac{\partial t_i}{\partial w} \right)_{1 \leq i, j \leq r},$$

$$\frac{\partial^2 \alpha}{\partial t \partial z} := \left( \frac{\partial^2 \alpha}{\partial t \partial z} \right)_{1 \leq i, j \leq r}, \quad \frac{\partial^2 \alpha}{\partial t \partial w} := \left( \frac{\partial^2 \alpha}{\partial t \partial w} \right)_{1 \leq i, j \leq r},$$

and so on, then we have

$$\begin{pmatrix}
\frac{\partial^2 \alpha}{\partial t \partial t} & \frac{\partial^2 \alpha}{\partial t \partial t} \\
\frac{\partial^2 \alpha}{\partial t \partial t} & \frac{\partial^2 \alpha}{\partial t \partial t}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial t}{\partial z} & \frac{\partial t}{\partial w} \\
\frac{\partial t}{\partial z} & \frac{\partial t}{\partial w}
\end{pmatrix}
= -\begin{pmatrix}
\frac{\partial^2 \alpha}{\partial t \partial z} & \frac{\partial^2 \alpha}{\partial t \partial w} \\
\frac{\partial^2 \alpha}{\partial t \partial z} & \frac{\partial^2 \alpha}{\partial t \partial w}
\end{pmatrix}.$$  

(5.7)

Now, it is easy to see that

$$\frac{\partial \alpha}{\partial t_i} (z, w, 0) = \frac{\bar{z}_i}{1 + \sum_{i=1}^{r} |z_j|^2 + \sum_{\mu=1}^{q} |w_\mu|^2}$$

and hence

$$\begin{align*}
\frac{\partial^2 \alpha}{\partial t_i \partial z_j} (0, w, 0) &= \frac{\delta_{ij}}{1 + \sum_{i=1}^{r} |w_\mu|^2}, \\
\frac{\partial^2 \alpha}{\partial t_i \partial w_\mu} (0, w, 0) &= \frac{\partial^2 \alpha}{\partial t_i \partial w_\mu} (0, w, 0) = 0
\end{align*}$$  

(5.8)

for $1 \leq i \leq r$ and $1 \leq \mu \leq q$. Since, by (5.1) and (4.3),

$$\frac{\partial^2 \alpha}{\partial t \partial t} (0, w, 0) = \frac{\mathcal{F}(w)}{1 + \sum_{\mu=1}^{q} |w_\mu|^2}, \quad \frac{\partial^2 \alpha}{\partial t \partial t} (0, w, 0) = \frac{-E_r}{1 + \sum_{\mu=1}^{q} |w_\mu|^2},$$

it follows from (5.7) and (5.8) that

$$\frac{\partial t}{\partial w} (0, w, 0) = \frac{\bar{t}}{\partial w} (0, w, 0) = 0,$$

(5.9)

and

$$\begin{align*}
\mathcal{F}(w) \cdot \frac{\partial t}{\partial z} (0, w, 0) - \frac{\partial \bar{t}}{\partial z} (0, w, 0) &= 0, \\
- \frac{\partial t}{\partial z} (0, w, 0) + \mathcal{F}(w) \cdot \frac{\partial \bar{t}}{\partial z} (0, w, 0) &= -E_r
\end{align*}$$
or equivalently
\[
\begin{align*}
\frac{\partial t}{\partial z}(0, w, 0) &= \left[ E_r - F(w)F(w) \right]^{-1}, \\
\frac{\partial t}{\partial z}(0, w, 0) &= F(w)\left[ E_r - F(w)F(w) \right]^{-1}
\end{align*}
\] (5.10)
for \(0 < \|w\| < \varepsilon\). Moreover, since
\[
\alpha(z, w, 0) = \frac{1}{1 + \sum_{i=1}^{r} |z_i|^2 + \sum_{\mu=1}^{q} |w_\mu|^2}
\]
we have
\[
\frac{\partial \alpha}{\partial z_i}(0, w, 0) = 0, \quad \frac{\partial \alpha}{\partial w_\mu}(0, w, 0) = \frac{-\bar{w}_\mu}{(1 + \sum_{\mu=1}^{q} |w_\mu|^2)^2} \tag{5.11}
\]
and
\[
\frac{\partial^2 \alpha}{\partial z_i \partial z_j}(0, w, 0) = \frac{-\delta_{ij}}{(1 + \sum_{\mu=1}^{q} |w_\mu|^2)^2}, \quad \frac{\partial^2 \alpha}{\partial z_i \partial \bar{w}_\mu}(0, w, 0) = 0,
\tag{5.12}
\]
for \(1 \leq i, j \leq r\) and \(1 \leq \mu, v \leq q\).

Therefore, by substituting (5.11) into (5.5) we have
\[
\frac{\partial \beta}{\partial z_i}(0, w) = 0, \quad \frac{\partial \beta}{\partial w_\mu}(0, w) = \frac{-\bar{w}_\mu}{(1 + \sum_{\mu=1}^{q} |w_\mu|^2)^2} \tag{5.13}
\]
and by substituting (5.9), and (5.12) into (5.6) we first obtain
\[
\frac{\partial^2 \beta}{\partial z_i \partial \bar{w}_\mu}(0, w) = 0 \tag{5.14}
\]
for \(1 \leq i \leq r\) and \(1 \leq \mu \leq q\). Moreover, by (5.6), (5.8), (5.10), and (5.12) we next obtain
\[
\left( \frac{\partial^2 \beta}{\partial z_i \partial z_j}(0, w) \right)_{1 \leq i, j \leq r} = \left[-E_r \left(1 + \sum_{\mu=1}^{q} |w_\mu|^2\right)^2 \right] \left[ E_r - F(w)F(w) \right]^{-1}
= \left[F(w)F(w) + \left( \sum_{\mu=1}^{q} |w_\mu|^2 \right) E_r \right] \left[ E_r - F(w)F(w) \right]^{-1},
\tag{5.15}
\]
where we have used the relation \((E - A)^{-1} - E = A(E - A)^{-1}\).

Now, if \(\delta_M\) is the Fubini–Study distance to the complex submanifold \(M\), it follows from (5.3) that \(\beta(z, w) = \cos^2 \delta_M(z, w)\) and
\[
\frac{\partial \beta}{\partial z_i}(z, w) = -2 \cos \delta_M(z, w) \sin \delta_M(z, w) \frac{\partial \delta_M}{\partial z_i}(z, w) \tag{5.16}
\]
for \((z, w) \in \Delta\) and \(1 \leq i \leq r\). Here, it follows from (5.4) that
\[
\cos \delta_M(0, w) = \frac{1}{\sqrt{1 + \sum_{\mu=1}^{q} |w_\mu|^2}}, \quad \sin \delta_M(0, w) = \frac{1}{\sqrt{1 + \sum_{\mu=1}^{q} |w_\mu|^2}} \tag{5.17}
\]
for $\|w\| < \varepsilon$. Hence, by (5.13), (5.16), and (5.17), we have
\[
\frac{\partial^2 \delta_M}{\partial z_i \partial \bar{z}_j}(0, w) = 0
\] (5.18)
for $0 < \|w\| < \varepsilon$ and $1 \leq i \leq r$. Moreover, by (5.16), (5.17), and (5.18), we have
\[
\frac{\partial^2 \delta_M}{\partial z_i \partial \bar{z}_j}(0, w) = \frac{1 + \sum_{\mu=1}^{q} |w_\mu|^2}{-2\sqrt{\sum_{\mu=1}^{q} |w_\mu|^2}} \cdot \frac{\partial^2 \beta}{\partial z_i \partial \bar{z}_j}(0, w)
\] (5.19)
for $1 \leq i \leq r$ and $1 \leq j \leq n$. Consequently, if we put $\varphi := -\log \delta_M$, we obtain
\[
\left\{\begin{array}{c}
\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \mid_{1 \leq i, j \leq r} = \frac{\mathcal{F}(w)\mathcal{F}(w) + \left(\sum_{\mu=1}^{q} |w_\mu|^2\right) E_r}{\sum_{\mu=1}^{q} |w_\mu|^2(1 + \sum_{\mu=1}^{q} |w_\mu|^2)} - 1,

\frac{\partial^2 \varphi}{\partial z_i \partial \bar{w}_v} \mid_{1 \leq i \leq r, 1 \leq v \leq q} = O_{r,q}
\end{array}\right.
\]
at $(0, w)$ for $0 < \|w\| < \varepsilon$. This completes the proof of Theorem 2.

6. A remark

Finally, we give a remark on the convexity property of the real-valued functions $\Phi$ and $\Psi$ found in Theorem 1. Let $z := (z_1, \ldots, z_n)$ be an inhomogeneous coordinate system on $\mathbb{C}^n \subset \mathbb{CP}^n$ and put $x := \sqrt{\sum_{i=1}^{n} |z_i|^2}$. Then $x$ and $\tan^{-1} x$ mean the Euclidean distance and the Fubini–Study distance of $z$ and $0$ in $\mathbb{C}^n$, respectively.

**Lemma 3.** Let
\[
F(x) := \frac{x}{\tan^{-1} x}
\]
for $x > 0$. Then $\lim_{x \to 0} F(x) = 1$ and $F'(x) > 0$, $F''(x) > 0$ for $x > 0$.

**Proof.** Since $\tan^{-1} x = x - x^3/3 + O(x^5)$, it follows that $F(x) = 1 + x^2/3 + O(x^4)$ and $\lim_{x \to 0} F(x) = 1$. Moreover,
\[
\lim_{x \to 0} F'(x) = 0, \quad F''(x) = \frac{2(x - \tan^{-1} x)}{(\tan^{-1} x)^3(1 + x^2)^2} > 0
\]
for $x > 0$, and hence $F'(x) > 0$ for $x > 0$. \qed

**Lemma 4.** Let
\[
G(x) := \frac{x - \tan^{-1} x}{x(\tan^{-1} x)^2}
\]
for $x > 0$. Then $\lim_{x \to 0} G(x) = 1/3$ and $G(x) > 1/3$ for $x > 0$.

**Proof.** Since $\tan^{-1} x = x - x^3/3 + O(x^5)$, it follows that $\lim_{x \to 0} G(x) = 1/3$. If we put $\varphi(x) := 3(x - \tan^{-1} x) - x(\tan^{-1} x)^2$, then
\[
\varphi(0) = \varphi'(0) = 0, \quad \varphi''(x) = \frac{4(x - \tan^{-1} x)}{(1 + x^2)^2} > 0
\]
for $x > 0$. Hence $\varphi'(x) > 0$ and $\varphi(x) > 0$ for $x > 0$. \qed
LEMMA 5. Let
\[ H(x) := \frac{x - \tan^{-1} x + x^2 \tan^{-1} x}{x (\tan^{-1} x)^2} \]
for \( x > 0 \). Then \( \lim_{x \to 0} H(x) = 4/3 \) and \( H'(x) > 0 \), \( H''(x) > 0 \) for \( x > 0 \).

Proof. Since \( H(x) = F(x) + G(x) \), it follows that \( \lim_{x \to 0} H(x) = 4/3 \) and \( H(x) > 4/3 \) for \( x > 0 \). Moreover, \( \lim_{x \to 0} H'(x) = 0 \) and
\[ H''(x) = \frac{2}{3} \cdot \frac{a(x, t) [(x^2 + 1)t - x] + b(x, t) [3(x - t) - xt^2]}{(1 + x^2)^2 x^4 t^4} \]
for \( x > 0 \), where \( t := \tan^{-1} x \) and
\[ a(x, t) := xt^3 > 0, \quad b(x, t) := 3x^2 + 2xt + t^2 + x^2 t^2 + 3x^3 t > 0 \]
for \( x > 0 \). It follows from Lemma 4 that \( 3(x - \tan^{-1} x) - x (\tan^{-1} x)^2 > 0 \) and it is easy to see that \( (x^2 + 1) \tan^{-1} x - x > 0 \) for \( x > 0 \). Hence \( H''(x) > 0 \) for \( x > 0 \). Since \( \lim_{x \to 0} H'(x) = 0 \), we have also \( H'(x) > 0 \) for \( x > 0 \). \( \square \)

LEMMA 6. The real-valued functions
\[ \Phi(x) := \frac{x}{2 \cdot \tan^{-1} x} \cdot \frac{1 + a^2}{1 - a^2 x^2} \quad \text{and} \quad \Psi(x) := \frac{x - \tan^{-1} x + x^2 \tan^{-1} x}{4 \cdot x (\tan^{-1} x)^2}, \]
appearing in Theorem 1 and Corollary 1, are convex on \( (0, 1/a) \) and \( (0, \infty) \), respectively, where \( a \geq 0 \), and \( \lim_{x \to 0} \Phi(x) = (1 + a^2)/2, \lim_{x \to 0} \Psi(x) = 1/3 \).

Proof. Since the function \( 1/(1 - a^2 x^2) \) is convex on \( (0, 1/a) \), Lemma 6 follows from Lemmas 3, 4, and 5. \( \square \)

The proposition stated in the introduction follows from Lemma 6 at once.

Remark 3. Since the function \( \tan x \) is convex on \( (0, \pi/2) \), the composite functions \( \Psi(\tan x) \), \( F(\tan x) \), and \( H(\tan x) \) are convex on \( (0, \pi/2) \). Although the function \( G(x) \) is neither increasing nor convex on \( (0, \infty) \), the composite function \( G(\tan x) \) is convex on \( (0, \pi) \). In fact, it has the Taylor expansion
\[ G(\tan x) = \frac{\tan x - x}{x^2 \tan x} = \frac{1 - x \cot x}{x^2} = \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{2^n B_n x^{2n}}{(2n)!} = \frac{1}{3} + \frac{1}{45} x^2 + \cdots \]
for \( |x| < \pi \), where \( B_n > 0, n \in \mathbb{N} \), are the Bernoulli numbers.

REFERENCES

Takeuchi’s equality for the Levi form


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