FINITE MULTIPLE ZETA VALUES, MULTIPLE ZETA FUNCTIONS AND MULTIPLE BERNOULLI POLYNOMIALS

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Abstract. We present explicit formulas for all finite multiple zeta values by introducing a multiple generalization of Bernoulli polynomials associated with finite multiple zeta values. Furthermore we show that these values are also described by special values of multiple zeta functions and multiple star analogues of the Hurwitz zeta function.

1. Introduction

Recently there has been a growing interest in finite multiple zeta values, which are defined as follows. Let $\mathcal{A} = (\prod_p \mathbb{Z}/p\mathbb{Z})/(\oplus_p \mathbb{Z}/p\mathbb{Z})$, where p runs over all primes. Then the finite multiple zeta value is defined as

$$\zeta^{\mathcal{A}}(k_1, \dots, k_r) = \left(\sum_{0 < n_1 < \dots < n_r < p} \frac{1}{n_1^{k_1} \dots n_r^{k_r}} \bmod p \right)_p \in \mathcal{A}$$

for $k_1, \dots, k_r \in \mathbb{Z}$. It is known that some of these values are written in terms of Bernoulli numbers (see [2] for example), and also that all these values are written in terms of multi-poly-Bernoulli numbers, which are defined as the coefficients of the Taylor expansion of multiple polylogarithms (see [3]).

The purpose of this article is to give an explicit description for all finite multiple zeta values in terms of the coefficients of the Taylor expansion of a certain generating function, which can be regarded as a generalization of Bernoulli numbers via a multiple analogue of Bernoulli polynomials. While in the case of multi-poly-Bernoulli numbers, their generating functions are given by multiple series, our generating functions are simple rational functions of exponentials. We also give their direct relations to special values of multiple zeta functions and multiple star analogues of the Hurwitz zeta function.

In the following, we state our main results. Here and hereafter the empty sum should be understood as zero and the empty product as one. Let

$$F_r(z_1, \dots, z_r; y) = \sum_{j=0}^r (-1)^{r-j} e^{y \sum_{l=j+1}^r z_l} \prod_{k=1}^j \frac{e^{\sum_{l=k}^j z_l}}{1 - e^{\sum_{l=k}^j z_l}} \prod_{k=j+1}^r \frac{1}{1 - e^{\sum_{l=j+1}^k z_l}}. \quad (1.1)$$

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For $N \in \mathbb{Z}_{\geq 0}$ and $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, let

$$S_N(k_1, \dots, k_r) = \sum_{0 < n_1 < \dots < n_r < N} n_1^{k_1} \dots n_r^{k_r}.$$

Then we have the following multiple analogue of Faulhaber's formula or Bernoulli's formula. Although this formula is an easy consequence of some residue calculus, and it might be already in the literature, we provide it with the proof for the reader's convenience. As a related result, we note that, for example, in [1] extended Bernoulli polynomials are introduced via certain recursion relations, and generalizations of Bernoulli's formula are given in terms of the sum of these polynomials.

THEOREM 1.1. *The function $F_r(z_1, \dots, z_r; y)$ is holomorphic on $\mathbb{C}$ in y and in a neighborhood of the origin on $\mathbb{C}^r$ in $(z_1, \dots, z_r)$ and we have*

$$F_r(z_1, \dots, z_r; y) = \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} P(k_1, \dots, k_r; y) \frac{z_1^{k_1}}{k_1!} \dots \frac{z_r^{k_r}}{k_r!}, \quad (1.2)$$

where $P(k_1, \dots, k_r; y) \in \mathbb{Q}[y]$ and its degree is $k_1 + \dots + k_r + r$ with the leading coefficient

$$\frac{1}{(k_1 + 1)(k_1 + k_2 + 2) \dots (k_1 + \dots + k_r + r)}. \quad (1.3)$$

When $(k_1, \dots, k_r) \neq (0, \dots, 0)$, $P(k_1, \dots, k_r; y)$ is divisible by y and $P(0, \dots, 0; y) = \binom{y-1}{r}$.

Furthermore for $N \in \mathbb{Z}_{\geq 0}$ and $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, with $(k_1, \dots, k_r, N) \neq (0, \dots, 0, 0)$, we have

$$S_N(k_1, \dots, k_r) = P(k_1, \dots, k_r; N).$$

As we will see below, the polynomials $P(k_1, \dots, k_r; y)$ can be regarded as multiple Bernoulli polynomials associated with finite multiple zeta values.

For $s_1, \dots, s_r \in \mathbb{C}$ with sufficiently large $\operatorname{Re} s_j$ ($j = 1, \dots, r$), the multiple zeta function is defined by

$$\zeta(s_1, \dots, s_r) = \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}, \quad (1.4)$$

and we define a multiple star analogue of the Hurwitz zeta function by

$$\zeta^*(s_1, \dots, s_r; a) = \sum_{0 \leq n_1 \leq \dots \leq n_r} \frac{1}{(a + n_1)^{s_1} \dots (a + n_r)^{s_r}} \quad (1.5)$$

for $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$. It is known that these zeta functions are analytically continued to meromorphic functions on the whole space $\mathbb{C}^r$ (see [4, 5] for example). For $N \in \mathbb{Z}_{\geq 0}$ and $s_1, \dots, s_r \in \mathbb{C}$, let

$$\zeta_N(s_1, \dots, s_r) = \sum_{0 < n_1 < \dots < n_r < N} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}. \quad (1.6)$$

Then, by definition, we have

$$S_N(k_1, \dots, k_r) = \zeta_N(-k_1, \dots, -k_r). \quad (1.7)$$

Let

$$Z(s_1, \dots, s_r; a) = \sum_{j=0}^r (-1)^{r-j} \zeta(s_1, \dots, s_j) \zeta^*(s_r, \dots, s_{j+1}; a),$$

where $\zeta(\emptyset)$ and $\zeta^*(\emptyset; a)$ should be understood as one.

THEOREM 1.2. For $N \in \mathbb{Z}_{\geq 1}$, we have

$$Z(s_1, \dots, s_r; N) = \zeta_N(s_1, \dots, s_r).$$

Hence $Z(s_1, \dots, s_r; N)$ is analytically continued to an entire function on the whole space $\mathbb{C}^r$.

We remark that the special case $N = 1$ of this theorem implies the functional relation $Z(s_1, \dots, s_r; 1) = 0$. This identity at positive integer points $(k_1, \dots, k_r)$ was rediscovered by several authors. For example, see [6, Remark 3.17] and the references therein.

For $k \in \mathbb{Z}$ and every prime p , let

$$k^{(p)} = \begin{cases} p - 1 - k & (k > 0), \\ -k & (k \leq 0). \end{cases}$$

THEOREM 1.3. For $k_1, \dots, k_r \in \mathbb{Z}$, we have

$$\begin{aligned} \zeta^{\mathcal{A}}(k_1, \dots, k_r) &= (P(p - 1 - k_1, \dots, p - 1 - k_r; p) \bmod p)_p \\ &= (Z(k_1 - p + 1, \dots, k_r - p + 1; p) \bmod p)_p, \end{aligned} \quad (1.8)$$

where we understand $P(p - 1 - k_1, \dots, p - 1 - k_r; p) = 0$ if one of $p - 1 - k_i$ is negative.

Moreover, for non-positive indices k_i in $\zeta^{\mathcal{A}}(k_1, \dots, k_r)$, the corresponding indices on the right-hand side can be fixed, that is,

$$\begin{aligned} \zeta^{\mathcal{A}}(k_1, \dots, k_r) &= (P(k_1^{(p)}, \dots, k_r^{(p)}; p) \bmod p)_p \\ &= (Z(-k_1^{(p)}, \dots, -k_r^{(p)}; p) \bmod p)_p. \end{aligned} \quad (1.9)$$

In particular, for $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, the expression (1.9) implies

$$\zeta^{\mathcal{A}}(-k_1, \dots, -k_r) = \begin{cases} (0)_p & (k_1, \dots, k_r) \neq (0, \dots, 0), \\ ((-1)^r)_p & (k_1, \dots, k_r) = (0, \dots, 0). \end{cases} \quad (1.10)$$

To state the above theorems in the case $r = 1$, let $\zeta(s)$ and $\zeta(s, a)$ be the Riemann zeta function and the Hurwitz zeta function, respectively, which are defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

and are analytically continued to meromorphic functions on the whole space $\mathbb{C}$ in s . Then we have the following trivial relation:

$$\zeta_N(s) = \zeta(s) - \zeta(s, N).$$

The formula (1.8) is a generalization of the following. For $k \in \mathbb{Z}$ and sufficiently large prime p ,

$$\begin{aligned}\zeta_p(k) &\equiv \sum_{n=1}^{p-1} n^{p-1-k} \pmod{p} \\ &= \frac{1}{p-k} (B_{p-k}(p) - B_{p-k}(1)),\end{aligned}$$

where $B_k(y)$ denotes the ordinary k th Bernoulli polynomial. In this case, with $k \geq 1$, the above is further equal to

$$\frac{1}{p-k} \sum_{j=0}^{p-k-1} \binom{p-k}{j} B_j p^{p-k-j} \equiv 0 \pmod{p}$$

because p does not appear in the denominator by the von Staudt–Clausen theorem.

The formulas (1.9) and (1.10) are a generalization of the following. For $k \geq 0$ and sufficiently large prime p ,

$$\begin{aligned}\zeta_p(-k) &= \frac{1}{k+1} (B_{k+1}(p) - B_{k+1}(1)) \\ &\equiv \frac{1}{k+1} (B_{k+1}(0) - B_{k+1}(1)) \pmod{p} \\ &= \begin{cases} 0 & (k \neq 0), \\ -1 & (k = 0). \end{cases}\end{aligned}$$

2. Proof of Theorem 1.1

Let $\delta > 0$ be sufficiently small and first we show that $F_r(z_1, \dots, z_r; y)$ is holomorphic in the domain $y \in \mathbb{C}$ and $(z_1, \dots, z_r) \in \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \delta\}^r$.

Fix j_0, j_1 with $1 \leq j_0 \leq j_1 \leq r$ and put $z_{j_0} + \dots + z_{j_1} = \epsilon$. Then we have

$$\begin{aligned}F_r(z_1, \dots, z_r; y) &= \sum_{\substack{j=0 \\ j \neq j_0-1, j_1}}^r (-1)^{r-j} e^{y \sum_{l=j+1}^r z_l} \prod_{k=1}^j \frac{e^{\sum_{l=k}^j z_l}}{1 - e^{\sum_{l=k}^j z_l}} \prod_{k=j+1}^r \frac{1}{1 - e^{\sum_{l=j+1}^k z_l}} \\ &\quad + (-1)^{r-j_0+1} e^{y(\epsilon + \sum_{l=j_1+1}^r z_l)} \prod_{k=1}^{j_0-1} \frac{e^{\sum_{l=k}^{j_0-1} z_l}}{1 - e^{\sum_{l=k}^{j_0-1} z_l}} \prod_{\substack{k=j_0 \\ k \neq j_1}}^r \frac{1}{1 - e^{\sum_{l=j_0}^k z_l}} \frac{1}{1 - e^\epsilon} \\ &\quad + (-1)^{r-j_1} e^{y \sum_{l=j_1+1}^r z_l} \prod_{\substack{k=1 \\ k \neq j_0}}^{j_1} \frac{e^{\sum_{l=k}^{j_1} z_l}}{1 - e^{\sum_{l=k}^{j_1} z_l}} \prod_{k=j_1+1}^r \frac{1}{1 - e^{\sum_{l=j_1+1}^k z_l}} \frac{e^\epsilon}{1 - e^\epsilon}.\end{aligned}$$

In the second term on the right-hand side, we have

$$\prod_{k=j_1+1}^r \frac{1}{1 - e^{\sum_{l=j_0}^k z_l}} = \prod_{k=j_1+1}^r \frac{1}{1 - e^{\sum_{l=j_1+1}^k z_l + \epsilon}}. \quad (2.1)$$

On the other hand, in the third term, we have

$$\prod_{k=1}^{j_0-1} \frac{e^{\sum_{l=k}^{j_1} z_l}}{1 - e^{\sum_{l=k}^{j_1} z_l}} = \prod_{k=1}^{j_0-1} \frac{e^{\sum_{l=k}^{j_0-1} z_l + \epsilon}}{1 - e^{\sum_{l=k}^{j_0-1} z_l + \epsilon}} \quad (2.2)$$

and

$$\begin{aligned} \prod_{k=j_0+1}^{j_1} \frac{e^{\sum_{l=k}^{j_1} z_l}}{1 - e^{\sum_{l=k}^{j_1} z_l}} &= \prod_{k=j_0+1}^{j_1} \frac{e^{-\sum_{l=j_0}^{k-1} z_l + \epsilon}}{1 - e^{-\sum_{l=j_0}^{k-1} z_l + \epsilon}} \\ &= (-1)^{j_1-j_0} \prod_{k=j_0}^{j_1-1} \frac{1}{1 - e^{\sum_{l=j_0}^k z_l - \epsilon}}. \end{aligned} \quad (2.3)$$

Rewriting the second term and the third term by use of (2.1), (2.2) and (2.3), we get

$$\begin{aligned} &\frac{(-1)^{r-j_0+1} e^{y \sum_{l=j_1+1}^r z_l}}{1 - e^\epsilon} \\ &\times \left(e^{y\epsilon} \prod_{k=1}^{j_0-1} \frac{e^{\sum_{l=k}^{j_0-1} z_l}}{1 - e^{\sum_{l=k}^{j_0-1} z_l}} \prod_{k=j_0}^{j_1-1} \frac{1}{1 - e^{\sum_{l=j_0}^k z_l}} \prod_{k=j_1+1}^r \frac{1}{1 - e^{\sum_{l=j_1+1}^k z_l + \epsilon}} \right. \\ &\left. - e^\epsilon \prod_{k=1}^{j_0-1} \frac{e^{\sum_{l=k}^{j_0-1} z_l + \epsilon}}{1 - e^{\sum_{l=k}^{j_0-1} z_l + \epsilon}} \prod_{k=j_0}^{j_1-1} \frac{1}{1 - e^{\sum_{l=j_0}^k z_l - \epsilon}} \prod_{k=j_1+1}^r \frac{1}{1 - e^{\sum_{l=j_1+1}^k z_l}} \right) \\ &= O(1) \end{aligned}$$

with respect to ϵ . Therefore we see that $F_r(z_1, \dots, z_r; y)$ is holomorphic in a neighborhood of $z_{j_0} + \dots + z_{j_1} = 0$. Since the codimension of the singularities of a meromorphic function cannot be more than one, we have the assertion about the holomorphy of $F_r(z_1, \dots, z_r; y)$.

From the holomorphy, we see that $F_r(z_1, \dots, z_r; y)$ has a Taylor expansion in a neighborhood of the origin with respect to $(z_1, \dots, z_r)$, and we have (1.2). It is clear from the definition that $P(k_1, \dots, k_r; y)$ is a polynomial in y . Next we show that its degree is $k_1 + \dots + k_r + r$ with the coefficient (1.3). Putting

$$f_j(z_1, \dots, z_r) = \prod_{k=1}^j \frac{(\sum_{l=k}^j z_l) e^{\sum_{l=k}^j z_l}}{e^{\sum_{l=k}^j z_l} - 1} \prod_{k=j+1}^r \frac{\sum_{l=j+1}^k z_l}{e^{\sum_{l=j+1}^k z_l} - 1},$$

we rewrite (1.1) as

$$F_r(z_1, \dots, z_r; y) = \sum_{j=0}^r (-1)^j f_j(z_1, \dots, z_r) e^{y \sum_{l=j+1}^r z_l} \prod_{k=1}^j \frac{1}{\sum_{l=k}^j z_l} \prod_{k=j+1}^r \frac{1}{\sum_{l=j+1}^k z_l}. \quad (2.4)$$

Note that $f_j(z_1, \dots, z_r)$ is holomorphic around the origin and $f_j(0, \dots, 0) = 1$. The polynomial $P(k_1, \dots, k_r; y)$ can be read off from the terms of total degree $k_1 + \dots + k_r + r$ in $z_1, \dots, z_r$ in the Taylor expansion of the numerators of (2.4), which implies that the highest degree of $P(k_1, \dots, k_r; y)$ in y is at most $k_1 + \dots + k_r + r$. Moreover, the

coefficient of $y^{k_1+\dots+k_r+r}$ comes from that of $z_1^{k_1} \dots z_r^{k_r}$ in

$$\frac{1}{(k_1 + \dots + k_r + r)!} \sum_{j=0}^{r-1} (-1)^j (z_{j+1} + \dots + z_r)^{k+r-1} \prod_{k=1}^j \frac{1}{\sum_{l=k}^j z_l} \prod_{k=j+1}^{r-1} \frac{1}{\sum_{l=j+1}^k z_l}.$$

Using binomial expansions and picking up the coefficients successively in $z_r^{k_r}, \dots, z_1^{k_1}$ in this order, we see that the coefficient of $z_1^{k_1} \dots z_r^{k_r}$ in the above comes only from the term for $j = 0$ and it is

$$\begin{aligned} & \frac{1}{(k_1 + \dots + k_r + r)!} \binom{k_1 + \dots + k_r + r - 1}{k_r} \binom{k_1 + \dots + k_{r-1} + r - 2}{k_{r-1}} \\ & \quad \dots \binom{k_1 + k_2 + 1}{k_2} \\ & = \frac{1}{(k_1 + 1)(k_1 + k_2 + 2) \dots (k_1 + \dots + k_r + r)} \frac{1}{k_1! \dots k_r!}, \end{aligned}$$

which shows (1.3).

As for the rest of the statement, we use the following integral representation of the gamma function for $s \in \mathbb{C}$,

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} \int_H e^{-nz} z^{s-1} dz, \quad (2.5)$$

where the Hankel contour H starts at $+\infty$, goes to sufficiently small $\delta > 0$, goes around the origin with radius δ , and goes back to $+\infty$. Then by substituting (2.5) into (1.6), we obtain

$$\begin{aligned} \zeta_N(s_1, \dots, s_r) &= \sum_{0 < n_1 < \dots < n_r < N} \left(\prod_{j=1}^r \frac{1}{\Gamma(s_j)(e^{2\pi i s_j} - 1)} \right) \int_{H^r} \prod_{j=1}^r e^{-n_j z_j} \prod_{l=1}^r z_l^{s_l-1} dz_l \\ &= \left(\prod_{j=1}^r \frac{1}{\Gamma(s_j)(e^{2\pi i s_j} - 1)} \right) \int_{H^r} G_{r,N}(e^{-z_1}, \dots, e^{-z_r}) \prod_{l=1}^r z_l^{s_l-1} dz_l, \end{aligned} \quad (2.6)$$

where, for $x_1, \dots, x_r \in \mathbb{C}$, we have put

$$G_{r,N}(x_1, \dots, x_r) = \sum_{0 < n_1 < \dots < n_r < N} x_1^{n_1} \dots x_r^{n_r}.$$

We have a simple description of $G_{r,N}(x_1, \dots, x_r)$, as follows.

PROPOSITION 2.1. *For $N \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned} & G_{r,N}(x_1, \dots, x_r) \\ &= \sum_{j=0}^r (-1)^{r-j} (x_{j+1} \dots x_r)^N \prod_{k=1}^j \frac{x_k \dots x_j}{1 - x_k \dots x_j} \prod_{k=j+1}^r \frac{1}{1 - x_{j+1} \dots x_k} - (-1)^r \delta_{N,0}. \end{aligned}$$

Proof. Let

$$H(t) = \frac{x_1 \dots x_r t}{1 - x_1 \dots x_r t} \dots \frac{x_{r-1} x_r t}{1 - x_{r-1} x_r t} \frac{x_r t}{1 - x_r t} \frac{t}{1 - t}.$$

Then for $t, x_1, \dots, x_r \in \mathbb{C}$ such that $|t|, |x_j \cdots x_r t| < 1$ ($j = 1, \dots, r$), the Taylor expansion yields

$$\begin{aligned} H(t) &= \sum_{\substack{m_1, \dots, m_r \geq 1 \\ m \geq 1}} x_1^{m_1} x_2^{m_1+m_2} \cdots x_r^{m_1+\cdots+m_r} t^{m_1+\cdots+m_r+m} \\ &= \sum_{N=0}^{\infty} G_{r,N}(x_1, \dots, x_r) t^N. \end{aligned}$$

By the residue theorem, we have

$$\begin{aligned} G_{r,N}(x_1, \dots, x_r) &= \operatorname{Res}_{t=0} \frac{H(t)}{t^{N+1}} dt \\ &= - \sum_{w \neq 0} \operatorname{Res}_{t=w} \frac{H(t)}{t^{N+1}} dt, \end{aligned} \quad (2.7)$$

where the last equality follows from the fact that the sum of all the residues of a rational function on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is zero. First we calculate the residue at infinity, which is

$$\begin{aligned} \operatorname{Res}_{t=\infty} \frac{H(t)}{t^{N+1}} dt &= - \operatorname{Res}_{u=0} H(u^{-1}) u^{N+1} \frac{du}{u^2} \\ &= - \operatorname{Res}_{u=0} \left(\prod_{k=0}^r \frac{x_{k+1} \cdots x_r}{u - x_{k+1} \cdots x_r} \right) u^{N-1} du \\ &= (-1)^r \delta_{N,0}. \end{aligned} \quad (2.8)$$

The poles other than the origin and infinity are $t = (x_{j+1} \cdots x_r)^{-1}$ for $j = 0, \dots, r$ and

$$\begin{aligned} &\operatorname{Res}_{t=(x_{j+1} \cdots x_r)^{-1}} \frac{H(t)}{t^{N+1}} dt \\ &= \operatorname{Res}_{t=(x_{j+1} \cdots x_r)^{-1}} t^{-N-1} \left(\prod_{k=0}^r \frac{x_{k+1} \cdots x_r t}{1 - x_{k+1} \cdots x_r t} \right) dt \\ &= t^{-N-1} \left(\prod_{\substack{k=0 \\ k \neq j}}^r \frac{x_{k+1} \cdots x_r t}{1 - x_{k+1} \cdots x_r t} \right) \frac{x_{j+1} \cdots x_r t}{(1 - x_{j+1} \cdots x_r t)^j} \Big|_{t=(x_{j+1} \cdots x_r)^{-1}} \\ &= (x_{j+1} \cdots x_r)^{-1-N-1} \left(\prod_{\substack{k=0 \\ k \neq j}}^r \frac{x_{k+1} \cdots x_r (x_{j+1} \cdots x_r)^{-1}}{1 - x_{k+1} \cdots x_r (x_{j+1} \cdots x_r)^{-1}} \right) \\ &\quad \times \frac{x_{j+1} \cdots x_r (x_{j+1} \cdots x_r)^{-1}}{-x_{j+1} \cdots x_r} \\ &= -(x_{j+1} \cdots x_r)^N \prod_{k=0}^{j-1} \frac{x_{k+1} \cdots x_j}{1 - x_{k+1} \cdots x_j} \prod_{k=j+1}^r \frac{1}{x_{j+1} \cdots x_k - 1} \\ &= -(-1)^{r-j} (x_{j+1} \cdots x_r)^N \prod_{k=1}^j \frac{x_k \cdots x_j}{1 - x_k \cdots x_j} \prod_{k=j+1}^r \frac{1}{1 - x_{j+1} \cdots x_k}. \end{aligned} \quad (2.9)$$

Substituting (2.8) and (2.9) into (2.7), we have the assertion. $\square$

We now return to the proof of Theorem 1.1. By (1.1) and Proposition 2.1, we have

$$G_{r,N}(e^{z_1}, \dots, e^{z_r}) = F_r(z_1, \dots, z_r; N) - (-1)^r \delta_{N,0}.$$

Therefore by the residue theorem in (2.6) with (1.2), and the limit

$$\lim_{s \rightarrow -m} \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} = \frac{(-1)^m m!}{2\pi i} \quad (m \in \mathbb{Z}_{\geq 0}),$$

we see that for $N \in \mathbb{Z}_{\geq 0}$ and $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$ with $(k_1, \dots, k_r, N) \neq (0, \dots, 0, 0)$,

$$\zeta_N(-k_1, \dots, -k_r) = P(k_1, \dots, k_r; N).$$

This implies

$$S_N(k_1, \dots, k_r) = P(k_1, \dots, k_r; N) \quad (2.10)$$

by (1.7). Together with the fact that $P(k_1, \dots, k_r; y)$ is a polynomial in y , (2.10) further implies that, for $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$ with $(k_1, \dots, k_r) \neq (0, \dots, 0)$, $P(k_1, \dots, k_r; 0) = 0$ and hence $P(k_1, \dots, k_r; y)$ is divisible by y . On the other hand, $S_N(0, \dots, 0) = \binom{N-1}{r}$ for $N > r$ implies $P(0, \dots, 0; y) = \binom{y-1}{r}$, which completes the proof of Theorem 1.1.

3. Proof of Theorems 1.2 and 1.3

To show Theorem 1.2, we use the well-known integral representations of multiple zeta functions for $\operatorname{Re} a > 0$ and sufficiently large $\operatorname{Re} s_j$ ($j = 1, \dots, r$), given as

$$\zeta(s_1, \dots, s_r) = \left(\prod_{j=1}^r \frac{1}{\Gamma(s_j)} \right) \int_0^\infty \cdots \int_0^\infty G_r(e^{-z_1}, \dots, e^{-z_r}) \prod_{l=1}^r z_l^{s_l-1} dz_l, \quad (3.1)$$

$$\zeta^\star(s_1, \dots, s_r; a) = \left(\prod_{j=1}^r \frac{1}{\Gamma(s_j)} \right) \int_0^\infty \cdots \int_0^\infty e^{-a(z_1 + \cdots + z_r)} G_r^\star(e^{-z_1}, \dots, e^{-z_r}) \prod_{l=1}^r z_l^{s_l-1} dz_l, \quad (3.2)$$

where, for $x_1, \dots, x_r \in \mathbb{C}$ with $|x_j| < 1$ ($j = 1, \dots, r$),

$$\begin{aligned} G_r(x_1, \dots, x_r) &= \sum_{0 < n_1 < \cdots < n_r} x_1^{n_1} \cdots x_r^{n_r} = \frac{x_1 \cdots x_r}{1 - x_1 \cdots x_r} \cdots \frac{x_{r-1} x_r}{1 - x_{r-1} x_r} \frac{x_r}{1 - x_r}, \\ G_r^\star(x_1, \dots, x_r) &= \sum_{0 \leq n_1 \leq \cdots \leq n_r} x_1^{n_1} \cdots x_r^{n_r} = \frac{1}{1 - x_1 \cdots x_r} \cdots \frac{1}{1 - x_{r-1} x_r} \frac{1}{1 - x_r}. \end{aligned} \quad (3.3)$$

These integral representations are obtained by substituting that of the gamma function for $\operatorname{Re} s > 0$,

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-nz} z^{s-1} dz,$$

into (1.4) and (1.5).

For $N \in \mathbb{Z}_{\geq 1}$, we have

$$G_{r,N} = (x_1, \dots, x_r) = \sum_{j=0}^r (-1)^{r-j} (x_{j+1} \cdots x_r)^N G_j(x_1, \dots, x_j) G_{r-j}^\star(x_r, \dots, x_{j+1}),$$

by Proposition 2.1 and (3.3). Hence by the integral representations (3.1) and (3.2), for $N \in \mathbb{Z}_{\geq 1}$ and sufficiently large $\operatorname{Re} s_j$ ($j = 1, \dots, r$), we have

$$\begin{aligned}
 & Z(s_1, \dots, s_r; N) \\
 &= \sum_{j=0}^r (-1)^{r-j} \zeta(s_1, \dots, s_j) \zeta^*(s_r, \dots, s_{j+1}; N) \\
 &= \sum_{j=0}^r (-1)^{r-j} \left(\prod_{k=1}^j \frac{1}{\Gamma(s_k)} \right) \int_0^\infty \cdots \int_0^\infty G_j(e^{-z_1}, \dots, e^{-z_j}) \prod_{l=1}^j z_l^{s_l-1} dz_l \\
 &\quad \times \left(\prod_{k=j+1}^r \frac{1}{\Gamma(s_k)} \right) \int_0^\infty \cdots \int_0^\infty e^{-N \sum_{l=j+1}^r z_l} G_{r-j}^*(e^{-z_r}, \dots, e^{-z_{j+1}}) \prod_{l=j+1}^r z_l^{s_l-1} dz_l \\
 &= \left(\prod_{k=1}^r \frac{1}{\Gamma(s_k)} \right) \int_0^\infty \cdots \int_0^\infty \sum_{j=0}^r (-1)^{r-j} (e^{-z_{j+1}} \cdots e^{-z_r})^N G_j(e^{-z_1}, \dots, e^{-z_j}) \\
 &\quad \times G_{r-j}^*(e^{-z_r}, \dots, e^{-z_{j+1}}) \prod_{l=1}^r z_l^{s_l-1} dz_l \\
 &= \left(\prod_{j=1}^r \frac{1}{\Gamma(s_j)} \right) \int_0^\infty \cdots \int_0^\infty G_{r,N}(e^{-z_1}, \dots, e^{-z_r}) \prod_{l=1}^r z_l^{s_l-1} dz_l \\
 &= \zeta_N(s_1, \dots, s_r),
 \end{aligned} \tag{3.4}$$

which implies Theorem 1.2.

For $k_1, \dots, k_r \in \mathbb{Z}$ and sufficiently large prime p , we have

$$\begin{aligned}
 \zeta_p(k_1, \dots, k_r) &= \sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \\
 &\equiv \sum_{0 < n_1 < \cdots < n_r < p} n_1^{p-1-k_1} \cdots n_r^{p-1-k_r} \pmod{p} \\
 &= S_p(p-1-k_1, \dots, p-1-k_r) \\
 &= P(p-1-k_1, \dots, p-1-k_r; p).
 \end{aligned} \tag{3.5}$$

The last expression of (3.5) with (3.4) shows (1.8). Similarly we have (1.9) because (3.5) with each $p-1-k_i$ replaced by $k_i^{(p)}$ holds. In particular, for $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$ and sufficiently large prime p , we have

$$\begin{aligned}
 \zeta_p(-k_1, \dots, -k_r) &= P(k_1, \dots, k_r; p) \\
 &\equiv \begin{cases} 0 & (k_1, \dots, k_r) \neq (0, \dots, 0) \\ (-1)^r & (k_1, \dots, k_r) = (0, \dots, 0) \end{cases} \pmod{p}
 \end{aligned}$$

due to Theorem 1.1, which implies (1.10) and completes the proof of Theorem 1.3.

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