COMBINATORIAL PROOFS OF IDENTITIES FOR SPECIAL VALUES OF ARAKAWA–KANEKO MULTIPLE ZETA FUNCTIONS

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Abstract. Alternative and simpler proofs of three identities between special values of multiple zeta functions of Arakawa–Kaneko type and multiple zeta(-star) values are given. The multiple integrals associated with 2-labeled partially ordered sets are used.

1. Introduction

In a recent paper [2], M. Kaneko and H. Tsumura introduced and studied a new kind of multiple zeta function,

$$\eta(k; s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}_k(1 - e^t)}{1 - e^t} t^{s-1} dt \quad (\text{Re}(s) > 1 - r),$$

(1)

which is called a ‘twin sibling’ of the Arakawa–Kaneko multiple zeta function [1]:

$$\xi(k; s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} t^{s-1} dt \quad (\text{Re}(s) > 0).$$

(2)

Here, for a positive integer $r$, $k = (k_1, \ldots, k_r)$ is a positive (multi-)index, i.e., an $r$-tuple of positive integers, and $s$ is a complex variable. And $\text{Li}_k(z)$ denotes the multiple polylogarithm of one variable:

$$\text{Li}_k(z) := \sum_{0 < n_1 < \cdots < n_r} \frac{z^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}.$$

Both $\eta(k; s)$ and $\xi(k; s)$ can be continued analytically to entire functions over $\mathbb{C}$, and considered to be a pair of generalizations of the Riemann zeta function:

$$\zeta(s) := \sum_{n>0} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \quad (\text{Re}(s) > 1).$$

In fact, they satisfy $\xi(1; s) = \eta(1; s) = s \zeta(s + 1)$ if $k = (1)$.

In [1, 2], for any positive index $k$, the special values of $\xi(k; s)$ and $\eta(k; s)$ at positive integers are analytically computed and written in terms of multiple zeta and zeta-star values, respectively. In this note, we give combinatorial and simpler proofs of their identities and also the duality, proved by S. Yamamoto in [6], of the single-variable case. These proofs are applications of integral representations of $\xi(k; m)$ and $\eta(k; m)$ at positive integer $m$ in terms of 2-labeled partially ordered sets (2-posets for short).

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2. Combinatorial proofs

First, we recall the basic properties of 2-posets and their associated integrals. Yamamoto introduced this theory in [5], and he obtained an alternative proof of a formula, given in [3, Theorem 2], which states a coincidence among the values of \( \xi(k; m) \) and multiple zeta-star \( \zeta^* \), as one of the applications of the theory.

**Definition 2.1.** A 2-poset is a pair \( X = (X, \delta_X) \) consisting of a finite poset \( X \) and a labeling map \( \delta_X : X \to \{0, 1\} \). A 2-poset \( X \) is called admissible if \( \delta_X(x) = 0 \) for all maximal elements \( x \in X \) and \( \delta_X(x) = 1 \) for all minimal elements \( x \in X \).

**Definition 2.2.** For an admissible 2-poset \( X \), we define the associated integral

\[
I(X) = \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta_X(x)}(t_x),
\]

where

\[
\Delta(X) = \{(t_x) \in [0, 1]^X \mid t_x < t_y \text{ if } x < y\}
\]

and

\[
\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1 - t}.
\]

**Proposition 2.1.** [5, Proposition 2.3]

(a) For a 2-poset \( X \), let \( X' \) denote its transpose, i.e., the 2-poset obtained by reversing the order on \( X \) and setting \( \delta_{X'}(x) = 1 - \delta_X(x) \). If \( X \) is an admissible 2-poset, then the transpose \( X' \) is admissible and

\[
I(X) = I(X').
\]

(b) For non-comparable elements \( a \) and \( b \) of a 2-poset \( X \), \( X^b_a \) denotes the 2-poset that is obtained from \( X \) by adjoining the relation \( a < b \). If \( X \) is an admissible 2-poset, then the 2-posets \( X^b_a \) and \( X^a_b \) are admissible and

\[
I(X) = I(X^b_a) + I(X^a_b).
\]

Note that the admissibility of a 2-poset corresponds to the convergence of the associated integral. We use Hasse diagrams to indicate 2-posets, with vertices \( \circ \) and \( \bullet \) corresponding to \( \delta_X(x) = 0 \) and \( 1 \), respectively. (See [5, §2] for precise details.) Moreover, we use the symbol \( \odot \) to represent the vertex associated with the differential form \( \omega_0(t) + \omega_1(t) = dt/(t(1 - t)) \).

The following representations of special values \( \eta(k; m) \) and \( \xi(k; m) \) are implicitly and essentially given in [2, 6].
THEOREM 2.1. (cf. [2, 6]) For any positive index \( k = (k_1, \ldots, k_r) \) and any positive integer \( m \), we have

\[
\eta(k; m) = (-1)^{r-1} I
\]

and

\[
\xi(k; m) = I
\]

Proof. We denote by \( k \preceq k' \) that \( k \) can be obtained from \( k' \) by replacing some commas of \( k' \) by pluses. The following formula is known as the Landen connection formula for the multiple polylogarithm [4, Proposition 9]; for any positive index \( k = (k_1, \ldots, k_r) \), we have

\[
\text{Li}_k\left(\frac{z}{z-1}\right) = (-1)^r \sum_{k \preceq k'} \text{Li}_{k'}(z).
\]

For positive integer \( m \), by applying the variable change \( x = 1 - e^{-t} \) and the Landen connection formula (7), we rewrite (1) and (2) as

\[
\eta(k; m) = \frac{(-1)^{r-1}}{(m-1)!} \sum_{k \preceq k'} \int_0^1 \text{Li}_{k'}(x)(-\log(1 - x))^{m-1} \frac{dx}{x}
\]

and

\[
\xi(k; m) = \frac{1}{(m-1)!} \int_0^1 \text{Li}_k(x)(-\log(1 - x))^{m-1} \frac{dx}{x}.
\]
respectively. By using the known representations of $\text{Li}_k(t)$ and $-\log(1-t)$ in terms of iterated integrals with $\omega_0(t)$ and $\omega_1(t)$ (cf. [5, §3]), we have

\[ \eta(k; m) = (-1)^{r-1} \frac{1}{(m-1)!} I \]

and

\[ \xi(k; m) = \frac{1}{(m-1)!} I \]

There are exactly $(m-1)!$ ways to impose a total order on the $m-1$ black vertices. Thus we obtain the identities (5) and (6).

We use Theorem 2.1 to give combinatorial proofs for certain identities obtained in [2, Theorem 2.5] and [6, Corollary 2.5]. To state the identities proved in [2, Theorem 2.5], we introduce some notation. For any non-negative index $j = (j_1, \ldots, j_r)$, i.e., any $r$-tuple of non-negative integers, the weight and the depth of $j$ are the integers $|j| = j_1 + \cdots + j_r$ and $d(j) = r$, respectively. The multiple zeta and zeta-star values of positive index $k = (k_1, \ldots, k_r)$ with $k_r \geq 2$ are positive real numbers defined by

\[ \zeta(k) = \zeta(k_1, \ldots, k_r) := \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \]

and

\[ \zeta^*(k) = \zeta^*(k_1, \ldots, k_r) := \sum_{0 < n_1 \leq \cdots \leq n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \]

respectively. The usual dual index of $k$ is denoted by $k'$ (cf. [5, equation (1.3)]). For a positive index $k$, we put $k_r := (k_1, \ldots, k_{r-1}, k_r + 1)$, and for non-negative index $j$ of the same depth $r$, we denote by $k + j$ the index obtained by the componentwise addition

\[ k + j := (k_1 + j_1, \ldots, k_r + j_r) \].
and by \( b(k; j) \) the quantity given by

\[
b(k; j) := \prod_{i=1}^{r} \left( k_i + j_i - 1 \right).
\]

It is known from [5], as a natural consequence of the iterated integral representation, that the multiple zeta and zeta-star values of positive index \( k = (k_1, \ldots, k_r) \) with \( k_r \geq 2 \) can be written in terms of integrals associated with 2-labeled totally ordered sets as follows:

\[
\zeta(k) = I \quad \text{and} \quad \zeta^*(k) = I.
\]

(8)

Now, we use Theorem 2.1 to prove the following two identities.

**Theorem 2.2.** [2, Theorem 2.5] For any positive index \( k = (k_1, \ldots, k_r) \) and any positive integer \( m \), we have

\[
\eta(k; m) = (-1)^{r-1} \sum_{|j| = m-1, d(j) = n} b((k_+)'; j) \xi^*((k_+)'+ j)
\]

and

\[
\xi(k; m) = \sum_{|j| = m-1, d(j) = n} b((k_+)'; j) \xi((k_+)'+ j).
\]

(9)

(10)

where both sums are over all non-negative indices \( j \) of weight \( m - 1 \) and depth \( n = d((k_+)') \).
Combinatorial proof. By applying (3) to the identity (5), we have

\[ \eta(k; m) = (-1)^{r-1} I \]

if we put \((k_+) = (k'_1, \ldots, k'_n)\). By identity (4), we have

\[ \eta(k; m) = (-1)^{r-1} \sum_{j_1=0}^{m-1} I \]

There are \(\binom{k'_1 + j_1 - 1}{j_1} \) ways to obtain totally ordered \(k'_1 + j_1 - 1\) white vertices between \(\bullet\) and the lowest \(\circ\), for each \(j_1\) \((0 \leq j_1 \leq m - 1)\). In a similar way, we have
\[ \eta(k; m) = (-1)^{r-1} \sum_{|j|=m-1, \, d(j)=n} b((k_+); j) I \]

By using (8), the inner sum on the right-hand side of the above identity is equal to \( \zeta^*(k_1' + j_1, k_2' + j_2, \ldots, k_n' + j_n) \); thus we obtain (9).

Identity (10) can be shown in a similar way.

Next, we consider the duality of \( \eta(k; m) \).

THEOREM 2.3. (Conjectured in [2, equation (36)] and proved in [6, Corollary 2.5])

For positive integers \( k \) and \( m \), we have

\[ \eta(k; m) = \eta(m; k). \] (11)

In [6], Yamamoto obtained the above identity for the more general case, namely, the multi-variable case. Here we treat only the single-variable case.

Combinatorial proof. By integrating the multiple integral of \( \eta(k; m) \) in Theorem 2.1 from the left to the right, we have

\[ \eta(k; m) = I \left( k \left( \begin{array}{c} \cdots \cr \cdots \cr \cdots \end{array} \right) m - 1 \right) \right) = \sum_{u_1 \leq \cdots u_k = v_m \geq \cdots \geq v_1} \frac{1}{u_1 \cdots u_k v_1 \cdots v_m} . \]

The multiple series on the right-hand side is symmetric under the exchange of \( k \) and \( m \). Thus we obtain (11).

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